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We show that an inclusion placed inside a dilute Stokesian suspension of microswimmers induces power-law number-density modulations and flows. These take a different form depending on whether the inclusion is held fixed by an external force—for example, an optical tweezer—or if it is free. When the inclusion is held in place, the far-field fluid flow is a Stokeslet, while the microswimmer density decays as  $1/r^{2+\epsilon}$ , with  $r$  the distance from the inclusion and  $\epsilon$  an anomalous exponent which depends on the symmetry of the inclusion and varies continuously as a function of a dimensionless number characterizing the relative amplitudes of the convective and diffusive effects. The angular dependence takes a nontrivial form which depends on the same dimensionless number. When the inclusion is free to move, the far-field fluid flow is a stresslet, and the microswimmer density decays as  $1/r^2$  with a simple angular dependence. These long-range modulations mediate long-range interactions between inclusions that we characterize.

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Statistical Physics**I. INTRODUCTION**

Active matter encompasses systems whose individual elements convert energy into directed motion on a microscopic scale [1–9]. When the dissipative conversion of energy is coupled to interactions between particles, a wealth of phenomena which is not exhibited by systems in the thermal equilibrium is observed. Similarly, when this breaking of time-reversal symmetry is coupled to interactions with external potentials, the resulting behavior is very different than that of equilibrium systems. Importantly, in equilibrium, when interactions are local, the Boltzmann weight implies that the effect of a localized external potential extends beyond its own support only out to a scale of the order of the correlation length. In stark contrast,

in active systems with local conservation laws, steady-state distributions are inherently nonlocal [9–13], which leads to long-ranged influences of external potentials. A particularly spectacular experimental manifestation is the response of active systems to asymmetric potentials placed in the middle of a chamber [14]. One finds that active particles accumulate on one side of the system as a result of a ratchetlike mechanism [15].

Much theoretical progress has been made in understanding the response of active matter to external potentials in *dry* active systems. In dry systems, momentum is not conserved, so that experimental realizations correspond, for example, to particles moving on a substrate [16], vibrating granular grains [17,18], and more. Significant attention has been given to the particle density in confining potentials [15,19–21] and in the vicinity of localized repulsive potentials [22–24], showing the generic tendency for active particles to accumulate close to walls and repulsive boundaries. Arguably equally significant is the observation that generic localized potentials (or inclusions) induce a universal long-range modulation of the density field

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[25–27] which decays  $\propto \mathbf{p} \cdot \mathbf{r}/r^d$  in  $d$  dimensions, with  $\mathbf{p}$  a vector characterizing the properties of the inclusion and  $\mathbf{r}$  is the distance from it. The behavior is a consequence of the emergence of ratchet currents from the interplay between the breaking of time-reversal symmetry and any asymmetry of the inclusion. The result has far-reaching consequences [13]. It implies that two inclusions placed in an active bath experience long-range interactions [25,26,28] and explains the sensitivity of the phase diagram of dry active systems to bulk [29] and boundary [30] disorder. In particular, quenched disorder generically leads to long-range correlations [29] in any dilute active system. Moreover, motility-induced phase separation [31–34] is destroyed by bulk disorder in dimensions  $d < 4$  and by boundary disorder in dimensions  $d < 3$ .

Despite the relevance of dry active matter to experiments, many realizations of active systems, biological or synthetic, comprise particles that self-propel in a viscous fluid. In such systems, termed “wet,” the conservation of momentum is known to lead to very different behaviors [1,35–39]. The dynamics of active particles in wet systems, which in this context are often called microswimmers, in the vicinity of walls and obstacles have been the subject of intense scrutiny [40–43]. However, the response to a localized inclusion has, to the best of our knowledge, remained unexplored. In this work, we investigate the

long-range effect of a localized inclusion by considering a dilute suspension of swimmers propelling in a three-dimensional viscous fluid, as depicted in Fig. 1. The presence of the ambient fluid mediates interactions between the particles, which are long range due to momentum conservation [44]. Direct, nonhydrodynamic, interactions between the swimmers are neglected but are taken into account between the swimmers and the obstacle as a short-ranged force field. As we show, the coupling to fluid flow can qualitatively alter the nature of the long-range effect and in ways not revealed by mere power counting.

We identify three cases of interest, corresponding to three different large-scale behaviors of the density field of the swimmers, depending on whether the inclusion is freely moving in the fluid or if it is held fixed by an external force—for instance, by optical tweezers—and depending on the internal symmetries of the inclusion. Our results are largely independent of the intrinsic complexity of the near-obstacle swimming motion. When the obstacle is freely moving, driven by the interactions with the swimming particles, hydrodynamic interactions have little impact on the far-field behavior of the density field, and the behavior of the dry case survives with modulations of the density field decaying as  $1/r^2$ . However, we predict a very different response when the obstacle is held fixed by an external force. In this case, the decay

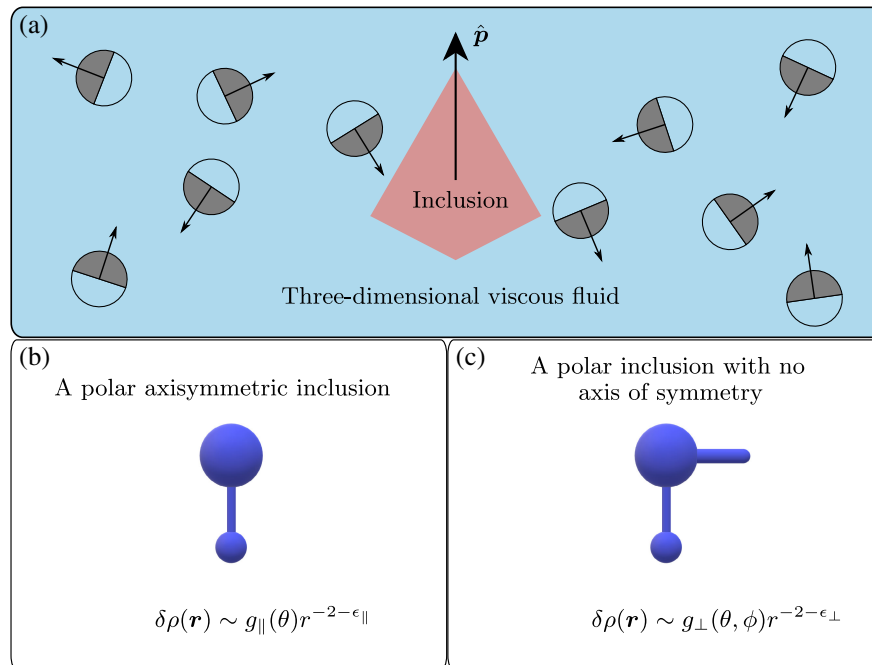


FIG. 1. (a) is a sketch of the system under consideration: self-propelled particles swimming in a three-dimensional Newtonian viscous fluid in the presence of a localized inclusion. The unit vector  $\hat{\mathbf{p}}$  is defined in Eq. (8) and points in the direction of the average force that must be exerted on the inclusion in order to maintain it fixed. (b) and (c) illustrate our key finding: A localized inclusion induces a long-range modulation of the density field, whose exponent depends on the symmetries of the inclusion. For fixed polar axisymmetric inclusions, we obtain  $\delta\rho(\mathbf{r}) \sim g_{\parallel}(\theta)r^{-2-\epsilon_{\parallel}}$ , while for those with no axis of symmetry, we get  $\delta\rho(\mathbf{r}) \sim g_{\perp}(\theta, \phi)r^{-2-\epsilon_{\perp}}$ , where  $g_{\parallel}(\theta)$ ,  $g_{\perp}(\theta, \phi)$ ,  $\epsilon_{\parallel}$ , and  $\epsilon_{\perp}$  are given in Eqs. (9)–(12).

exponent depends on the symmetries of the object and on a Péclet number, called  $\lambda$  in the following, that compares the relative amplitude of hydrodynamic to diffusive effects and whose mathematical expression is given in Eq. (7). We find that obstacles with a polarity that also defines an axis of (possibly discrete) rotational symmetry induce density modulations decaying as  $1/r^{2+\epsilon_{\parallel}}$  with  $\epsilon_{\parallel} > 0$ , while less symmetric obstacles induce density modulations decaying as  $1/r^{2+\epsilon_{\perp}}$  with  $\epsilon_{\perp} < 0$ . Lastly, obstacles with no polarity induce, as in the dry case, shorter-ranged density modulations. Notably, we expect density modulations induced by spherical obstacles to decay faster than a power law.

We begin in Sec. II by presenting a heuristic approach to the effect of hydrodynamic interactions on the behavior of the number-density field far away from a localized inclusion. The range of results we obtain are stated at the end of this section. This heuristic is supported by the use of a microscopic model of squirmers that we present in Sec. III and for which we derive, in a mean-field approximation, the equation obeyed by the steady-state density profile of the swimming particles. We solve this equation in the far field in Sec. IV, using an asymptotic expansion of the second kind [45,46]. We obtain the decay exponent and associated angular dependence of the density field perturbatively in the parameter  $\lambda$ . An alternative route to these results, based on the renormalization group, is presented in Appendix C. Finally, before concluding, we build in Sec. V on the previous sections to derive the far-field interaction between two inclusions in a bath of swimmers. Throughout, vectors are denoted in bold  $\mathbf{p}$  or in component notation  $p^{\alpha}$ , and  $\hat{\mathbf{p}}$  is the unit vector  $\hat{\mathbf{p}} = \mathbf{p}/|\mathbf{p}|$ .

## II. HEURISTIC ARGUMENTS

Before turning to a systematic derivation, we start by presenting the physical picture that underlies the results. It is useful to first consider the dry case. In this case, the localized asymmetric object, through a ratchet effect, acts as a pump on the active particles. Since the active particles diffuse on large scales, the steady-state density  $\rho(\mathbf{r})$  is controlled by the equation  $D\partial_{\alpha}\partial^{\alpha}\rho(\mathbf{r}) = -\partial_{\alpha}C^{\alpha}(\mathbf{r})$ . Here,  $D$  is a diffusion constant, the boundary conditions are  $\rho(\mathbf{r}) \rightarrow \rho_0$  as  $r \equiv |\mathbf{r}| \rightarrow \infty$ , and  $C^{\alpha}(\mathbf{r})$  is a current term localized in the vicinity of the obstacle which accounts for near-field effects. Taking  $\mathbf{r} = 0$  as the position of the obstacle, it is easy to check that the known far-field behavior, described in the introduction, is captured by this equation as long as  $c^{\alpha} = \int d\mathbf{r}C^{\alpha}(\mathbf{r})$  is finite. The addition of a three-dimensional viscous fluid, because of the long-range nature of hydrodynamic interactions, then modifies the diffusive behavior of the swimmers according to

$$D\partial_{\alpha}\partial^{\alpha}\rho(\mathbf{r}) - \partial_{\alpha}(\bar{v}^{\alpha}(\mathbf{r})\rho(\mathbf{r})) = -\partial_{\alpha}C^{\alpha}(\mathbf{r}), \quad (1)$$

where  $\bar{\mathbf{v}}(\mathbf{r})$  is an effective long-ranged convective flow generated by the combined effect of the swimmers and the

object. In Sec. III, we show that Eq. (1) can be derived from a mean-field microscopic model of swimmers. Note that if the obstacle is moving, we assume that it does so on a timescale that is slow enough that the density  $\rho(\mathbf{r})$  can be taken to be in a steady state.

While the microscopic derivation also makes the form of the velocity field  $\bar{\mathbf{v}}(\mathbf{r})$  explicit, it can be understood intuitively using momentum conservation. Denote by  $\mathbf{F}_{\text{swim} \rightarrow \text{fluid}}^i$  the force exerted on the fluid by the swimmer labeled by  $i$ . Since its inertia is negligible, and in the absence of nonhydrodynamic interactions between swimmers, momentum conservation implies that  $\mathbf{F}_{\text{swim} \rightarrow \text{fluid}}^i = -\mathbf{F}_{\text{swim} \rightarrow \text{obs}}^i$ , where  $\mathbf{F}_{\text{swim} \rightarrow \text{obs}}^i$  is the force exerted by swimmer  $i$  on the obstacle. By assumption, the latter is nonzero only for particles in the vicinity of the obstacle. Denote now by  $\mathbf{F}_{\text{fluid} \rightarrow \text{obs}}$  the force exerted by the fluid on the obstacle. The total force exerted by the combined effect of the swimmers and the obstacle on the fluid, denoted by  $\mathbf{f}$ , is therefore

$$\mathbf{f} = -\left(\mathbf{F}_{\text{fluid} \rightarrow \text{obs}} + \sum_i \mathbf{F}_{\text{swim} \rightarrow \text{obs}}^i\right). \quad (2)$$

In the far field, this induces a viscous flow, corresponding to a force monopole localized at  $\mathbf{r} = 0$  with amplitude  $\mathbf{f}$ . It follows that two distinct cases need to be distinguished, depending on whether the obstacle is held fixed externally or not.

If the obstacle is held fixed by an external force, momentum is injected locally into the system, and  $\mathbf{f} = \mathbf{F}_{\text{ext}}$  with  $\mathbf{F}_{\text{ext}}$  the force exerted by the external observer. Accordingly, the effective flow in Eq. (1) behaves as a Stokeslet on large scales, and we find

$$\bar{v}^{\alpha}(\mathbf{r}) \simeq \frac{1}{8\pi\eta} J^{\alpha\beta}(\mathbf{r}) \overline{\mathbf{F}_{\text{ext}}^{\beta}}, \quad (3)$$

where the overline denotes a steady-state average of  $\mathbf{F}_{\text{ext}}$  which on symmetry grounds is nonzero for a polar obstacle. Here,

$$J^{\alpha\beta}(\mathbf{r}) = \frac{\delta^{\alpha\beta}}{r} + \frac{r^{\alpha}r^{\beta}}{r^3} \quad (4)$$

is the fundamental solution of the Stokes equation in the presence of a force monopole. Note that the flow  $\bar{\mathbf{v}}(\mathbf{r})$  decreases as  $r^{-1}$  away from the obstacle. A second case of interest is that of a free obstacle. Here, the total momentum is conserved and  $\mathbf{f} = 0$  so that the leading order far-field effective flow is that of a force dipole:

$$\bar{v}^{\alpha}(\mathbf{r}) \simeq \frac{1}{8\pi\eta} \partial_{\gamma} J^{\alpha\beta}(\mathbf{r}) Q^{\gamma\beta}, \quad (5)$$

with  $Q^{\gamma\beta}$  the effective average dipole strength. In this case,  $\bar{\mathbf{v}}(\mathbf{r})$  decays as  $r^{-2}$ .

As we now argue, the difference in the decay of the velocity field between these two cases results in drastically

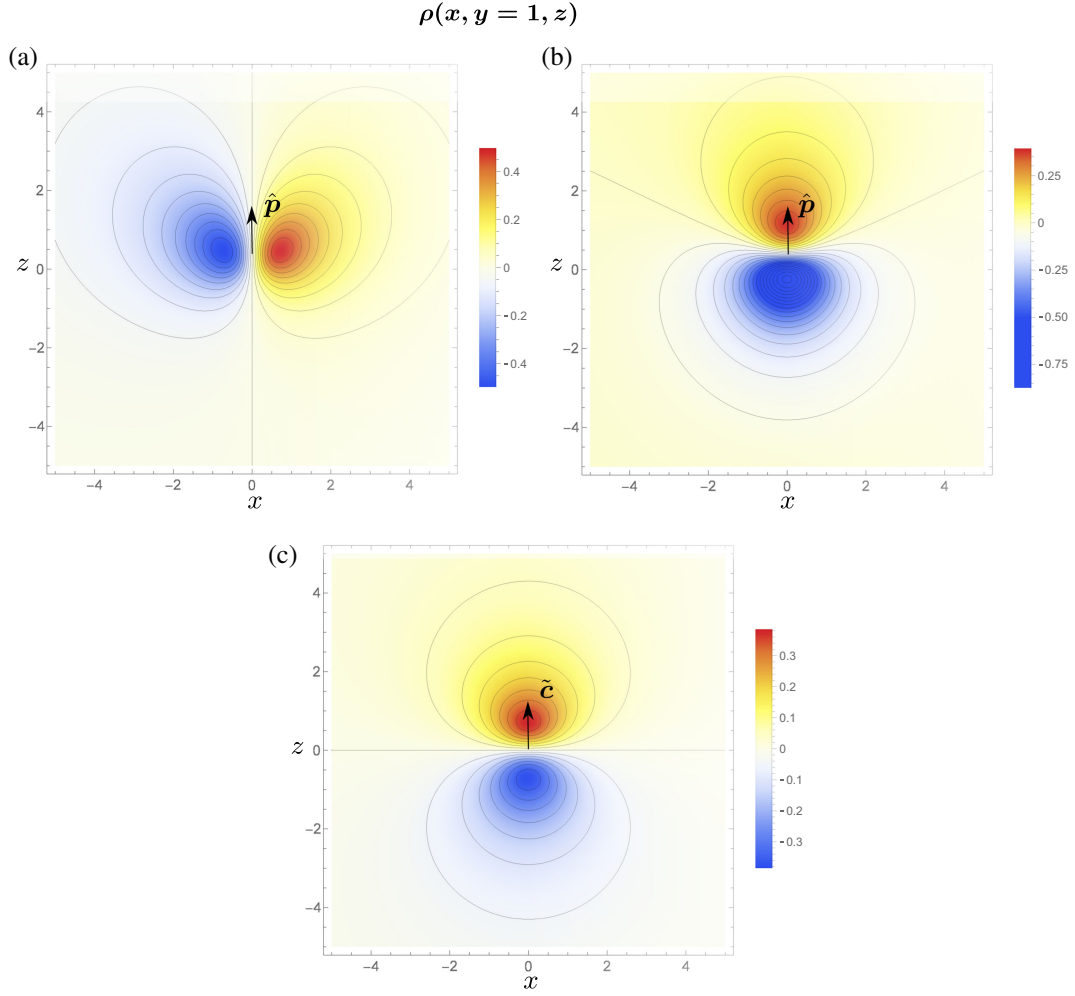


FIG. 2. Far-field density profile in a two-dimensional section  $\rho(x, y = 1, z)$ , up to a multiplicative constant, for the three different cases: fixed inclusion with no axis of symmetry, fixed polar inclusion with an axis of symmetry, and freely moving inclusion. (a) Fixed inclusion with no axis of symmetry. The vector  $\hat{\mathbf{p}}$  giving the direction of the force monopole is taken along the  $z$  axis. The  $x$  axis is defined such that the phase  $\phi_0$  vanishes in spherical coordinates of axis  $(x, y, z)$ . (b) Fixed polar inclusion with an axis of symmetry. The vector  $\hat{\mathbf{p}}$  giving the direction of the force monopole is taken along the  $z$  axis. In both (a) and (b), we use the second-order expansion in  $\lambda$  in Eqs. (10)–(12) and plot the results taking  $\lambda = 1$ . (c) Freely moving polar inclusion. The vector  $\tilde{\mathbf{c}}^\alpha$  entering Eq. (13) is taken along the  $z$  axis.

different behaviors for the density field which, in general, cannot be inferred using simple power counting. This can be understood through the following asymptotic arguments. Denote  $\delta\rho(\mathbf{r}) \equiv \rho(\mathbf{r}) - \rho_0$  such that  $\delta\rho(\mathbf{r}) \rightarrow 0$  as  $r \rightarrow \infty$ . In the far field, we replace the localized current  $C^\alpha(\mathbf{r})$  by  $c^\alpha\delta(\mathbf{r})$  and the velocity field by  $\tilde{v}^\alpha(\mathbf{r}) = Ar^{-\chi}g^\alpha(\hat{\mathbf{r}})$ , with  $g^\alpha(\hat{\mathbf{r}})$  controlling the angular dependence. Here,  $\chi$  is treated as a variable, and we keep in mind that  $\chi = 1$  corresponds to an externally held obstacle and  $\chi = 2$  to the freely moving one. The parameter  $A$  measures the strength of the hydrodynamic term and can be read from Eq. (3) for a fixed obstacle and Eq. (5) for a free obstacle. Since the flow field is incompressible, we have

$$D\Delta\delta\rho - Ar^{-\chi}\mathbf{g}(\hat{\mathbf{r}}) \cdot \nabla\delta\rho = -\mathbf{c} \cdot \nabla\delta(\mathbf{r}). \quad (6)$$

Now, note that if  $\chi > 1$ , the convection term decays faster at infinity than the diffusive one, rendering the former irrelevant on large length scales. However, both have the same amplitude when  $\chi = 1$ , indicating that the convection term is marginal in the renormalization group sense and could modify the far-field decay of the density [47]. With this in mind, we find the following behaviors for fixed and free obstacles embedded in three-dimensional active suspensions. The results are depicted in Fig. 2 in the three cases of interest that we identify.

### A. Fixed obstacle

We treat the hydrodynamic coupling using an intermediate asymptotic expansion of the second kind [46] in Sec. IV and a renormalization group analysis in Appendix C. We find that

the decay of the density field exhibits an anomalous exponent and an angular dependence which depend on the dimensionless parameter  $\lambda$  which quantifies the relative amplitude of the diffusive and convective terms:

$$\lambda = \frac{\overline{F_{\text{ext}}}}{8\pi\eta D} \quad (7)$$

and on the unit vector:

$$\hat{\boldsymbol{p}} = \frac{\overline{\boldsymbol{F}_{\text{ext}}}}{|\overline{\boldsymbol{F}_{\text{ext}}}|}, \quad (8)$$

which points along the force monopole. Note that local injection of angular momentum leads to flow fields decaying as  $r^{-2}$ , which is why, following the reasoning below Eq. (6), the large-scale behavior of the density field is insensitive to the total external torque exerted on the obstacle, if any. A striking feature is that the anomalous exponent and the angular dependence also depend on the symmetry of the obstacle. Our results are expressed as a perturbative expansion in powers of  $\lambda$ , which is relevant for dilute suspensions where  $\lambda$  is small [because the force in the numerator of Eq. (7) scales as the density of active particles at low density].

For obstacles for which the vector  $\hat{\boldsymbol{p}}$  defines an axis of (possibly discrete) rotational symmetry, we obtain

$$\delta\rho(\boldsymbol{r}) \sim \frac{g_{\parallel}(\theta)}{r^{2+\epsilon_{\parallel}}} \quad \text{with} \quad \epsilon_{\parallel} = \frac{\lambda^2}{3} + O(\lambda^4), \quad (9)$$

where  $\theta$  is the angle between  $\hat{\boldsymbol{r}}$  and  $\hat{\boldsymbol{p}}$ . The density field, therefore, decays *faster* than in the absence of hydrodynamic interactions. The angular dependence is given to order of  $O(\lambda^2)$  by

$$g_{\parallel}(\theta) = \cos\theta - \frac{\lambda}{4}(3 - 5\cos^2\theta) + \frac{3}{4}\lambda^2\cos^3\theta. \quad (10)$$

However, for obstacles with no axis of symmetry, the density field also depends on the azimuthal angle  $\phi$  of spherical coordinates of axis  $\hat{\boldsymbol{p}}$  and features a different exponent:

$$\delta\rho(\boldsymbol{r}) \sim \frac{g_{\perp}(\theta, \phi)}{r^{2+\epsilon_{\perp}}} \quad \text{with} \quad \epsilon_{\perp} = -\frac{\lambda^2}{12} + O(\lambda^4), \quad (11)$$

showing that the decay is *slower* than in the absence of hydrodynamic interactions. To order of  $O(\lambda^2)$ , the angular dependence is given by

$$g_{\perp}(\theta, \phi) = \cos(\phi + \phi_0) \sin(\theta) \times \left( 1 + \frac{5\lambda}{4}\cos\theta + \frac{3}{4}\lambda^2\cos^2\theta \right), \quad (12)$$

where, for a given choice of reference axis for the azimuthal angle, the phase  $\phi_0$  depends on the precise shape of the inclusion. Note that, even though  $\lambda$  is defined to be positive,  $\bar{v}^{\alpha}(\boldsymbol{r})/D \simeq \lambda J^{\alpha\beta}(\boldsymbol{r})\hat{\boldsymbol{p}}^{\beta}$  is formally left invariant

under the joint transformation  $\hat{\boldsymbol{p}} \rightarrow -\hat{\boldsymbol{p}}$  and  $\lambda \rightarrow -\lambda$ , therefore explaining why corrections to the  $-2$  exponent in Eqs. (9)–(11) appear only to second order in powers of  $\lambda$ .

## B. Free obstacles

As discussed after Eq. (6), the coupling to the fluid flow in Eq. (1) is irrelevant at large scales. The density field, thus, behaves as in a purely diffusive (dry) theory:

$$\delta\rho(\boldsymbol{r}) \simeq \frac{1}{4\pi D} \frac{r^{\alpha}}{r^3} \tilde{c}^{\alpha}, \quad (13)$$

where  $\tilde{c}^{\alpha}$  depends on the near-field details of the system and is generically nonzero for polar obstacles. The spatial decay exponent  $-2$  is universal, and the nonuniversal vector  $\tilde{c}^{\alpha}$  is contracted with a universal angular dependence. Note that for an obstacle with no polarity, even if fixed, we have by symmetry  $\overline{\boldsymbol{F}_{\text{ext}}} = 0$  and so  $\lambda = 0$ . In this case, hydrodynamic effects are, thus, irrelevant on large scales, similarly to the case of freely moving obstacles with arbitrary shape. Additionally,  $\tilde{c}$  also vanishes by symmetry. We, therefore, expect density modulations to be governed by the next-order term in the multipole expansion of the diffusion equation with a localized current at  $\boldsymbol{r} = 0$ , leading to  $\delta\rho(\boldsymbol{r}) \sim r^{-3}$  at large distances. Also note that Eq. (13) strictly holds only if the orientation of the obstacle is constrained during motion. If its orientation rotates at a slow rate—either from fluctuations or from a ratchet effect—we expect the result in Eq. (13) to be screened beyond a length scale given by the typical distance run by diffusion during the persistence time of the orientation.

In the next sections, we derive the above results in a systematic manner starting from a microscopic model of spherical squirmers in the presence of a localized obstacle.

## III. MICROSCOPIC MODEL

We consider a fluid which obeys the Stokes equation

$$\eta\Delta\boldsymbol{v}(\boldsymbol{r}) - \nabla P(\boldsymbol{r}) = 0 \quad \text{and} \quad \nabla \cdot \boldsymbol{v}(\boldsymbol{r}) = 0, \quad (14)$$

where  $\boldsymbol{v}(\boldsymbol{r})$  and  $P(\boldsymbol{r})$  are the flow and pressure fields, respectively, at position  $\boldsymbol{r}$ . The fluid contains spherical squirmers of radius  $a$ , labeled by  $i = 1 \dots N$ , with centers of mass at  $\boldsymbol{x}_i$ . Each squirmer imposes, in a frame of reference moving with it, a velocity field  $\boldsymbol{v}_{s,i}(\boldsymbol{r}, \boldsymbol{u}_i)$  on its surface. Here,  $\boldsymbol{u}_i$  is a unit vector characterizing the orientation of the squirmer, and we assume that  $\boldsymbol{v}_{s,i}(\boldsymbol{r}, \boldsymbol{u}_i)$  has a polar asymmetry determined by  $\boldsymbol{u}_i$ . We assume that the swimmers are dilute enough so that they interact only through hydrodynamics and that contact interactions between them can be neglected. The fluid also contains an obstacle that interacts with the swimmers both through hydrodynamics, by imposing a no-slip boundary condition on its surface, and directly through short-range external forces  $\boldsymbol{F}(\boldsymbol{x}_i - \boldsymbol{x}_0)$  and torques with respect to their center  $\boldsymbol{\Gamma}(\boldsymbol{x}_i - \boldsymbol{x}_0, \boldsymbol{u}_i)$ , with

$\mathbf{x}_0$  the center of mass of the obstacle. Denoting by  $\dot{\mathbf{x}}_0$  and  $\boldsymbol{\omega}$  ( $\dot{\mathbf{x}}_i$  and  $\boldsymbol{\omega}_i$ ) the translation and angular velocity, respectively, of the obstacle (swimmer  $i$ ), the above implies the boundary conditions on the surface of the obstacle,  $\partial\Omega$ :

$$\mathbf{v}(\mathbf{r})|_{\partial\Omega} = \dot{\mathbf{x}}_0 + \boldsymbol{\omega} \wedge (\mathbf{r} - \mathbf{x}_0), \quad (15)$$

and on the surface of each swimmer,  $\partial\Omega_i$ :

$$\mathbf{v}(\mathbf{r})|_{\partial\Omega_i} = \dot{\mathbf{x}}_i + \boldsymbol{\omega}_i \wedge (\mathbf{r} - \mathbf{x}_i) + \mathbf{v}_{s,i}(\mathbf{r}, \mathbf{u}_i). \quad (16)$$

The translation and angular velocities  $\dot{\mathbf{x}}_i$  and  $\boldsymbol{\omega}_i$  of the swimmers are such that the total force and torque exerted on each of them (by the fluid flow and the obstacle) vanish:

$$-\int_{\partial\Omega_i} dS n^\mu \sigma^{\mu\nu} + F^\nu(\mathbf{x}_i - \mathbf{x}_0) = 0 \quad (17)$$

and

$$-a\epsilon_{\alpha\mu\nu} \int_{\partial\Omega_i} dS n^\mu n^\gamma \sigma^{\gamma\nu} + \Gamma^\alpha(\mathbf{x}_i - \mathbf{x}_0, \mathbf{u}_i) = 0, \quad (18)$$

where  $\mathbf{n}$  is an outward-pointing normal vector to the surface of the swimmers and  $\sigma^{\mu\nu}(\mathbf{r}) = \eta(\partial_\mu v^\nu(\mathbf{r}) + \partial_\nu v^\mu(\mathbf{r})) - P(\mathbf{r})\delta^{\mu\nu}$  is the stress tensor. We consider both the cases

where the obstacle is held fixed externally, in which case  $\dot{\mathbf{x}} = 0$  and  $\boldsymbol{\omega} = 0$ , and the case where it is free to move. For the latter, the force-free condition reads

$$-\int_{\partial\Omega} dS n^\mu \sigma^{\mu\nu} - \sum_i F_i^\nu(\mathbf{x}_i - \mathbf{x}_0) = 0, \quad (19)$$

and we assume that the motion is adiabatic so that the obstacle is much slower than the relaxation time of the squirmers' dynamics. In the remainder of this section, we compute the average far-field fluid flow generated by the swimmers' suspension. We then use this average flow to build a mean-field model for the swimmers' dynamics, from which we recover Eq. (1).

### A. The average fluid flow

We start by computing the average fluid flow generated by the suspension. To do so, we use the boundary-integral representation of the Stokes equation (see Chap. 2 in Ref. [48]) and express  $\mathbf{v}(\mathbf{r})$  in terms of the velocity and stress tensor at the boundary of the domain which is composed of the surfaces of the obstacle and of the swimmers. We obtain

$$\begin{aligned} 8\pi\eta v^\alpha(\mathbf{r}) &= \int_{\partial\Omega} dS n^\rho \sigma^{\rho\beta}(\mathbf{r}') J^{\beta\alpha}(\mathbf{r} - \mathbf{r}') - \eta \int_{\partial\Omega} dS v^\beta(\mathbf{r}') n^\gamma T^{\beta\gamma\alpha}(\mathbf{r} - \mathbf{r}') \\ &+ \sum_i \left[ \int_{\partial\Omega_i} dS n^\rho \sigma^{\rho\beta}(\mathbf{r}') J^{\beta\alpha}(\mathbf{r} - \mathbf{r}') - \eta \int_{\partial\Omega_i} dS v^\beta(\mathbf{r}') n^\gamma T^{\beta\gamma\alpha}(\mathbf{r} - \mathbf{r}') \right], \end{aligned} \quad (20)$$

where

$$T^{\alpha\beta\gamma}(\mathbf{r}) = -6 \frac{r^\alpha r^\beta r^\gamma}{r^5} \quad (21)$$

generates the stress tensor corresponding to a Stokeslet solution and where  $\mathbf{r}'$  denotes the integration variable of the different surface integrals. While the velocity field  $\mathbf{v}(\mathbf{r})$  is prescribed at the different surfaces over which the integrals are performed, the stress tensor  $\sigma^{\mu\nu}$  is not and, in principle, needs to be solved for. Equation (20) is, thus, implicit. It is nonetheless a useful starting point for determining the far-field flow. To proceed, we use first the boundary conditions of the Stokes equation. From Eq. (15), we note using Gauss's theorem that

$$\begin{aligned} &\int_{\partial\Omega} dS v^\beta(\mathbf{r}') n^\gamma T^{\beta\gamma\alpha}(\mathbf{r} - \mathbf{r}') \\ &= \int_{\partial\Omega} dS n^\gamma T^{\beta\gamma\alpha}(\mathbf{r} - \mathbf{r}') \left[ \dot{x}_0^\beta + \epsilon_{\beta\nu\delta} \omega^\nu (r'^\delta - x_0^\delta) \right] \\ &= - \int_{\Omega} d\mathbf{r}' \partial_\gamma T^{\beta\gamma\alpha}(\mathbf{r} - \mathbf{r}') \left[ \dot{x}_0^\beta + \epsilon_{\beta\nu\delta} \omega^\nu (r'^\delta - x_0^\delta) \right] \\ &+ \int_{\Omega} d\mathbf{r}' \epsilon_{\beta\nu\delta} \omega^\nu \delta^{\delta\gamma} T^{\beta\gamma\alpha}(\mathbf{r} - \mathbf{r}') = 0, \end{aligned} \quad (22)$$

where we take advantage of the fact that  $T^{\beta\gamma\alpha}(\mathbf{r})$  is symmetric [see Eq. (21)] and that  $\partial_\gamma T^{\beta\gamma\alpha}(\mathbf{r} - \mathbf{r}') = \delta^{\alpha\beta} \delta(\mathbf{r} - \mathbf{r}')$ , which is a consequence of momentum conservation in the Stokes equation. Because the point  $\mathbf{r}$  lies outside  $\Omega$ , this leads to the result of Eq. (22). Similar considerations also imply that

$$\begin{aligned} &\int_{\partial\Omega_i} dS n^\gamma T^{\beta\gamma\alpha}(\mathbf{r} - \mathbf{r}') v^\beta(\mathbf{r}') \\ &= \int_{\partial\Omega_i} dS n^\gamma T^{\beta\gamma\alpha}(\mathbf{r} - \mathbf{r}') v_{s,i}^\beta(\mathbf{r}', \mathbf{u}_i), \end{aligned} \quad (23)$$

so that only the contribution from the surface velocity survives. Using these, we obtain

$$\begin{aligned} 8\pi\eta v^\alpha(\mathbf{r}) &= \int_{\partial\Omega} dS n^\rho \sigma^{\rho\beta}[\{\mathbf{x}_i, \mathbf{u}_i\}](\mathbf{r}') J^{\beta\alpha}(\mathbf{r} - \mathbf{r}') \\ &+ \sum_i \int_{\partial\Omega_i} dS n^\rho \sigma^{\rho\beta}[\{\mathbf{x}_i, \mathbf{u}_i\}](\mathbf{r}') J^{\beta\alpha}(\mathbf{r} - \mathbf{r}') \\ &- \sum_i \eta \int_{\partial\Omega_i} dS n^\mu T^{\mu\nu\alpha}(\mathbf{r} - \mathbf{r}') v_{s,i}^\nu(\mathbf{r}', \mathbf{u}_i), \end{aligned} \quad (24)$$

where the argument  $\{\mathbf{x}_i, \mathbf{u}_i\}$  emphasizes that the stress tensor  $\sigma^{\rho\beta}(\mathbf{r}')$  is a function of the positions and orientations of all the swimmers.

We now evaluate the average flow  $\bar{v}^\alpha(\mathbf{r})$ , where the overline, as before, denotes an average over the many-body distribution  $P[\{\mathbf{x}_i, \mathbf{u}_i\}]$  of the swimmers' positions and orientations. As noted previously, the motion of the obstacle is neglected. For simplicity, we thus consider  $\mathbf{x}_0 = 0$  in the following. For any point  $\mathbf{r}'$  on the surface of the obstacle, we denote accordingly  $\bar{\sigma}_{\text{obs}}^{\rho\beta}(\mathbf{r}')$  the average stress tensor at that point. Next, for any unit vector  $\mathbf{n}$ , we introduce the average stress tensor on a swimmer's surface, at a location  $\mathbf{an}$  with respect to its center:

$$\bar{\sigma}_{\text{swim}}^{\rho\beta}(\mathbf{x}', \mathbf{x}' + \mathbf{an}) \equiv \langle \sigma^{\rho\beta}[\{\mathbf{x}_i, \mathbf{u}_i\}](\mathbf{x}' + \mathbf{an}) \rangle_{\mathbf{x}'}, \quad (25)$$

where we denote by  $\langle \cdots \rangle_{\mathbf{x}'}$  a many-body average conditioned on the presence of a swimmer centered at  $\mathbf{x}'$ , so that  $\mathbf{x}' + \mathbf{an}$  lies on the surface of one of the swimmers. Lastly, using the same notations, we introduce

$$\bar{v}_{\text{surf}}^\nu(\mathbf{x}', \mathbf{x}' + \mathbf{an}) \equiv \langle v_{s,j}^\nu(\mathbf{x}' + \mathbf{an}, \mathbf{u}) \rangle_{\mathbf{x}'}, \quad (26)$$

the average surface velocity at  $\mathbf{x}' + \mathbf{an}$  on the surface of a swimmer centered at  $\mathbf{x}'$ . Using these definitions and denoting by  $\rho(\mathbf{x}) = \langle \sum_i \delta(\mathbf{x} - \mathbf{x}_i) \rangle$  the mean density of swimmers, the average flow can, thus, be written as

$$\begin{aligned} 8\pi\eta\bar{v}^\alpha(\mathbf{r}) = & \int_{\partial\Omega} dS n^\rho \bar{\sigma}_{\text{obs}}^{\rho\beta}(\mathbf{r}') J^{\beta\alpha}(\mathbf{r} - \mathbf{r}') + \int d\mathbf{x}' \rho(\mathbf{x}') \int d\mathbf{na}^2 n^\rho \bar{\sigma}_{\text{swim}}^{\rho\beta}(\mathbf{x}', \mathbf{x}' + \mathbf{an}) J^{\beta\alpha}(\mathbf{r} - \mathbf{x}' - \mathbf{an}) \\ & - \int d\mathbf{x}' \rho(\mathbf{x}') \int d\mathbf{na}^2 \eta n^\mu T^{\mu\nu\alpha}(\mathbf{r} - \mathbf{x}' - \mathbf{an}) \bar{v}_{\text{surf}}^\nu(\mathbf{x}', \mathbf{x}' + \mathbf{an}). \end{aligned} \quad (27)$$

Equation (27) can now be used for a multipole expansion. Since  $T^{\mu\nu\alpha}(\mathbf{r}) \sim r^{-2}$  while  $J^{\beta\alpha}(\mathbf{r}) \sim r^{-1}$ , we obtain to leading order in the far field

$$\begin{aligned} \bar{v}^\alpha(\mathbf{r}) \simeq & \frac{1}{8\pi\eta} J^{\beta\alpha}(\mathbf{r}) \left[ \int_{\partial\Omega} dS n^\rho \bar{\sigma}_{\text{obs}}^{\rho\beta}(\mathbf{r}') \right. \\ & \left. + \int d\mathbf{x}' \rho(\mathbf{x}') \int d\mathbf{na}^2 n^\rho \bar{\sigma}_{\text{swim}}^{\rho\beta}(\mathbf{x}', \mathbf{x}' + \mathbf{an}) \right]. \end{aligned} \quad (28)$$

By definition,  $\int d\mathbf{na}^2 n^\rho \bar{\sigma}_{\text{swim}}^{\rho\beta}(\mathbf{x}', \mathbf{x}' + \mathbf{an})$  is minus the average force exerted by the fluid on a swimmer at position  $\mathbf{x}'$  and is, therefore, equal, using the force-balance condition in Eq. (17), to the force exerted by the obstacle on that swimmer, that is,  $\int d\mathbf{na}^2 n^\rho \bar{\sigma}_{\text{swim}}^{\rho\beta}(\mathbf{x}', \mathbf{x}' + \mathbf{an}) = F^\beta(\mathbf{x}')$ . We, therefore, get

$$\bar{v}^\alpha(\mathbf{r}) \simeq \frac{1}{8\pi\eta} J^{\beta\alpha}(\mathbf{r}) \left[ -\overline{F_{\text{fluid} \rightarrow \text{obs}}^\beta} + \int d\mathbf{x}' \rho(\mathbf{x}') F^\beta(\mathbf{x}') \right], \quad (29)$$

where  $\overline{F_{\text{fluid} \rightarrow \text{obs}}^\beta} \equiv -\int_{\partial\Omega} dS n^\rho \bar{\sigma}_{\text{obs}}^{\rho\beta}(\mathbf{r}')$  is the average force exerted by the fluid on the obstacle. The term between brackets thus reads, up to a minus sign, as the average for exerted by the fluid on the obstacle plus the average force exerted by the swimmers on the obstacle and is, therefore, equal to  $\overline{F_{\text{ext}}}$ , the average external force exerted on the obstacle. This justifies Eq. (3). As expected from the heuristic argument in Sec. II, a fixed obstacle embedded in a suspension of swimmers generates a far-field fluid flow that behaves as a Stokeslet. In addition, if the obstacle is (adiabatically) moving under force-free conditions, meaning that the total momentum of the system is conserved, the effective force monopole  $\overline{F_{\text{ext}}}$  vanishes. A higher-order multipole expansion then shows that  $\bar{v}^\alpha(\mathbf{r})$  behaves as the velocity field generated by a force dipole which decays as  $r^{-2}$ ; see Eq. (5). The effective force dipole is given by

$$\begin{aligned} Q_{\text{eff}}^{\beta\gamma} = & \int_{\partial\Omega} dS n^\rho \bar{\sigma}_{\text{obs}}^{\rho\beta}(\mathbf{r}') r'^\gamma + \int d\mathbf{x}' \rho(\mathbf{x}') x'^\gamma F^\beta(\mathbf{x}') + \int d\mathbf{x}' \rho(\mathbf{x}') \int d\mathbf{na}^3 n^\rho n^\gamma \bar{\sigma}_{\text{swim}}^{\rho\beta}(\mathbf{x}', \mathbf{x}' + \mathbf{an}) \\ & + \int d\mathbf{x}' \rho(\mathbf{x}') \int d\mathbf{na}^2 \eta \left[ n^\gamma \bar{v}_{\text{surf}}^\beta(\mathbf{x}', \mathbf{x}' + \mathbf{an}) + n^\beta \bar{v}_{\text{surf}}^\gamma(\mathbf{x}', \mathbf{x}' + \mathbf{an}) \right]. \end{aligned} \quad (30)$$

## B. Mean-field approximation

With the expression for the mean flow at hand, we can now turn to derive the drift-diffusion equation (1). We use a mean-field approximation, where we consider the motion of a single swimmer in a steady inhomogeneous

background flow identified with the average flow  $\bar{\mathbf{v}}(\mathbf{x})$  derived above. For that swimmer, the equations of motion read

$$\dot{\mathbf{x}} = \mu \mathbf{F}(\mathbf{x}) + v_0 \mathbf{u} + \bar{\mathbf{v}}(\mathbf{x}) \quad (31)$$

together with

$$\dot{\mathbf{u}} = \left( \mu_r \mathbf{\Gamma}(\mathbf{x}, \mathbf{u}) + \frac{1}{2} \nabla \wedge \bar{\mathbf{v}}(\mathbf{x}) \right) \wedge \mathbf{u} + \text{noise}, \quad (32)$$

where the noise is taken for simplicity to be of the run-and-tumble type [49]. Here,  $\mu = 1/(6\pi\eta a)$  is the mobility of a sphere of radius  $a$ , and  $\mu_r = 1/(8\pi\eta a^3)$  is the corresponding rotational mobility. Also,  $v_0$  is the self-propulsion speed of an isolated swimmer, which is given by

$$v_0 = -\frac{1}{4\pi a^2} \int dS \mathbf{v}_s(\mathbf{r}, \mathbf{u}) \cdot \mathbf{u}. \quad (33)$$

Henceforth, to ease the notations, we use  $\bar{\omega}(\mathbf{x}) \equiv (1/2)\nabla \wedge \bar{\mathbf{v}}(\mathbf{x})$ . These equations have been derived in Ref. [50] in the absence of an external force  $\mathbf{F} = 0$  and torque  $\mathbf{\Gamma} = 0$  and in the absence of a background flow  $\bar{\mathbf{v}}(\mathbf{x}) = 0$ . The results of Ref. [50] generalize to Eqs. (31) and (32), as we show in Appendix A, for swimmers much smaller than the scale of variation of  $\bar{\mathbf{v}}(\mathbf{x})$ .

Our interest is in the steady-state density profile generated by the dynamics in Eqs. (31) and (32). Let  $\psi(\mathbf{x}, \mathbf{u})$  be the steady-state distribution. It is a solution of

$$\begin{aligned} 0 = & -\nabla_{\mathbf{x}} \cdot ([\mu \mathbf{F}(\mathbf{x}) + v_0 \mathbf{u} + \bar{\mathbf{v}}(\mathbf{x})] \psi(\mathbf{x}, \mathbf{u})) \\ & + \frac{1}{\tau} \sum_i \left( \int d\mathbf{u}' \psi(\mathbf{x}, \mathbf{u}') - \psi(\mathbf{x}, \mathbf{u}) \right) \\ & - \nabla_{\mathbf{u}} \cdot ([(\mu_r \mathbf{\Gamma}(\mathbf{x}, \mathbf{u}) + \bar{\omega}(\mathbf{x})) \wedge \mathbf{u}] \psi(\mathbf{x}, \mathbf{u})). \end{aligned}$$

We introduce the density  $\rho(\mathbf{x}) = \int d\mathbf{u} \psi(\mathbf{x}, \mathbf{u})$ , polarity  $m^\mu(\mathbf{x}) = \int d\mathbf{u} u^\mu \psi(\mathbf{x}, \mathbf{u})$ , and nematic tensor  $Q^{\alpha\beta}(\mathbf{x}) = \int d\mathbf{u} [u^\alpha u^\beta - (\delta^{\alpha\beta}/3)] \psi(\mathbf{x}, \mathbf{u})$ . Upon integrating Eq. (34) over  $\mathbf{u}$ , we get

$$-\partial_\alpha [\mu F^\alpha(\mathbf{x}) \rho(\mathbf{x}) + v_0 m^\alpha(\mathbf{x}) + \bar{v}^\alpha(\mathbf{x}) \rho(\mathbf{x})] = 0. \quad (34)$$

Multiplying Eq. (34) by  $u^\beta$  and integrating it again over  $\mathbf{u}$  yields

$$\begin{aligned} \frac{m^\beta}{\tau} = & -\frac{v_0}{3} \partial_\beta \rho - \partial_\alpha [\mu F^\alpha m^\beta + v_0 Q^{\alpha\beta} + \bar{v}^\alpha(\mathbf{x}) m^\beta(\mathbf{x})] \\ & + \epsilon_{\beta\mu\nu} \left[ \mu_r \int d\mathbf{u} u^\nu \Gamma^\mu(\mathbf{x}, \mathbf{u}) \psi(\mathbf{x}, \mathbf{u}) + \bar{\omega}^\mu(\mathbf{x}) m^\nu(\mathbf{x}) \right], \end{aligned} \quad (35)$$

which can be used in Eq. (34) to give

$$\begin{aligned} \frac{v_0^2 \tau}{3} \partial_\alpha \partial^\alpha \rho(\mathbf{x}) - \partial_\alpha [\bar{v}^\alpha(\mathbf{x}) \rho(\mathbf{x})] = & \partial_\alpha \left\{ \mu F^\alpha(\mathbf{x}) \rho(\mathbf{x}) + v_0 \tau \epsilon_{\alpha\mu\nu} \left[ \mu_r \int d\mathbf{u} u^\nu \Gamma^\mu(\mathbf{x}, \mathbf{u}) \psi(\mathbf{x}, \mathbf{u}) + \bar{\omega}^\mu(\mathbf{x}) m^\nu(\mathbf{x}) \right] \right\} \\ & - v_0 \tau \partial^\alpha \partial^\beta [\mu F^\alpha m^\beta + v_0 Q^{\alpha\beta} + \bar{v}^\alpha(\mathbf{x}) m^\beta(\mathbf{x})]. \end{aligned} \quad (36)$$

Therefore, we find that the equation satisfied by the density field can be written as a drift-diffusion equation with sources as in Eq. (1), where  $C^\alpha(\mathbf{x}) = C_1^\alpha(\mathbf{x}) + C_2^\alpha(\mathbf{x})$  with

$$\begin{aligned} C_1^\alpha(\mathbf{x}) = & -\mu F^\alpha(\mathbf{x}) \rho(\mathbf{x}) + v_0 \tau \mu \partial^\beta [F^\alpha(\mathbf{x}) m^\beta(\mathbf{x})] \\ & - v_0 \tau \mu_r \epsilon_{\alpha\mu\nu} \int d\mathbf{u} u^\nu \Gamma^\mu(\mathbf{x}, \mathbf{u}) \psi(\mathbf{x}, \mathbf{u}) \end{aligned} \quad (37)$$

and

$$\begin{aligned} C_2^\alpha(\mathbf{x}) = & -v_0 \tau \epsilon_{\alpha\mu\nu} \bar{\omega}^\mu(\mathbf{x}) m^\nu(\mathbf{x}) \\ & + v_0 \tau \partial^\beta [v_0 Q^{\alpha\beta} + \bar{v}^\alpha(\mathbf{x}) m^\beta(\mathbf{x})]. \end{aligned} \quad (38)$$

It is clear that the integral of  $C_1^\alpha(\mathbf{x})$  is finite, since the force and torque fields  $\mathbf{F}(\mathbf{x})$  and  $\mathbf{\Gamma}(\mathbf{u}, \mathbf{x})$  are short ranged. To bridge the gap with Eq. (1) and the discussion in Sec. II, we now argue that

$$c_2^\alpha = \int d\mathbf{x} C_2^\alpha(\mathbf{x}) \quad (39)$$

is also finite. Since we cannot solve the whole hierarchy of angular moments, we proceed by self-consistency assuming that  $c_2^\alpha$  exists. As we discuss in Sec. II and is shown in the following section, the density field decays faster

than  $x^{-1}$ . The polarity  $m^\alpha(\mathbf{x})$  then decays faster than  $x^{-2}$ , since it is proportional to density gradients; see Eq. (35). Accordingly, we expect that  $Q^{\alpha\beta}(\mathbf{x})$  decays faster than  $O(x^{-3})$ . In fact, successive moments of the orientation decay faster and faster, which can be shown in any truncation of the hierarchy of angular moments. Therefore, we expect that  $C_2^\alpha(\mathbf{x})$  decays faster than  $x^{-4}$  and is indeed integrable, thereby closing the self-consistency argument.

#### IV. FAR-FIELD DECAY OF THE DENSITY FIELD

In this section, we derive the far-field density decay when the obstacle is held fixed. To do so, we use a similarity solution, close to what is done, for example, for the Barenblatt equation; see Chap. 10 in Ref. [45] and Chap. 3 in Ref. [46]. Even though the Barenblatt equation features a time dependence that ours does not, in both cases, the large-scale behavior of the partial differential equation under study is mapped, by choosing a suitable ansatz, to an ordinary differential equation from which the anomalous exponent is obtained by solving a nonlinear eigenvalue problem. For completeness, the same results are derived using a renormalization group procedure in Appendix C. In the far field, we look for a solution of



$$D\Delta\delta\rho - \nabla \cdot [\bar{\mathbf{v}}(\mathbf{r})\delta\rho] = -\mathbf{c} \cdot \nabla\delta(\mathbf{r}), \quad (40)$$

where the convective flow, derived in Eq. (27), follows the scale-free form given in Eq. (3) at large distances. We work with spherical coordinates with polar angle  $\theta$  such that  $\cos\theta = \hat{\mathbf{p}} \cdot \hat{\mathbf{r}}$ , where  $\hat{\mathbf{p}}$ , defined in Eq. (8), points along the force monopole and with an azimuthal angle  $\phi$ . Dimensional analysis then shows that

$$\delta\rho(\mathbf{r}) = \frac{1}{r^2} \frac{|\mathbf{c}|}{D} \mathcal{F}\left(\frac{\ell}{r}, \theta, \phi\right), \quad (41)$$

where  $\ell$  is a microscopic length scale emerging from the near-field behavior of the velocity field. We first decompose  $\mathcal{F}$  into Fourier modes:

$$\mathcal{F}\left(\frac{\ell}{r}, \theta, \phi\right) = \sum_{m=-\infty}^{+\infty} e^{im\phi} f_m\left(\frac{\ell}{r}, \theta\right). \quad (42)$$

In the far field, with  $r$  much larger than any microscopic length scale, we write each Fourier mode as a product  $f_m(\ell/r, \theta) \propto g_m(\theta)r^{-\epsilon_m}$ , and we find using Eq. (6) that the angular functions satisfy

$$\frac{1}{\sin\theta} \partial_\theta(\sin\theta \partial_\theta g_m) + \lambda \sin\theta \partial_\theta g_m + g_m \left[ (2 + \epsilon_m)(1 + \epsilon_m) + 2\lambda(2 + \epsilon_m) \cos\theta - \frac{m^2}{\sin^2\theta} \right] = 0, \quad (43)$$

where  $\lambda$  is defined in Eq. (7). The exponent  $\epsilon_m$  is then fixed by requiring that Eq. (43) has a well-behaved solution at the boundaries of the interval  $\cos\theta = \pm 1$ . For a freely moving obstacle, meaning when  $\lambda = 0$ , or, equivalently, in the absence of hydrodynamic interactions, the set of possible exponents  $\epsilon_m$  are integers such that  $\epsilon_m \geq |m| - 1$ . Since the source term in Eq. (40) is a derivative of a delta function, the far-field decay of the density field is dominated by the modes  $m = 0$  and  $m = \pm 1$ , with exponents  $\epsilon_{0,\pm 1} = 0$ , meaning  $\delta\rho(\mathbf{r}) \sim r^{-2}$ . The solution  $\epsilon_0 = -1$  is indeed ignored, as it corresponds to a delta function source. This reproduces the well-known Eq. (13) for the solution of the Laplace equation in the presence of a localized current.

When  $\lambda > 0$  and small, the far-field decay of the density field is also dominated by the modes  $m = 0, \pm 1$ . Indeed, as is clear in the following, higher modes  $|m| \geq 2$  correspond to decay exponents close to  $|m| - 1 \geq 1$  when  $\lambda$  is small and, therefore, contribute only as subleading corrections in the far-field compared to the modes  $m = 0, \pm 1$ .

To characterize these modes, it is naively tempting to postulate  $\epsilon_{0,\pm 1} = 0$  and solve for  $g_m(\theta)$  using a perturbation theory in  $\lambda$ . However, solutions of this form inevitably diverge at one of the end points  $\cos\theta = \pm 1$ , to order of  $O(\lambda^2)$ , as we show in Appendix B. This signals the presence of an anomalous exponent  $\epsilon_m \neq 0$ .

We now evaluate the exponents  $\epsilon_m$  and the angular functions  $g_m(\theta)$  perturbatively in  $\lambda$  using  $\epsilon_m = \lambda\epsilon_m^{(1)} + \lambda^2\epsilon_m^{(2)} + O(\lambda^3)$  and  $g_m(\theta) = g_m^{(0)}(\theta) + \lambda g_m^{(1)}(\theta) + \lambda^2 g_m^{(2)}(\theta) + O(\lambda^3)$ . Requiring that  $g_m(\theta)$  remains finite to second order in  $\lambda$  at  $\cos\theta = \pm 1$  yields the anomalous exponents

$$\begin{aligned} \epsilon_0 &= \frac{1}{3}\lambda^2 + O(\lambda^4), \\ \epsilon_{\pm 1} &= -\frac{1}{12}\lambda^2 + O(\lambda^4) \end{aligned} \quad (44)$$

and the angular function to order of  $O(\lambda^2)$ :

$$\begin{aligned} g_0(\theta) &\propto \cos\theta - \frac{\lambda}{4}(3 - 5\cos^2\theta) + \frac{3\lambda^2 \cos^3\theta}{4}, \\ g_{\pm 1}(\theta) &\propto \sin\theta \left( 1 + \frac{5}{4}\lambda \cos\theta + \frac{3}{4}\lambda^2 \cos^2\theta \right). \end{aligned} \quad (45)$$

The above equations can then be used to obtain the results presented in Sec. II. For a generic polar obstacle, the far-field density is governed by the slowest  $m = \pm 1$  modes, and we identify  $\epsilon_\perp \equiv \epsilon_{\pm 1}$ . We, thus, recover Eqs. (11) and (12), where in Eq. (12) the dependence on the azimuthal angle from Eq. (42) is included. In contrast, if the obstacle possesses an axis of symmetry, whose direction  $\hat{\mathbf{p}}$  must be pointing along [51], the modes  $m = \pm 1$  must vanish, and the far-field decay is, thus, governed by the  $m = 0$  mode. This holds whether the rotational symmetry is continuous or discrete. Hence, we identify  $\epsilon_\parallel \equiv \epsilon_0$  and get Eqs. (9) and (10). It is, in principle, straightforward to extend this procedure to arbitrary order in  $\lambda$ .

## V. INTERACTIONS BETWEEN BODIES

Since an inclusion generates a long-range density modulation and a long-range fluid flow in the system, it affects the neighborhood of other inclusions. This leads to long-range interactions, mediated by the swimmers and the viscous fluid, that we explore in this section. Such long-range mediated interactions are well-known between particles, passive or active, embedded in a viscous fluid [44,52] and have been recently calculated for passive inclusions in “dry” active systems [25]. In the case we consider here, both the hydrodynamic field and the active particles mediate the interactions.

In this section, we derive the long-range mediated interactions that emerge between two inclusions immersed in a three-dimensional suspension of self-propelling particles, in two simple cases. First, we describe the dynamics (within an adiabatic approximation) of two inclusions that are pinned at one point but free to rotate around this point.

Second, we discuss the effective interactions between two freely moving inclusions. We assume that the inclusions are polar and, for simplicity, with an axis of symmetry. The extension to other cases is straightforward even if tedious.

### A. Two fixed polar obstacles

We consider two fixed inclusions, at position  $\mathbf{r}_1$  and  $\mathbf{r}_2$  and denoted in the following by 1 and 2. Asymptotically, when the distance  $|\mathbf{r}_1 - \mathbf{r}_2|$  goes to infinity, each inclusion has to be held in place by an average force, denoted  $\bar{\mathbf{F}}_1$  for inclusion 1 and  $\bar{\mathbf{F}}_2$  for inclusion 2, in order to maintain their position fixed. Note that, due to the axisymmetry of the obstacles, there is no need to exert an average torque in order to prevent them from rotating.

We now consider a case where these two obstacles are pinned at points  $\mathbf{r}_1$  and  $\mathbf{r}_2$  but each free to rotate around that pinning point. We assume that the pinning points lie on the axis of symmetry of the corresponding inclusion. When  $|\mathbf{r}_1 - \mathbf{r}_2|$  is large but finite, the presence of obstacle 1 induces a far-field fluid flow around obstacle 2, which influences its orientation. We treat the dynamics within the adiabatic approximation so that at each time the two inclusions behave as fixed force monopoles, and we use the conventions  $\hat{\mathbf{p}}_2(t) = \bar{\mathbf{F}}_2(t)/|\bar{\mathbf{F}}_2|$ ,  $\hat{\mathbf{p}}_1(t) = \bar{\mathbf{F}}_1(t)/|\bar{\mathbf{F}}_1|$ , and  $\hat{\mathbf{r}}_{21} = (\mathbf{r}_2 - \mathbf{r}_1)/|\mathbf{r}_2 - \mathbf{r}_1|$ . Neglecting fluctuations, the dynamics of the orientation of the first inclusion reads

$$\frac{d\hat{\mathbf{p}}_1}{dt} = \bar{\omega}_1(\hat{\mathbf{p}}_1, \hat{\mathbf{p}}_2, \mathbf{r}_1, \mathbf{r}_2) \wedge \hat{\mathbf{p}}_1, \quad (46)$$

where  $\bar{\omega}_1(\hat{\mathbf{p}}_1, \hat{\mathbf{p}}_2, \mathbf{r}_1, \mathbf{r}_2)$  is the average angular velocity of obstacle 1 at orientation  $\hat{\mathbf{p}}_1$  in the presence of (the far-away) obstacle 2 with fixed orientation  $\hat{\mathbf{p}}_2$ . The impact resulting from variations in swimmer density modulations (scaling as  $\sim |\mathbf{r}_2 - \mathbf{r}_1|^{-2+\epsilon}$ ) is minimal when compared to the fluid flow (scaling as  $\sim |\mathbf{r}_2 - \mathbf{r}_1|^{-1}$ ), at least perturbatively in  $\lambda$ . Therefore, to leading order in the distance  $|\mathbf{r}_1 - \mathbf{r}_2|$ , the angular velocity can be expressed as a linear response to the Stokeslet flow  $\mathbf{v}_2(\hat{\mathbf{p}}_2, \mathbf{r}_1, \mathbf{r}_2)$  generated by obstacle 2 at point  $\mathbf{r}_1$  in the absence of obstacle 1, as in Ref. [53]:

$$\bar{\omega}_1^{\mu\nu}(\hat{\mathbf{p}}_1, \hat{\mathbf{p}}_2, \mathbf{r}_1, \mathbf{r}_2) = M_1^{\mu\nu}(\hat{\mathbf{p}}_1) v_2^{\nu}(\hat{\mathbf{p}}_2, \mathbf{r}_1, \mathbf{r}_2). \quad (47)$$

Here,  $M_1^{\mu\nu}(\hat{\mathbf{p}}_1)$  is the linear-response tensor of the average angular velocity of obstacle 1 to a uniform background flow. Note that the pinning of obstacle 1 breaks Galilean invariance, therefore coupling the dynamics of  $\hat{\mathbf{p}}_1(t)$  to the fluid flow  $\mathbf{v}_2(\hat{\mathbf{p}}_2, \mathbf{r}_1, \mathbf{r}_2)$  itself and not only to its gradients (other instances in which Galilean invariance is explicitly broken in active suspensions, therefore leading to possible alignment with the local suspension velocity, include confined suspensions and suspensions on substrates [54–56]). By symmetry, the linear-response tensor must be antisymmetric in the indices  $(\mu, \nu)$  and invariant under rotations around  $\hat{\mathbf{p}}_1$ . This yields

$$M_1^{\mu\nu}(\hat{\mathbf{p}}_1) = -\gamma_1 \epsilon_{\mu\nu\alpha} \hat{p}_1^{\alpha}, \quad (48)$$

with  $\gamma_1$  an object-dependent coefficient that depends on the near-field properties of the active suspension in the vicinity of obstacle 1. Note that  $\gamma_1 > 0$  implies that, in a steady uniform background flow,  $\hat{\mathbf{p}}_1$  aligns with the flow, while it antialigns with it if  $\gamma_1 < 0$ . Therefore, one has

$$\frac{d\hat{\mathbf{p}}_1}{dt} = \frac{\gamma_1 |\bar{\mathbf{F}}_2|}{8\pi\eta|\mathbf{r}_1 - \mathbf{r}_2|} \hat{\mathbf{p}}_1 \wedge [(\hat{\mathbf{p}}_2 + (\hat{\mathbf{p}}_2 \cdot \hat{\mathbf{r}}_{12})\hat{\mathbf{r}}_{12}) \wedge \hat{\mathbf{p}}_1]. \quad (49)$$

Accordingly, the dynamics of  $\hat{\mathbf{p}}_2(t)$  follows from

$$\frac{d\hat{\mathbf{p}}_2}{dt} = \frac{\gamma_2 |\bar{\mathbf{F}}_1|}{8\pi\eta|\mathbf{r}_1 - \mathbf{r}_2|} \hat{\mathbf{p}}_2 \wedge [(\hat{\mathbf{p}}_1 + (\hat{\mathbf{p}}_1 \cdot \hat{\mathbf{r}}_{12})\hat{\mathbf{r}}_{12}) \wedge \hat{\mathbf{p}}_2]. \quad (50)$$

The lack of reciprocity in the interactions between the two inclusions visible in Eqs. (49) and (50) is a trademark of interactions mediated by active baths [25,26,57]. When  $\gamma_1 > 0$  and  $\gamma_2 > 0$ , the effective interactions drive alignment between the two directors in the direction separating the two inclusions, meaning  $\mathbf{p}_1 = \mathbf{p}_2 = \pm \hat{\mathbf{r}}_{12}$  in the steady state. Furthermore, when both  $\gamma_1 < 0$  and  $\gamma_2 < 0$ , the effective interactions lead to antialignment between the two directors in the direction separating the two inclusions, meaning  $\mathbf{p}_1 = -\mathbf{p}_2 = \pm \hat{\mathbf{r}}_{12}$ . None of these equilibrium points is stable when  $\gamma_1 \gamma_2 < 0$ . In fact, numerical solutions of the joint dynamics Eqs. (49) and (50) show that interactions between two such freely rotating bodies generically lead to complex trajectories of  $\hat{\mathbf{p}}_1$  and  $\hat{\mathbf{p}}_2$ ; see Fig. 3. The dynamics are rich depending on the initial conditions and their study, including the influence of noise

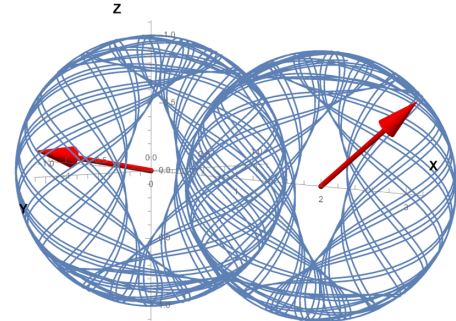


FIG. 3. Examples of complex trajectories induced by the interactions between an aligning ( $\gamma_1 > 0$ ) and an antialigning ( $\gamma_2 < 0$ ) freely rotating polar object embedded in a suspension of microswimmers. The center of the first obstacle is located at the origin, and that of obstacle 2 is on the  $x$  axis. The instantaneous position of the two directors  $\hat{\mathbf{p}}_1$  and  $\hat{\mathbf{p}}_2$  at some time  $t > 0$  is depicted by two red arrows, while the solid blue lines represent the past trajectories starting from a random initial condition at  $t = 0$ . For such generic initial conditions, the trajectory of each director seem to densely cover a portion of the sphere at large times. Here,  $\gamma_1 = -\gamma_2$ .

on the dynamics Eqs. (49) and (50) or in the presence of more than two bodies, is left for future work.

Broadly speaking, the above phenomenology was already identified in the dynamics of pinned inclusions in suspensions of dry active particles [25,26], albeit with slightly different dynamics. We stress, however, that momentum conservation leads to much longer-ranged effective interactions. In fact, the effective interactions in Eqs. (49) and (50) decay as  $O(|\mathbf{r}_1 - \mathbf{r}_2|^{-1})$ , whereas they were shown to decay as  $O(|\mathbf{r}_1 - \mathbf{r}_2|^{-3})$  in three-dimensional dry systems [25]. This difference could have striking consequences on the behavior of ensembles of pinned embedded inclusions.

### B. Freely moving bodies

Next, consider the case of two freely moving obstacles. Let  $\mathbf{u}_1$  ( $\mathbf{u}_2$ ) denote the average velocity of obstacle 1 (obstacle 2) when in isolation. Then, the far-field density decay around obstacle 2 follows from Eq. (13) and reads

$$\rho(\mathbf{r}) = \rho_0 + \delta\rho_2(\mathbf{r}) \quad \text{with} \quad \delta\rho_2(\mathbf{r}) \simeq \frac{1}{4\pi D} \frac{(\mathbf{r} - \mathbf{r}_2) \cdot \tilde{\mathbf{c}}_2}{|\mathbf{r} - \mathbf{r}_2|^3}. \quad (51)$$

The same result holds around obstacle 1 upon replacing  $\tilde{\mathbf{c}}_2$  by  $\tilde{\mathbf{c}}_1$  and  $\mathbf{r}_2$  by  $\mathbf{r}_1$ . For what follows, we introduce  $v_1(\rho_0)$  and  $v_2(\rho_0)$  the average speed of obstacles 1 and 2, respectively, which are scalar functions of the bulk density  $\rho_0$ .

There are two sources for the interaction between the inclusions. First, there is a contribution from the fluid flow created by one inclusion in the vicinity of the other. The other one comes from the change in swimmers' density in the vicinity of one inclusion due to the presence of the other. Both contributions scale in the same manner with the distance between the inclusions.

We denote the changes in the average velocity of each obstacle by  $\mathbf{u}_1 + \delta\mathbf{u}_1$  and  $\mathbf{u}_2 + \delta\mathbf{u}_2$  for obstacles 1 and 2, respectively. To leading order in the far field,  $\delta\mathbf{u}_1$  is given by the sum of the two contributions discussed above. First, due to the presence of object 2, the apparent bulk density of swimmers around obstacle 1 is perturbed, going from  $\rho_0$  to  $\rho_0 + \delta\rho_2(\mathbf{r}_1)$ . This scalar perturbation modifies the speed of obstacle 1 but not the propulsion direction. The second contribution emerges from the coupling to the fluid flow generated by object 2, which behaves as the one generated by a force dipole  $Q_2^{\alpha\beta}$  at position  $\mathbf{r}_2$ . These two contributions scale as  $|\mathbf{r}_1 - \mathbf{r}_2|^{-2}$  and yield

$$\delta u_1^\alpha = \frac{1}{8\pi\eta} \partial_\gamma J^{\alpha\beta}(\mathbf{r}_1 - \mathbf{r}_2) Q_2^{\gamma\beta} + \hat{u}_1^\alpha v_1'(\rho_0) \delta\rho_2(\mathbf{r}_1). \quad (52)$$

Because obstacle 2 is polar with an axis of symmetry, we have  $\tilde{\mathbf{c}}_2 = \chi_2 \hat{\mathbf{u}}_2$  with  $\chi_2$  a parameter which depends on

near-field properties of the suspension close to obstacle 2. Furthermore, we have

$$Q_2^{\gamma\beta} = \kappa_2 \left( \hat{u}_2^\gamma \hat{u}_2^\beta - \frac{\delta^{\gamma\beta}}{3} \right), \quad (53)$$

with  $\kappa_2$  also depending on the near-field properties of the suspension close to obstacle 2. Hence, to leading order, the effective interactions between the two bodies take the form

$$\delta u_1^\alpha = -\frac{\kappa_2 r_{21}^\alpha}{8\pi\eta |\mathbf{r}_2 - \mathbf{r}_1|^2} \left( 1 - 3(\hat{\mathbf{r}}_{12} \cdot \hat{\mathbf{u}}_2)^2 \right) - \chi_2 \hat{u}_1^\alpha \frac{v_1'(\rho_0)}{4\pi D} \frac{\hat{\mathbf{r}}_{21} \cdot \hat{\mathbf{u}}_2}{|\mathbf{r}_1 - \mathbf{r}_2|^2} \quad (54)$$

and correspondingly for the shift  $\delta\mathbf{u}_2$  in the velocity of object 2. The first term is a swimmer-swimmer interaction, showing that passive bodies embedded in an active suspension partly behave as swimming particles themselves. The second term, however, does not correspond to a swimmer-swimmer interaction but is akin to the far-field interactions emerging between two passive bodies embedded in a medium of dry self-propelled particles [25].

## VI. CONCLUSION

In this paper, we studied the long-range effect of a localized obstacle on a three-dimensional suspension of active swimmers. First, we showed that hydrodynamic interactions can lead to striking deviations from earlier results obtained in the dry case when the obstacle is held fixed by an external force so that there is a net average flux of momentum injected into the system. In that case, the far-field density modulations of the swimmers decay with an exponent that depends continuously on the relative amplitude of hydrodynamic and diffusive contributions. The exponent also depends on the internal symmetry of the obstacle: A polar obstacle with an axis of symmetry induces density modulations that decay faster than in the absence of hydrodynamic interactions, while an obstacle with no axis of symmetry induces modulations that decay slower than in the dry case. In both cases, we have a perturbative prediction for the exponent in terms of the independently measurable quantities  $|\bar{\mathbf{F}}_{\text{ext}}|$ ,  $\eta$ , and  $D$ . In particular,  $|\bar{\mathbf{F}}_{\text{ext}}|$  can be read off from the leading far-field decay of the hydrodynamic velocity. The case of a freely moving inclusion is closer to earlier studies on the dry problem. There, hydrodynamic interactions are irrelevant far away from the obstacle, and the  $-2$  exponent is recovered [25]. As argued in Sec. II, these predictions emerge from a competition between diffusive effects and convective transport due to the local injection of momentum in the vicinity of the obstacle. We believe this scenario is generic enough for our results to robustly extend beyond the presently studied case of spherical squirmers and be appraised in experiments on synthetic or biological microswimmers. We stress that our predictions rely on the

three-dimensional nature of the surrounding fluid flow. In fact, in the vicinity of a container's wall, acting as a momentum sink, the flow field around a localized momentum source decays faster (as  $\sim 1/r^2$ ) when compared to three-dimensional bulk fluids. In such a case, following the dimensional analysis of Eq. (6), we therefore expect hydrodynamic interactions to be irrelevant far away from a localized obstacle, even if it is held fixed. Note, however, that we expect effects similar to the ones described here if the motion of the microswimmers is limited near the interface between two immiscible viscous fluids or inside a two-dimensional fluid layer in a three-dimensional viscous fluid.

In addition, we have also described the effective long-range interactions, mediated by the active suspension, between two far-away localized objects. If freely moving, the effective interactions between the two objects lead to a modification of their average propulsion velocity. This modification decays as the distance between the two objects squared and can be expressed as the sum of two contributions. The first one is akin to the hydrodynamic interactions existing between two force dipoles. The second contribution has the same form as the effective interactions mediated by a bath of dry self-propelled particles [25]. When their center of mass is held fixed, effective torques emerge that decay as the inverse of the distance between the two obstacles. Depending on the details, these can lead to alignment, antialignment, or complex trajectories.

We believe this study opens the way for a quantitative description of many phenomena, including the effect of disorder on suspensions of microswimmers [13,29,30,58] and the interactions of inclusions with confining walls [59].

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## APPENDIX A: DYNAMICS OF AN ISOLATED SWIMMER

The dynamics of an isolated squirmer, a spherical particle that self-propels in a viscous fluid by imposing

a nonzero surface flow in its frame of reference, has been derived in Ref. [50]. In this appendix, we extend their derivation to the case where an external force and torque are imposed on the squirmer. Because of the linearity of the Stokes equation, the resulting velocity is the sum of the self-propulsion of the isolated squirmer and of the translation velocity of a passive sphere of the same size driven by the external force.

The squirmer motion is a combination of translation with velocity  $\dot{\mathbf{x}}$  and solid rotation with angular velocity  $\boldsymbol{\omega}$ . The equation governing the fluid flow reads

$$\eta\Delta\mathbf{v} - \nabla P = 0 \quad (\text{A1})$$

together with

$$\nabla \cdot \mathbf{v} = 0 \quad (\text{A2})$$

and the boundary conditions

$$\mathbf{v}|_{\partial\Omega}(\mathbf{r}) = \dot{\mathbf{x}} + \mathbf{v}_s(\mathbf{r}, \mathbf{u}) + a\boldsymbol{\omega} \wedge \mathbf{n} \quad \text{and} \quad \mathbf{v}|_{\infty} = 0, \quad (\text{A3})$$

with  $\mathbf{v}_s(\mathbf{r}, \mathbf{u})$  the local surface velocity imposed by the swimmer in its frame of reference and  $\mathbf{n}$  is the local outward-pointing normal to the squirmer's surface  $\partial\Omega$ . We recall that  $\mathbf{v}_s(\mathbf{r}, \mathbf{u})$  has a polarity, that is, a vectorial asymmetry, determined by  $\mathbf{u}$ . The translation velocity  $\dot{\mathbf{x}}$  is fixed by the force-balance condition

$$\int_{\partial\Omega} dS n^\beta \sigma^{\alpha\beta}(\mathbf{r}') = F^\alpha, \quad (\text{A4})$$

and the angular velocity  $\boldsymbol{\omega}$  is fixed by the torque-balance condition

$$a\epsilon_{\rho\alpha\beta} \int_{\partial\Omega} dS n^\alpha n^\mu \sigma^{\mu\beta} = \Gamma^\rho. \quad (\text{A5})$$

In order to obtain  $\dot{\mathbf{x}}$  and  $\boldsymbol{\omega}$ , we apply the Lorentz reciprocal theorem. Let  $\hat{\mathbf{v}}$  and  $\hat{\boldsymbol{\sigma}}$  be the velocity flow and the stress tensor, respectively, of another solution of the Stokes equation which is regular over the domain  $\mathbb{R}^3/\Omega$ . The Lorentz reciprocal theorem then states that

$$\int_{\partial\Omega} \mathbf{n} \cdot \hat{\boldsymbol{\sigma}} \cdot \mathbf{v} = \int_{\partial\Omega} \mathbf{n} \cdot \boldsymbol{\sigma} \cdot \hat{\mathbf{v}}. \quad (\text{A6})$$

First, in order to get the squirmer's translation velocity, we choose  $\hat{\mathbf{v}}, \hat{\boldsymbol{\sigma}}$  to be the flow generated by a translation at velocity  $\mathbf{U}$  of the sphere  $\Omega$  by an external force  $\hat{\mathbf{F}}$ . The no-slip boundary condition then reads  $\hat{\mathbf{v}}|_{\partial\Omega} = \hat{\mathbf{U}}$ . We, therefore, obtain

$$\hat{\mathbf{F}} \cdot \dot{\mathbf{x}} + \int_{\partial\Omega} \mathbf{n} \cdot \hat{\boldsymbol{\sigma}} \cdot (\mathbf{v}_s + a\boldsymbol{\omega} \wedge \mathbf{n}) = \mathbf{F} \cdot \hat{\mathbf{U}}. \quad (\text{A7})$$

For a sphere of radius  $a$ , it leads to

$$\dot{\mathbf{x}} = \frac{1}{6\pi\eta a} \mathbf{F} - \frac{1}{4\pi a^2} \int_{\partial\Omega} dS \mathbf{v}_s(\mathbf{r}, \mathbf{u}), \quad (\text{A8})$$

independently of the angular velocity  $\boldsymbol{\omega}$ , since  $\mathbf{n} \cdot \hat{\boldsymbol{\sigma}}$  is constant along the surface of the sphere. We then recover Eq. (31), in the absence of a background flow, with the self-propulsion speed

$$v_0 = -\frac{1}{4\pi a^2} \int_{\partial\Omega} dS \mathbf{v}_s(\mathbf{r}, \mathbf{u}) \cdot \mathbf{u} \quad (\text{A9})$$

and the mobility  $\mu = 1/(6\pi\eta a)$ . In order to obtain  $\boldsymbol{\omega}$ , we apply the Lorentz reciprocal theorem by considering  $\hat{\mathbf{v}}, \hat{\boldsymbol{\sigma}}$  to be the flow generated by a solid rotation at angular velocity  $\hat{\boldsymbol{\omega}}$  of  $\Omega$ . On  $\partial\Omega$ , we have  $\hat{\mathbf{v}} = a\hat{\boldsymbol{\omega}} \wedge \mathbf{n}$  and  $\mathbf{n} \cdot \hat{\boldsymbol{\sigma}} = 3\eta\hat{\boldsymbol{\omega}} \wedge \mathbf{n}$ ; see Ref. [50]. We, therefore, obtain

$$\begin{aligned} 3\eta\epsilon_{\alpha\beta\gamma} \int_{\partial\Omega} dS n^\gamma (\dot{x}^\alpha + v_s^\alpha + a\epsilon_{\alpha\mu\nu}\omega^\mu n^\nu) \\ = a\epsilon_{\alpha\beta\gamma} \int_{\partial\Omega} dS n^\rho \sigma^{\rho\alpha} n^\gamma, \end{aligned} \quad (\text{A10})$$

yielding

$$\boldsymbol{\omega} = \frac{1}{8\pi\eta a^3} \mathbf{F} - \frac{3}{8\pi a^3} \int_{\partial\Omega} dS \mathbf{n} \wedge \mathbf{v}_s(\mathbf{r}, \mathbf{u}). \quad (\text{A11})$$

The equation of motion for the director  $\mathbf{u}$  then reads

$$\dot{\mathbf{u}} = \boldsymbol{\omega} \wedge \mathbf{u} = \frac{1}{8\pi\eta a^3} \mathbf{F} \wedge \mathbf{u}, \quad (\text{A12})$$

since the second term of Eq. (A11) points along  $\mathbf{u}$  by symmetry. We, therefore, recover the noiseless version of Eq. (32), without the background flow, with the angular mobility  $\mu_r = 1/(8\pi\eta a^3)$ . In the presence of a background flow  $\bar{\mathbf{v}}$ , the equations of motion can be found by considering the same Stokes equation imposing that at large distances the flow is equal to the background one,  $\mathbf{v}_\infty(\mathbf{r}) = \bar{\mathbf{v}}(\mathbf{r})$ . One can then obtain a formulation similar to Eqs. (A1)–(A3), with a vanishing fluid flow at infinity, by considering  $\hat{\mathbf{v}}(\mathbf{r}) = \mathbf{v}(\mathbf{r}) - \mathbf{v}_\infty(\mathbf{r})$ . At the surface  $\partial\Omega$ , the corresponding boundary condition reads

$$\hat{\mathbf{v}}|_{\partial\Omega}(\mathbf{r}) = \dot{\mathbf{x}} + v_s(\mathbf{r}, \mathbf{u}) + a\boldsymbol{\omega} \wedge \mathbf{n} - \bar{\mathbf{v}}(\mathbf{r}). \quad (\text{A13})$$

By denoting  $\mathbf{x}$  the position of the swimmer, one can then expand  $\bar{\mathbf{v}}(\mathbf{r})$  around  $\bar{\mathbf{v}}(\mathbf{x})$  to first order in the radius  $a$ . Equations (31) and (32) in the main text then follow from the application of the Lorentz reciprocal theorem as above.

## APPENDIX B: SINGULARITY OF THE ANGULAR DEPENDENCE WHEN $\epsilon_m = 0$

In this appendix, we consider the mode  $m = 0$  as an example. By incorrectly assuming that  $\epsilon_0 = 0$ , one obtains an equation for the angular dependence:

$$\frac{1}{\sin\theta} \partial_\theta(\sin\theta \partial_\theta g_0) + \lambda \sin\theta \partial_\theta g_0 + g_0[2 + 4\lambda \cos\theta] = 0. \quad (\text{B1})$$

We now look for a perturbative solution in powers of the coupling constant  $\lambda$  as  $g_0(\theta) = g_0^{(0)}(\theta) + \lambda g_0^{(1)}(\theta) + \lambda^2 g_0^{(2)}(\theta) + \dots$ . To leading order, we get

$$g_0^{(0)}(\theta) = c_1^{(0)} \cos\theta + c_2^{(0)} \left[ \frac{\cos\theta}{2} \log\left(\frac{1+\cos\theta}{1-\cos\theta}\right) - 1 \right], \quad (\text{B2})$$

with  $c_1^{(0)}$  and  $c_2^{(0)}$  two integration constants. We set  $c_2^{(0)} = 0$  to prevent divergence at  $\cos\theta = \pm 1$  and choose  $c_1^{(0)} = -1/4\pi$  to match known results for the Green function of the diffusion operator. Accordingly, to first order, we obtain

$$\begin{aligned} g_0^{(1)}(\theta) = \frac{-10\cos^2\theta + 3\cos\theta \log\left(\frac{1+\cos\theta}{1-\cos\theta}\right)}{32\pi} + c_1^{(1)} \cos\theta \\ + c_2^{(1)} \left[ \frac{\cos\theta}{2} \log\left(\frac{1+\cos\theta}{1-\cos\theta}\right) - 1 \right], \end{aligned} \quad (\text{B3})$$

with  $c_1^{(1)}$  and  $c_2^{(1)}$  two new integration constants. We then set  $c_2^{(1)} = -3/16\pi$  for the solution to be well behaved as  $\cos\theta = \pm 1$ . The integration constant  $c_1^{(1)}$  is left undetermined so that

$$g_0^{(1)}(\theta) = \frac{16\pi c_1^{(1)} \cos\theta - 5\cos^2\theta + 3}{16\pi}. \quad (\text{B4})$$

Using this, we then evaluate  $g_0^{(2)}(\theta)$  to find

$$\begin{aligned} g_0^{(2)}(\theta) = \frac{60\pi c_1^{(1)} \cos^2\theta - 18\pi c_1^{(1)} \cos\theta \log\left(\frac{1+\cos\theta}{1-\cos\theta}\right) - 9\cos^3\theta + 2\cos\theta \log(1-\cos^2\theta)}{48\pi} \\ + c_1^{(2)} \cos\theta + c_2^{(2)} \left[ \frac{\cos\theta}{2} \log\left(\frac{1+\cos\theta}{1-\cos\theta}\right) - 1 \right], \end{aligned} \quad (\text{B5})$$

with  $c_1^{(2)}$  and  $c_2^{(2)}$  two new integration constants. Hence, removing the log-divergence at both  $\cos\theta = 1$  and  $\cos\theta = -1$  requires

$$18\pi c_1^{(1)} - 24\pi c_2^{(2)} + 2 = 18\pi c_1^{(1)} - 24\pi c_2^{(2)} - 2 = 0, \quad (\text{B6})$$

which is impossible, so that no well-behaved solution can be found. This signals the emergence of a correction of the scaling dimension to order of  $O(\lambda^2)$ .

### APPENDIX C: RENORMALIZATION GROUP TREATMENT OF EQ. (40)

In this appendix, we apply a perturbative renormalization group treatment to Eq. (40) to find the far-field decay of the density field. By linearity, this amounts to finding  $K_\mu(\mathbf{r})$ , where

$$\Delta K_\mu(\mathbf{r}) - \lambda \partial_\alpha (v_\ell^\alpha(\mathbf{r}) K_\mu(\mathbf{r})) = -\partial_\mu \delta(\mathbf{r}) \quad (\text{C1})$$

and where  $\mathbf{v}_\ell(\mathbf{r}) \equiv \bar{\mathbf{v}}(\mathbf{r})/(\lambda D)$  is such that

$$v_\ell^\alpha(\mathbf{r}) \simeq J^{\alpha\beta}(\mathbf{r}) \hat{p}^\beta, \quad (\text{C2})$$

at large distances. For the sake of the renormalization group argument, the velocity field  $\mathbf{v}_\ell(\mathbf{r})$  is explicitly built from a microscopic length scale  $\ell$  as follows. First, we assume that the velocity field  $v_\ell^\alpha(\mathbf{r})$  can be expressed from a force density  $q_\ell^\beta(\mathbf{r})$ , so that

$$v_\ell^\alpha(\mathbf{r}) = \int d\mathbf{r}' J^{\alpha\beta}(\mathbf{r} - \mathbf{r}') q_\ell^\beta(\mathbf{r}'), \quad (\text{C3})$$

with

$$\int d\mathbf{r} q_\ell^\beta(\mathbf{r}) = \hat{p}^\beta. \quad (\text{C4})$$

Then, we assume that the force density depends on a microscopic length scale  $\ell$  through a scaling function  $q^\beta$  according to

$$q_\ell^\beta(\mathbf{r}) = \frac{1}{\ell^3} q^\beta\left(\frac{\mathbf{r}}{\ell}\right). \quad (\text{C5})$$

We now look for a perturbative solution of Eq. (C1) and study its behavior in the asymptotic regime where  $\ell/|\mathbf{r}| \ll 1$ . For any  $\mathbf{r}$  finite, we obtain the solution up to order of  $O(\lambda^2)$  as

$$\begin{aligned} K_\mu(\mathbf{r}) &= -\frac{1}{4\pi} \int d\mathbf{r}' \frac{1}{|\mathbf{r} - \mathbf{r}'|} (-\partial'_\mu \delta(\mathbf{r}') + \lambda \partial'_\alpha (v_\ell^\alpha(\mathbf{r}') K_\mu(\mathbf{r}')) \\ &= -\frac{1}{4\pi} \frac{r^\mu}{r^3} + \frac{\lambda}{4\pi} \int d\mathbf{r}' v_\ell^\alpha(\mathbf{r}') K_\mu(\mathbf{r}') \frac{r^\alpha - r'^\alpha}{|\mathbf{r} - \mathbf{r}'|^3} \\ &= -\frac{1}{4\pi} \frac{r^\mu}{r^3} - \frac{1}{4\pi} \frac{\lambda}{4\pi} \int d\mathbf{r}' v_\ell^\alpha(\mathbf{r}') \frac{r'^\mu r^\alpha - r'^\alpha}{r'^3 |\mathbf{r} - \mathbf{r}'|^3} \\ &\quad - \frac{1}{4\pi} \left(\frac{\lambda}{4\pi}\right)^2 \int d\mathbf{r}' v_\ell^\alpha(\mathbf{r}') \int d\mathbf{r}'' v_\ell^\beta(\mathbf{r}'') \frac{r'^{\mu\beta}}{r'^3} \\ &\quad \times \frac{r'^\beta - r''^\beta}{|\mathbf{r}' - \mathbf{r}''|^3} \frac{r^\alpha - r'^\alpha}{|\mathbf{r} - \mathbf{r}'|^3} + O(\lambda^3). \end{aligned} \quad (\text{C6})$$

In the following, we investigate the fate of this expansion in the far-field regime and use a renormalization group treatment to infer the anomalous scaling exponents.

#### 1. First order

To first order in  $\lambda$ , we have

$$K_\mu(\mathbf{r}) = -\frac{1}{4\pi} \frac{r^\mu}{r^3} - \frac{1}{4\pi} \frac{\lambda}{4\pi} \int d\mathbf{r}' v_\ell^\alpha(\mathbf{r}') \frac{r'^\mu r^\alpha - r'^\alpha}{r'^3 |\mathbf{r} - \mathbf{r}'|^3} + O(\lambda^2). \quad (\text{C7})$$

We define

$$\begin{aligned} I_1^\mu(\ell, \mathbf{r}) &\equiv \int d\mathbf{r}' v_\ell^\alpha(\mathbf{r}') \frac{r'^\mu r^\alpha - r'^\alpha}{r'^3 |\mathbf{r} - \mathbf{r}'|^3} \\ &= \int d\mathbf{r}' \int d\mathbf{r}'' J^{\alpha\beta}(\mathbf{r}' - \mathbf{r}'') \frac{1}{\ell^3} q^\beta\left(\frac{\mathbf{r}''}{\ell}\right) \frac{r'^\mu r^\alpha - r'^\alpha}{r'^3 |\mathbf{r} - \mathbf{r}'|^3}. \end{aligned} \quad (\text{C8})$$

The latter can be brought to the scaling form of Eq. (41) by using dimensionless integration variables  $\mathbf{r}'' \rightarrow \ell \mathbf{r}''$  and  $\mathbf{r}' \rightarrow r \mathbf{r}'$  and using  $J^{\alpha\beta}(\kappa \mathbf{r}) = \kappa^{-1} J^{\alpha\beta}(\mathbf{r})$  for any positive number  $\kappa > 0$ :

$$\begin{aligned} I_1^\mu(\ell, \mathbf{r}) &= \frac{1}{r^2} \int d\mathbf{r}' \int d\mathbf{r}'' J^{\alpha\beta}\left(\mathbf{r}' - \frac{\ell}{r} \mathbf{r}''\right) q^\beta(\mathbf{r}'') \frac{r'^\mu \hat{r}^\alpha - r'^\alpha}{r'^3 |\hat{\mathbf{r}} - \mathbf{r}'|^3} \\ &= \frac{1}{r^2} \hat{I}_1^\mu\left(\frac{\ell}{r}, \hat{\mathbf{r}}\right) \end{aligned} \quad (\text{C9})$$

with

$$\hat{I}_1^\mu(\epsilon, \hat{\mathbf{r}}) = \int d\mathbf{r}' \int d\mathbf{r}'' J^{\alpha\beta}(\mathbf{r}' - \epsilon \mathbf{r}'') q^\beta(\mathbf{r}'') \frac{r'^\mu \hat{r}^\alpha - r'^\alpha}{r'^3 |\hat{\mathbf{r}} - \mathbf{r}'|^3}. \quad (\text{C10})$$

We now prove that the limit  $\epsilon \rightarrow 0$  of the above integral exists. This amounts to showing that there is no anomalous scaling to first order in  $\lambda$ . To do so, we first split the integral between a near-field and a far-field contribution:

$$\hat{I}_1^\mu(\epsilon, \hat{\mathbf{r}}) = J_1^\mu(\epsilon, \hat{\mathbf{r}}) + J_2^\mu(\epsilon, \hat{\mathbf{r}}), \quad (\text{C11})$$

with

$$\begin{aligned} J_1^\mu(\epsilon, \hat{\mathbf{r}}) &= \int_0^{\sqrt{\epsilon}} d\mathbf{r}' \int d\hat{\mathbf{r}}'' \int d\mathbf{r}'' J^{\alpha\beta}(\mathbf{r}' - \epsilon \mathbf{r}'') q^\beta(\mathbf{r}'') \hat{r}^\mu \\ &\quad \times \frac{\hat{r}^\alpha - r'^\alpha}{|\hat{\mathbf{r}} - \mathbf{r}'|^3} \end{aligned} \quad (\text{C12})$$

and

$$J_2^\mu(\epsilon, \hat{\mathbf{r}}) = \int_{\sqrt{\epsilon}}^{+\infty} dr' \int d\hat{\mathbf{r}}' \int d\mathbf{r}'' J^{\alpha\beta}(\mathbf{r}' - \epsilon\mathbf{r}'') q^\beta(\mathbf{r}'') \hat{r}'^\mu \times \frac{\hat{r}^\alpha - r'^\alpha}{|\hat{\mathbf{r}} - \mathbf{r}'|^3}. \quad (\text{C13})$$

We can now evaluate the far-field  $\epsilon \ll 1$  behavior of these integrals. Disregarding contributions vanishing when  $\epsilon \rightarrow 0$ , we obtain for the first one

$$J_1^\mu(\epsilon, \hat{\mathbf{r}}) = \int_0^{1/\sqrt{\epsilon}} dr' \int d\hat{\mathbf{r}}' \int d\mathbf{r}'' J^{\alpha\beta}(\mathbf{r}' - \mathbf{r}'') q^\beta(\mathbf{r}'') \hat{r}'^\mu \times \frac{\hat{r}^\alpha - \epsilon r'^\alpha}{|\hat{\mathbf{r}} - \epsilon\mathbf{r}'|^3} \simeq \hat{J}_1^{\alpha\mu} \hat{r}^\alpha, \quad (\text{C14})$$

with the tensor

$$\hat{J}_1^{\alpha\mu} \equiv \lim_{L \rightarrow \infty} \int_0^L dr' \int d\hat{\mathbf{r}}' \int d\mathbf{r}'' J^{\alpha\beta}(\mathbf{r}' - \mathbf{r}'') q^\beta(\mathbf{r}'') \hat{r}'^\mu. \quad (\text{C15})$$

We note that the above integral superficially seems logarithmically divergent as  $L \rightarrow \infty$ . Nonetheless, this divergence is prevented, because the integral over the unit vector  $\hat{\mathbf{r}}'$  vanishes at large distances. The tensor  $\hat{J}_1^{\alpha\mu}$  is a nonuniversal correction, as it depends on the whole force distribution  $q^\beta(\mathbf{r})$ . To leading order in the far field, the second integral becomes

$$J_2^\mu(\epsilon, \hat{\mathbf{r}}) \simeq \hat{J}_2^{\beta\mu}(\hat{\mathbf{r}}) \hat{p}^\beta \quad (\text{C16})$$

with the tensor

$$\hat{J}_2^{\beta\mu}(\hat{\mathbf{r}}) = \lim_{L \rightarrow 0} \int_L^{+\infty} dr' \int d\hat{\mathbf{r}}' J^{\alpha\beta}(\mathbf{r}') \hat{r}'^\mu \frac{\hat{r}^\alpha - r'^\alpha}{|\hat{\mathbf{r}} - \mathbf{r}'|^3}. \quad (\text{C17})$$

Therefore, to leading order in the far field and to order of  $O(\lambda)$  in the perturbation expansion, the solution reads

$$K_\mu(\mathbf{r}) = -\frac{1}{4\pi r^2} \hat{r}^\mu - \frac{1}{4\pi r^2} \frac{\lambda}{4\pi} \left( \hat{J}_1^{\alpha\mu} \hat{r}^\alpha + \hat{J}_2^{\alpha\mu}(\hat{\mathbf{r}}) \hat{p}^\alpha \right) + O(\lambda^2) \\ = -\frac{1}{4\pi r^2} \left( \delta^{\alpha\mu} + \frac{\lambda}{4\pi} \hat{J}_1^{\alpha\mu} \right) \left( \hat{r}^\alpha + \frac{\lambda}{4\pi} \hat{J}_2^{\beta\alpha}(\hat{\mathbf{r}}) \hat{p}^\beta \right) \\ + O(\lambda^2) \quad (\text{C18})$$

and takes the form of a nonuniversal (tensorial) amplitude multiplied by a universal angular dependence. We now evaluate  $\hat{J}_2^{\beta\alpha}(\hat{\mathbf{r}})$ . Using isotropy, we can decompose

$$\hat{J}_2^{\beta\alpha}(\hat{\mathbf{r}}) = A_1 \delta^{\beta\alpha} + A_2 \hat{r}^\beta \hat{r}^\alpha, \quad (\text{C19})$$

with

$$A_1 = \frac{1}{2} \hat{J}_2^{\beta\alpha}(\hat{\mathbf{r}}) (\delta^{\beta\alpha} - \hat{r}^\beta \hat{r}^\alpha) \quad (\text{C20})$$

and

$$A_2 = -\frac{1}{2} \hat{J}_2^{\beta\alpha}(\hat{\mathbf{r}}) (\delta^{\beta\alpha} - 3\hat{r}^\beta \hat{r}^\alpha). \quad (\text{C21})$$

Then,

$$A_1 = \frac{1}{2} \int_0^{+\infty} dr' \int d\hat{\mathbf{r}}' \frac{(\delta^{\gamma\beta} + \hat{r}'^\gamma \hat{r}'^\beta) \hat{r}'^\alpha}{r' |\hat{\mathbf{r}} - \hat{\mathbf{r}}'|^3} \\ \times (\hat{r}'^\gamma - r' \hat{r}'^\gamma) (\delta^{\beta\alpha} - \hat{r}^\beta \hat{r}^\alpha) \\ = \frac{1}{2} \int_0^{+\infty} dr' \int d\hat{\mathbf{r}}' \frac{(\hat{r}'^\gamma - r' \hat{r}'^\gamma) \hat{r}'^\alpha}{r' |\hat{\mathbf{r}} - \hat{\mathbf{r}}'|^3} \\ \times (\delta^{\gamma\alpha} - \hat{r}'^\gamma \hat{r}^\alpha + \hat{r}'^\gamma \hat{r}'^\alpha - \hat{r}^\alpha \hat{r}'^\gamma (\hat{\mathbf{r}} \cdot \hat{\mathbf{r}}')) \\ = \frac{1}{2} \int_0^{+\infty} dr' \int d\hat{\mathbf{r}}' \frac{(\hat{\mathbf{r}} \cdot \hat{\mathbf{r}}') - (\hat{\mathbf{r}} \cdot \hat{\mathbf{r}}')^3 - 2r' + 2r' (\hat{\mathbf{r}} \cdot \hat{\mathbf{r}}')^2}{r' |\hat{\mathbf{r}} - \hat{\mathbf{r}}'|^3} \\ = \pi \int_0^{+\infty} dr' \int_{-1}^{-1} dw \frac{w - w^3 - 2r' + 2r' w^2}{r' (1 + r'^2 - 2r' w)^{3/2}} \\ = -3\pi. \quad (\text{C22})$$

Furthermore,

$$A_2 = -\frac{1}{2} \int_0^{+\infty} dr' \int d\hat{\mathbf{r}}' \frac{(\delta^{\gamma\beta} + \hat{r}'^\gamma \hat{r}'^\beta) \hat{r}'^\alpha}{r' |\hat{\mathbf{r}} - \hat{\mathbf{r}}'|^3} \\ \times (\hat{r}'^\gamma - r' \hat{r}'^\gamma) (\delta^{\beta\alpha} - 3\hat{r}^\beta \hat{r}^\alpha) \\ = -\frac{1}{2} \int_0^{+\infty} dr' \int d\hat{\mathbf{r}}' \frac{(\hat{r}'^\gamma - r' \hat{r}'^\gamma) \hat{r}'^\alpha}{r' |\hat{\mathbf{r}} - \hat{\mathbf{r}}'|^3} \\ \times (\delta^{\gamma\alpha} - 3\hat{r}'^\gamma \hat{r}^\alpha + \hat{r}'^\gamma \hat{r}'^\alpha - 3\hat{r}^\alpha \hat{r}'^\gamma (\hat{\mathbf{r}} \cdot \hat{\mathbf{r}}')) \\ = \frac{1}{2} \int_0^{+\infty} dr' \int d\hat{\mathbf{r}}' \frac{(\hat{\mathbf{r}} \cdot \hat{\mathbf{r}}') + 3(\hat{\mathbf{r}} \cdot \hat{\mathbf{r}}')^3 + 2r' - 6r' (\hat{\mathbf{r}} \cdot \hat{\mathbf{r}}')^2}{r' |\hat{\mathbf{r}} - \hat{\mathbf{r}}'|^3} \\ = \pi \int_0^{+\infty} dr' \int_{-1}^{-1} dw \frac{w + 3w^3 + 2r' - 6r' w^2}{r' (1 + r'^2 - 2r' w)^{3/2}} \\ = 5\pi. \quad (\text{C23})$$

Then, to first order in  $O(\lambda)$  and to leading order in the far field, we have

$$K_\mu(\mathbf{r}) = -\frac{1}{4\pi r^2} \left( \delta^{\alpha\mu} + \frac{\lambda}{4\pi} \hat{J}_1^{\alpha\mu} \right) \\ \times \left( \hat{r}^\alpha - \frac{3\lambda}{4} \hat{p}^\alpha + \frac{5\lambda}{4} (\hat{\mathbf{p}} \cdot \hat{\mathbf{r}}) \hat{r}^\alpha \right) + O(\lambda^2). \quad (\text{C24})$$

## 2. Second order

To second order, we need to compute the following integral:

$$I_2^\mu(\ell, \mathbf{r}) \equiv \int d\mathbf{r}' v_\ell^\alpha(\mathbf{r}') \int d\mathbf{r}'' v_\ell^\beta(\mathbf{r}'') \frac{r''^\mu r'^\beta - r''^\beta r'^\mu}{r''^3 |\mathbf{r}' - \mathbf{r}''|^3} \frac{r^\alpha - r'^\alpha}{|\mathbf{r} - \mathbf{r}'|^3}, \quad (\text{C25})$$

which enters Eq. (C6). Note that

$$\begin{aligned} I_2^\mu(\ell, \mathbf{r}) &= \int d\mathbf{r}' v_\ell^\alpha(\mathbf{r}') I_1^\mu(\ell, \mathbf{r}') \frac{r^\alpha - r'^\alpha}{|\mathbf{r} - \mathbf{r}'|^3} \\ &= \int d\mathbf{r}' v_\ell^\alpha(\mathbf{r}') \frac{1}{r'^2} \hat{I}_1^\mu\left(\frac{\ell}{r'}, \hat{\mathbf{r}}'\right) \frac{r^\alpha - r'^\alpha}{|\mathbf{r} - \mathbf{r}'|^3}, \end{aligned} \quad (\text{C26})$$

as it appears from the definitions of  $I_1^\mu(\ell, \mathbf{r}')$  and  $\hat{I}_1^\mu(\ell/r', \hat{\mathbf{r}}')$  in Eqs. (C8) and (C9), respectively. This expression can then be brought to the scaling form of Eq. (41) using the same changes of variables as in Eq. (C9):

$$\begin{aligned} I_2^\mu(\ell, \mathbf{r}) &= \int d\mathbf{r}' v_\ell^\alpha(\mathbf{r}') \frac{1}{r'^2} \hat{I}_1^\mu\left(\frac{\ell}{r'}, \hat{\mathbf{r}}'\right) \frac{r^\alpha - r'^\alpha}{|\mathbf{r} - \mathbf{r}'|^3} \\ &= \int d\mathbf{r}' \int d\mathbf{r}'' J^{\alpha\beta}(\mathbf{r}' - \ell\mathbf{r}'') q^\beta(\mathbf{r}'') \frac{1}{r'^2} \hat{I}_1^\mu\left(\frac{\ell}{r'}, \hat{\mathbf{r}}'\right) \\ &\quad \times \frac{r^\alpha - r'^\alpha}{|\mathbf{r} - \mathbf{r}'|^3} \\ &= \frac{1}{r^2} \hat{I}_2^\mu\left(\frac{\ell}{r}, \hat{\mathbf{r}}\right), \end{aligned} \quad (\text{C27})$$

with

$$\begin{aligned} \hat{I}_2^\mu(\epsilon, \hat{\mathbf{r}}) &= \int d\mathbf{r}' \int d\mathbf{r}'' J^{\alpha\beta}(\mathbf{r}' - \epsilon\mathbf{r}'') q^\beta(\mathbf{r}'') \frac{1}{r'^2} \hat{I}_1^\mu\left(\frac{\epsilon}{r'}, \hat{\mathbf{r}}'\right) \\ &\quad \times \frac{\hat{r}^\alpha - r'^\alpha}{|\hat{\mathbf{r}} - \mathbf{r}'|^3}. \end{aligned} \quad (\text{C28})$$

Again, we split the integral between a far-field and a near-field contribution:

$$\hat{I}_2^\mu(\epsilon, \hat{\mathbf{r}}) = K_1^\mu(\epsilon, \hat{\mathbf{r}}) + K_2^\mu(\epsilon, \hat{\mathbf{r}}), \quad (\text{C29})$$

with

$$\begin{aligned} K_1^\mu(\epsilon, \hat{\mathbf{r}}) &= \int_0^{\sqrt{\epsilon}} d\mathbf{r}' \int d\mathbf{r}'' \int d\mathbf{r}''' J^{\alpha\beta}(\mathbf{r}' - \epsilon\mathbf{r}''') q^\beta(\mathbf{r}''') \\ &\quad \times \hat{I}_1^\mu\left(\frac{\epsilon}{r'}, \hat{\mathbf{r}}'\right) \frac{\hat{r}^\alpha - r'^\alpha}{|\hat{\mathbf{r}} - \mathbf{r}'|^3} \end{aligned} \quad (\text{C30})$$

and

$$\begin{aligned} K_2^\mu(\epsilon, \hat{\mathbf{r}}) &= \int_{\sqrt{\epsilon}}^{+\infty} d\mathbf{r}' \int d\mathbf{r}'' \int d\mathbf{r}''' J^{\alpha\beta}(\mathbf{r}' - \epsilon\mathbf{r}''') q^\beta(\mathbf{r}''') \\ &\quad \times \hat{I}_1^\mu\left(\frac{\epsilon}{r'}, \hat{\mathbf{r}}'\right) \frac{\hat{r}^\alpha - r'^\alpha}{|\hat{\mathbf{r}} - \mathbf{r}'|^3}. \end{aligned} \quad (\text{C31})$$

We now investigate the behavior of both contributions when  $\epsilon \ll 1$ , neglecting vanishing corrections as  $\epsilon \rightarrow 0$ . For the second integral  $K_2^\mu(\epsilon, \hat{\mathbf{r}})$ , we have

$$\begin{aligned} K_2^\mu(\epsilon, \hat{\mathbf{r}}) &\simeq \hat{p}^\beta \int_{\sqrt{\epsilon}}^{+\infty} d\mathbf{r}' \int d\mathbf{r}'' J^{\alpha\beta}(\mathbf{r}'') \hat{I}_1^\mu(0, \hat{\mathbf{r}}') \frac{\hat{r}^\alpha - r'^\alpha}{|\hat{\mathbf{r}} - \mathbf{r}'|^3} \\ &\simeq \hat{p}^\beta \int_{\sqrt{\epsilon}}^{+\infty} d\mathbf{r}' \int d\mathbf{r}'' J^{\alpha\beta}(\mathbf{r}'') \left[ \hat{J}_1^{\gamma\mu}(\hat{\mathbf{r}}') + \hat{J}_2^{\gamma\mu}(\hat{\mathbf{r}}') p^\gamma \right] \\ &\quad \times \frac{\hat{r}^\alpha - r'^\alpha}{|\hat{\mathbf{r}} - \mathbf{r}'|^3} \\ &\simeq \hat{J}_1^{\gamma\mu} \hat{J}_2^{\beta\gamma}(\hat{\mathbf{r}}) \hat{p}^\beta + \hat{p}^\beta \hat{p}^\gamma \int_{\sqrt{\epsilon}}^{+\infty} d\mathbf{r}' \int d\mathbf{r}'' J^{\alpha\beta}(\mathbf{r}'') \\ &\quad \times \hat{J}_2^{\gamma\mu}(\hat{\mathbf{r}}') \frac{\hat{r}^\alpha - r'^\alpha}{|\hat{\mathbf{r}} - \mathbf{r}'|^3}, \end{aligned} \quad (\text{C32})$$

where we use the far-field expression of  $J_2(\epsilon, \hat{\mathbf{r}})$  in Eq. (C16) to get the first term on the right-hand side of the last equality. Crucially, because  $J^{\alpha\beta}(\mathbf{r}) \sim r^{-1}$  and the angular integral does not vanish at short distances, the second term diverges logarithmically when  $\epsilon \rightarrow 0$  and, therefore, contributes to the renormalization of the anomalous dimension to order of  $O(\lambda^2)$ . We now focus on these diverging contributions which can be obtained by replacing the integrand by its small  $r'$  behavior and using any finite number as an upper bound for the integral over  $r'$ , now represented as  $\int_{\sqrt{\epsilon}} d\mathbf{r}'$ . As the logarithmically divergent part is insensitive to the upper bound, we get

$$\begin{aligned} \hat{p}^\beta \hat{p}^\gamma \int_{\sqrt{\epsilon}}^{+\infty} d\mathbf{r}' \int d\mathbf{r}'' J^{\alpha\beta}(\mathbf{r}'') \hat{J}_2^{\gamma\mu}(\hat{\mathbf{r}}') \frac{\hat{r}^\alpha - r'^\alpha}{|\hat{\mathbf{r}} - \mathbf{r}'|^3} &\sim \hat{r}^\alpha \hat{p}^\beta \hat{p}^\gamma \int_{\sqrt{\epsilon}} d\mathbf{r}' \int d\mathbf{r}'' J^{\alpha\beta}(\mathbf{r}'') \hat{J}_2^{\gamma\mu}(\hat{\mathbf{r}}') \\ &\sim \hat{r}^\alpha \hat{p}^\beta \hat{p}^\gamma \int_{\sqrt{\epsilon}} d\mathbf{r}' \frac{1}{r'} \int d\mathbf{r}'' (\delta^{\alpha\beta} + \hat{r}^\alpha \hat{r}^\beta) (-3\pi\delta^{\gamma\mu} + 5\pi\hat{r}^\gamma \hat{r}^\mu) \\ &\sim \ln \epsilon \left( \frac{10\pi^2}{3} \hat{p}^\mu(\hat{\mathbf{r}} \cdot \hat{\mathbf{p}}) - \frac{2\pi^2}{3} \hat{r}^\mu \right). \end{aligned} \quad (\text{C33})$$

To get the far-field angular dependence up to order of  $O(\lambda^2)$ , it is further necessary to keep track of the terms in  $K_2(\epsilon, \hat{\mathbf{r}})$  that remain finite as  $\epsilon \rightarrow 0$ . To do so, we introduce



$$\begin{aligned}
Q^{\beta\gamma\mu}(\hat{\mathbf{r}}) &= \int_{\sqrt{\epsilon}}^{+\infty} dr' \int d\hat{\mathbf{r}}' J^{\alpha\beta}(\mathbf{r}') J_2^{\gamma\mu}(\hat{\mathbf{r}}') \frac{\hat{\mathbf{r}}^\alpha - r'^\alpha}{|\hat{\mathbf{r}} - \mathbf{r}'|^3} \\
&= \int_{\sqrt{\epsilon}}^{+\infty} dr' \int d\hat{\mathbf{r}}' J^{\alpha\beta}(\mathbf{r}') [-3\pi\delta^{\gamma\mu} + 5\pi\hat{r}'^\gamma \hat{r}'^\mu] \frac{\hat{\mathbf{r}}^\alpha - r'^\alpha}{|\hat{\mathbf{r}} - \mathbf{r}'|^3} \\
&= -3\pi\delta^{\gamma\mu} \int_{\sqrt{\epsilon}}^{+\infty} dr' \int d\hat{\mathbf{r}}' J^{\alpha\beta}(\mathbf{r}') \frac{\hat{\mathbf{r}}^\alpha - r'^\alpha}{|\hat{\mathbf{r}} - \mathbf{r}'|^3} + 5\pi \int_{\sqrt{\epsilon}}^{+\infty} dr' \int d\hat{\mathbf{r}}' J^{\alpha\beta}(\mathbf{r}') \frac{\hat{\mathbf{r}}^\alpha - r'^\alpha}{|\hat{\mathbf{r}} - \mathbf{r}'|^3} \hat{r}'^\gamma \hat{r}'^\mu \\
&= -3\pi\delta^{\gamma\mu} Q_1^\beta(\hat{\mathbf{r}}) + 5\pi Q_2^{\beta\gamma\mu}(\hat{\mathbf{r}}),
\end{aligned} \tag{C34}$$

with

$$Q_1^\beta(\hat{\mathbf{r}}) = \int_{\sqrt{\epsilon}}^{+\infty} dr' \int d\hat{\mathbf{r}}' J^{\alpha\beta}(\mathbf{r}') \frac{\hat{\mathbf{r}}^\alpha - r'^\alpha}{|\hat{\mathbf{r}} - \mathbf{r}'|^3} \tag{C35}$$

and

$$Q_2^{\beta\gamma\mu}(\hat{\mathbf{r}}) = \int_{\sqrt{\epsilon}}^{+\infty} dr' \int d\hat{\mathbf{r}}' J^{\alpha\beta}(\mathbf{r}') \frac{\hat{\mathbf{r}}^\alpha - r'^\alpha}{|\hat{\mathbf{r}} - \mathbf{r}'|^3} \hat{r}'^\gamma \hat{r}'^\mu. \tag{C36}$$

By symmetry, the vector  $Q_1^\beta(\hat{\mathbf{r}})$  points along  $\hat{\mathbf{r}}^\beta$ , meaning  $Q_1^\beta(\hat{\mathbf{r}}) = Q_1 \hat{\mathbf{r}}^\beta$  with

$$\begin{aligned}
Q_1 &= \int_{\sqrt{\epsilon}}^{+\infty} dr' \int d\hat{\mathbf{r}}' J^{\alpha\gamma}(\mathbf{r}') \frac{\hat{\mathbf{r}}^\alpha - r'^\alpha}{|\hat{\mathbf{r}} - \mathbf{r}'|^3} \hat{\mathbf{r}}^\gamma \\
&= 2\pi \int_{\sqrt{\epsilon}}^{+\infty} dr' \int_{-1}^1 dw \frac{1 - 2r'w + w^2}{r'(1 + r'^2 - 2r'w)^{3/2}} \\
&= -\frac{8\pi}{9}(1 + 3\ln\epsilon).
\end{aligned} \tag{C37}$$

The computation of  $Q_2^{\beta\gamma\mu}(\hat{\mathbf{r}})$  is more tedious. We note that  $Q_2^{\beta\gamma\mu}(\hat{\mathbf{r}})$  is symmetric under exchange of the indices  $(\gamma, \mu)$ .

This leads to the decomposition

$$Q_2^{\beta\gamma\mu}(\hat{\mathbf{r}}) = B_1 \hat{\mathbf{r}}^\beta \hat{\mathbf{r}}^\gamma \hat{\mathbf{r}}^\mu + B_2 \hat{\mathbf{r}}^\beta \delta^{\gamma\mu} + B_3 (\hat{\mathbf{r}}^\gamma \delta^{\beta\mu} + \hat{\mathbf{r}}^\mu \delta^{\beta\gamma}). \tag{C38}$$

We now evaluate  $B_1$ ,  $B_2$ , and  $B_3$  using the identities

$$\begin{aligned}
Q_2^{\beta\gamma\mu}(\hat{\mathbf{r}}) \hat{\mathbf{r}}^\beta \hat{\mathbf{r}}^\gamma \hat{\mathbf{r}}^\mu &= B_1 + B_2 + 2B_3, \\
Q_2^{\beta\gamma\mu}(\hat{\mathbf{r}}) \hat{\mathbf{r}}^\beta \delta^{\gamma\mu} &= B_1 + 3B_2 + 2B_3, \\
Q_2^{\beta\gamma\mu}(\hat{\mathbf{r}}) \hat{\mathbf{r}}^\gamma \delta^{\beta\mu} &= B_1 + B_2 + 4B_3,
\end{aligned} \tag{C39}$$

from which we obtain

$$\begin{aligned}
B_1 &= \frac{1}{2} Q_2^{\beta\gamma\mu}(\hat{\mathbf{r}}) [5\hat{\mathbf{r}}^\beta \hat{\mathbf{r}}^\gamma \hat{\mathbf{r}}^\mu - \hat{\mathbf{r}}^\beta \delta^{\gamma\mu} - 2\hat{\mathbf{r}}^\gamma \delta^{\beta\mu}], \\
B_2 &= \frac{1}{2} Q_2^{\beta\gamma\mu}(\hat{\mathbf{r}}) [-\hat{\mathbf{r}}^\beta \hat{\mathbf{r}}^\gamma \hat{\mathbf{r}}^\mu + \hat{\mathbf{r}}^\beta \delta^{\gamma\mu}], \\
B_3 &= \frac{1}{2} Q_2^{\beta\gamma\mu}(\hat{\mathbf{r}}) [-\hat{\mathbf{r}}^\beta \hat{\mathbf{r}}^\gamma \hat{\mathbf{r}}^\mu + \hat{\mathbf{r}}^\gamma \delta^{\beta\mu}].
\end{aligned} \tag{C40}$$

Hence, we have

$$\begin{aligned}
B_1 &= \frac{1}{2} \int_{\sqrt{\epsilon}}^{+\infty} dr' \int d\hat{\mathbf{r}}' J^{\alpha\beta}(\mathbf{r}') \frac{\hat{\mathbf{r}}^\alpha - r'^\alpha}{|\hat{\mathbf{r}} - \mathbf{r}'|^3} \hat{r}'^\gamma \hat{r}'^\mu [5\hat{\mathbf{r}}^\beta \hat{\mathbf{r}}^\gamma \hat{\mathbf{r}}^\mu - \hat{\mathbf{r}}^\beta \delta^{\gamma\mu} - 2\hat{\mathbf{r}}^\gamma \delta^{\beta\mu}] \\
&= \frac{1}{2} \int_{\sqrt{\epsilon}}^{+\infty} dr' \int d\hat{\mathbf{r}}' \frac{\hat{\mathbf{r}}^\beta - \hat{r}'^\beta (2r' - (\hat{\mathbf{r}} \cdot \hat{\mathbf{r}}'))}{r' |\hat{\mathbf{r}} - \mathbf{r}'|^3} (5\hat{\mathbf{r}}^\beta (\hat{\mathbf{r}} \cdot \hat{\mathbf{r}}')^2 - \hat{\mathbf{r}}^\beta - 2\hat{r}'^\beta (\hat{\mathbf{r}} \cdot \hat{\mathbf{r}}')) \\
&= \pi \int_{\sqrt{\epsilon}}^{+\infty} dr' \int_{-1}^1 dw \frac{-1 - 10w^3 r' + 5w^4 + 6wr'}{r'(1 + r'^2 - 2wr')^{3/2}} \\
&= \frac{12\pi}{5}.
\end{aligned} \tag{C41}$$

Similarly,

$$\begin{aligned}
B_2 &= \frac{1}{2} \int_{\sqrt{\epsilon}}^{+\infty} dr' \int d\hat{\mathbf{r}}' J^{\alpha\beta}(\mathbf{r}') \frac{\hat{\mathbf{r}}^\alpha - r'^\alpha}{|\hat{\mathbf{r}} - \mathbf{r}'|^3} \hat{r}'^\gamma \hat{r}'^\mu [-\hat{\mathbf{r}}^\beta \hat{\mathbf{r}}^\gamma \hat{\mathbf{r}}^\mu + \hat{\mathbf{r}}^\beta \delta^{\gamma\mu}] \\
&= \frac{1}{2} \int_{\sqrt{\epsilon}}^{+\infty} dr' \int d\hat{\mathbf{r}}' \frac{\hat{\mathbf{r}}^\beta - \hat{r}'^\beta (2r' - (\hat{\mathbf{r}} \cdot \hat{\mathbf{r}}'))}{r' |\hat{\mathbf{r}} - \mathbf{r}'|^3} (-\hat{\mathbf{r}}^\beta (\hat{\mathbf{r}} \cdot \hat{\mathbf{r}}')^2 + \hat{\mathbf{r}}^\beta)
\end{aligned}$$

$$\begin{aligned}
 &= \pi \int_{\sqrt{\epsilon}}^{+\infty} dr' \int_{-1}^1 dw \frac{(1-w^2)(1-w(2r'-w))}{r'(1+r'^2-2wr')^{3/2}} \\
 &= -\frac{\pi}{5} \left( \frac{12}{5} + 4 \ln \epsilon \right). \tag{C42}
 \end{aligned}$$

Finally, we have

$$\begin{aligned}
 B_3 &= \frac{1}{2} \int_{\sqrt{\epsilon}}^{+\infty} dr' \int d\hat{r}' J^{\alpha\beta}(\mathbf{r}') \frac{\hat{r}'^\alpha - r'^\alpha}{|\hat{\mathbf{r}} - \mathbf{r}'|^3} \hat{r}'^\gamma \hat{r}'^\mu [-\hat{r}'^\beta \hat{r}'^\gamma \hat{r}'^\mu + \hat{r}'^\gamma \delta^{\beta\mu}] \\
 &= \frac{1}{2} \int_{\sqrt{\epsilon}}^{+\infty} dr' \int d\hat{r}' \frac{\hat{r}'^\beta - \hat{r}'^\beta(2r' - (\hat{\mathbf{r}} \cdot \hat{\mathbf{r}}'))}{r' |\hat{\mathbf{r}} - \mathbf{r}'|^3} \left( -\hat{r}'^\beta (\hat{\mathbf{r}} \cdot \hat{\mathbf{r}}')^2 + \hat{r}'^\beta (\hat{\mathbf{r}} \cdot \hat{\mathbf{r}}') \right) \\
 &= \pi \int_{\sqrt{\epsilon}}^{+\infty} dr' \int_{-1}^1 dw \frac{(2r' - w)(w^3 - w)}{r'(1+r'^2-2wr')^{3/2}} \\
 &= -\frac{\pi}{5} \left( \frac{208}{45} + \frac{2}{3} \ln \epsilon \right). \tag{C43}
 \end{aligned}$$

This leads to the following expression for  $K_2^\mu(\epsilon, \hat{\mathbf{r}})$ , where we keep all the terms that do not vanish as  $\epsilon \rightarrow 0$ :

$$\begin{aligned}
 K_2^\mu(\epsilon, \hat{\mathbf{r}}) &= \hat{p}^\beta \hat{J}_1^{\gamma\mu} \hat{J}_2^{\beta\gamma}(\hat{\mathbf{r}}) + \hat{p}^\beta \hat{p}^\gamma Q^{\beta\gamma\mu}(\hat{\mathbf{r}}) \\
 &= \hat{p}^\beta \hat{J}_1^{\gamma\mu} \hat{J}_2^{\beta\gamma}(\hat{\mathbf{r}}) + \hat{p}^\beta \hat{p}^\gamma [-3\pi \hat{r}^\beta \delta^{\gamma\mu} Q_1 + 5\pi(B_1 \hat{r}^\beta \hat{r}'^\gamma \hat{r}'^\mu + B_2 \hat{r}^\beta \delta^{\gamma\mu} + B_3(\hat{r}'^\gamma \delta^{\beta\mu} + \hat{r}'^\mu \delta^{\beta\gamma}))] \\
 &= \hat{p}^\beta \hat{J}_1^{\gamma\mu} \hat{J}_2^{\beta\gamma}(\hat{\mathbf{r}}) + \pi^2 \left( \frac{10}{3} \log(\epsilon) - \frac{196}{45} \right) \hat{p}^\mu (\hat{\mathbf{p}} \cdot \hat{\mathbf{r}}) + 12\pi^2 \hat{r}'^\mu (\hat{\mathbf{p}} \cdot \hat{\mathbf{r}})^2 - \pi^2 \left( \frac{208}{45} + \frac{2}{3} \ln \epsilon \right) \hat{r}'^\mu. \tag{C44}
 \end{aligned}$$

Next, we expand  $K_1^\mu(\epsilon, \hat{\mathbf{r}})$ , defined in Eq. (C30), when  $\epsilon \ll 1$  and disregarding all the terms that vanish as  $\epsilon \rightarrow 0$ . First, we obtain

$$\begin{aligned}
 K_1^\mu(\epsilon, \hat{\mathbf{r}}) &= \int_0^{\sqrt{\epsilon}} dr' \int d\hat{r}' \int d\mathbf{r}'' J^{\alpha\beta}(\mathbf{r}' - \epsilon \mathbf{r}'') q^\beta(\mathbf{r}'') \hat{I}_1^\mu \left( \frac{\epsilon}{r'}, \hat{\mathbf{r}}' \right) \frac{\hat{r}'^\alpha - r'^\alpha}{|\hat{\mathbf{r}} - \mathbf{r}'|^3} \\
 &= \int_0^{1/\sqrt{\epsilon}} dr' \int d\hat{r}' \int d\mathbf{r}'' J^{\alpha\beta}(\mathbf{r}' - \mathbf{r}'') q^\beta(\mathbf{r}'') \hat{I}_1^\mu \left( \frac{1}{r'}, \hat{\mathbf{r}}' \right) \frac{\hat{r}'^\alpha - \epsilon r'^\alpha}{|\hat{\mathbf{r}} - \epsilon \mathbf{r}'|^3} \\
 &\simeq \hat{r}'^\alpha \int_0^{1/\sqrt{\epsilon}} dr' \int d\hat{r}' \int d\mathbf{r}'' J^{\alpha\beta}(\mathbf{r}' - \mathbf{r}'') q^\beta(\mathbf{r}'') \hat{I}_1^\mu \left( \frac{1}{r'}, \hat{\mathbf{r}}' \right). \tag{C45}
 \end{aligned}$$

This last integral splits into two contributions, a finite nonuniversal contribution, denoted  $\hat{K}_1^{\alpha\mu}$  in the following, and a universal logarithmically divergent contribution coming from the large distance behavior of the integral over  $r'$ . The latter can be obtained by replacing the integrand in Eq. (C45) by its large  $r'$  leading-order behavior [the integral over  $\mathbf{r}''$  then gives  $\hat{p}^\beta$  following Eq. (C4)] and using the expression for  $I_1^\mu(0, \hat{\mathbf{r}}')$  derived in Eqs. (C14)–(C17). In doing so, the lower bound in the  $r'$  integral should be set to any strictly positive number, which we represent by  $\int^{1/\sqrt{\epsilon}} dr'$ . The logarithmically divergent contribution as  $\epsilon \rightarrow 0$  is indeed insensitive to this bound. We, therefore, obtain

$$\begin{aligned}
 \hat{p}^\beta \int^{1/\sqrt{\epsilon}} dr' \int d\hat{r}' J^{\alpha\beta}(\mathbf{r}') \hat{I}_1^\mu(0, \hat{\mathbf{r}}') &= \hat{p}^\beta \int^{1/\sqrt{\epsilon}} dr' \int d\hat{r}' J^{\alpha\beta}(\mathbf{r}') \left[ \hat{J}_1^{\gamma\mu} \hat{r}'^\gamma + \hat{J}_2^{\gamma\mu}(\hat{\mathbf{r}}') \hat{p}^\gamma \right] \\
 &\simeq \hat{p}^\beta \hat{p}^\gamma \int^{1/\sqrt{\epsilon}} dr' \int d\hat{r}' J^{\alpha\beta}(\mathbf{r}') \hat{J}_2^{\gamma\mu}(\hat{\mathbf{r}}') \\
 &\simeq p^\beta p^\gamma \int_{\sqrt{\epsilon}} dr' \int d\hat{r}' J^{\alpha\beta}(\mathbf{r}') \hat{J}_2^{\gamma\mu}(\hat{\mathbf{r}}'). \tag{C46}
 \end{aligned}$$

Note that the term proportional to  $\hat{J}_1^{\gamma\mu}$  that appears in the first line in Eq. (C46) does not contribute to the singular part, because the corresponding angular integral vanishes. Comparison with Eq. (C33) then shows that the singular part of

$K_1^\mu(\epsilon, \hat{\mathbf{r}})$  and that of  $K_2^\mu(\epsilon, \hat{\mathbf{r}})$  are identical. We, therefore, obtain in the far field

$$\begin{aligned} \hat{I}_2(\epsilon, \hat{\mathbf{r}}) &= \left( \hat{K}_1^{\alpha\mu} - \delta^{\alpha\mu} \frac{4}{3} \pi^2 \ln \epsilon - \delta^{\alpha\mu} \frac{208}{45} \pi^2 \right) \hat{r}^\alpha + \hat{p}^\beta \hat{J}_1^{\gamma\mu} \hat{J}_2^{\beta\gamma}(\hat{\mathbf{r}}) \\ &+ \pi^2 \left( \frac{20}{3} \log(\epsilon) - \frac{196}{45} \right) \hat{p}^\mu (\hat{\mathbf{p}} \cdot \hat{\mathbf{r}}) + 12\pi^2 \hat{r}^\mu (\hat{\mathbf{p}} \cdot \hat{\mathbf{r}})^2. \end{aligned} \quad (\text{C47})$$

Altogether, Eqs. (C24) and (C47) lead to the following expression for the expansion to second order  $O(\lambda^2)$  of the solution of Eq. (C1), in the far field:

$$K_\mu(\mathbf{r}) = -\frac{1}{4\pi r^2} M^{\alpha\mu}(r) \left[ \hat{r}^\alpha - \frac{\lambda}{4} (3\hat{p}^\alpha - 5\hat{r}^\alpha (\hat{\mathbf{p}} \cdot \hat{\mathbf{r}})) + \frac{3}{4} \lambda^2 \hat{r}^\alpha (\hat{\mathbf{p}} \cdot \hat{\mathbf{r}})^2 \right] + O(\lambda^3), \quad (\text{C48})$$

with the tensor

$$M^{\alpha\mu}(r) = \delta^{\alpha\mu} \left( 1 - \frac{\lambda^2}{12} \ln \epsilon - \frac{13\lambda^2}{45} \right) + \frac{\lambda}{4\pi} \hat{J}_1^{\alpha\mu} + \left( \frac{\lambda}{4\pi} \right)^2 \hat{K}_1^{\alpha\mu} + \lambda^2 \left( \frac{5}{12} \log(\epsilon) - \frac{49}{45} \right) \hat{p}^\alpha \hat{p}^\mu. \quad (\text{C49})$$

### 3. Renormalization group equations

The far-field density decay is governed by Eq. (40), from which we get  $\delta\rho(\mathbf{r}) = c^\mu K_\mu(\mathbf{r})/D$ . In the following, we show that, as in the treatment in the main text, we obtain two different anomalous dimensions depending on whether the polar obstacle has an axis of symmetry or not.

#### a. Polar obstacle with an axis of symmetry

If the obstacle has an axis of symmetry, the latter is necessarily along  $\hat{\mathbf{p}}$  so that  $c^\mu = c \hat{p}^\mu$ . Accordingly, by symmetry, we obtain  $\hat{J}_1^{\alpha\mu} \hat{p}^\mu = j_1 \hat{p}^\alpha$  and  $\hat{K}_1^{\alpha\mu} \hat{p}^\mu = k_1 \hat{p}^\alpha$ , where  $j_1$  and  $k_1$  are constants which depend on near-field properties of the velocity field. We, therefore, get

$$\begin{aligned} c^\mu M^{\alpha\mu}(r) &= c \hat{p}^\alpha \left[ 1 + \frac{\lambda^2}{3} \lambda^2 \ln \epsilon - \frac{62}{45} \lambda^2 + \frac{\lambda}{4\pi} j_1 + \left( \frac{\lambda}{4\pi} \right)^2 k_1 \right] \\ &= c \hat{p}^\alpha \left[ 1 - \frac{62}{45} \lambda^2 + \frac{\lambda}{4\pi} j_1 + \left( \frac{\lambda}{4\pi} \right)^2 k_1 \right] \\ &\times \left( 1 + \frac{\lambda^2}{3} \ln \epsilon \right) + O(\lambda^3), \end{aligned} \quad (\text{C50})$$

which leads to the following expression for the density field:

$$\begin{aligned} \delta\rho(\mathbf{r}) &\propto \frac{m_\parallel(r)}{r^2} \left[ \cos\theta - \frac{\lambda}{4} (3 - 5\cos^2\theta) + \frac{3}{4} \lambda^2 \cos^3\theta \right] \\ &+ O(\lambda^3), \end{aligned} \quad (\text{C51})$$

with the function  $m_\parallel(r)$  given by

$$m_\parallel(r) = 1 + \frac{\lambda^2}{3} \ln \frac{\ell}{r}. \quad (\text{C52})$$

Note that Eq. (C51) reproduces the angular dependence of Eq. (10). This perturbative expansion is the first step of a renormalization group treatment done by introducing an arbitrary length scale  $r'$  and writing

$$m_\parallel(r) = m_\parallel(r') \left( 1 + \frac{\lambda^2}{3} \ln \frac{r'}{r} \right), \quad (\text{C53})$$

which is valid up to order of  $O(\lambda^2)$ . The renormalization group equation  $\partial_{r'} m_\parallel(r) = 0$ , therefore, becomes

$$\partial_{r'} m_\parallel(r') + \frac{\lambda^2}{3r'} m_\parallel(r') = 0, \quad (\text{C54})$$

since the term scaling as  $O(\lambda^2 \partial_{r'} m_\parallel(r'))$  can be neglected to the considered order. Equation (C54) finally leads to

$$\delta\rho(\mathbf{r}) \propto \frac{1}{r^{2+\lambda^2/3}}, \quad (\text{C55})$$

which reproduces the result of Eq. (9).

#### b. Obstacle with no axis of symmetry

The situation is different when the obstacle does not have an axis of symmetry. In that case, we decompose  $\mathbf{c} = c_\parallel \mathbf{p} + \mathbf{c}_\perp$  with  $\mathbf{p} \cdot \mathbf{c}_\perp = 0$ . We isolate the logarithmically diverging contributions and split the different terms according to

$$\begin{aligned} \delta\rho(\mathbf{r}) = & -\frac{c_{\parallel}}{4\pi D r^2} \left(1 + \frac{\lambda^2}{3} \ln \frac{\ell}{r}\right) \left(\cos\theta - \frac{\lambda}{4}(3 - 5\cos^2\theta) + \frac{3}{4}\lambda^2 \cos^3\theta\right) \\ & - \frac{1}{4\pi D r^2} \left(1 - \frac{\lambda^2}{12} \ln \frac{\ell}{r}\right) c_{\perp}^{\alpha} \left[\hat{r}^{\alpha} - \frac{\lambda}{4}(3\hat{p}^{\alpha} - 5\hat{r}^{\alpha}(\hat{\mathbf{p}} \cdot \hat{\mathbf{r}})) + \frac{3}{4}\lambda^2 \hat{r}^{\alpha}(\hat{\mathbf{p}} \cdot \hat{\mathbf{r}})^2\right] \\ & - \frac{1}{4\pi D r^2} c^{\mu} \left[-\frac{13\lambda^2}{45} \delta^{\mu\alpha} + \frac{\lambda}{4\pi} \hat{J}_1^{\alpha\mu} + \left(\frac{\lambda}{4\pi}\right)^2 \hat{K}_1^{\alpha\mu} - \frac{49}{45} \lambda^2 \hat{p}^{\alpha} \hat{p}^{\mu}\right] \left[\hat{r}^{\alpha} - \frac{\lambda}{4}(3\hat{p}^{\alpha} - 5\hat{r}^{\alpha}(\hat{\mathbf{p}} \cdot \hat{\mathbf{r}})) + \frac{3}{4}\lambda^2 \hat{r}^{\alpha}(\hat{\mathbf{p}} \cdot \hat{\mathbf{r}})^2\right]. \end{aligned} \quad (\text{C56})$$

Up to order of  $O(\lambda^2)$ , we, therefore, obtain

$$\begin{aligned} \delta\rho(\mathbf{r}) = & -\frac{c_{\parallel}}{4\pi D r^2} \left(\frac{\ell}{r}\right)^{\lambda^2/3} \left(\cos\theta - \frac{\lambda}{4}(3 - 5\cos^2\theta) + \frac{3}{4}\lambda^2 \cos^3\theta\right) \\ & - \frac{1}{4\pi D r^2} \left(\frac{\ell}{r}\right)^{-\lambda^2/12} c_{\perp}^{\alpha} \left[\hat{r}^{\alpha} - \frac{\lambda}{4}(3\hat{p}^{\alpha} - 5\hat{r}^{\alpha}(\hat{\mathbf{p}} \cdot \hat{\mathbf{r}})) + \frac{3}{4}\lambda^2 \hat{r}^{\alpha}(\hat{\mathbf{p}} \cdot \hat{\mathbf{r}})^2\right] \\ & - \frac{1}{4\pi D r^2} c^{\mu} \left[-\frac{13\lambda^2}{45} \delta^{\mu\alpha} + \frac{\lambda}{4\pi} \hat{J}_1^{\alpha\mu} + \left(\frac{\lambda}{4\pi}\right)^2 \hat{K}_1^{\alpha\mu} - \frac{49}{45} \lambda^2 \hat{p}^{\alpha} \hat{p}^{\mu}\right] \left[\hat{r}^{\alpha} - \frac{\lambda}{4}(3\hat{p}^{\alpha} - 5\hat{r}^{\alpha}(\hat{\mathbf{p}} \cdot \hat{\mathbf{r}})) + \frac{3}{4}\lambda^2 \hat{r}^{\alpha}(\hat{\mathbf{p}} \cdot \hat{\mathbf{r}})^2\right]. \end{aligned} \quad (\text{C57})$$

Hence, the second line of the right-hand side dominates in the far field, and we obtain

$$\delta\rho(\mathbf{r}) \propto \frac{1}{r^{2-\lambda^2/12}} \cos(\phi + \phi_0) \sin\theta \left(1 + \frac{5}{4}\lambda \cos\theta + \frac{3}{4}\lambda^2 \cos^2\theta\right), \quad (\text{C58})$$

which reproduces Eqs. (11) and (12) and where the phase  $\phi_0$  is such that  $\hat{\mathbf{r}} \cdot \mathbf{c}_{\perp} = |\mathbf{c}_{\perp}| \sin\theta \cos(\phi + \phi_0)$ .

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- [1] M. Cristina Marchetti, Jean-François Joanny, Sriram Ramaswamy, Tanniemola B. Liverpool, Jacques Prost, Madan Rao, and R. Aditi Simha, *Hydrodynamics of soft active matter*, *Rev. Mod. Phys.* **85**, 1143 (2013).
- [2] Sriram Ramaswamy, *The mechanics and statistics of active matter*, *Annu. Rev. Condens. Matter Phys.* **1**, 323 (2010).
- [3] Jacques Prost, Frank Jülicher, and Jean-François Joanny, *Active gel physics*, *Nat. Phys.* **11**, 111 (2015).
- [4] Clemens Bechinger, Roberto Di Leonardo, Hartmut Löwen, Charles Reichhardt, Giorgio Volpe, and Giovanni Volpe, *Active particles in complex and crowded environments*, *Rev. Mod. Phys.* **88**, 045006 (2016).
- [5] Lokrshi Prawar Dadhichi, Ananyo Maitra, and Sriram Ramaswamy, *Origins and diagnostics of the nonequilibrium character of active systems*, *J. Stat. Mech.* (2018) 123201.
- [6] Étienne Fodor and M. Cristina Marchetti, *The statistical physics of active matter: From self-catalytic colloids to living cells*, *Physica (Amsterdam)* **504A**, 106 (2018).
- [7] Hugues Chaté, *Dry aligning dilute active matter*, *Annu. Rev. Condens. Matter Phys.* **11**, 189 (2020).
- [8] Julien Tailleur, Gerhard Gompper, M. Cristina Marchetti, Julia M. Yeomans, and Christophe Salomon, in *Active Matter and Nonequilibrium Statistical Physics*, Proceedings of the Les Houches Summer School, Volume 112, 2018 (Oxford University, New York, 2022).
- [9] Jérémy O'Byrne, Yariv Kafri, Julien Tailleur, and Frédéric van Wijland, *Time irreversibility in active matter, from micro to macro*, *Nat. Rev. Phys.* **4**, 167 (2022).

- [10] Alexandre P. Solon, Yaouen Fily, Aparna Baskaran, Mickael E. Cates, Yariv Kafri, Mehran Kardar, and J. Tailleur, *Pressure is not a state function for generic active fluids*, *Nat. Phys.* **11**, 673 (2015).
- [11] Eric Woillez, Yariv Kafri, and Vivien Lecomte, *Nonlocal stationary probability distributions and escape rates for an active Ornstein-Uhlenbeck particle*, *J. Stat. Mech.* (2020) 063204.
- [12] Eric Woillez, Yongfeng Zhao, Yariv Kafri, Vivien Lecomte, and Julien Tailleur, *Activated escape of a self-propelled particle from a metastable state*, *Phys. Rev. Lett.* **122**, 258001 (2019).
- [13] Omer Granek, Yariv Kafri, Mehran Kardar, Sunghan Ro, Julien Tailleur, and Alexandre Solon, *Colloquium: Inclusions, boundaries, and disorder in scalar active matter*, *Rev. Mod. Phys.* **96**, 031003 (2024).
- [14] Peter Galajda, Juan Keymer, Paul Chaikin, and Robert Austin, *A wall of funnels concentrates swimming bacteria*, *J. Bacteriol.* **189**, 8704 (2007).
- [15] Julien Tailleur and M. E. Cates, *Sedimentation, trapping, and rectification of dilute bacteria*, *Europhys. Lett.* **86**, 60002 (2009).
- [16] Charles Wolgemuth, Egbert Hoiczyk, Dale Kaiser, and George Oster, *How myxobacteria glide*, *Curr. Biol.* **12**, 369 (2002).
- [17] Vijay Narayan, Sriram Ramaswamy, and Narayanan Menon, *Long-lived giant number fluctuations in a swarming granular nematic*, *Science* **317**, 105 (2007).
- [18] Julien Deseigne, Olivier Dauchot, and Hugues Chaté, *Collective motion of vibrated polar disks*, *Phys. Rev. Lett.* **105**, 098001 (2010).
- [19] Barath Ezhilan, Roberto Alonso-Matilla, and David Saintillan, *On the distribution and swim pressure of*

- run-and-tumble particles in confinement*, *J. Fluid Mech.* **781**, R4 (2015).
- [20] Caleb G. Wagner, Michael F. Hagan, and Aparna Baskaran, *Steady-state distributions of ideal active Brownian particles under confinement and forcing*, *J. Stat. Mech.* (2017) 043203.
- [21] Naftali R. Smith, Pierre Le Doussal, Satya N. Majumdar, and Grégory Schehr, *Exact position distribution of a harmonically confined run-and-tumble particle in two dimensions*, *Phys. Rev. E* **106**, 054133 (2022).
- [22] A. Kaiser, H. H. Wensink, and H. Löwen, *How to capture active particles*, *Phys. Rev. Lett.* **108**, 268307 (2012).
- [23] Nitin Kumar, Rahul Kumar Gupta, Harsh Soni, Sriram Ramaswamy, and A. K. Sood, *Trapping and sorting active particles: Motility-induced condensation and smectic defects*, *Phys. Rev. E* **99**, 032605 (2019).
- [24] Thibaut Arnoux de Pirey and Frédéric van Wijland, *A run-and-tumble particle around a spherical obstacle: The steady-state distribution far-from-equilibrium*, *J. Stat. Mech.* (2023) 093202.
- [25] Yongjoo Baek, Alexandre P. Solon, Xinpeng Xu, Nikolai Nikola, and Yariv Kafri, *Generic long-range interactions between passive bodies in an active fluid*, *Phys. Rev. Lett.* **120**, 058002 (2018).
- [26] Omer Granek, Yongjoo Baek, Yariv Kafri, and Alexandre P. Solon, *Bodies in an interacting active fluid: Far-field influence of a single body and interaction between two bodies*, *J. Stat. Mech.* (2020) 063211.
- [27] Thomas Speck and Ashreya Jayaram, *Vorticity determines the force on bodies immersed in active fluids*, *Phys. Rev. Lett.* **126**, 138002 (2021).
- [28] Ran Ni, Martien A. Cohen Stuart, and Peter G. Bolhuis, *Tunable long-range forces mediated by self-propelled colloidal hard spheres*, *Phys. Rev. Lett.* **114**, 018302 (2015).
- [29] Sunghan Ro, Yariv Kafri, Mehran Kardar, and Julien Tailleur, *Disorder-induced long-ranged correlations in scalar active matter*, *Phys. Rev. Lett.* **126**, 048003 (2021).
- [30] Ydan Ben Dor, Sunghan Ro, Yariv Kafri, Mehran Kardar, and Julien Tailleur, *Disordered boundaries destroy bulk phase separation in scalar active matter*, *Phys. Rev. E* **105**, 044603 (2022).
- [31] J. Tailleur and M. E. Cates, *Statistical mechanics of interacting run-and-tumble bacteria*, *Phys. Rev. Lett.* **100**, 218103 (2008).
- [32] Yaouen Fily and M. Cristina Marchetti, *Athermal phase separation of self-propelled particles with no alignment*, *Phys. Rev. Lett.* **108**, 235702 (2012).
- [33] Gabriel S. Redner, Michael F. Hagan, and Aparna Baskaran, *Structure and dynamics of a phase-separating active colloidal fluid*, *Phys. Rev. Lett.* **110**, 055701 (2013).
- [34] Michael E. Cates and Julien Tailleur, *Motility-induced phase separation*, *Annu. Rev. Condens. Matter Phys.* **6**, 219 (2015).
- [35] R. Aditi Simha and Sriram Ramaswamy, *Hydrodynamic fluctuations and instabilities in ordered suspensions of self-propelled particles*, *Phys. Rev. Lett.* **89**, 058101 (2002).
- [36] R. Voituriez, Jean-François Joanny, and Jacques Prost, *Spontaneous flow transition in active polar gels*, *Europhys. Lett.* **70**, 404 (2005).
- [37] Ananyo Maitra, *Two-dimensional long-range uniaxial order in three-dimensional active fluids*, *Nat. Phys.* **19**, 733 (2023).
- [38] Adriano Tiribocchi, Raphael Wittkowski, Davide Marenduzzo, and Michael E. Cates, *Active model h: Scalar active matter in a momentum-conserving fluid*, *Phys. Rev. Lett.* **115**, 188302 (2015).
- [39] Aparna Baskaran and M. Cristina Marchetti, *Statistical mechanics and hydrodynamics of bacterial suspensions*, *Proc. Natl. Acad. Sci. U.S.A.* **106**, 15567 (2009).
- [40] Daisuke Takagi, Jérémie Palacci, Adam B. Braunschweig, Michael J. Shelley, and Jun Zhang, *Hydrodynamic capture of microswimmers into sphere-bound orbits*, *Soft Matter* **10**, 1784 (2014).
- [41] Saverio E. Spagnolie, Gregorio R. Moreno-Flores, Denis Bartolo, and Eric Lauga, *Geometric capture and escape of a microswimmer colliding with an obstacle*, *Soft Matter* **11**, 3396 (2015).
- [42] Gao-Jin Li and Arezoo M. Ardekani, *Hydrodynamic interaction of microswimmers near a wall*, *Phys. Rev. E* **90**, 013010 (2014).
- [43] Jens Elgeti and Gerhard Gompper, *Microswimmers near surfaces*, *Eur. Phys. J. Special Topics* **225**, 2333 (2016).
- [44] B. U. Felderhof, *Hydrodynamic interaction between two spheres*, *Physica (Amsterdam)* **89A**, 373 (1977).
- [45] Nigel Goldenfeld, *Lectures on Phase Transitions and the Renormalization Group* (CRC Press, Boca Raton, 2018).
- [46] Grigory Isaakovich Barenblatt, *Scaling, Self-Similarity, and Intermediate Asymptotics: Dimensional Analysis and Intermediate Asymptotics* (Cambridge University Press, Cambridge, England, 1996), No. 14.
- [47] Indeed, when  $A = 0$ , the equation is left invariant by rescaling space according to  $\mathbf{r} \rightarrow \mathbf{r}' = b^{-1}\mathbf{r}$  and the density field according to  $\delta\rho(\mathbf{r}) \rightarrow \delta\rho'(\mathbf{r}') = b^2\delta\rho(\mathbf{r})$ . Implementing the same rescaling in Eq. (6), we obtain the following equation:
- $$D\Delta'\delta\rho' + Ab^{1-\chi}\nabla \cdot (\mathbf{r}'^{-\chi}\mathbf{g}(\hat{\mathbf{r}})\delta\rho') = -\mathbf{c} \cdot \nabla'\delta(\mathbf{r}').$$
- Here, coupling to hydrodynamics is irrelevant, in the sense of the renormalization group, if  $\chi > 1$  and is marginal for  $\chi = 1$ .
- [48] Constantine Pozrikidis, *Boundary Integral and Singularity Methods for Linearized Viscous Flow* (Cambridge University Press, Cambridge, England, 1992).
- [49] The same derivation can be repeated with rotational diffusion with the conclusions unchanged.
- [50] Howard A. Stone and Aravinthan D. T. Samuel, *Propulsion of microorganisms by surface distortions*, *Phys. Rev. Lett.* **77**, 4102 (1996).
- [51] If the obstacle has an axis of rotational symmetry, and the rest of the system is completely isotropic (which assumes that the bulk suspension does not exhibit orientational order), then any nonvanishing vector built from the steady-state distributions of stresses and positions of the active particles must be along this axis.
- [52] Eric Lauga and Thomas R. Powers, *The hydrodynamics of swimming microorganisms*, *Rep. Prog. Phys.* **72**, 096601 (2009).
- [53] Rahul Chajwa, Narayanan Menon, Sriram Ramaswamy, and Rama Govindarajan, *Waves, algebraic growth, and*

- clumping in sedimenting disk arrays*, *Phys. Rev. X* **10**, 041016 (2020).
- [54] Ananyo Maitra, Pragya Srivastava, M. Cristina Marchetti, Sriram Ramaswamy, and Martin Lenz, *Swimmer suspensions on substrates: Anomalous stability and long-range order*, *Phys. Rev. Lett.* **124**, 028002 (2020).
- [55] Tommaso Brotto, Jean-Baptiste Caussin, Eric Lauga, and Denis Bartolo, *Hydrodynamics of confined active fluids*, *Phys. Rev. Lett.* **110**, 038101 (2013).
- [56] Nitin Kumar, Harsh Soni, Sriram Ramaswamy, and A. K. Sood, *Flocking at a distance in active granular matter*, *Nat. Commun.* **5**, 4688 (2014).
- [57] Suropriya Saha, Sriram Ramaswamy, and Ramin Golestanian, *Pairing, waltzing and scattering of chemotactic active colloids*, *New J. Phys.* **21**, 063006 (2019).
- [58] Ydan Ben Dor, Eric Woillez, Yariv Kafri, Mehran Kardar, and Alexandre P. Solon, *Ramifications of disorder on active particles in one dimension*, *Phys. Rev. E* **100**, 052610 (2019).
- [59] Ydan Ben Dor, Yariv Kafri, Mehran Kardar, and Julien Tailleur, *Passive objects in confined active fluids: A localization transition*, *Phys. Rev. E* **106**, 044604 (2022).