arXiv:2301.03159v1 [math.FA] 9 Jan 2023

NUMERICAL RADIUS INEQUALITIES AND ESTIMATION OF ZEROS OF POLYNOMIALS

SUVENDU JANA[1], PINTU BHUNIA[2] and KALLOL PAUL[2]

Abstract. Let A be a bounded linear operator defined on a complex Hilbert space and let $|A|=(A^*A)^{1/2}$ be the positive square root of A. Among other refinements of the well known numerical radius inequality $w^2(A)\leq \frac{1}{2}\|A^*A+AA^*\|$, we show that

$$\begin{aligned} w^2(A) &\leq \frac{1}{4}w^2\left(|A|+i|A^*|\right)+\frac{1}{8}\left\||A|^2+|A^*|^2\right\|+\frac{1}{4}w\left(|A||A^*|\right)\\ &\leq \frac{1}{2}\|A^*A+AA^*\|. \end{aligned}$$

Also, we develop inequalities involving numerical radius and spectral radius for the sum of the product operators, from which we derive the following inequalities

$$w^p(A)\leq \frac{1}{\sqrt{2}}w(|A|^p+i|A^*|^p)\leq \|A\|^p$$

for all $p\geq 1$. Further, we derive new bounds for the zeros of complex polynomials.

1. Introduction

Let $\mathscr{H}$ be a complex Hilbert space with usual inner product $\langle\cdot,\cdot\rangle$ and the corresponding norm $\|\cdot\|$ induced by the inner product. Let $\mathbb{B}(\mathscr{H})$ denote the C^*-algebra of all bounded linear operators on $\mathscr{H}$. For $A\in\mathbb{B}(\mathscr{H})$, $|A|=(A^*A)^{1/2}$ is the positive square root of A. The numerical range of A, denoted as $W(A)$, is defined by $W(A)=\{\langle Ax,x\rangle : x\in\mathscr{H}, \|x\|=1\}$. Let $\|A\|$, $r(A)$ and $w(A)$ denote the operator norm, the spectral radius and the numerical radius of A, respectively. Recall that $w(A)=\sup\{|\langle Ax,x\rangle| : x\in\mathscr{H}, \|x\|=1\}$. The numerical radius $w(\cdot)$ defines a norm on $\mathbb{B}(\mathscr{H})$, (is equivalent to the operator norm $\|\cdot\|$) is satisfying the following inequality

$$\frac{1}{2}\|A\|\leq w(A)\leq \|A\|. \tag{1.1}$$

The first inequality becomes equality if $A^2=0$ and the second one turns into equality if A is normal. Similar as the operator norm, numerical radius also satisfies the power

2020 *Mathematics Subject Classification.* Primary 47A12, 26C10 Secondary 47A30, 30C15.

Key words and phrases. Numerical radius, Operator norm, Frobenius companion matrix, Zeros of a polynomial.

Pintu Bhunia would like to thank UGC, Govt. of India for the financial support in the form of SRF under the mentorship of Prof. Kallol Paul.

inequality:

$$w(A^n) \leq w^n(A) \text{ for every } n = 1, 2, 3, \ldots. \tag{1.2}$$

It is well known that for $A \in \mathbb{B}(\mathscr{H})$,

$$r(A) \leq w(A). \tag{1.3}$$

The inequality (1.3) is sharp. In fact, if A is normal, then $r(A) = w(A) = \|A\|$. For $A, B \in \mathbb{B}(\mathscr{H})$, we have $r(AB) = r(BA)$ and $r(A^n) = r^n(A)$ for every positive integer n. Over the years many eminent mathematicians have studied various refinements of (1.1) and obtained various bounds for the zeros of a complex polynomial, we refer the readers to [2, 3, 5, 7, 15, 18, 22, 23] and the references therein. In [14], Kittaneh improved the inequalities in (1.1) to prove that

$$\frac{1}{4}\|A^*A + AA^*\| \leq w^2(A) \leq \frac{1}{2}\|A^*A + AA^*\|. \tag{1.4}$$

In this article, we develop new refinements of the second inequality in (1.4). We obtain inequalities involving numerical radius and spectral radius of the sum of the product operators, from which we achieve a nice refinement of the classical inequality $w(A) \leq \|A\|$. As application of the numerical radius inequalities, we give new bounds for the zeros of a complex monic polynomial which improve on the existing ones.

2. Numerical radius inequalities

We begin the section with the following lemmas.

Lemma 2.1. [13]*(Generalized Cauchy-Schwarz inequality) If $A \in \mathcal{B}(\mathscr{H})$ and $0 \leq \alpha \leq 1$, then*

$$|\langle Ax, y\rangle|^2 \leq \langle |A|^{2\alpha}x, x\rangle\langle |A^*|^{2(1-\alpha)}y, y\rangle$$

for all $x, y \in \mathscr{H}$.

Lemma 2.2. [21]*(Holder-McCarthy inequality) Let $A \in \mathcal{B}(\mathscr{H})$ be positive. Then the following inequalities hold:*

$$\langle A^r x, x\rangle \geq \|x\|^{2(1-r)}\langle Ax, x\rangle^r, \quad \text{when } r \geq 1$$

$$\langle A^r x, x\rangle \leq \|x\|^{2(1-r)}\langle Ax, x\rangle^r, \quad \text{when } 0 \leq r \leq 1$$

for any $x \in \mathscr{H}$.

Lemma 2.3. [9]*(Buzano's inequality) Let $x, e, y \in \mathscr{H}$ with $\|e\| = 1$, then*

$$|\langle x, e\rangle\langle e, y\rangle| \leq \frac{1}{2}\left(\|x\|\|y\| + |\langle x, y\rangle|\right).$$

Now, we are in a position to present our results. First we develop the following upper bound for the numerical radius.

Theorem 2.4. *Let $A \in \mathcal{B}(\mathscr{H})$. Then*

$$w^2(A) \leq \frac{1}{4}w^2\left(|A| + i|A^*|\right) + \frac{1}{8}\left\||A|^2 + |A^*|^2\right\| + \frac{1}{4}w\left(|A||A^*|\right).$$

Proof. Let $x \in \mathscr{H}$ with $\|x\| = 1$. Then we have

$$\begin{aligned}
& |\langle Ax, x\rangle|^2 \\
\leq\ & \langle |A|x, x\rangle\langle |A^*|x, x\rangle \ (\textit{by Lemma 2.1}) \\
\leq\ & \frac{1}{4}\left(\langle |A|x, x\rangle + \langle |A^*|x, x\rangle\right)^2 \\
=\ & \frac{1}{4}\left(\langle |A|x, x\rangle^2 + \langle |A^*|x, x\rangle^2 + 2\langle |A|x, x\rangle\langle |A^*|x, x\rangle\right) \\
\leq\ & \frac{1}{4}\left\{|\langle |A|x, x\rangle + i\langle |A^*|x, x\rangle|^2 + \||A|x\|\||A^*|x\| + |\langle |A|x, |A^*|x\rangle|\right\} \ (\textit{by Lemma 2.3}) \\
\leq\ & \frac{1}{4}\left\{|\langle (|A| + i|A^*|)x, x\rangle|^2 + \frac{1}{2}\||A|x\|^2 + \frac{1}{2}\||A^*|x\|^2 + |\langle |A^*||A|x, x\rangle|\right\} \\
\leq\ & \frac{1}{4}w^2\left(|A| + i|A^*|\right) + \frac{1}{8}\left\||A|^2 + |A^*|^2\right\| + \frac{1}{4}w\left(|A||A^*|\right).
\end{aligned}$$

Taking supremum over all $x \in \mathcal{H}$ with $\|x\| = 1$, we get the desired inequality. □

Clearly, we see that

$$\begin{aligned}
& \frac{1}{4}w^2\left(|A| + i|A^*|\right) + \frac{1}{8}\left\||A|^2 + |A^*|^2\right\| + \frac{1}{4}w\left(|A||A^*|\right) \\
\leq\ & \frac{1}{4}\left\||A|^2 + |A^*|^2\right\| + \frac{1}{8}\left\||A|^2 + |A^*|^2\right\| + \frac{1}{4}\left\||A||A^*|\right\| \\
=\ & \frac{3}{8}\left\||A|^2 + |A^*|^2\right\| + \frac{1}{4}\left\|A^2\right\| \\
\leq\ & \frac{3}{8}\left\||A|^2 + |A^*|^2\right\| + \frac{1}{8}\left\||A|^2 + |A^*|^2\right\| \\
=\ & \frac{1}{2}\left\||A|^2 + |A^*|^2\right\|.
\end{aligned}$$

Thus, we would like to remark that the upper bound obtained in Theorem 2.4 refines the second inequality in (1.4). Next result reads as follows.

Theorem 2.5. *Let $X, Y \in \mathbb{B}(\mathscr{H})$, and $0 \leq \alpha \leq 1$, $0 \leq \beta \leq 1$. Then for each $x \in \mathscr{H}$ with $\|x\| = 1$,*

$$\begin{aligned}
& |\langle Xx, x\rangle\langle Yx, x\rangle| \\
\leq\ & \frac{1}{4}\left\|\alpha|X|^2 + (1-\alpha)|X^*|^2 + \beta|Y|^2 + (1-\beta)|Y^*|^2\right\| + \frac{1}{8}\left\||X|^2 + |Y^*|^2\right\| + \frac{1}{4}w(YX).
\end{aligned}$$

Proof. We have

$$
\begin{aligned}
& |\langle Xx, x\rangle\langle Yx, x\rangle| \\
\leq\ & \frac{1}{4}\{|\langle Xx, x\rangle| + |\langle Yx, x\rangle|\}^2 \\
=\ & \frac{1}{4}\left\{|\langle Xx, x\rangle|^2 + |\langle Yx, x\rangle|^2 + 2|\langle Xx, x\rangle||\langle Yx, x\rangle|\right\} \\
\leq\ & \frac{1}{4}\left\{\langle |X|^{2\alpha}x, x\rangle\langle |X^*|^{2(1-\alpha)}x, x\rangle + \langle |Y|^{2\beta}x, x\rangle\langle |Y^*|^{2(1-\beta)}x, x\rangle + 2|\langle Xx, x\rangle||\langle x, Y^*x\rangle|\right\} \\
& \quad (\textit{using Lemma 2.1}) \\
\leq\ & \frac{1}{4}\left\{\langle |X|^2x, x\rangle^{\alpha}\langle |X^*|^2x, x\rangle^{(1-\alpha)} + \langle |Y|^2x, x\rangle^{\beta}\langle |Y^*|^2x, x\rangle^{(1-\beta)} + \|Xx\|\|Y^*x\| + |\langle Xx, Y^*x\rangle|\right\} \\
& \quad (\textit{using Lemma 2.2 and Lemma 2.3}) \\
\leq\ & \frac{1}{4}\left\{\alpha\langle |X|^2x, x\rangle + (1-\alpha)\langle |X^*|^2x, x\rangle + \beta\langle |Y|^2x, x\rangle + (1-\beta)\langle |Y^*|^2x, x\rangle\right\} \\
& + \frac{1}{4}\left\{\frac{1}{2}\left(\langle |X|^2x, x\rangle + \langle |Y^*|^2x, x\rangle\right) + |\langle YXx, x\rangle|\right\} \\
\leq\ & \frac{1}{4}\|\alpha|X|^2 + (1-\alpha)|X^*|^2 + \beta|Y|^2 + (1-\beta)|Y^*|^2\| + \frac{1}{8}\||X|^2 + |Y^*|^2\| + \frac{1}{4}w(YX).
\end{aligned}
$$

□

Applying the inequality in Theorem 2.5 we derive the following upper bound for the numerical radius.

Corollary 2.6. *If $A \in \mathcal{B}(\mathscr{H})$, then*

$$w^2(A) \leq \frac{1}{4}\left\|\mu|A|^2 + (2-\mu)|A^*|^2\right\| + \frac{1}{8}\left\||A|^2 + |A^*|^2\right\| + \frac{1}{4}w(A^2),$$

for $0 \leq \mu \leq 2$.

Proof. Putting $X = Y = A$ in Theorem 2.5, and then taking supremum over all $x \in \mathscr{H}$ with $\|x\| = 1$, we get

$$w^2(A) \leq \frac{1}{4}\left\|(\alpha+\beta)|A|^2 + (2-\alpha-\beta)|A^*|^2\right\| + \frac{1}{8}\left\||A|^2 + |A^*|^2\right\| + \frac{1}{4}w(A^2),$$

for $0 \leq \alpha, \beta \leq 1$. This implies the desired bound. □

It follows from Corollary 2.6 that

$$w^2(A) \leq \frac{1}{4}\min_{\mu\in[0,2]}\left\|\mu|A|^2 + (2-\mu)|A^*|^2\right\| + \frac{1}{8}\left\||A|^2 + |A^*|^2\right\| + \frac{1}{4}w(A^2). \tag{2.1}$$

Remark 2.7. Clearly, We have

$$\begin{aligned}
&\min_{\mu\in[0,2]} \frac{1}{4}\left\|\mu|A|^2+(2-\mu)|A^*|^2\right\| + \frac{1}{8}\left\||A|^2+|A^*|^2\right\| + \frac{1}{4}w(A^2)\\
\leq\ & \frac{1}{4}\||A|^2+|A^*|^2\| + \frac{1}{8}\||A|^2+|A^*|^2\| + \frac{1}{4}w(A^2) \ \ (\textit{by taking } \mu=1)\\
=\ & \frac{3}{8}\||A|^2+|A^*|^2\| + \frac{1}{4}w(A^2)\\
\leq\ & \frac{3}{8}\||A|^2+|A^*|^2\| + \frac{1}{4}w^2(A)\\
\leq\ & \frac{3}{8}\||A|^2+|A^*|^2\| + \frac{1}{8}\||A|^2+|A^*|^2\| \ \ (\textit{using the second inequality of } (1.4))\\
=\ & \frac{1}{2}\||A|^2+|A^*|^2\|.
\end{aligned}$$

Thus, we would like to remark that inequality (2.1) is stronger than that in (1.4). We also note that the minimum value is not always attained for $\mu = 1$. For example, consider the matrix $A = \begin{pmatrix} 0 & 1 & 0\\ 0 & 0 & 2\\ 0 & 0 & 0\end{pmatrix}$. Then, $\min_{\mu\in[0,2]} \|\mu|A|^2+(2-\mu)|A^*|^2\| = \frac{32}{7}$ for $\mu = \frac{8}{7}$, and we see that

$$\begin{aligned}
\frac{1}{4}\min_{\mu\in[0,2]}\left\|\mu|A|^2+(2-\mu)|A^*|^2\right\| + \frac{1}{8}\left\||A|^2+|A^*|^2\right\| + \frac{1}{4}w(A^2) &= \frac{113}{56} \approx 2.01785714\\
&< \frac{5}{2} = \frac{1}{2}\||A|^2+|A^*|^2\|.
\end{aligned}$$

To prove our next result we need the following two lemmas. First one is a generalization of the inequality in Lemma 2.1, and the second one is known as Bohr's inequality.

Lemma 2.8. *([19, Th. 5]) Let $A, B \in \mathbb{B}(\mathscr{H})$ with $|A|B = B^*|A|$. Let f, g be two non-negative continuous functions on $[0,\infty)$ such that $f(t)g(t) = t$ for all $t \geq 0$. Then*

$$|\langle ABx, y\rangle| \leq r(B)\|f(|A|)x\|\|g(|A^*|)y\|,$$

for all $x, y \in \mathscr{H}$.

Lemma 2.9. *([24]) For $i = 1, 2, \cdots, n$, let $a_i \geq 0$. Then*

$$\left(\sum_{i=1}^{n} a_i\right)^p \leq n^{p-1}\sum_{i=1}^{n} a_i^p,$$

for all $p \geq 1$.

By using the above lemmas we prove the following inequality involving numerical radius and spectral radius.

Theorem 2.10. *Let $A_i, B_i \in \mathbb{B}(\mathscr{H})$ be such that $|A_i|B_i = B_i^*|A_i|$ for $i = 1, 2, \cdots, n$. Then*

$$w^p\left(\sum_{i=1}^n A_iB_i\right) \leq \frac{n^{p-1}}{\sqrt{2}} w\left(\sum_{i=1}^n r^p(B_i)\left(f^{2p}(|A_i|) + ig^{2p}(|A_i^*|)\right)\right),$$

for all $p \geq 1$.

Proof. Let $x \in \mathscr{H}$ with $\|x\| = 1$. Then we have

$$\begin{aligned}
& \left|\langle\left(\sum_{i=1}^n A_iB_i\right)x, x\rangle\right|^p \\
= & \left|\sum_{i=1}^n \langle A_iB_ix, x\rangle\right|^p \\
\leq & \left(\sum_{i=1}^n |\langle A_iB_ix, x|\right)^p \\
\leq & \left(\sum_{i=1}^n r(B_i)\|f(|A_i|)x\|\|g(|A_i^*|)x\|\right)^p \quad (\textit{by Lemma 2.8}) \\
= & \left(\sum_{i=1}^n r(B_i)\langle f^2(|A_i|)x, x\rangle^{\frac{1}{2}}\langle g^2(|A_i^*|)x, x\rangle^{\frac{1}{2}}\right)^p \\
\leq & \left(\sum_{i=1}^n r(B_i)\frac{\langle f^2(|A_i|)x, x\rangle + \langle g^2(|A_i^*|)x, x\rangle}{2}\right)^p \\
\leq & n^{p-1}\sum_{i=1}^n r^p(B_i)\left(\frac{\langle f^2(|A_i|)x, x\rangle + \langle g^2(|A_i^*|)x, x\rangle}{2}\right)^p \quad (\textit{by Lemma 2.9}) \\
\leq & \frac{n^{p-1}}{2}\sum_{i=1}^n r^p(B_i)\left(\langle f^2(|A_i|)x, x\rangle^p + \langle g^2(|A_i^*|)x, x\rangle^p\right) \quad (\textit{by convexity of } f(t) = t^p) \\
\leq & \frac{n^{p-1}}{2}\sum_{i=1}^n r^p(B_i)\left(\langle f^{2p}(|A_i|)x, x\rangle + \langle g^{2p}(|A_i^*|)x, x\rangle\right) \quad (\textit{by Lemma 2.2}) \\
\leq & \frac{n^{p-1}}{\sqrt{2}}\left|\sum_{i=1}^n r^p(B_i)\left(\langle f^{2p}(|A_i|)x, x\rangle + i\langle g^{2p}(|A_i^*|)x, x\rangle\right)\right| \\
& \qquad\qquad (\textit{as } |a+b| \leq \sqrt{2}|a+ib| \textit{ for all } a, b \in \mathbb{R}) \\
\leq & \frac{n^{p-1}}{\sqrt{2}}\left|\sum_{i=1}^n r^p(B_i)\langle\left(f^{2p}(|A_i|) + ig^{2p}(|A_i^*|)\right)x, x\rangle\right| \\
\leq & \frac{n^{p-1}}{\sqrt{2}} w\left(\sum_{i=1}^n r^p(B_i)\left(f^{2p}(|A_i|) + ig^{2p}(|A_i^*|)\right)\right).
\end{aligned}$$

Now, taking supremum over all $x \in \mathscr{H}$, $\|x\| = 1$ we get,

$$w^p\left(\sum_{i=1}^n A_iB_i\right) \le \frac{n^{p-1}}{\sqrt{2}} w\left(\sum_{i=1}^n r^p(B_i)\left(f^{2p}(|A_i|) + ig^{2p}(|A_i^*|)\right)\right).$$

as desired.

□

Observe that the inequality in Theorem 2.10 indeed does not depend on the number n of summands in the case $p = 1$. In particular, considering $p = n = 1$, $A_1 = A$, $B_1 = B$, $f(t) = g(t) = \sqrt{t}$ in Theorem 2.10, we get the following corollary.

Corollary 2.11. *Let $A, B \in \mathbb{B}(\mathscr{H})$ be such that $|A|B = B^*|A|$. Then*

$$w(AB) \le \frac{1}{\sqrt{2}} r(B) w(|A| + i|A^*|).$$

In particular, for $B = I$ we have the following inequality (also obtained in [8]):

$$w(A) \le \frac{1}{\sqrt{2}} w(|A| + i|A^*|). \tag{2.2}$$

Note that the bound (2.2) refines that in (1.4), see [8, Remark 2.16]. Again, considering $B_i = I$ for $i = 1, 2, \cdots, n$ in Theorem 2.10 we have the following inequality for the sum of operators.

Corollary 2.12. *Let $A_i \in \mathbb{B}(\mathscr{H})$ for $i = 1, 2, \cdots, n$, and let f, g be two non-negative continuous functions on $[0, \infty)$ such that $f(t)g(t) = t$ for all $t \ge 0$. Then*

$$w^p\left(\sum_{i=1}^n A_i\right) \le \frac{n^{p-1}}{\sqrt{2}} w\left(\sum_{i=1}^n \left(f^{2p}(|A_i|) + ig^{2p}(|A_i^*|)\right)\right),$$

for all $p \ge 1$.

In particular, for $n = 1$ and $f(t) = g(t) = \sqrt{t}$ in Corollary 2.12, we get the following upper bound for the numerical radius.

Corollary 2.13. *If $A \in \mathbb{B}(\mathscr{H})$, then*

$$w^p(A) \le \frac{1}{\sqrt{2}} w(|A|^p + i|A^*|^p),$$

for all $p \ge 1$.

It is easy to verify that $\frac{1}{\sqrt{2}} w(|A|^p + i|A^*|^p) \le \|A\|^p$ for all $p \ge 1$. Therefore, we would like to remark that Corollary 2.13 improves the classical bound $w(A) \le \|A\|$ for all $p \ge 1$.

At the end of this section, we give a sufficient condition for the equality of $w(A) = \frac{1}{2}\|A^*A + AA^*\|^{1/2}$. For this purpose first we note the following known lemma.

Lemma 2.14. [17] *Let $A, B \in \mathbb{B}(\mathscr{H})$ be positive. Then, $\|A+B\| = \|A\| + \|B\|$ if and only if $\|AB\| = \|A\|\|B\|$.*

Theorem 2.15. *Let $A \in \mathbb{B}(\mathscr{H})$. Then $\|A\|^4 = \|\Re^2(A)\Im^2(A)\|$ implies*

$$w^2(A) = \frac{1}{4}\|A^*A + AA^*\|.$$

Proof. We have

$$\begin{aligned} \|A\|^4 &= \|\Re^2(A)\Im^2(A)\| \leq \|\Re^2(A)\|\|\Im^2(A)\| = \|\Re(A)\|^2\|\Im(A)\|^2 \\ &\leq \frac{1}{2}\left(\|\Re(A)\|^4 + \|\Im(A)\|^4\right) \leq \max\left(\|\Re(A)\|^4, \|\Im(A)\|^4\right) \\ &\leq w^4(A) \leq \|A\|^4. \end{aligned}$$

This implies that

$$\|\Re^2(A)\Im^2(A)\| = \|\Re(A)\|^2\|\Im(A)\|^2. \tag{2.3}$$

Also, we have

$$\frac{1}{2}\left(\|\Re(A)\|^4 + \|\Im(A)\|^4\right) = \max\left(\|\Re(A)\|^4, \|\Im(A)\|^4\right) = w^4(A). \tag{2.4}$$

This implies that

$$\|\Re(A)\| = \|\Im(A)\| = w(A). \tag{2.5}$$

Now, by using lemma 2.14, it follows from the identity (2.3) that

$$\begin{aligned} \frac{1}{2}\|\Re^2(A) + \Im^2(A)\| &= \frac{1}{2}\left(\|\Re^2(A)\| + \|\Im^2(A)\|\right) \\ &= \frac{1}{2}\left(\|\Re(A)\|^2 + \|\Im(A)\|^2\right) \\ &= \|\Re(A)\|^2 = w^2(A) \ (\textit{using } (2.5)). \end{aligned}$$

This completes the proof. □

It should be mentioned here that the converse of Theorem 2.15 is not true, in general. For example, we consider $A = \begin{pmatrix} 0 & 3 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$. Then, $w^2(A) = \frac{1}{4}\|A^*A + AA^*\| = \frac{9}{4}$, however $\|A\|^4 \neq \|\Re^2(A)\Im^2(A)\|$.

3. Estimation of zeros of polynomials

Suppose $p(z) = z^n + a_n z^{n-1} + \ldots + a_2 z + a_1$ is a complex monic polynomial of degree $n \geq 2$ and $a_1 \neq 0$. Location of the zeros of $p(z)$ have been obtained by applying numerical radius inequalities to Frobenius companion matrix of the polynomial $p(z)$. The Frobenius companion matrix of the polynomial $p(z)$ is given by

$$C_p = \begin{pmatrix} -a_n & -a_{n-1} & \cdots & -a_2 & -a_1 \\ 1 & 0 & \cdots & 0 & 0 \\ 0 & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & 1 & 0 \end{pmatrix}.$$

The characteristic polynomial of C_p is the polynomial $p(z)$. Thus, the zeros of $p(z)$ are exactly the eigenvalues of C_p, see [12, p. 316]. The square of C_p is given by

$$C_p^2 = \begin{pmatrix} b_n & b_{n-1} & \cdots & b_3 & b_2 & b_1 \\ -a_n & -a_{n-1} & \cdots & -a_3 & -a_2 & -a_1 \\ 1 & 0 & \cdots & 0 & 0 & 0 \\ 0 & 1 & \cdots & 0 & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & \cdots & 1 & 0 & 0 \end{pmatrix},$$

where $b_j = a_n a_j - a_{j-1}$ for $j = 1, 2, \ldots, n$, with $a_0 = 0$.
Also,

$$C_p^3 = \begin{pmatrix} c_n & c_{n-1} & \cdots & c_4 & c_3 & c_2 & c_1 \\ b_n & b_{n-1} & \cdots & b_4 & b_3 & b_2 & b_1 \\ -a_n & -a_{n-1} & \cdots & -a_4 & -a_3 & -a_2 & -a_1 \\ 1 & 0 & \cdots & 0 & 0 & 0 & 0 \\ 0 & 1 & \cdots & 0 & 0 & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \cdots & 1 & 0 & 0 & 0 \end{pmatrix},$$

where $b_j = a_n a_j - a_{j-1}$ and $c_j = -a_n b_j + a_{n-1} a_j - a_{j-2}$ for $j = 1, 2, \ldots, n$, with $a_0 = a_{-1} = 0$, and

$$C_p^4 = \begin{pmatrix} d_n & d_{n-1} & ..... & d_5 & d_4 & d_3 & d_2 & d_1 \\ c_n & c_{n-1} & ..... & c_5 & c_4 & c_3 & c_2 & c_1 \\ b_n & b_{n-1} & ..... & b_5 & b_4 & b_3 & b_2 & b_1 \\ -a_n & -a_{n-1} & .... & -a_5 & -a_4 & -a_3 & -a_2 & -a_1 \\ 1 & 0 & ... & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & ... & 0 & 0 & 0 & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & .... & 1 & 0 & 0 & 0 & 0 \end{pmatrix},$$

where $b_j = a_n a_j - a_{j-1}$, $c_j = -a_n b_j + a_{n-1}a_j - a_{j-2}$, and $d_j = -a_n c_j - a_{n-1}b_{j-1} + a_{n-2}a_j - a_{j-3}$ for $j = 1, 2, \dots, n$, with $a_0 = a_{-1} = a_{-2} = 0$.

The exact value of $\|C_p\|$ is well known (see in [18]), it is given by

$$\|C_p\| = \sqrt{\frac{\alpha + 1 + \sqrt{(\alpha+1)^2 - 4|a_1|^2}}{2}}, \tag{3.1}$$

where $\alpha = \sum_{j=1}^n |a_j|^2$.

An estimation of $\|C_p^2\|$ obtained in [16] is as follows

$$\|C_p^2\| \leq \sqrt{\frac{\delta + 1 + \sqrt{(\delta-1)^2 + 4\delta'}}{2}}, \tag{3.2}$$

where $\delta = \frac{1}{2}\left(\alpha + \beta + \sqrt{(\alpha-\beta)^2 + 4|\gamma|^2}\right)$ and $\delta' = \frac{1}{2}\left(\alpha' + \beta' + \sqrt{(\alpha'-\beta')^2 + 4|\gamma'|^2}\right)$, $\alpha = \sum_{j=1}^n |a_j|^2$, $\beta = \sum_{j=1}^n |b_j|^2$, $\alpha' = \sum_{j=3}^n |a_j|^2$, $\beta' = \sum_{j=3}^n |b_j|^2$, $\gamma = -\sum_{j=1}^n \bar{a}_j b_j$, $\gamma' = -\sum_{j=3}^n \bar{a}_j b_j$.

We note that

$$\|C_p^2\|^{\frac{1}{2}} \leq \left(\sqrt{\frac{\delta + 1 + \sqrt{(\delta-1)^2 + 4\delta'}}{2}}\right)^{1/2} \leq \sqrt{\frac{\alpha + 1 + \sqrt{(\alpha+1)^2 - 4|a_1|^2}}{2}} = \|C_p\|.$$

Motivated by the above estimation, here we will obtain an estimation of $\|C_p^4\|^{1/4}$. For this purpose first we note the following norm inequality for the sum of two positive operators.

Lemma 3.1. [17] *If $A, B \in \mathbb{B}(\mathscr{H})$ are positive, then*

$$\|A + B\| \leq \frac{1}{2}\left(\|A\| + \|B\| + \sqrt{(\|A\| - \|B\|)^2 + 4\left\|A^{\frac{1}{2}}B^{\frac{1}{2}}\right\|^2}\right).$$

Now, we are in a position to obtain an estimation of $\|C_p^4\|^{1/4}$. Let $C_p^4 = R + S + T$, where

$$R = \begin{pmatrix} d_n & d_{n-1} & ..... & d_5 & d_4 & d_3 & d_2 & d_1 \\ c_n & c_{n-1} & ..... & c_5 & c_4 & c_3 & c_2 & c_1 \\ 0 & 0 & ..... & 0 & 0 & 0 & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & .... & 0 & 0 & 0 & 0 & 0 \end{pmatrix},$$

$$S = \begin{pmatrix} 0 & 0 & ..... & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & ..... & 0 & 0 & 0 & 0 & 0 \\ b_n & b_{n-1} & ..... & b_5 & b_4 & b_3 & b_2 & b_1 \\ -a_n & -a_{n-1} & .... & -a_5 & -a_4 & -a_3 & -a_2 & -a_1 \\ 0 & 0 & ... & 0 & 0 & 0 & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & .... & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

and

$$T = \begin{pmatrix} 0 & 0 & ..... & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & ..... & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & ..... & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & .... & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & ... & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & ... & 0 & 0 & 0 & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & .... & 1 & 0 & 0 & 0 & 0 \end{pmatrix}.$$

Now,

$$\begin{aligned} \|C_p^4\|^2 &= \|R+S+T\|^2 \\ &= \|(R+S+T)^*(R+S+T)\| \\ &= \|R^*R + S^*S + T^*T\| \; (\textit{since } R^*S = R^*T = S^*R = S^*T = T^*R = T^*S = 0) \\ &\leq \|R^*R + S^*S\| + \|T^*T\| \\ &\leq \frac{1}{2}\left(\|R\|^2 + \|S\|^2 + \sqrt{(\|R\|^2 - \|S\|^2)^2 + 4\|RS^*\|^2} \right) + 1 \; (\textit{using Lemma } 3.1). \end{aligned}$$

By simple calculations, we have

$$\begin{aligned} \|R\|^2 &= \|R^*R\| = \|RR^*\| \\ &= \frac{1}{2}\left(\alpha_1 + \beta_1 + \sqrt{(\alpha_1 - \beta_1)^2 + 4|\gamma_1|^2} \right) = \delta_1, \end{aligned}$$

where $\alpha_1=\sum_{j=1}^n|d_j|^2$, $\beta_1=\sum_{j=1}^n|c_j|^2$, $\gamma_1=\sum_{j=1}^n d_j\bar{c_j}$,

$$\begin{aligned}\|S\|^2 &= \|S^*S\|=\|SS^*\|\\ &= \frac{1}{2}\left(\alpha+\beta+\sqrt{(\alpha-\beta)^2+4|\gamma|^2}\right)=\delta,\end{aligned}$$

where $\alpha=\sum_{j=1}^n|a_j|^2$, $\beta=\sum_{j=1}^n|b_j|^2$, $\gamma=-\sum_{j=1}^n b_j\bar{a_j}$,

$$\begin{aligned}&\|RS^*\|^2\\ =&\ \frac{1}{2}\left(|\gamma_2|^2+|\gamma_3|^2+|\gamma_4|^2+|\gamma_5|^2+\sqrt{\left((|\gamma_2|^2+|\gamma_3|^2)-(|\gamma_4|^2+|\gamma_5|^2)\right)^2+4|\gamma_2\bar{\gamma_4}+\gamma_3\bar{\gamma_5}|^2}\right)\\ =&\ \delta_2,\end{aligned}$$

where $\gamma_2=\sum_{j=1}^n d_j\bar{b_j}$, $\gamma_3=\sum_{j=1}^n d_j\bar{a_j}$, $\gamma_4=\sum_{j=1}^n c_j\bar{b_j}$, $\gamma_5=\sum_{j=1}^n c_j\bar{a_j}$.
Therefore,

$$\|C_p^4\|\leq\sqrt{\frac{1}{2}\left(\delta_1+\delta+\sqrt{(\delta_1-\delta)^2+4\delta_2}\right)}+1. \tag{3.3}$$

We observe that the estimation of $\|C_p^4\|^{1/4}$ in (3.3) is incomparable with the existing estimation of $\|C_p^2\|^{1/2}$ in (3.2). In the following theorem we derive an upper bound for the spectral radius of the Frobenius companion matrix C_p, by using the estimations in (3.2) and (3.3).

Theorem 3.2. *The following inequality holds:*

$$r(C_p)\leq\left\{\frac{1}{4}\left(\frac{\delta+1+\sqrt{(\delta-1)^2+4\delta'}}{2}\right)+\frac{3}{4}\left(\frac{1}{2}\left(\delta_1+\delta+\sqrt{(\delta_1-\delta)^2+4\delta_2}\right)+1\right)^{\frac{1}{2}}\right\}^{\frac{1}{4}},$$

where $\delta'=\frac{1}{2}\left(\alpha'+\beta'+\sqrt{(\alpha'-\beta')^2+4|\gamma'|^2}\right)$,
$\delta=\frac{1}{2}\left(\alpha+\beta+\sqrt{(\alpha-\beta)^2+4|\gamma|^2}\right)$,
$\delta_1=\frac{1}{2}\left(\alpha_1+\beta_1+\sqrt{(\alpha_1-\beta_1)^2+4|\gamma_1|^2}\right)$,
$\delta_2=\frac{1}{2}\left(|\gamma_2|^2+|\gamma_3|^2+|\gamma_4|^2+|\gamma_5|^2+\sqrt{\left((|\gamma_2|^2+|\gamma_3|^2)-(|\gamma_4|^2+|\gamma_5|^2)\right)^2+4|\gamma_2\bar{\gamma_4}+\gamma_3\bar{\gamma_5}|^2}\right)$,
$\alpha'=\sum_{j=3}^n|a_j|^2$, $\beta'=\sum_{j=3}^n|b_j|^2$, $\gamma'=-\sum_{j=3}^n\bar{a_j}b_j$,
$\alpha=\sum_{j=1}^n|a_j|^2$, $\beta=\sum_{j=1}^n|b_j|^2$, $\gamma=-\sum_{j=1}^n b_j\bar{a_j}$,
$\alpha_1=\sum_{j=1}^n|d_j|^2$, $\beta_1=\sum_{j=1}^n|c_j|^2$, $\gamma_1=\sum_{j=1}^n d_j\bar{c_j}$,
$\gamma_2=\sum_{j=1}^n d_j\bar{b_j}$, $\gamma_3=\sum_{j=1}^n d_j\bar{a_j}$, $\gamma_4=\sum_{j=1}^n c_j\bar{b_j}$, $\gamma_5=\sum_{j=1}^n c_j\bar{a_j}$.

Proof. Let $A\in\mathbb{B}(\mathscr{H})$. Putting $A=A^2$ in the inequality $w^2(A)\leq\frac{1}{4}\|A^*A+AA^*\|+\frac{1}{2}w(A^2)$ (see [1, Th. 2.4]), we get

$$w^2(A^2)\ \leq\ \frac{1}{4}\left\||A^2|^2+|(A^*)^2|^2\right\|+\frac{1}{2}w(A^4).$$

It follows that

$$r^2(A) = r(A^2) \leq w(A^2) \leq \left\{ \frac{1}{4} \left\| |A^2|^2 + |(A^*)^2|^2 \right\| + \frac{1}{2} w(A^4) \right\}^{\frac{1}{2}},$$

i.e.,

$$r(A) \leq \left\{ \frac{1}{4} \left\| |A^2|^2 + |(A^*)^2|^2 \right\| + \frac{1}{2} w(A^4) \right\}^{\frac{1}{4}}. \tag{3.4}$$

Now, it follows from (3.4) and the inequality $\|C_p^* C_p + C_p C_p^*\| \leq \|C_p\|^2 + \|C_p^2\|$ (see [6, Remark 3.9]) that

$$\begin{aligned} r(C_p) &\leq \left\{ \frac{1}{4} \left\| |C_p^2|^2 + |(C_p^*)^2|^2 \right\| + \frac{1}{2} w(C_p^4) \right\}^{\frac{1}{4}} \\ &\leq \left\{ \frac{1}{4} (\|C_p^2\|^2 + \|C_p^4\|) + \frac{1}{2} \|C_p^4\| \right\}^{\frac{1}{4}} \\ &\leq \left\{ \frac{1}{4} \left\|C_p^2\right\|^2 + \frac{3}{4} \left\|C_p^4\right\| \right\}^{\frac{1}{4}}. \end{aligned}$$

Therefore, the required inequality follows by using the estimations in (3.2) and (3.3). □

By using the fact $|\lambda_j(C_p)| \leq r(C_p)$, where $\lambda_j(C_p)$ is the j-th eigenvalue of C_p, we infer the following estimation for the zeros of the polynomial $p(z)$.

Theorem 3.3. *If z is any zero of $p(z)$, then*

$$|z| \leq \left\{ \frac{1}{4} \left(\frac{\delta + 1 + \sqrt{(\delta-1)^2 + 4\delta'}}{2} \right) + \frac{3}{4} \left(\frac{1}{2} \left(\delta_1 + \delta + \sqrt{(\delta_1 - \delta)^2 + 4\delta_2} \right) + 1 \right)^{\frac{1}{2}} \right\}^{\frac{1}{4}},$$

where δ, δ_1, δ_2 and δ' are same as in Theorem 3.2.

Applying the spectral mapping theorem, we conclude that if z is any zero of $p(z)$ then $|z| \leq \|C_p^4\|^{\frac{1}{4}}$. Thus, by using the inequality (3.3) we achieve another new estimation for the zeros of $p(z)$.

Theorem 3.4. *If z is any zero of $p(z)$, then*

$$|z| \leq \left\{ \frac{1}{2} \left(\delta_1 + \delta + \sqrt{(\delta_1 - \delta)^2 + 4\delta_2} \right) + 1 \right\}^{\frac{1}{8}},$$

where δ, δ_1 and δ_2 are given in Theorem 3.2.

Again, putting $A = A^2$ in the inequality $w(A) \leq \frac{1}{2} \left(\|A\| + \|A^2\|^{\frac{1}{2}} \right)$ (see [16, Th. 1]), and proceeding as (3.4), we get

$$r(A) \leq \left\{ \frac{1}{2} \|A^2\| + \frac{1}{2} \|A^4\|^{\frac{1}{2}} \right\}^{\frac{1}{2}}. \tag{3.5}$$

Proceeding similarly as in Theorem 3.2 we obtain the following estimation by using the inequalities in (3.5), (3.2) and (3.3).

Theorem 3.5. *If z is any zero of $p(z)$, then*

$$|z| \leq \left\{ \frac{1}{2}\sqrt{\frac{\delta+1+\sqrt{(\delta-1)^2+4\delta'}}{2}} + \frac{1}{2}\left(\frac{1}{2}\left(\delta_1+\delta+\sqrt{(\delta_1-\delta)^2+4\delta_2}\right)+1\right)^{\frac{1}{4}} \right\}^{\frac{1}{2}},$$

where δ, δ_1, δ_1 and δ' are given in Theorem 3.2.

Finally, we compare the bounds obtained here for the zeros of $p(z)$ with the existing ones. First we note some well known existing bounds. Let z be any zero of $p(z)$. Then

Linden [20] obtained that

$$|z| \leq \frac{|a_n|}{n} + \left(\frac{n-1}{n}\left(n-1+\sum_{j=1}^{n}|a_j|^2 - \frac{|a_n|^2}{n}\right)\right)^{\frac{1}{2}}.$$

Montel [11, Th. 3] obtained that

$$|z| \leq \max\{1, |a_1|+\cdots+|a_n|\}.$$

Cauchy [12] obtained that

$$|z| \leq 1+\max\{|a_1|,\cdots,|a_n|\}.$$

Kittaneh [15] proved that

$$|z| \leq \frac{1}{2}\left(|a_n|+1+\sqrt{(|a_n|-1)^2+4\sqrt{\sum_{j=1}^{n-1}|a_j|^2}}\right).$$

Fujii and Kubo [10] obtained that

$$|z| \leq \cos\frac{\pi}{n+1} + \frac{1}{2}\left(|a_n|+\sqrt{\sum_{j=1}^{n}|a_j|^2}\right).$$

Bhunia and Paul [4, Th. 2.6] proved that

$$|z|^2 \leq \cos^2\frac{\pi}{n+1} + |a_{n-1}| + \frac{1}{4}\left(|a_n|+\sqrt{\alpha}\right)^2 + \frac{1}{2}\sqrt{\alpha-|a_n|^2} + \frac{1}{2}\sqrt{\alpha},$$

where $\alpha = \sum_{j=1}^{n}|a_j|^2$.

We consider a polynomial $p(z) = z^3+z^2+\frac{1}{2}z+1$. Different upper bounds for the modulus of the zeros of this polynomial, mentioned above, are as shown in the following table.

Linden [20]	1.9492
Montel[11]	2.5
Cauchy[12]	2
Kittaneh[15]	2.0547
Fujii and Kubo[10]	1.9571
Bhunia and Paul[4]	1.96761

However, Theorem 3.3 gives $|z| \leq 1.38047091798$, Theorem 3.4 gives $|z| \leq 1.3798438819$ and Theorem 3.5 gives $|z| \leq 1.381095966$, which are better than the above mentioned bounds.

Statements & Declarations:

Funding. The authors declare that no funds, grants, or other support were received during the preparation of this manuscript.

Competing interests. The authors have no relevant financial or non-financial interests to disclose.

Data availability statements. Data sharing not applicable to this article as no datasets were generated or analysed during the current study.

Author Contributions. All authors have contributed equally in the preparation of the manuscript.

References

1. A. Abu-Omar and F. Kittaneh, Upper and lower bounds for numerical radius with an application to involution operators, Rocky Mountain J. Math. 45 (2015), no. 4, 1055–1064.
2. P. Bhunia, S.S. Dragomir, M.S. Moslehian and K. Paul, Lectures on Numerical Radius Inequalities, Infosys Science Foundation Series in Mathematical Sciences, Springer, (2022). https://doi.org/10.1007/978-3-031-13670-2
3. P. Bhunia, S. Jana, M.S. Moslehian and K. Paul, Improved inequalities for the numerical radius via Cartesian decomposition, Funct. Anal. Appl. (2022), to appear.
4. P. Bhunia and K. Paul, Annular bounds for the zeros of a polynomial from companion matrices, Adv. Oper. Theory 7 (2022), no. 1, Paper No. 8, 19 pp.
5. P. Bhunia and K. Paul, Proper improvement of well-known numerical radius inequalities and their applications. Results Math. 76 (2021), no. 4, Paper No. 177, 12 pp.
6. P. Bhunia and K. Paul, Furtherance of numerical radius inequality of Hilbert space operators, Arch. Math. (Basel) 117 (2021), no. 5, 537–546.
7. P. Bhunia, S. Bag and K. Paul, Bounds for zeros of a polynomial using numerical radius of Hilbert space operators, Ann. Funct. Anal. 12 (2021), no. 2, Paper No. 21, 14 pp.
8. P. Bhunia and K. Paul, New upper bounds for the numerical radius of Hilbert space operators, Bull. Sci. Math. 167 (2021), Paper No. 102959, 11 pp.
9. M.L. Buzano, Generalizzatione della diseguaglianza di Cauchy-Schwarz, Rend. Sem. Mat. Univ. e Politech. Torino 31(1971/73) (1974) 405–409.

10. M. Fujii and F. Kubo, Buzano's inequality and bounds for roots of algebraic equations, Proc. Amer. Math. Soc. 117 (1993), no. 2, 359–361.
11. M. Fujii and F. Kubo, Operator norms as bounds for roots of algebraic equations, Proc. Japan Acad. Sci. 49 (1973), 805–808.
12. R.A. Horn and C.R. Johnson, Matrix Analysis, Cambridge, 1985.
13. T. Kato, Notes on some inequalities for linear operators, Math. Ann. 125 (1952), 208–212.
14. F. Kittaneh, Numerical radius inequalities for Hilbert space operators, Studia Math. 168 (2005), no. 1, 73–80.
15. F. Kittaneh, Bounds for the zeros of polynomials from matrix inequalities, Arch. Math. (Basel) 81 (2003), no. 5, 601–608.
16. F. Kittaneh, A numerical radius inequality and an estimate for the numerical radius of the Frobenius companion matrix, Studia Math. 158 (2003), no. 1, 11–17.
17. F. Kittaneh, Norm inequalities for sums of positive operators, J. Operator Theory 48 (2002), 95–103.
18. F. Kittaneh, Singular values of companion matrices and bounds on zeros of polynomials, SIAM J. Matrix Anal. Appl. 16 (1995), 333–340.
19. F. Kittaneh, Notes on some inequalities for Hilbert space operators, Publ. Res. Inst. Math. Sci. 24 (1988), no. 2, 283–293.
20. H. Linden, Bounds for zeros of polynomials using traces and determinants, Seminarberichte Fachb. Math. FeU Hagen. 69 (2000), 127–146.
21. C.A. McCarthy, C_p , Israel J. Math. 5 (1967), 249–271.
22. S. Sahoo, N.C. Rout and M. Sababheh, Some extended numerical radius inequalities, Linear Multilinear Algebra 69 (2021), no. 5, 907–920.
23. Y. Seo, Numerical radius inequalities related to the geometric means of negative power, Oper. Matrices 13 (2019), no. 2, 489–493.
24. M.P. Vasić and D.J. Kečkić, Some inequalities for complex numbers, Math. Balkanica 1 (1971), 282–286.

[1] Department of Mathematics, Mahisadal Girls' College, Purba Medinipur 721628, West Bengal, India

Email address: janasuva8@gmail.com

[2] Department of Mathematics, Jadavpur University, Kolkata 700032, West Bengal, India

Email address: pintubhunia5206@gmail.com

Email address: kalloldada@gmail.com; kallol.paul@jadavpuruniversity.in