A DILATION THEORETIC APPROACH TO APPROXIMATION BY INNER FUNCTIONS

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ABSTRACT. Using results from theory of operators on a Hilbert space, we prove approximation results for matrix-valued holomorphic functions on the unit disc and the unit bidisc. The essential tools are the theory of unitary dilation of a contraction and the realization formula for functions in the unit ball of H^∞. We first prove a generalization of a result of Carathéodory. This generalization has many applications. A uniform approximation result for matrix-valued holomorphic functions which extend continuously to the unit circle is proved using the Potapov factorization. This generalizes a theorem due to Fisher. Approximation results are proved for matrix-valued functions for whom a naturally associated kernel has finitely many negative squares. This uses the Krein-Langer factorization. Approximation results for J-contractive meromorphic functions where J induces an indefinite metric on $\mathbb{C}^N$ are proved using the Potapov-Ginzburg Theorem. Moreover, approximation results for holomorphic functions on the unit disc with values in certain other domains of interest are also proved.

1. INTRODUCTION

Let $M_N(\mathbb{C})$ be the Banach algebra of all $N \times N$ complex matrices with the operator norm. For $\Omega = \mathbb{D}$ or $\Omega = \mathbb{D}^2$, a holomorphic function $F : \Omega \to M_N(\mathbb{C})$ is called *rational* if every entry is a rational function with the poles off Ω and is called *inner* if the boundary values of the function on the unit circle/torus are unitary matrices almost everywhere.

Carathéodory, in his study of holomorphic functions from the open unit disc $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$ of the complex plane to the closed unit disc $\overline{\mathbb{D}}$, proved the following theorem, see Section 284 in [18]. Later, Rudin generalized this to functions taking values in $\overline{\mathbb{D}}$ but defined on the polydisc $\mathbb{D}^n$, see [33].

Theorem (Carathéodory and Rudin)**.** Let Ω denote the open unit disc $\mathbb{D}$ or the bidisc $\mathbb{D}^2$. Any holomorphic function $\varphi : \Omega \to \overline{\mathbb{D}}$ can be approximated (uniformly on compact subsets) by rational inner functions.

Such a function φ is said to be in the *Schur class*. There is a proof of this theorem through the fact that any solvable Pick-Nevanlinna interpolation problem has a rational inner solution. This technique carries over to matrix-valued functions.

For decades now, the theory of bounded operators on Hilbert spaces has been successfully used to give new proofs of complex analytic theorems. Two prominent examples are Sarason's approach to H^∞ interpolation [34] and Agler's proof of Lempert's theorem [3]. See also [7].

We shall give a new proof of the theorem above in a more general setting, viz., when the target set $\overline{\mathbb{D}}$ is replaced by certain compact sets of interest in higher dimension. This includes matrix-valued functions. We shall use the *state space method*, a term coined in [26], motivated by the huge contribution of linear system theory to function theoretic operator theory by the transfer function realization formula for operator-valued holomorphic functions on appropriate domains in $\mathbb{C}$ or $\mathbb{C}^n$. See [16] and [27]. The second tool in our kitty is a dilation theorem due to O. Nevanlinna [31], greatly popularized later by Levy and Shalit in [30].

Carathéodory's (and Rudin's) theorem is striking because the approximants map the unit disc *onto* itself whereas the approximated function is only required to map the unit disc *into* itself. In our proof using the "state space method", the idea is to start with the fact that any Schur class function has a realization

$$\varphi(z) = A + zB(I - zD)^{-1}C$$

with the associated *system matrix (colligation)* $\begin{pmatrix} A & B \\ C & D \end{pmatrix}$ contractive and then produce approximants φ_m in terms of unitary colligations $\begin{pmatrix} A & B_m \\ C_m & D_m \end{pmatrix}$. We can ensure that these unitary colligations act on finite dimensional spaces, thereby making φ_m rational and inner. The convergence question is converted into showing a matrix convergence:

$$B_m D_m^k C_m \to B D^k C \quad \text{as} \quad m \to \infty, \quad \text{for all } k \geq 1.$$

Carathéodory's theorem for matrix-valued functions and an appealing characterization of matrix-valued rational inner functions on the unit disc by Potapov lead us to a generalization of Fisher's theorem. Using the Blaschke product description of scalar rational inner functions, Fisher proved the following well-known result in [25].

Theorem (Fisher)**.** Let f be analytic on $\mathbb{D}$, continuous on $\overline{\mathbb{D}}$, and bounded by one. Then f may be uniformly approximated on $\overline{\mathbb{D}}$ by convex combinations of finite Blaschke products.

Potapov showed in [32] that any $N \times N$ matrix-valued rational inner function Φ is of the form

$$\Phi(z) = U \prod_{m=1}^{M} \left(b_{\alpha_m}(z) P_m + (I_{\mathbb{C}^N} - P_m) \right) \text{ for } z \in \mathbb{D},$$

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where M is a natural number, U is an $N \times N$ unitary matrix, the P_m are projections onto certain subspaces of $\mathbb{C}^N$, the α_m are points in the open unit disc and

$$b_\alpha(z) := \frac{z - \alpha}{1 - \overline{\alpha}z} \text{ for } \alpha \in \mathbb{D}$$

stands for a Blaschke factor. Such functions came to be known as Blaschke-Potapov products with a function of the form $b_\alpha P_\mathcal{M} + (I_{\mathbb{C}^N} - P_\mathcal{M})$ being called a Blaschke-Potapov factor because b_α is a Blaschke factor.

As one of the principal applications of Theorem 2.6, we shall reap a *uniform* approximation result for matrix-valued holomorphic functions on $\mathbb{D}$ which are continuous on $\overline{\mathbb{D}}$ as well. This generalizes Fisher's theorem. The crucial input which makes this possible is the Blaschke-Potapov formula. This is done in Section 3.

Theorem 2.6 has further applications. The fact that a matrix-valued contractive holomorphic function F satisfies $I \geq F(z)F(z)^*$ as well as, equivalently,

$$K_F(z, w) = \frac{I - F(z)F(w)^*}{1 - z\overline{w}} \succeq 0,$$

where $K \succeq 0$ for a kernel means that it is positive semi-definite, leads to generalizations in two different directions. Relaxing the positivity condition, we prove the following in Section 4. The proof of this uses the Krein-Langer Theorem.

One way to study non self adjoint operators is through their characteristic functions. This inexorably leads to J-contractive functions, where $J \in \mathbb{C}^{N \times N}$ is a signature matrix, i.e., $J = J^{-1} = J^*$, see page 62 of [17] for example. We have approximation results for J-contractive meromorphic functions as well as for functions whose kernel corresponding to J (analogous to K_F above, but now J replacing the identity operator) has finitely many negative squares. The terminologies are explained in the relevant section.

We also have two results about functions taking values into the symmetrized bidisc Γ or into the tetrablock $\overline{\mathbb{E}}$. The sets Γ and $\overline{\mathbb{E}}$ as well as the Γ-inner functions and the $\overline{\mathbb{E}}$-inner functions will be described in the context in the final section when we prove the results.

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2. Approximation by dilation

We start with a proposition which is a slight improvement of Lemma 6.2 of [28]. It has the same proof and the proof also carries verbatim in the case the function F takes its values in rectangular matrices instead of square ones.

Proposition 2.1. *Any holomorphic function $F : \mathbb{D}^n \to M_N(\mathbb{C})$ with $\|F(z)\| < 1$ for all $z \in \mathbb{D}^n$, can be approximated (uniformly on compact subsets) by matrix-valued polynomials P_m with $\|P_m\|_{\infty, \overline{\mathbb{D}^n}} < 1$, for all $m \geq 1$.*

We now quote a useful tool.

Theorem 2.2 (Realization formula for the disc). *Let $F : \mathbb{D} \to M_N(\mathbb{C})$ be a rational function such that $\|F\|_\infty \leq 1$. Then there exist a positive integer d and a contractive matrix*

$$\begin{bmatrix} A & B \\ C & D \end{bmatrix} : \mathbb{C}^N \oplus \mathbb{C}^d \to \mathbb{C}^N \oplus \mathbb{C}^d$$

such that

$$F(z) = A + zB(I - zD)^{-1}C.$$

This finite-dimensional realization formula is the disc version of the celebrated Kalman-Yakubovich-Popov lemma; see [21] for an indefinite version of it and [28], Proposition 4.2 for a recent proof. These two proofs give different points of view. See also [15].

The next result is the most crucial step towards proving the main theorem.

Theorem 2.3. *Any rational function $F : \mathbb{D} \to M_N(\mathbb{C})$ with $\|F(z)\| \leq 1$ for all $z \in \mathbb{D}$ can be approximated (uniformly on compact subsets) by $M_N(\mathbb{C})$-valued rational inner functions.*

Proof. The sequence of $M_N(\mathbb{C})$-valued rational inner functions which approximates F will actually be constructed by mixing two ingredients. First we invoke the Realization Formula, viz., Theorem 2.2 and set some notations. Let T denote the contraction

$$\begin{bmatrix} A & B \\ C & D \end{bmatrix} : \mathbb{C}^N \oplus \mathbb{C}^d \to \mathbb{C}^N \oplus \mathbb{C}^d$$

with D_{T^*} and D_T being the defect operators $(I - TT^*)^{1/2}$ and $(I - T^*T)^{1/2}$ respectively. Let

$$D_{T^*} = \begin{bmatrix} S_1 & S_2 \\ S_3 & S_4 \end{bmatrix} \text{ and } D_T = \begin{bmatrix} T_1 & T_2 \\ T_3 & T_4 \end{bmatrix}$$

as operators from $\mathbb{C}^N \oplus \mathbb{C}^d$ into itself. Let $\mathcal{H} := \mathbb{C}^N \oplus \mathbb{C}^d$. The second ingredient is a finite dilation of the contraction T, i.e., for any $m \geq 1$, a space $\mathcal{H}_m$ consisting of the direct sum of $(m+1)$ copies of $\mathcal{H}$ and a unitary U_m on it such that $T^j = P_{\mathcal{H}} U_m^j|_{\mathcal{H}}$ for $j = 1, \ldots, m$. This idea originated with [31], see also [30]. A sequence of functions F_m induced by the unitaries U_m will be the approximating sequence.

To that end, consider the space

$$\mathcal{K}_m \overset{\text{def}}{=} \mathbb{C}^d \oplus \mathcal{H} \oplus \cdots \oplus \mathcal{H} \oplus \mathbb{C}^N \oplus \mathbb{C}^d,$$

where $\mathcal{H}$ occurs $(m - 1)$ times. Now, consider the block operator matrix

$$U_m := \begin{bmatrix} A & B & 0 & \ldots & 0 & S_1 & S_2 \\ C & D & 0 & \ldots & 0 & S_3 & S_4 \\ T_1 & T_2 & 0 & \ldots & 0 & -A^* & -C^* \\ T_3 & T_4 & 0 & \ldots & 0 & -B^* & -D^* \\ 0 & 0 & I_{\mathcal{H}} & \ldots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \ldots & I_{\mathcal{H}} & 0 & 0 \end{bmatrix}$$

acting on the space $\mathbb{C}^N \oplus \mathcal{K}_m$. A straightforward calculation will show that U_m is a unitary matrix. This U_m is our $\begin{bmatrix} A & B_m \\ C_m & D_m \end{bmatrix}$ alluded to in the introduction. We note that

$$B_m = \begin{bmatrix} B & 0 & \dots & 0 & S_1 & S_2 \end{bmatrix}, \quad C_m = \begin{bmatrix} C & T_1 & T_3 & 0 & \dots & 0 \end{bmatrix}^t,$$

and

$$D_m = \begin{bmatrix} D & 0 & \dots & 0 & S_3 & S_4 \\ T_2 & 0 & \dots & 0 & -A^* & -C^* \\ T_4 & 0 & \dots & 0 & -B^* & -D^* \\ 0 & I_{\mathcal{H}} & \dots & 0 & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & \dots & I_{\mathcal{H}} & 0 & 0 \end{bmatrix}.$$

For fixed $k \geq 1$, we shall show that

$$B_m D_m^k C_m = B D^k C \tag{2.1}$$

for all $m \geq k + 2$. First note that for $m \geq 3$, the matrix $D_m C_m : \mathbb{C}^N \to \mathbb{C}^d \oplus \mathbb{C}^N \oplus \mathbb{C}^d \oplus \mathbb{C}^N \oplus \mathbb{C}^d \oplus \dots \oplus \mathbb{C}^N \oplus \mathbb{C}^d$ is given by

$$\begin{bmatrix} DC & T_2 C & T_4 C & T_1 & T_3 & 0 & \dots & 0 \end{bmatrix}^t.$$

Also, a simple calculation gives the following

$$D_m^k C_m = \begin{bmatrix} DC & * & * & * & \dots & * & 0 & 0 \end{bmatrix}^t, \quad \text{for } m \geq k + 2,$$

where the asterisk symbols mean that certain matrices are there which do not enter later computation. A matrix multiplication then yields (2.1).

To summarize, we have proved that there is a sequence of finite-dimensional Hilbert spaces $\mathcal{H}_m$, viz., the direct sum of $m + 1$ copies of $\mathcal{H}$ and a sequence of unitary matrices U_m on them satisfying a convergence property as follows.

$$\mathcal{H}_m = \mathbb{C}^N \oplus \mathcal{K}_m \text{ and } \begin{bmatrix} A & B_m \\ C_m & D_m \end{bmatrix} : \mathbb{C}^N \oplus \mathcal{K}_m \to \mathbb{C}^N \oplus \mathcal{K}_m$$

and $B_m D_m^k C_m \to B D^k C$ (in norm) for all $k \geq 1$. We are ready to define the approximants.

Consider the matrix-valued functions F_m defined as

$$F_m(z) = A + z B_m (I - z D_m)^{-1} C_m.$$

The functions F_m are rational inner because $\begin{bmatrix} A & B_m \\ C_m & D_m \end{bmatrix}$ are unitary matrices. Fix a compact set $S \subset \mathbb{D}$. For given $\epsilon > 0$, there exists $M_0 \in \mathbb{N}$ such that

$$|z|^l < \epsilon \qquad \text{for all} \quad l \geq M_0 \text{ and } z \in S.$$

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Now,

$$\|F(z) - F_m(z)\|$$

$$=\|z \sum_{k\geq 0}^{\infty}(B_m D_m^k C_m - BD^k C z^k)\|$$

$$\leq |z| \sum_{k\geq 0}^{\infty} \|B_m D_m^k C_m - BD^k C\| |z|^k$$

$$=|z| \sum_{k\geq 0}^{M_0-1} \|B_m D_m^k C_m - BD^k C\| |z|^k + |z| \sum_{k\geq M_0}^{\infty} \|B_m D_m^k C_m - BD^k C\| |z|^k$$

$$\leq |z| \sum_{k\geq 0}^{M_0-1} \|B_m D_m^k C_m - BD^k C\| |z|^k + \epsilon \frac{2|z|}{1-|z|}$$

$$=\epsilon \frac{2|z|}{1-|z|} \quad (\text{ for all } m \geq M_0 + 1).$$

Therefore, the sequence of rational inner functions F_m converges uniformly on compact subsets of $\mathbb{D}$. $\square$

Theorem 2.4. *Any holomorphic function $F : \mathbb{D} \to M_N(\mathbb{C})$ with $\|F(z)\| \leq 1$ for all $z \in \mathbb{D}$ can be approximated (uniformly on compact subsets) by $M_N(\mathbb{C})$-valued rational inner functions.*

Proof. By maximum norm principle, [Theorem 2, [19]], either $\|F(z)\| < 1$ for all $z \in \mathbb{D}$, or $\|F(z)\| \equiv 1$.
   Case-1: $\|F(z)\| < 1$ for all $z \in \mathbb{D}$.
In this case, Proposition 2.1 and Theorem 2.3 together will give us an approximation of F by matrix-valued rational inner functions.
   Case-2: $\|F(z)\| \equiv 1$.
By Theorem 4 of [19], there are $N \times N$ constant unitary matrices U and V, and an analytic function $G : \mathbb{D} \to M_{N-1}$ with $\|G(z)\| \leq 1$ for all $z \in \mathbb{D}$, such that

$$F(z) = U \begin{bmatrix} 1 & 0 \\ 0 & G(z) \end{bmatrix} V. \tag{2.2}$$

So, if $N = 2$, then Caratheodory's Theorem together with the equation (2.2) will give us an approximation of F by matrix-valued rational inner functions. Inductively, we can prove the result for $N > 2$. $\square$

The approximation theorem above continues to hold for matrix-valued functions on the bidisc. We shall outline the proof below. The finite dimensional realization formula we need has recently been proven by Knese in [28].

Theorem 2.5 (Realization formula for the bidisc)**.** *Let $F : \mathbb{D}^2 \to M_N(\mathbb{C})$ be a rational function such that $\|F\|_\infty \leq 1$. Then there exist positive integers d_1, d_2 and*

a contractive matrix

$$\begin{bmatrix} A & B \\ C & D \end{bmatrix} : \mathbb{C}^N \oplus \mathbb{C}^d \to \mathbb{C}^N \oplus \mathbb{C}^d \qquad with\ d = d_1 + d_2$$

such that with the notation $Z = z_1 I_{d_1} \oplus z_2 I_{d_2}$, *we have*

$$F(z_1, z_2) = A + BZ(I - DZ)^{-1}C.$$

Theorem 2.6. *Any holomorphic function* $F : \mathbb{D}^2 \to M_N(\mathbb{C})$ *with* $\|F(z_1, z_2)\| \leq 1$ *for all* $(z_1, z_2) \in \mathbb{D}^2$ *can be approximated (uniformly on compact subsets) by* $M_N(\mathbb{C})$-*valued rational inner functions.*

Proof. Let $F : \mathbb{D}^2 \to M_N(\mathbb{C})$ be a holomorphic map with $\|F(z_1, z_2)\| \leq 1$ for all $(z_1, z_2) \in \mathbb{D}^2$. Then in view of Lemma 6.1 of [28], it is enough to consider the case when $\|F(z_1, z_2)\| < 1$ for all $(z_1, z_2) \in \mathbb{D}^2$. Now by Proposition 2.1, we can take F to be a polynomial. Invoke Theorem 2.5 to get positive integers d_1, d_2 and a contractive matrix

$$T = \begin{bmatrix} A & B \\ C & D \end{bmatrix} : \mathbb{C}^N \oplus \mathbb{C}^d \to \mathbb{C}^N \oplus \mathbb{C}^d \qquad with\ d = d_1 + d_2$$

such that with the notation $Z = z_1 I_{d_1} \oplus z_2 I_{d_2}$, we have

$$F(z_1, z_2) = A + BZ(I - DZ)^{-1}C.$$

Consider the m-unitary dilation $\left(\begin{smallmatrix} A & B_m \\ C_m & D_m \end{smallmatrix} \right)$ of T on $\mathbb{C}^N \oplus \mathcal{K}_m$. A matrix multiplications then yields that

$$B_m Z_m (D_m Z_m)^k C_m = BZ(DZ)^k C, \qquad for\ m \geq k + 2$$

where $Z_m = \mathrm{diag}\,(Z, *, *, \ldots, *)$ be any diagonal operator acting on $\mathcal{K}_m$ and the asterisk symbols stand for diagonal matrices whose diagonal entries are either z_1 or z_2 or $e^{i\theta}$ for some θ. Consider the matrix-valued rational inner functions F_m on $\mathbb{D}^2$ defined as

$$F_m(z_1, z_2) = A + B_m Z_m (I - D_m Z_m)^{-1} C_m.$$

A similar argument as in the case of the disc will give that the sequence of rational inner functions F_m converges to F uniformly on compact subsets of $\mathbb{D}^2$.

$\square$

Remark 2.7. A comment about the case of the polydisc $\mathbb{D}^n$ is in order for $n > 2$. Let φ be a function from the Schur-Agler class, i.e., φ is in H_n^∞ in the notation of [2]. Let $\{z_1, z_2, \ldots\}$ be a countable dense subset of $\mathbb{D}^n$. Consider for every $m \geq 1$, the solvable Pick-Nevanlinna interpolation data $\{(z_1, \varphi(z_1)), \ldots, (z_m, \varphi(z_m))\}$. It is known that this has a rational inner solution φ_m from the Schur-Agler class. Montel's theorem then proves that there is a subsequence of $\{\varphi_m\}$ converging to φ uniformly over compact subsets of $\mathbb{D}^n$. This technique carries over to matrix-valued functions of Schur-Agler class. This matrix-valued version of Rudin's result is not known if φ is in Schur class because the Schur class is bigger than the Schur-Agler class. Also, the state space method cannot be applied to prove the result even for

the Schur-Agler class because a finite realization for rational inner functions on the polydisc is not known. This is a limitation for the state space method.

3. The convex hull of matrix-valued rational inner functions on the disc

From now on, our functions will be on $\mathbb{D}$. It follows from Potapov's work that every matrix-valued rational inner function is holomorphic in a neighbourhood of the closed unit disc $\overline{\mathbb{D}}$. In this section, we shall give a description of the closed convex hull of the matrix-valued rational inner functions generalizing the theorem in [25].

Let F be an $M_N(\mathbb{C})$-valued function which is holomorphic in $\mathbb{D}$ and continuous on $\overline{\mathbb{D}}$. For $0 \leq r \leq 1$, set

$$F_r(z) := F(rz) \quad (z \in \overline{\mathbb{D}}). \tag{3.1}$$

Clearly, F_r is holomorphic in $\mathbb{D}$ and continuous on $\overline{\mathbb{D}}$ for any $0 \leq r \leq 1$. The following two lemmas follow from direct calculations.

Lemma 3.1. *Let Φ, Ψ be two $M_N(\mathbb{C})$-valued rational inner functions. Suppose for some fixed $r \in [0, 1]$, Φ_r, Ψ_r can be written as convex combination of rational inner functions, then $(\Phi\Psi)_r$ can also be written as convex combination of rational inner functions.*

Lemma 3.2. *Let Φ be an $M_N(\mathbb{C})$-valued rational inner functions and $U \in M_N(\mathbb{C})$ be a unitary. Suppose for some fixed $r \in [0, 1]$, Φ_r can be written as convex combination of rational inner functions, then $U\Phi_r$ can also be written as convex combination of rational inner functions.*

Lemma 3.3. *If Φ is any $M_N(\mathbb{C})$-valued rational inner function, then for any $0 \leq r \leq 1$, Φ_r can be written as convex combination of $M_N(\mathbb{C})$-valued rational inner functions.*

Proof. Note that if φ is a scalar-valued rational inner function and P is an orthogonal projection of $\mathbb{C}^N$ onto some subspace, then the matrix-valued function $\varphi P + (I_{\mathbb{C}^N} - P)$ is also rational inner. For a Blaschke factor b and for any $0 \leq r \leq 1$, it follows from [25] that b_r, as defined in (3.1) can be written as a convex combination of scalar-valued rational inner functions. So the $M_N(\mathbb{C})$-valued holomorphic function $b_r P + (I_{\mathbb{C}^N} - P)$ can be written as a convex combination of $M_N(\mathbb{C})$-valued rational inner functions. The rest of the proof follows from Lemma 3.1 and Lemma 3.2. □

Lemma 3.4. *Let F be an $M_N(\mathbb{C})$-valued function which is holomorphic in $\mathbb{D}$ and continuous on $\overline{\mathbb{D}}$. Then F_r converges uniformly to F on $\overline{\mathbb{D}}$ as $r \to 1$.*

Proof. If F is scalar-valued, then it follows from Mergelyan's theorem. Since $M_N(\mathbb{C})$ is finite dimensional, all norms on $M_N(\mathbb{C})$ are equivalent. So there exists a positive constant c_N such that

$$\|A\| \leq c_N \max_{i,j} |a_{ij}|$$

for all $A = [a_{ij}]_{N \times N}$, where $\|A\|$ is the operator norm of matrix A. Let $F = [F_{ij}]_{N \times N}$. Let $\epsilon > 0$ be given. Since each F_{ij} is scalar-valued, there exists r close to 1 such that

$$|F_{ij}(z) - F_{ij}(rz)| < \frac{\epsilon}{c_N}$$

for all $z \in \overline{\mathbb{D}}$ and for all i, j. So we get

$$\|F(z) - F(rz)\| \leq c_N \max_{i,j} |F_{ij}(z) - F_{ij}(rz)| < \epsilon$$

for all $z \in \overline{\mathbb{D}}$. This concludes the proof. $\square$

We are now ready with the generalization of Fisher's theorem.

Theorem 3.5. *Let F be an $M_N(\mathbb{C})$-valued function which is holomorphic in $\mathbb{D}$ and continuous on $\overline{\mathbb{D}}$. Suppose $\|F(z)\| \leq 1$ for all $z \in \mathbb{D}$. Then F can be uniformly approximated on $\overline{\mathbb{D}}$ by convex combinations of $M_N(\mathbb{C})$-valued rational inner functions.*

Proof. Let $\epsilon > 0$ be given. By Lemma 3.4, there exists $r \in (0, 1)$ such that

$$\|F - F_r\|_{\infty, \overline{\mathbb{D}}} < \frac{\epsilon}{2}.$$

Let $\mathbb{D}_r$ be the closed unit ball of radius r centered at 0. By Theorem 2.6, there exists an $M_N(\mathbb{C})$-valued rational inner function Φ such that

$$\|F - \Phi\|_{\infty, \mathbb{D}_r} < \frac{\epsilon}{2}.$$

This implies

$$\|F_r - \Phi_r\|_{\infty, \overline{\mathbb{D}}} < \frac{\epsilon}{2}.$$

So we get

$$\|F - \Phi_r\|_{\infty, \overline{\mathbb{D}}} < \epsilon.$$

By Lemma 3.3, it follows that Φ_r itself is a convex combination of $M_N(\mathbb{C})$-valued rational inner functions. $\square$

4. Relaxing analyticity

4.1. **Meromorphic functions.**

Theorem 4.1. *Let F be an $M_N(\mathbb{C})$-valued meromorphic function on $\mathbb{D}$. Suppose the kernel $K_F(z, w) = \frac{I - F(z)F(w)^*}{1 - z\overline{w}}$ has finitely many negative squares. Let $A(F) \subset \mathbb{D}$ be the set on which F is analytic. Then F can be approximated uniformly on compact subsets of $A(F)$ by rational functions which are unitary matrix-valued on the unit circle. Moreover if F is continuous on the unit circle, then F can be approximated uniformly on the unit circle $\mathbb{T}$ by convex combinations of quotient of matrix-valued rational inner functions.*

Proof. There is a remarkable factorization of operator-valued functions, due to Krein and Langer, for those functions F which satisfy that K_F has finitely many negative squares, see [29], [22]. Since our function is matrix-valued, the Krein-Langer factorization in this context says that there exists a Blaschke-Potapov product B of degree k and a matrix-valued holomorphic function on the disc L such that

$$F(z) = B(z)^{-1}L(z)$$

and $\|L(z)\| \leq 1$ for all $z \in \mathbb{D}$. We apply Theorem 2.6 to get a sequence $\{L_m\}$ of matrix-valued rational inner functions converging to L uniformly on compact subsets of $\mathbb{D}$. Then, the sequence $B(z)^{-1}L_m(z)$ does the job.

In the case when F is continuous on the unit circle, we use holomorphicity of B in a neighbourhood of $\overline{\mathbb{D}}$ to conclude that the L obtained above is continuous on $\mathbb{T}$ and $\|L(z)\| \leq 1$ for all $z \in \mathbb{T}$. Now we invoke Theorem 3.5 to get a sequence of convex combinations of matrix-valued rational inner functions $\{L_m\}$ such that L_m converges to L uniformly on $\overline{\mathbb{D}}$. Define

$$F_m(z) = B(z)^{-1}L_m(z).$$

Consider

$$\|F(z) - F_m(z)\| = \|B(z)^{-1}(L(z) - L_m(z))\| \leq \|B(z)^{-1}\|\|L(z) - L_m(z)\|.$$

Since B is continuous on $\mathbb{T}$, F_m converges to F uniformly on $\mathbb{T}$.

If we apply the right Krein-Langer factorization, then

$$F(z) = R(z)\tilde{B}(z)^{-1}.$$

By a similar calculation $R_m(z)\tilde{B}(z)^{-1}$ will approximate F uniformly on $\mathbb{T}$ where R_m approximates R as in Theorem 3.5. That completes the proof of Theorem 4.1. $\square$

4.2. *J*-contractive functions.

In a new direction of generalization, we consider the case of indefinite metric in the coefficient space $\mathbb{C}^N$, that is kernels of the form

$$\frac{J - F(z)JF(w)^*}{1 - z\overline{w}}$$

where $J \in \mathbb{C}^{N \times N}$ is a signature matrix. Such a matrix is unitarily equivalent to J_0 defined by

$$J_0 = \begin{bmatrix} I_p & 0 \\ 0 & -I_q \end{bmatrix}, \quad p + q = N,$$

with $J_0 = I_N$ if $q = 0$ and $J_0 = -I_N$ if $p = 0$. We are interested in the case $p > 0$, $q > 0$. In the sequel we focus on the case $J = J_0$. The formulas presented are valid for arbitrary J (for which $p > 0$ and $q > 0$ in the corresponding J_0).

We will use the Potapov-Ginzburg transform (see [11, 17]), which allows to reduce to the case $J = J_0 = I_N$. Following [9], we set

$$P = \frac{I_N + J_0}{2} \quad \text{and} \quad Q = \frac{I_N - J_0}{2}.$$

For $J = J_0$ at hand, we have

$$P = \begin{bmatrix} I_p & 0 \\ 0 & 0 \end{bmatrix} \quad \text{and} \quad Q = \begin{bmatrix} 0 & 0 \\ 0 & I_q \end{bmatrix}$$

and

$$P + QF(z) = \begin{bmatrix} I_p & 0 \\ F_{21}(z) & F_{22}(z) \end{bmatrix}.$$

Writing $F = \begin{bmatrix} F_{11} & F_{12} \\ F_{21} & F_{22} \end{bmatrix}$ we will assume that $\det F_{22} \not\equiv 0$.

Definition 4.2. The Potapov-Ginzburg transform of F is given by

$$\Sigma(z) = (PF(z) + Q)(P + QF(z))^{-1},$$

at those points where the inverse exists, with inverse given by

$$F(z) = (P - \Sigma(z)Q)^{-1}(\Sigma(z)P - Q).$$

The following formulas hold. See [9, p. 66], [10].

$$
\begin{aligned}
\Sigma(z) &= (P - F(z)Q)^{-1}(F(z)P - Q) \\
F(z) &= (Q + P\Sigma(z))(P + Q\Sigma(z))^{-1} \\
I_N - \Sigma(z)\Sigma(w)^* &= (P - F(z)Q)^{-1}(J_0 - F(z)J_0F(w)^*)(P - F(w)Q)^{-*} \quad (4.1) \\
I_N - \Sigma(w)^*\Sigma(z) &= (P + QF(w))^{-*}(J_0 - F(w)^*J_0F(z))(P + QF(z))^{-1}
\end{aligned}
$$

A function F meromorphic in $\mathbb{D}$ is called J_0-*contractive* if

$$F(z)J_0F(z)^* \leq J_0$$

at each point of analyticity of F in $\mathbb{D}$. Such a function is in particular of bounded type in $\mathbb{D}$ and admits non-tangential limits almost everywhere on the unit circle. A matrix A is called J_0-*unitary* if $AJ_0A^* = J_0$. A rational function F will be called J_0-*inner* if the limiting values exist and are J_0-unitary everywhere on the unit circle except possibly at a finite number of points. In the following theorem, we mention a special case first for the sake of better exposition.

Theorem 4.3.

(1) *Let F be J_0-contractive, with domain of analyticity $A(F) \subset \mathbb{D}$. Then, F can be approximated uniformly on compact subsets by rational J_0-inner functions.*

(2) *Let F be meromorphic in the open unit disc with domain of analyticity $A(F) \subset \mathbb{D}$ such that the kernel*

$$\frac{J_0 - F(z)J_0F(w)^*}{1 - z\overline{w}}$$

has a finite number of negative squares in $A(F)$. Then F can be approximated uniformly on compact subsets of $A(F)$ by rational J_0-inner functions.

Proof. The Potapov-Ginzburg transform of F exists by [23, Theorem 1.1, p. 14]. By (4.1), Σ is contractive and meromorphic in the open unit disc, and hence contractive and analytic there (the contractivity implies that the isolated singularities of F are removable). Applying Theorem 2.6 to Σ we can write $\Sigma = \lim_{m\to\infty} B_m$, where the B_m are finite Blaschke products and where the convergence is uniform on compact subsets of the open unit disc. Writing $B_m = ((B_m)_{ij})_{i,j=1}^2$ where $(B_m)_{22}$ is $\mathbb{C}^{q\times q}$-valued, we have in particular

$$\lim_{m\to\infty} \det(B_m)_{22} = \det \Sigma_{22}$$

and in particular $\det B_m \not\equiv 0$ for m large enough. It follows that the inverse Potapov-Ginzburg transforms, say F_m, of the B_m exist for such m. The functions F_m are rational and J_0-inner. That completes the proof of part (1).

We now consider the case of negative squares and recall that its Potapov-Ginzburg transform, say Σ, is well defined (see e.g. [12, Theorem 6.8]). It follows from (4.1) that

$$\frac{I_N - \Sigma(z)\Sigma(w)^*}{1 - z\overline{w}} = (P - F(z)Q)^{-1}\frac{J_0 - F(z)J_0F(w)^*}{1 - z\overline{w}}(P - F(w)Q)^{-*}$$

and in particular the kernel $\frac{I_N - \Sigma(z)\Sigma(w)^*}{1 - z\overline{w}}$ has a finite number of negative squares in the open unit disc. We apply Theorem 4.1 to Σ, and we get an approximation for F by taking the inverse Potapov-Ginzburg transform. Thus, we have proved part (2) of Theorem 4.3. □

5. Γ-valued and $\overline{\mathbb{E}}$-valued functions

Let Ω be a bounded polynomially convex domain. The *distinguished boundary* $b\Omega$ is the smallest closed subset of $\overline{\Omega}$ on which every continuous function on $\overline{\Omega}$ that is analytic in Ω attains its maximum modulus.

Definition 5.1. A rational $\overline{\Omega}-$ inner function is a rational analytic map $x : \mathbb{D} \to \overline{\Omega}$ with the property that x maps $\mathbb{T}$ into the distinguished boundary $b\Omega$ of Ω. The degree, $\deg(x)$, of a rational $\overline{\Omega}-$ inner function is defined to be the maximum of degree of each components.

This section deals with functions which take values into the symmetrized bidisc

$$\Gamma = \{(z + w, zw) : |z| \le 1, |w| \le 1\}$$

or into the tetrablock

$$\overline{\mathbb{E}} = \{(a_{11}, a_{22}, \det(A)) : A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \text{ satisfies } \|A\| \le 1\}.$$

The sets Γ and $\overline{\mathbb{E}}$ are non-convex and polynomially convex domains. The symmetrized bidisc was introduced by Agler and Young in [8] and the tetrablock was introduced by Abouhajar, White and Young in [1]. A great deal of function theory and operator theory has been done on these two domains. The following criteria will be useful. Let $\mathbb{G}$ be the open symmetrized bidisc.

Proposition 5.2. [8] *Let $(s, p) \in \mathbb{C}^2$. The point $(s, p) \in \mathbb{G}$ (respectively Γ) if and only if*

$$|s| < (respectively \ \leq)2, \ and \ |s - \overline{s}p| < (respectively \ \leq)1 - |p|^2.$$

The point $(s, p) \in b\mathbb{G}$ if and only if $|s| \leq 2, |p| = 1, \ and \ s = \overline{s}p$.

There are similar criteria about the tetrablock.

Proposition 5.3. [1] *Let $(x_1, x_2, x_3) \in \mathbb{C}^3$. The point $(x_1, x_2, x_3) \in \mathbb{E}$ (respectively $\overline{\mathbb{E}}$) if and only if*

$$|x_1 - \overline{x_2}x_3| + |x_2 - \overline{x_1}x_3| < (respectively \ \leq)1 - |x_3|^2.$$

The point $(x_1, x_2, x_3) \in b\mathbb{E}$ if and only if $x_1 = \overline{x_2}x_3|, |x_3| = 1, \ and \ |x_2| \leq 1$.

Algebraic and geometric aspects of rational $\Gamma-$ inner functions were studied in [6]. For details about rational $\Gamma-$ inner functions and rational $\overline{\mathbb{E}}-$ functions, see [5, 6, 13, 14].

Proposition 5.4. *Any holomorphic function $h = (s, p) : \mathbb{D} \to \Gamma$ can be approximated (uniformly on compact subsets) by rational Γ-inner functions.*

Proof. Let $h = (s, p) : \mathbb{D} \to \Gamma$ be a holomorphic function. Invoke Proposition 6.1 of [4] to obtain an analytic function $F : \mathbb{D} \to M_2(\mathbb{C})$ with $\|F(\lambda\| \leq 1$ for all $\lambda \in \mathbb{D}$ such that

$$h = (\mathrm{tr}\, F, \det F).$$

By Theorem 2.6, there exists a sequence of matrix-valued rational inner functions $\{F_m\}$ on $\mathbb{D}$ which approximates F uniformly on compact subsets of $\mathbb{D}$. For each $m \in \mathbb{N}$, consider the holomorphic functions $h_m : \mathbb{D} \to \Gamma$ defined as

$$h_m := (\mathrm{tr}\, F_m, \det F_m).$$

It is easy to see that h_m are rational functions.

To prove that h_m are Γ-inner functions, we only need to make the elementary observation that for a unitary matrix A, the eigenvalues λ_1 and λ_2 lie in $\mathbb{T}$. So, $(\mathrm{tr}\, A, \det A) = (\lambda_1 + \lambda_2, \lambda_1\lambda_2) \in b\Gamma$. Since F_m are inner, $F_m(\lambda)$ are unitaries a.e. on the circle. Thus, h_m are Γ-inner functions.

Since F_m converges to F uniformly on compact subsets, it follows that $(F_m)_{ij}$ converges to F_{ij} uniformly on compact subsets. Therefore, h_m converges to h uniformly on compact subsets of $\mathbb{D}$. □

We remark that the method of proof of Carathéodory's theorem through Pick-Nevanlinna interpolation can also be applied to approximate holomorphic functions from $\mathbb{D}$ into the symmetrized bidisc because of a result of Costara, see Theorem 4.2 in [20].

Proposition 5.5. *Any holomorphic function $x = (x_1, x_2, x_3) : \mathbb{D} \to \overline{\mathbb{E}}$ can be approximated (uniformly on compact subsets) by rational $\overline{\mathbb{E}}$-inner functions.*

Proof. Let $x = (x_1, x_2, x_3)$ be as in the above Theorem. By Lemma 7 of [24], there exists an analytic function $F : \mathbb{D} \to M_2(\mathbb{C})$ with $\|F(\lambda)\| \leq 1$ for all $\lambda \in \mathbb{D}$ such that

$$x = (F_{11}, F_{22}, \det F)$$

where $F = [F_{ij}]_{i,j=1}^2$. Again by Theorem 2.6, there exists a sequence of matrix-valued rational inner functions $\{F_m\}$ on $\mathbb{D}$ which approximates F uniformly on compact subsets of $\mathbb{D}$. For $m \in \mathbb{N}$, define the holomorphic maps $x_m : \mathbb{D} \to \overline{\mathbb{E}}$ by

$$x_m = ((F_m)_{11}, (F_m)_{22}, \det F_m).$$

Now we shall prove that this x_m will do our job. It is easy to see that x_m are rational functions. Now we shall prove that x_m are $\overline{\mathbb{E}}$-inner functions. Since F_m are inner, $F_m(\lambda)$ are unitaries a.e. on the circle. It follows that $x_m(\lambda) \in b\mathbb{E}$ a.e. $\lambda \in \mathbb{T}$, see Theorem 7.1 of [1]. Thus, x_m are rational $\overline{\mathbb{E}}$-inner functions.

Since F_m converges uniformly on compact subsets to F, $(F_m)_{11}, (F_m)_{22}$, and $\det F_m$ converges uniformly on compact subsets to F_{11}, F_{22} and $\det F$ respectively. Hence x_m converges uniformly on compact subsets of $\mathbb{D}$ to x. This completes the proof. $\qquad\square$

References

[1] A. A. Abouhajar, M. C. White and N. J. Young, *A Schwarz lemma for a domain related to μ-synthesis*, J. Geom. Anal. 17 (2007), no. 4, 717–750.

[2] J. Agler, *On the representation of certain holomorphic functions defined on a polydisc*, in: Topics in operator theory: Ernst D. Hellinger memorial volume, Oper. Theory Adv. Appl., 48, pages 47-66. Birkhäuser Verlag, Basel, 1990.

[3] J. Agler, *Operator theory and the Carathéodory metric*, Invent. Math. 101 (1990), no. 2, 483–500.

[4] J. Agler, Z. A. Lykova and N. J. Young, *N. J. A case of μ-synthesis as a quadratic semidefinite program*, SIAM J. Control Optim. 51 (2013), no. 3, 2472-2508.

[5] J. Agler, Z. A. Lykova and N. J. Young, *Finite Blaschke products and the construction of rational Γ-inner functions*, J. Math. Anal. Appl. 447 (2017), no. 2, 1163-1196.

[6] J. Agler, Z. A. Lykova and N. J. Young, *Algebraic and geometric aspects of rational Γ-inner functions*, Adv. Math. 328 (2018), 133–159.

[7] J. Agler, J. E. McCarthy and N. J. Young, *A Carathéodory theorem for the bidisk via Hilbert space methods*, Math. Ann. 352 (2012), no. 3, 581-624.

[8] J. Agler and N. J. Young, *A commutant lifting theorem for a domain in $\mathbb{C}^2$ and spectral interpolation,* J. Funct. Anal. 161 (1999), no. 2, 452-477.

[9] D. Alpay, *Algorithme de Schur, espaces à noyau reproduisant et théorie des systèmes*, volume 6 of Panoramas et Synthèses [Panoramas and Syntheses], Société Mathématique de France, Paris, 1998.

[10] D. Alpay, *The Schur algorithm, reproducing kernel spaces and system theory*, American Mathematical Society, Providence, RI, 2001. Translated from the 1998 French original by Stephen S. Wilson, Panoramas et Synthèses.

[11] D. Alpay, A. Dijksma, J. Rovnyak and H. de Snoo, *Schur functions, operator colligations, and reproducing kernel Pontryagin spaces*, Oper. Theory Adv. Appl. 96, Birkhäuser Verlag, Basel, 1997.

[12] D. Alpay and H. Dym. *On applications of reproducing kernel spaces to the Schur algorithm and rational J-unitary factorization*, in: Oper. Theory Adv. Appl. 18, pages 89–159. Birkhäuser Verlag, Basel, 1986.

[13] H. O. Alshammari and Z. A. Lykova, *Interpolation by holomorphic maps from the disc to the tetrablock*, J. Math. Anal. Appl. 498 (2021), no. 2, Paper No. 124951, 36 pp.

[14] O. M. O. Alsalhi and Z. A. Lykova, *Rational tetra-inner functions and the special variety of the tetrablock*, J. Math. Anal. Appl. 506 (2022), no. 1, Paper No. 125534, 52 pp.

[15] J. A. Ball, I. Gohberg and L. Rodman, *Interpolation of rational matrix functions*, Oper. Theory Adv. Appl. 45. Birkhäuser Verlag, Basel, 1990.

[16] H. Bart, I. Gohberg and M. A. Kaashoek, *Minimal factorization of matrix and operator functions*, Oper. Theory Adv. Appl. 1, Birkhäuser Verlag, Basel, 1979.

[17] M. S. Brodskiĭ, *Triangular and Jordan representations of linear operators*, American Mathematical Society, Providence, R.I., 1971. Translated from the Russian by J. M. Danskin, Translations of Mathematical Monographs, Vol. 32.

[18] C. Carathéodory, *Theory of functions of a complex variable, Volume Two*, Chelsea Publishing Company, New York, 1954.

[19] A. A. Condori, *Maximum principles for matrix-valued analytic functions*, Amer. Math. Monthly 127 (2020), no. 4, 331-343.

[20] C. Costara, *On the spectral Nevanlinna-Pick problem*, Studia Math. 170 (2005), no. 1, 23–55.

[21] B. Dickinson, Ph. Delsarte, Y. Genin and Y. Kamp, *Minimal realizations of pseudo–positive and pseudo–bounded rational matrices*, IEEE Transactions on Circuits and Systems, 32 (1985), no. 6, 603–605.

[22] A. Dijksma, H. Langer and H. S. de Snoo, *Characteristic functions of unitary operator colligations in π_k-spaces*, in: Operator theory and systems (Amsterdam, 1985), Oper. Theory Adv. Appl. 19, pages 125– 194, Birkhäuser, Basel, 1986.

[23] H. Dym, *J–contractive matrix functions, reproducing kernel Hilbert spaces and interpolation*, CBMS Regional Conference Series in Mathematics, 71. Published for the Conference Board of the Mathematical Sciences, Washington, DC; by the American Mathematical Society, Providence, RI, 1989.

[24] A. Edigarian, L. Kosiński and W. Zwonek, *The Lempert theorem and the tetrablock*, J. Geom. Anal. 23 (2013), no. 4, 1818–1831.

[25] S. Fisher, *The convex hull of the finite Blaschke products,* Bull. Amer. Math. Soc. 74 (1968), 1128–1129.

[26] I. Gohberg and M. A. Kaashoek, *The state space method for solving singular integral equations*, Mathematical System Theory, 509-523, Springer, Berlin, 1991.

[27] I. Gohberg, P. Lancaster and L. Rodman, *Matrix polynomials*, Academic Press, New York-London,1982.

[28] G. Knese, *Kummert's approach to realization on the bidisc*, Indiana Univ. Math. J. 70 (2021), no. 6, 2369-2403.

[29] M. G. Krein and H. Langer, *Über die verallgemeinerten Resolventen und die charakteristische Funktion eines isometrischen Operators im Raume Π_k*, (German) Hilbert space operators and operator algebras (Proc. Internat. Conf., Tihany, 1970), 353–399. Colloq. Math. Soc. János Bolyai, 5. North-Holland, Amsterdam, 1972.

[30] E. Levy and O.R. Shalit, *Dilation theory in finite dimensions: the possible, the impossible and the unknown*, Rocky Mountain J. Math. 44 (2014), no. 1, 203–221.

[31] O. Nevanlinna, *Matrix valued versions of a result of von Neumann with an application to time discretization*, J. Comput. Appl. Math. 12&13 (1985), 475-489.

[32] V. P. Potapov, *The multiplicative structure of J-nonexpansive matrix functions*, Trudy Moskov. Mat. Obšč. (1955), 125-236 (in Russian); English trasl. in: Amer. Math. Soc. Transl. (2) 15(1960), 131–243.

[33] W. Rudin, *Function theory in polydiscs*, W. A. Benjamin, Inc., 1969.

[34] D. Sarason, *Generalized interpolation in H^∞*, Trans. Amer. Math. Soc. 127 (1967), 179–203.

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