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Spanning caterpillar in biconvex bipartite graphs

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ABSTRACT

A bipartite graph $G = (A, B, E)$ is said to be a biconvex bipartite graph if there exist orderings $<_A$ in A and $<_B$ in B such that the neighbors of every vertex in A are consecutive with respect to $<_B$ and the neighbors of every vertex in B are consecutive with respect to $<_A$. A *caterpillar* is a tree that will result in a path upon deletion of all the leaves. In this paper, we prove that *there exists a spanning caterpillar in any connected biconvex bipartite graph*. Besides being interesting on its own, this structural result has other consequences. For instance, this directly resolves the *burning number conjecture* for biconvex bipartite graphs.

1. Introduction

We consider finite, simple, and undirected graphs throughout the paper. Let $G = (V, E)$ be a graph with the set of vertices V and the set of edges E. For basic graph theoretic notations and definitions, we refer to [7].

A graph is *connected* if, for any pair of vertices, there exists a path between them. A graph is *acyclic* if it does not contain a cycle. A *tree* is a connected acyclic graph. A *star* on n vertices, denoted as $K_{1,n-1}$, is a tree with exactly $n-1$ leaves. A *caterpillar* is a tree that will result in a path upon deletion of all the leaves. Henceforth, we refer to this path as *residual path*. In other words, a *caterpillar* is a tree having a residual path P such that every vertex that is not in P is adjacent to a vertex in P. A graph is said to be a *permutation graph* if its vertices correspond to the elements of a permutation σ such that there exists an edge between two vertices i and j if and only if either $i < j$ and $\sigma(j) < \sigma(i)$ or $j < i$ and $\sigma(i) < \sigma(j)$. A subgraph H of a graph G is said to be a *spanning subgraph* if $V(H) = V(G)$. The following lemma on trees was given by Driscoll et al. [4].

Lemma 1.1 (*[4]*). *Every tree on at most six vertices is a caterpillar.*

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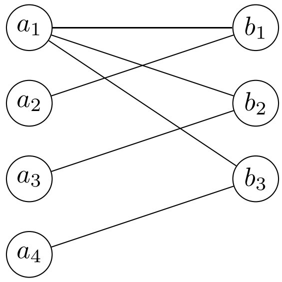

Fig. 1. A convex bipartite graph having no spanning caterpillar.

A graph $G = (V, E)$ is said to be *bipartite* if the vertex set V can be partitioned into two partite sets A and B such that every edge in E has its one end-point in A and the other end-point in B. Let $<_A$ be an ordering of A and let $<_B$ be an ordering of B. For any two distinct vertices $a_i, a_j \in A$, by $a_i <_A a_j$, we mean a_i appears before a_j in the ordering $<_A$. Similarly, for any two distinct vertices $b_i, b_j \in B$, by $b_i <_B b_j$, we mean b_i appears before b_j in the ordering $<_B$. The set of all neighbors of a vertex v in G is denoted as $N_G(v)$ or simply $N(v)$ if G is clear from the context.

A bipartite graph $G = (A, B, E)$ is said to be a *chain graph* if there exist orderings $<_A$ of A and $<_B$ of B such that $N(u) \subseteq N(v)$ whenever $u <_A v$ and $N(a) \supseteq N(b)$ whenever $a <_B b$. A bipartite graph $G = (A, B, E)$ is said to be a *convex bipartite* graph if there exists an ordering of vertices, say, $<_B$ of B such that the neighbors of every vertex in A are consecutive with respect to $<_B$. A bipartite graph $G = (A, B, E)$ is said to be a *biconvex bipartite graph* if there exist orderings $<_A$ of A and $<_B$ of B such that the neighbors of every vertex in A are consecutive with respect to $<_B$ and the neighbors of every vertex in B are consecutive with respect to $<_A$.

By a result of Spinrad et al. [6], the class of bipartite permutation graphs is a subclass of the class of biconvex bipartite graphs. Further, it is easy to see that the class of chain graphs is a subclass of the class of biconvex bipartite graphs. The following observation follows from the definition of a biconvex bipartite graph.

Observation 1.2. *Let $G = (A, B, E)$ be a biconvex bipartite graph, $x, y \in A$ (respectively, $x, y \in B$), and $z \in N(x) \cap N(y)$. If $w \in A$ (respectively, $w \in B$) such that $x <_A w <_A y$ (respectively, $x <_B w <_B y$), then by consecutive property of the neighbors of z, we have that z is adjacent to w, i.e., $z \in N(w)$.*

Let $G = (A, B, E)$ be a biconvex bipartite graph. Two edges a_ib_s and a_jb_r in G are said to be *cross edges* if either $a_i <_A a_j$ and $b_r <_B b_s$, or $a_j <_A a_i$ and $b_s <_B b_r$. A biconvex ordering of G (pair $<_A$ and $<_B$) is said to be a *straight ordering* (in short, an S-ordering) if, whenever a pair of cross edges a_ib_s and a_jb_r exists in G, then at least one of the edges a_ib_r and a_jb_s also exists. A biconvex ordering of G that is also an S-ordering is said to be a *biconvex S-ordering* of G. Abbas and Stewart [1] studied the structure of biconvex bipartite graphs and gave the following theorem.

Theorem 1.3 ([1])**.** *Every connected biconvex bipartite graph has a biconvex S-ordering.*

A path between a pair of vertices u and v in G is said to be a *straight path* (in short, an S-path) if it does not contain any cross edges. Abbas and Stewart [1] also proved the following theorem on the existence of S-paths.

Theorem 1.4 ([1])**.** *Let G be a connected biconvex bipartite graph with a biconvex S-ordering and let u and v be any two vertices in G. Then there exists a shortest (u, v)-path in G that is an S-path.*

An *asteroidal triple* in a graph G is an independent set (a set of vertices with no edge between any pair) of three vertices in G if every two of them have a path between them avoiding the neighborhood of the third one. If there is no asteroidal triple in a graph, then the graph is said to be *asteroidal triple-free* (*AT-free*, for short). By a result of Corneil et al. [3], we have that there exists a spanning caterpillar in every AT-free graph. One can see that biconvex bipartite graphs can have an asteroidal triple, i.e., biconvex bipartite graphs need not be AT-free. Here, we present a similar structural existential result on connected biconvex bipartite graphs. In particular, we prove the following theorem.

Theorem 1.5. *If G is a connected biconvex bipartite graph, then there exists a spanning caterpillar in G.*

Note that a similar statement of Theorem 1.5 does not hold for convex bipartite graphs that are not biconvex. For instance, Fig. 1 illustrates an example of a connected convex bipartite graph that does not have a spanning caterpillar. One can verify that the graph in Fig. 1 is not a biconvex bipartite graph.

2. Proof of Theorem 1.5

This section is all about proving our main result, i.e., Theorem 1.5.

Proof. Let G be the given biconvex bipartite graph of order n, with A and B as the partite sets of cardinalities n_1 and n_2. Let $<_A = (a_1, a_2, \dots, a_{n_1})$ and $<_B = (b_1, b_2, \dots, b_{n_2})$ be the biconvex S-ordering of G (existence of such a pair $<_A$ and $<_B$ follows from Theorem 1.3). Suppose $n \leq 6$. Since graph G is connected, it has a spanning tree T. By Lemma 1.1, we have that T is a caterpillar, as desired. Thus we can assume that $n \geq 7$. Depending on the cardinalities of n_1 and n_2, we have the following cases.

Case 1. *$n_1 = 1$ or $n_2 = 1$.*

Without loss of generality, say $n_1 = 1$, i.e., $A = \{a_1\}$. Then since $n \geq 7$, we have $n_2 \geq 6$. Since G is connected, a_1 is adjacent to every vertex in B. Since G is bipartite, there is no edge between two vertices in B, implying that G is a star, i.e., $G = K_{1,n_2}$. Hence, if we delete all the leaves in G (which itself is a tree), we get P_1, i.e., a path on one vertex, which implies that G is a caterpillar, as desired.

Case 2. *$n_1 \geq 2$ and $n_2 \geq 2$*

Let a_f be the first neighbor of b_1 and let a_l be the last neighbor of b_{n_2} with respect to $<_A$. Note that as $n_1 \geq 2$ and $n_2 \geq 2$, we have $a_1 \neq a_{n_1}$ and $b_1 \neq b_{n_2}$. Depending on $N(a_1) \cap N(a_{n_1})$ and $N(b_1) \cap N(b_{n_2})$, we have the following cases.

Case 2.1. *$N(a_1) \cap N(a_{n_1}) \neq \emptyset$ or $N(b_1) \cap N(b_{n_2}) \neq \emptyset$.*

Without loss of generality, let $N(a_1) \cap N(a_{n_1}) \neq \emptyset$. Let b_c be a common neighbor of a_1 and a_{n_1}. Then by Observation 1.2, we have that every vertex in A is adjacent to b_c. Now, depending on $N(b_1) \cap N(b_{n_2})$, we have the following cases.

Case 2.1.1. *$N(b_1) \cap N(b_{n_2}) \neq \emptyset$.*

Let a_c be a common neighbor of b_1 and b_{n_2}. Then by Observation 1.2, we have that every vertex in B is adjacent to a_c. Now, we consider $P = a_c b_c$. Then every vertex of G that is not in P is adjacent to either a_c or b_c implying that G has a spanning caterpillar with P being the residual path.

Case 2.1.2. *$N(b_1) \cap N(b_{n_2}) = \emptyset$.*

Since b_c is adjacent to every vertex in A, the fact that $N(b_1) \cap N(b_{n_2}) = \emptyset$ will imply that b_1, b_c and b_{n_2} are three distinct vertices in B and $a_f \neq a_l$. Now, we consider $P = b_1 a_f b_c a_l b_{n_2}$. Let v be a vertex of G that is not in P. If $v \in A$, then v is adjacent to b_c. Otherwise, $v \in B$ and by Observation 1.2, v is adjacent to a_f or a_l implying that G has a spanning caterpillar with P being the residual path.

Case 2.2. *$N(a_1) \cap N(a_{n_1}) = \emptyset$ and $N(b_1) \cap N(b_{n_2}) = \emptyset$.*

Let Q be a shortest path from b_1 to b_{n_2} in G which is also an S-path (recall that an S-path is a path without cross edges). Since G is connected, the existence of Q follows from Theorem 1.4. Let $Q = b'_1 a'_1 b'_2 \dots b'_{k-1} a'_{k-1} b'_k$ for some positive integer k. Here we have $b'_1 = b_1$ and $b'_k = b_{n_2}$. Moreover, every vertex of the form a'_i is in A and every vertex of the form b'_i is in B. Further, since $N(b_1) \cap N(b_{n_2}) = \emptyset$, we have $a'_1 \neq a'_{k-1}$, implying that $k \geq 3$. Since Q is an S-path, we have:

$$a'_1 <_A a'_2 <_A \cdots <_A a'_k \text{ and } b_1 = b'_1 <_B b'_2 <_B \cdots <_B b'_k = b_{n_2}$$

Recall that a_f is the first neighbor of b_1 and a_l is the last neighbor of b_{n_2}. If $a_f <_A a'_1$, then add the edge $a_f b_1$ (same as $a_f b'_1$) to the path Q and label $a'_0 = a_f$. Note that otherwise, we have $a_f = a'_1$. If $a'_{k-1} <_A a_l$, then add the edge $a_l b_{n_2}$ (same as $a_l b'_k$) to the path Q and label $a'_k = a_l$. Note that otherwise, we have $a_l = a'_{k-1}$. Let this modified path be P. In other words, the path P is defined as follows:

$$P = \begin{cases} a'_0 b'_1 a'_1 b'_2 \dots b'_{k-1} a'_{k-1} b'_k a'_k, & \text{if } a'_0 = a_f <_A a'_1 \text{ and } a'_{k-1} <_A a_l = a'_k \\ a'_0 b'_1 a'_1 b'_2 \dots b'_{k-1} a'_{k-1} b'_k, & \text{if } a'_0 = a_f <_A a'_1 \text{ and } a'_{k-1} = a_l \\ b'_1 a'_1 b'_2 \dots b'_{k-1} a'_{k-1} b'_k a'_k, & \text{if } a_f = a'_1 \text{ and } a'_{k-1} <_A a_l = a'_k \\ b'_1 a'_1 b'_2 \dots b'_{k-1} a'_{k-1} b'_k, & \text{otherwise} \end{cases}$$

Let b_r be a vertex in B but not in P. Then b_r is sandwiched between two vertices b'_i and b'_{i+1} for some i with $1 \leq i \leq k-1$ (i.e., $b'_i <_B b_r <_B b'_{i+1}$) such that $b'_i, b'_{i+1} \in V(P)$. Since the vertex a'_i in P is adjacent to both b'_i and b'_{i+1}, by Observation 1.2, a'_i is also adjacent to b_r. Since b_r was chosen arbitrarily, we can infer that every vertex of B that is not in P is adjacent to a vertex in P. Let a_s be a vertex in A but not in P such that $a_f <_A a_s <_A a_l$. Then analogously one can show that by Observation 1.2, b'_{i+1} is also adjacent to a_s. Since a_s was chosen arbitrarily, we can infer that every vertex of A that is not in P, appearing in between a_f and a_l is adjacent to a vertex of P.

Let A_0 be the set of all vertices in A that appear before a_f in the ordering $<_A$. Let A_1 be the set of all vertices in A that appear after a_l in the ordering $<_A$. Note that every vertex in G that is not in $A_0 \cup A_1$ is either in P or is adjacent to a vertex in P. Hence, if $A_0 = A_1 = \emptyset$, then P itself is the residual path leading to the caterpillar. Therefore, we can assume that $A_0 \neq \emptyset$ or $A_1 \neq \emptyset$. Further, we have the following cases.

Case 2.2.1. *Exactly one among A_0 and A_1 is non-empty.*

Without loss of generality, assume that $A_1 \neq \emptyset$. Thus we have $A_0 = \emptyset$. Then clearly, $a_{n_1} \in A_1$ and is the last vertex in A_1. Suppose a_{n_1} is adjacent to some vertex b'_i in P. Since b'_i is adjacent to both a'_{i-1} and a_{n_1}, by Observation 1.2, b'_i is also adjacent to every vertex in between a'_{i-1} and a_{n_1}. In particular, b'_i is adjacent to every vertex in A_1. Therefore, since $A_0 = \emptyset$, we are sure that P is the residual path which yields the required caterpillar. Hence, we can assume that a_{n_1} is not adjacent to any vertex in P.

Let $\tilde{b}$ be the last neighbor of a_{n_1} (for the case $A_0 \neq \emptyset$ and $A_1 = \emptyset$, we consider $\tilde{b}$ to be the first neighbor of a_1). Since a_{n_1} is not adjacent to any vertex of P, we have $\tilde{b} \notin V(P)$. In particular, $\tilde{b} \neq b_1$ and $\tilde{b} \neq b_{n_2}$. Therefore, $\tilde{b}$ is sandwiched between two vertices b'_i and b'_{i+1} for some i with $1 \leq i \leq k-1$ (i.e., $b'_i <_B \tilde{b} <_B b'_{i+1}$) such that $b'_i, b'_{i+1} \in V(P)$. Since the vertex a'_i in P is adjacent to both b'_i and b'_{i+1}, by Observation 1.2, a'_i is also adjacent to $\tilde{b}$.

Now, obtain a path P_1 from P by replacing the vertex b'_{i+1} by $\tilde{b}$. We call this a *vertex replacement operation* and denote it as $R(b'_{i+1}, \tilde{b})$. Observe that any vertex in $G - V(P)$ which has b'_{i+1} as the only neighbor in P is either in between a'_i and a'_{i+1}, or in A_1. Since the vertex $\tilde{b}$ in P_1 is adjacent to both a'_i and a'_{n_1}, by Observation 1.2, $\tilde{b}$ is also adjacent to every vertex in between a'_i and a'_{n_1}. This also includes all the vertices in A_1 and all the vertices in between a'_i and a'_{i+1}. Further, every vertex in B is either in P_1 or is adjacent to some vertex in P_1, since $V(P_1) \cap A = V(P) \cap A$. Therefore, we have that every vertex of G which is not in P_1 is adjacent to a vertex of P_1, implying that G has a spanning caterpillar with P_1 being the residual path.

Case 2.2.2. *Both A_0 and A_1 are non-empty.*

Recall that A_0 is the set of all vertices in A that appear before a_f in the ordering $<_A$ and A_1 is the set of all vertices in A that appear after a_l in the ordering $<_A$. Let $G_0 = G - A_1$ and let $G_1 = G - A_0$. Notice that in graph G_0, we have $A_1 = \emptyset$ and in graph G_1, we have $A_0 = \emptyset$. Observe that in order to obtain G_0 and G_1 from G, we are deleting some consecutive vertices with respect to $<_A$. Hence, one can see that the graphs G_0 and G_1 remain biconvex bipartite. From the choice of A_0 and A_1, it is clear that the graphs G_0 and G_1 are connected. Therefore, by using the arguments in Case 2.2.1 individually, we obtain paths P_0 in G_0 and P_1 in G_1 of the corresponding caterpillars. Let $R(x_0, y_0)$ be the vertex replacement operation (refer Case 2.2.1 for the definition) performed to obtain P_0 from P and let $R(x_1, y_1)$ be the vertex replacement operation performed to obtain P_1 from P. By the steps in Case 2.2.1, we clearly have that y_0 is the first neighbor of a_1, y_1 is the last neighbor of a_{n_1}, $x_0 <_B y_0$, and $y_1 <_B x_1$.

Now, we obtain a new path $\tilde{P}$ by performing both the replacements $R(x_0, y_0)$ and let $R(x_1, y_1)$ in the path P. Observe that if $y_0 <_B y_1$, then the replacements R_0 and R_1 are independent of each other. In that case, one can see that every vertex in $G - V(\tilde{P})$ is adjacent to a vertex of $\tilde{P}$. This will in turn imply that G has a spanning caterpillar with $\tilde{P}$ being the residual path. So, the rest of the proof boils down to proving that $y_0 <_B y_1$.

Observe that since $N(a_1) \cap N(a_{n_1}) = \emptyset$, we have $y_0 \neq y_1$. By way of contradiction, assume that $y_1 <_B y_0$. Then since $a_1 <_A a_{n_1}$ and $y_1 <_B y_0$, we have a pair of cross edges a_1y_0 and $a_{n_1}y_1$. Since the ordering considered for G is an S-ordering, we have $a_1y_1 \in E(G)$ or $a_{n_1}y_0 \in E(G)$. In the former case, y_1 is a common neighbor of a_1 and a_{n_1}, and in the latter case, y_0 is a common neighbor of a_1 and a_{n_1}. Therefore, in either case, we have a contradiction to the fact that $N(a_1) \cap N(a_{n_1}) = \emptyset$. Hence, we have $y_0 <_B y_1$, as desired.

This completes the proof of Theorem 1.5. ∎

3. Impact on graph burning

Apart from the structural significance, Theorem 1.5 would be useful in several aspects. Here, we provide an implication of the result on a well-known conjecture, called the *burning number conjecture*.

For a graph G, the *burning number*, $b(G)$, is the minimum number of iterations required to inflame (or burn) the whole graph while in each iteration the fire spreads from all burned vertices to their neighbors and one additional vertex can be burned. A sequence of vertices $B = (b_1, b_2, \dots, b_k)$ is said to be a *burning sequence* of G if the whole graph can be burned in k steps by burning the vertices in B sequentially.

The concept of burning number was coined by Bonato et al. [2]. They conjectured as follows:

Conjecture 3.1 ([2])**.** *For any connected graph G of order n, $b(G) \leq \lceil \sqrt{n} \rceil$.*

This is called the *burning number conjecture* in the literature and is widely worked upon from thereon. Hiller et al. [5] proved the conjecture for caterpillars.

Theorem 3.2 ([5])**.** *If G is a caterpillar of order n, then $b(G) \leq \lceil \sqrt{n} \rceil$.*

It is clear that burning a spanning tree of a graph is sufficient to burn the entire graph. Therefore, we have the following corollary of Theorems 1.5 and 3.2.

Corollary 3.3. *If G is a connected biconvex bipartite graph of order n, then $b(G) \leq \lceil \sqrt{n} \rceil$.*

Hence, we can conclude that Conjecture 3.1 is true for the class of biconvex bipartite graphs. Consequently, the conjecture is true for any subclass of biconvex bipartite graphs. In particular, we can also conclude that Conjecture 3.1 is true for the bipartite permutation graphs and the chain graphs.

4. Conclusion

In this paper, we proved the existence of a spanning caterpillar in connected biconvex bipartite graphs. This structural existence is useful in studying several problems on biconvex bipartite graphs. One such instance is provided in the paper by applying this result in proving the burning number conjecture for biconvex bipartite graphs. It would be interesting to see whether some other problems can be addressed by using this result.

Data availability

No data was used for the research described in the article.

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