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Identities of the multi-variate independence polynomials from heaps theory

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Abstract. We study and derive identities for the multi-variate independence polynomials from the perspective of heaps theory. Using the inversion formula and the combinatorics of partially commutative algebras we show how the multi-variate version of Godsil type identity as well as the fundamental identity can be obtained from weight preserving bijections. Finally, we obtain a multi-variate identity involving connected bipartite subgraphs similar to the Christoffel–Darboux type identities obtained by Bencs.

Keywords. Independence polynomial of a graph; Cartier–Foata monoids; heaps.

1. Introduction

Let $\mathcal{G}$ be a finite simple connected graph with vertex set $V(\mathcal{G})$. A subset of $V(\mathcal{G})$ is said to be *independent* if it does not include two adjacent vertices and by convention, we allow the empty subset to be independent. The *multi-variate independence polynomial* of $\mathcal{G}$ is defined as

$$I(\mathcal{G}, \mathbf{x}) := \sum_{S} (-1)^{|S|} \prod_{v \in S} x_v,$$

where the sum runs over all independent subsets S of $V(\mathcal{G})$. The aim of this article is to approach certain identities for multi-variate independence polynomials using the inversion formula from heaps theory.

To explain our motivations and results, we need some terminologies. One can associate a monoid called the *Cartier–Foata monoid* to the graph $\mathcal{G}$ (see [5]). This monoid is generated by the vertices of $\mathcal{G}$ and the defining relations are given by $uv = vu$ if $u, v \in V(\mathcal{G})$ and there is no edge between them. One can prove that the Cartier–Foata monoid of $\mathcal{G}$ is equivalent to the monoid of heaps with pieces in $V(\mathcal{G})$ and the concurrency relation is determined by $\mathcal{G}$ (see [13]). The fundamental result of Viennot's general theory of heaps is the inversion lemma (see, for example, [13] and [4, Theorem 2.1]) which gives a closed formula for the generating function of heaps with all maximal pieces in some fixed subset.

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Even though heaps give a geometric interpretation of the elements of the Cartier–Foata monoid, we prefer to work with the Cartier–Foata monoid itself in this paper. Fix a subset K of $V(\mathcal{G})$, and consider the set $\mathcal{P}_K^{\emptyset}(\mathcal{G})$ that consists of all elements in the monoid that can only end with one of the v's from K (see Section 2 for more details). We can assign a weight to each element of $\mathcal{P}_K^{\emptyset}(\mathcal{G})$ as follows: given $\mathbf{w} = u_1 \cdots u_r \in \mathcal{P}_K^{\emptyset}(\mathcal{G})$, define $\mathrm{wt}(\mathbf{w}) = \prod_{i=1}^{r} x_{u_i} \in \mathbb{C}[x_v : v \in V(\mathcal{G})]$. Then the generating function of $\mathcal{P}_K^{\emptyset}(\mathcal{G})$ is simply given by

$$\sum_{\mathbf{w} \in \mathcal{P}_K^{\emptyset}(\mathcal{G})} \mathrm{wt}(\mathbf{w}) = \frac{I(\mathcal{G} - K, \mathbf{x})}{I(\mathcal{G}, \mathbf{x})},$$

where $\mathcal{G} - K$ is the graph obtained from $\mathcal{G}$ by removing the vertices in K. The motivation of this work comes from a Godsil's type identity that has been proved in [3] for one-variable independence polynomials; recall that the one-variable independence polynomial is obtained by evaluating $x_v = -x$ for all $v \in V(\mathcal{G})$ in the multi-variate independence polynomial. Given a vertex $u \in G$, Bencs constructed a rooted (stable path) tree $(\mathcal{T}, u')$ such that

$$\frac{I(\mathcal{G} - u, x)}{I(\mathcal{G}, x)} = \frac{I(\mathcal{T} - u', x)}{I(\mathcal{T}, x)}. \tag{1.1}$$

Godsil's original identity was stated for matching polynomials [7] and was one of the key ingredients in proving that the matching polynomial is real rooted. Furthermore, the importance of this identity is also highlighted in [12] where the authors prove the existence of infinite families of regular bipartite Ramanujan graphs of every degree greater than 2. It is not hard to prove the multi-variate version of equation (1.1) (the proof goes along the same lines as the proof of [3, Theorem 2.3]). However, both sides of the multi-variate version of equation (1.1) are the generating functions of certain words from the Cartier–Foata monoid of $\mathcal{G}$. More precisely, the left-hand side of equation (1.1) corresponds to the generating function of $\mathcal{P}_u^{\emptyset}(\mathcal{G})$ and the right-hand side corresponds to the generating function of $\mathcal{P}_{u'}^{\emptyset}(\mathcal{T})$. So, we have the following natural questions:

- Is there any natural weight preserving bijective map from $\mathcal{P}_u^{\emptyset}(\mathcal{G})$ onto $\mathcal{P}_{u'}^{\emptyset}(\mathcal{T})$ that gives the multi-variate version of equation (1.1)?
- Using the method of finding weight preserving bijections, is one able to give new proofs of existing identities, generalize them to the multi-variate case and obtain new identities?

We answer the first question affirmatively in this paper. We will also use our approach to get more identities and prove existing identities for multi-variate independence polynomial of $\mathcal{G}$. In particular, we prove a new multi-variate identity equation (4.4) involving connected bipartite subgraphs similar to the Christoffel–Darboux type identities obtained by Bencs [2]. This identity seems to be new in the literature.

2. Independence polynomials and word decompositions

2.1. By the given sets $A_1, \dots, A_k$, we denote $A_1 \dot\cup \cdots \dot\cup A_k$ by the disjoint union of $A_1, \dots, A_k$. All our graphs in this paper are assumed to be finite simple connected graphs, i.e., they are connected and contain only finitely many vertices and edges and have no loops and multiple edges. We use $\mathcal{G}, \mathcal{T}, \mathcal{H}, \dots$ to denote the graphs and $S, V, H, \dots$ to denote the set of vertices.

2.2. Let $\mathcal{G}$ be a finite simple connected graph. The vertex set and edge set of $\mathcal{G}$ are denoted as $V(\mathcal{G})$ and $E(\mathcal{G})$ respectively. We denote by $e(u, v)$ the edge between the vertices u and v of $\mathcal{G}$. For $u \in V(\mathcal{G})$, we denote by $N_{\mathcal{G}}(u)$ the neighbourhood of u in $\mathcal{G}$, $d_{\mathcal{G}}(u) := |N_{\mathcal{G}}(u)|$ the degree of u in $\mathcal{G}$ and set $N_{\mathcal{G}}[u] = N_{\mathcal{G}}(u) \cup \{u\}$. For a subset $S \subseteq V(\mathcal{G})$, we denote by $N_{\mathcal{G}}(S) := \cup_{v \in S} N_{\mathcal{G}}(v)$ and by $\mathcal{G}[S]$ the subgraph of $\mathcal{G}$ spanned by the vertices in S. Let $\mathcal{P}^{\emptyset}(\mathcal{G})$ denote the partially commutative monoid of $\mathcal{G}$ which is generated by the elements of $V(\mathcal{G})$ modulo the relations

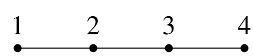
$$uv = vu \iff e(u, v) \notin E(\mathcal{G}).$$

If $\mathcal{C}^{\emptyset}(\mathcal{G})$ denotes the commutative monoid generated by $V(\mathcal{G})$, we have a canonical monoid morphism $\pi_{\mathcal{G}} : \mathcal{P}^{\emptyset}(\mathcal{G}) \to \mathcal{C}^{\emptyset}(\mathcal{G})$. We set $\mathcal{P}(\mathcal{G}) := \mathcal{P}^{\emptyset}(\mathcal{G})\backslash\{\mathrm{pt}\}$ where we think of the extra point in $\mathcal{P}^{\emptyset}(\mathcal{G})$ as the empty word and introduce further

$$\begin{aligned}
\mathcal{P}_{v_1,\ldots,v_r}(\mathcal{G}) &= \{\mathbf{w} \in \mathcal{P}(\mathcal{G}) : \mathrm{IA}(\mathbf{w}) \subseteq \{v_1, \ldots, v_r\}\},\\
\mathcal{P}^{c}_{v_1,\ldots,v_r}(\mathcal{G}) &= \{\mathbf{w} \in \mathcal{P}(\mathcal{G}) : \mathrm{IA}(\mathbf{w}) = \{v_1, \ldots, v_r\}\},\\
\mathcal{P}^{\emptyset}_{v_1,\ldots,v_r}(\mathcal{G}) &= \mathcal{P}_{v_1,\ldots,v_r}(\mathcal{G}) \cup \{\mathrm{pt}\},
\end{aligned}$$

i.e., $\mathcal{P}_{v_1,\ldots,v_r}(\mathcal{G})$ consists of all words that can only end with one of the v_i's. For a word $\mathbf{w} = v_1 \cdots v_r \in \mathcal{P}(\mathcal{G})$, we write $|\mathbf{w}| = r$ for the length of $\mathbf{w}$ and set $v(\mathbf{w}) = |\{1 \leq j \leq r : v_j = v\}|$ for a vertex $v \in V(\mathcal{G})$. The *initial alphabet* of $\mathbf{w}$ is the multiset denoted by $\mathrm{IA}_{\mathrm{m}}(\mathbf{w})$ and defined by $v \in \mathrm{IA}_{\mathrm{m}}(\mathbf{w})$ (counted with multiplicities) if and only if $\mathbf{w} = \mathbf{u}v$ for some $\mathbf{u} \in \mathcal{P}(\mathcal{G})$. We denote the underlying set by $\mathrm{IA}(\mathbf{w})$.

Example. Let us take $\mathcal{G}$ to be the path graph P_4 (and we keep this as our running example to explain all our terminologies and results):



Take $\mathbf{w} = 342111 \in \mathcal{P}(\mathcal{G})$, then

$$|\mathbf{w}| = 6,\ 1(\mathbf{w}) = 3,\ 2(\mathbf{w}) = 3(\mathbf{w}) = 4(\mathbf{w}) = 1,\ \mathrm{IA}_{\mathrm{m}}(\mathbf{w}) = \{1, 1, 1, 4\}$$
$$\text{and } \mathrm{IA}(\mathbf{w}) = \{1, 4\}.$$

2.3. Given $\mathbf{w} \in \mathcal{P}_u(\mathcal{G})$, it has been shown in [1, Proposition 4.3] that there exists unique words $\mathbf{w}_1, \ldots, \mathbf{w}_{u(\mathbf{w})} \in \mathcal{P}(\mathcal{G})$ such that

$$\mathbf{w} = \mathbf{w}_1 \cdots \mathbf{w}_{u(\mathbf{w})}, \quad \mathrm{IA}_m(\mathbf{w}_i) = \{u\} \quad \text{for all } 1 \leq i \leq u(\mathbf{w}). \tag{2.1}$$

If $u(\mathbf{w}) > 1$, we refer to the decomposition above as the *initial-alphabet*-decomposition or simply *ia*-decomposition of $\mathbf{w}$.

Example. Consider $G = P_4$ and $\mathbf{w} = 32111 \in \mathcal{P}_1(\mathcal{G})$. It is easy to see that $1(\mathbf{w}) = 3$ and the *ia*-decomposition of $\mathbf{w}$ is $32111 = (321)(1)(1)$.

2.4. We shall define now the so-called neighborhood decomposition. We write

$$N_{\mathcal{G}}(u, \mathbf{w}) = \{v \in N_{\mathcal{G}}(u) : v(\mathbf{w}) > 0\}, \quad d_{\mathcal{G}}(u, \mathbf{w}) = |N_{\mathcal{G}}(u, \mathbf{w})|.$$

PROPOSITION. *Let* $\mathbf{w} \in \mathcal{P}_u(\mathcal{G})$ *with* $u(\mathbf{w}) = 1$ *and write* $N_{\mathcal{G}}(u, \mathbf{w}) = \{u_1 < \cdots < u_d\}$, *where* $d = d_{\mathcal{G}}(u, \mathbf{w})$. *Then there exists an unique* $\mathbf{w}_1, \ldots, \mathbf{w}_d \in \mathcal{P}^{\emptyset}(\mathcal{G})$ *such that*

(*i*) $\mathbf{w} = \mathbf{w}_1 \cdots \mathbf{w}_d u$.
(*ii*) *If* $\mathbf{w}_i \in \mathcal{P}(\mathcal{G})$, *then* $\mathrm{IA}(\mathbf{w}_i) = \{u_i\}$ *for all* $1 \le i \le d$.
(*iii*) $u_i \notin \mathbf{w}_j$ *for all* $i < j$.

Proof. We proceed by induction on d where the $d = 1$ case is obviously true. So we can assume that $d > 1$. We choose $\mathbf{u}_1, \mathbf{u}_2$ such that $\mathbf{w} = \mathbf{u}_1\mathbf{u}_2$ and $|\mathbf{u}_2|$ is maximal with the property that $u_1 \notin \mathbf{u}_2$. This forces $\mathrm{IA}(\mathbf{u}_1) = \{u_1\}$. Since $d_{\mathcal{G}}(u, \mathbf{u}_2) < d_{\mathcal{G}}(u, \mathbf{w})$, we can use induction to get the required decomposition for $\mathbf{u}_2$. This gives the decomposition for $\mathbf{w}$ with the properties (i)–(iii) once we set $\mathbf{w}_1 = \mathbf{u}_1$. The rest of the proof is concerned with the uniqueness. Assume that $\mathbf{w} = \mathbf{w}'_1 \cdots \mathbf{w}'_d u$ is another decomposition satisfying the conditions (i)–(iii) of the lemma. Write $\mathbf{w} = \mathbf{w}'_1\mathbf{u}'$, then we have $u_1 \notin \mathbf{u}'$. However, the choice of $\mathbf{w}_1$ implies $|\mathbf{w}_1| \le |\mathbf{w}'_1|$ and $\mathbf{u}'$ is a subword of $\mathbf{u}_2$. This forces $|\mathbf{w}_1| = |\mathbf{w}'_1|$, since $\mathrm{IA}(\mathbf{w}'_1) = \{u_1\}$. Hence $\mathbf{u}' = \mathbf{u}_2$ and $\mathbf{w}_1 = \mathbf{w}'_1$. Now a simple induction argument completes the proof. □

For $\mathbf{w} \in \mathcal{P}_u(\mathcal{G})$ with $u(\mathbf{w}) = 1$, we refer to the decomposition of Proposition 2.4 as the *neighbourhood*-decomposition or simply *nbd*-decomposition of $\mathbf{w}$.

Example. Consider $\mathcal{G} = P_4$ and take the word $\mathbf{w} = 311432$, then $2(\mathbf{w}) = 1$ and $N_{\mathcal{G}}(2, \mathbf{w}) = \{1, 3\}$. The *nbd*-decomposition of $\mathbf{w}$ is $\mathbf{w} = 311432 = (11)(343)2$.

2.5. A subset S of $V(\mathcal{G})$ is said to be *independent* if there is no edge between the elements of S in the graph $\mathcal{G}$. We denote by $\mathcal{I}_{\mathcal{G}}$ the set of independent subsets of $\mathcal{G}$ and note that we have $\emptyset, \{v\} \in \mathcal{I}_{\mathcal{G}}$ for each $v \in V(\mathcal{G})$. The *multi-variate independence polynomial* of $\mathcal{G}$ is defined as

$$I(\mathcal{G}, \mathbf{x}) := \sum_{S \in \mathcal{I}_{\mathcal{G}}} (-1)^{|S|} \prod_{v \in S} x_v$$

and we view it as an element in $\mathbb{C}[x_v : v \in V(\mathcal{G})]$, the polynomial algebra over $\mathbb{C}$ generated by the commuting variables $\{x_v : v \in V(\mathcal{G})\}$. The aim of this article is to approach certain identities for multi-variate independence polynomials using the inversion formula from heap theory. We need the following trivial identifications.

Lemma. Let $S \subseteq V(\mathcal{G})$ *and* $\{K_1, \ldots, K_s\}$ *be the set of non-empty independent subsets of the graph* $\mathcal{G}[S]$.

(1) *We have a bijection*

$$\mathcal{P}^c_{K_1}(\mathcal{G}) \dot\cup \cdots \dot\cup \mathcal{P}^c_{K_s}(\mathcal{G}) \to \mathcal{P}_S(\mathcal{G}). \tag{2.2}$$

(2) *For any independent subset* $K \neq \emptyset$ *of* S, *we have a bijection*

$$\varphi_K : \mathcal{P}^c_K(\mathcal{G}) \to \mathcal{P}^{\emptyset}_{N_{\mathcal{G}}[K]}(\mathcal{G}), \quad \mathbf{w} \mapsto \frac{\mathbf{w}}{\prod_{y \in K} y}.$$

Proof. We first show that the left-hand side of (2.2) is a disjoint union. Let $\mathbf{w} \in \mathcal{P}^c_{K_1}(\mathcal{G}) \cap \mathcal{P}^c_{K_2}(\mathcal{G})$ and $u \in K_1 \backslash K_2$. Then we have $\mathbf{w} = \mathbf{w}'u$ and thus $u \in \mathrm{IA}(\mathbf{w}) = K_2$ which is a contradiction. So the left-hand side is disjoint. The identity map

$$\mathrm{Id}_{K_i} : \mathcal{P}^c_{K_i}(\mathcal{G}) \to \mathcal{P}_S(\mathcal{G})$$

for all $i \in \{1, \ldots, s\}$ induces the desired map (2.2) which is clearly bijective. In order to show the second part, we first note that the map is well defined. If $z \in \mathrm{IA}(\varphi_K(\mathbf{w}))$ but $z \notin N_{\mathcal{G}}(K)$, then we would also have $z \in \mathrm{IA}(\mathbf{w}) = K$. Hence $z \in N_{\mathcal{G}}[K]$. The map is bijective because the inverse map is simply given by multiplication with $\prod_{y \in K} y$. □

2.6. The inversion lemma from heap theory [13, Proposition 5.3] states that

$$\frac{I(\mathcal{G} - S, \mathbf{x})}{I(\mathcal{G}, \mathbf{x})} = \sum_{\mathbf{w} = v_1 \cdots v_r \in \mathcal{P}^{\emptyset}_S(\mathcal{G})} x_{v_1} \cdots x_{v_r}, \quad S \subseteq V(\mathcal{G}).$$

Using the inversion lemma, one can derive certain well-known identities of independence polynomials and extend them to the multi-variate version. For example, Lemma 2.5 simply implies that (keeping the same notation)

$$I(\mathcal{G} - S, \mathbf{x}) - I(\mathcal{G}, \mathbf{x}) = \sum_{i=1}^{s} \Big(\prod_{v \in K_i} x_v \Big) \, I(\mathcal{G} - N_{\mathcal{G}}[K_i], \mathbf{x}) \tag{2.3}$$

which is known as the fundamental identity if S is singleton. The importance of the identity can be seen, for example, in [6] where the authors proved that independence polynomials of claw free graphs are real-rooted by using (2.3) when S is a clique. The single variable version of the above identity is the main result of [10].

3. Weight preserving bijection and Godsil's identity

3.1. Here we recall the construction of a rooted tree associated with $(\mathcal{G}, u)$, where $u \in V(\mathcal{G})$, which is important in Godsil type identity (originally it is stated for the matching polynomial; see [8] and also [3]) which relates the independence polynomial of $\mathcal{G}$ to that of the tree. The constructed tree is called a stable-path tree of $\mathcal{G}$. For more details, we refer the reader to [3] and for an example, see Fig. 1. Let $V(\mathcal{G}) = \{1, \ldots, n\}$ be an enumeration of the vertices of $\mathcal{G}$ and let $N_{\mathcal{G}}(u) = \{u_1 < \cdots < u_d\}$, where $u \in V(\mathcal{G})$ and $d := d_{\mathcal{G}}(u)$. For each vertex $u \in V(\mathcal{G})$, we will recursively associate a rooted tree $(\mathcal{T}_{\mathcal{G}}, u')$ and a surjective graph homomorphism

$$\ell_{\mathcal{G}} : V(\mathcal{T}_{\mathcal{G}}) \to V(\mathcal{G}), \; u' \mapsto u$$

as follows: If $d = 0$, then $\mathcal{G}$ is a single vertex and we set $T_{\mathcal{G}} = \{u'\}$ as the tree with one vertex u'. If $d \geq 1$, we let $\mathcal{G}_i$ be the connected component of $\mathcal{G}[V(\mathcal{G}) \backslash \{u, u_1, u_2, \ldots, u_{i-1}\}]$ containing u_i and we take the induced total ordering on $V(\mathcal{G}_i)$ that comes from $V(\mathcal{G})$. Now we have by induction the family of rooted trees $(\mathcal{T}_{\mathcal{G}_i}, u_i')$ and the graph homomorphisms

$$\ell_{\mathcal{G}_i} : V(\mathcal{T}_{\mathcal{G}_i}) \to V(\mathcal{G}_i), \quad u_i' \mapsto u_i.$$

Finally we take the disjoint union of rooted trees $(\mathcal{T}_{\mathcal{G}_i}, u_i')$ and a new vertex u', and join the vertex u' with the vertices u_i', $1 \leq i \leq d$. Clearly the graph $(\mathcal{T}_{\mathcal{G}}, u')$ obtained in this way

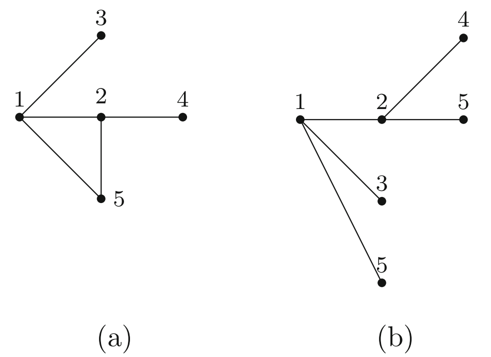


Figure 1. A graph with its stable-path tree. **(a)** A graph $\mathcal{G}$ with labeled vertices and **(b)** the graph $T_{\mathcal{G},1}$.

is a rooted tree. Define the map $\ell_{\mathcal{G}} : V(\mathcal{T}_{\mathcal{G}}) \to V(\mathcal{G})$ by $\ell_{\mathcal{G}}(u') = u$ and $\ell_{\mathcal{G}}(v) = \ell_{\mathcal{G}_i}(v)$ if $v \in V(\mathcal{T}_{\mathcal{G}_i})$. This is clearly a surjective graph homomorphism and the map $\ell_{\mathcal{G}}$ induces a partial ordering on $V(\mathcal{T}_{\mathcal{G}})$ as follows: for $v_1, v_2 \in V(\mathcal{T}_{\mathcal{G}})$, we have

$$v_1 \geq v_2 \iff \ell_{\mathcal{G}}(v_1) \geq \ell_{\mathcal{G}}(v_2).$$

We extend this partial order to a total ordering on $V(\mathcal{T}_{\mathcal{G}})$. The extension of $\ell_{\mathcal{G}}$ to $\mathcal{C}(\mathcal{T}_{\mathcal{G}})$ is again denoted as $\ell_{\mathcal{G}}$.

3.2. We freely use the notations that were developed in the earlier sections. We now state and prove the following result.

Theorem 1. *Let $\mathcal{G}$ be a finite, simple and connected graph. Then there exists a bijection $\varphi_{\mathcal{G}} : \mathcal{P}_u^{\emptyset}(\mathcal{G}) \to \mathcal{P}_{u'}^{\emptyset}(\mathcal{T}_{\mathcal{G}})$ such that $|\varphi_{\mathcal{G}}(\mathbf{w})| = |\mathbf{w}|$ and*

$$\begin{tikzcd}
\mathcal{P}_u^{\emptyset}(\mathcal{G}) \arrow[r, "\varphi_{\mathcal{G}}"] \arrow[d, "\pi_{\mathcal{G}}"'] & \mathcal{P}_{u'}^{\emptyset}(\mathcal{T}_{\mathcal{G}}) \arrow[d, "\pi_{T_{\mathcal{G}}}"] \\
\mathcal{C}^{\emptyset}(\mathcal{G}) & \mathcal{C}^{\emptyset}(\mathcal{T}_{\mathcal{G}}) \arrow[l, "\ell_{\mathcal{G}}"]
\end{tikzcd}$$

is a commutative diagram.

Proof. We recursively construct the map $\varphi_{\mathcal{G}}$ and its inverse $\psi_{\mathcal{G}}$. If $|V(\mathcal{G})| = 1$, then we set $\varphi_{\mathcal{G}}(u) = u'$ and $\psi_{\mathcal{G}}(u') = u$. So assume that $|V(\mathcal{G})| > 1$ and let $\varphi_{\mathcal{H}}$ be the required map for all finite, connected graphs with $|V(\mathcal{H})| < |V(\mathcal{G})|$. We first consider the case $\mathbf{w} \in \mathcal{P}_u(\mathcal{G})$ with $u(\mathbf{w}) = 1$ and recall that we have the *nbd*-decomposition $\mathbf{w} = \mathbf{w}_1 \cdots \mathbf{w}_d u$, by Proposition 2.4 where we abbreviate $d = d(u, \mathbf{w})$ in the rest of the proof. From the conditions (ii) and (iii) of Proposition 2.4, it is clear that $\mathbf{w}_i \in \mathcal{P}_{u_i}^{\emptyset}(\mathcal{G}_i)$ for all $1 \leq i \leq d$. Now since $|V(\mathcal{G}_i)| < |V(\mathcal{G})|$, we obtain by induction a family of bijective maps $\varphi_{\mathcal{G}_i} : \mathcal{P}_{u_i}^{\emptyset}(\mathcal{G}_i) \to \mathcal{P}_{u_i'}^{\emptyset}(\mathcal{T}_{\mathcal{G}_i})$ satisfying the required properties for all $1 \leq i \leq d$. We define

$$\varphi_{\mathcal{G}}(\mathbf{w}) = \varphi_{\mathcal{G}_1}(\mathbf{w}_1)\varphi_{\mathcal{G}_2}(\mathbf{w}_2) \cdots \varphi_{\mathcal{G}_d}(\mathbf{w}_d)u' \tag{3.1}$$

Since the decomposition $\mathbf{w} = \mathbf{w}_1 \cdots \mathbf{w}_d u$ is unique, the above map is well-defined. Clearly the map $\varphi_{\mathcal{G}}$ preserves the *nbd*-decomposition, i.e., the decomposition in (3.1) is the *nbd*-decomposition of $\varphi_{\mathcal{G}}(\mathbf{w})$.

Now we extend this map using the *ia*-decomposition of $\mathbf{w} \in \mathcal{P}_u(\mathcal{G})$ with $u(\mathbf{w}) > 1$. We have $\mathbf{w} = \mathbf{w}_1 \cdots \mathbf{w}_{u(\mathbf{w})}$ satisfying $\mathbf{w}_i \in \mathcal{P}_u(\mathcal{G})$ and $u(\mathbf{w}_i) = 1$ for all $1 \le i \le u(\mathbf{w})$. We extend $\varphi_{\mathcal{G}}$ as follows:

$$\varphi_{\mathcal{G}}(\mathbf{w}) = \varphi_{\mathcal{G}}(\mathbf{w}_1)\varphi_{\mathcal{G}}(\mathbf{w}_2) \cdots \varphi_{\mathcal{G}}(\mathbf{w}_{u(\mathbf{w})}).$$

Again $\varphi_{\mathcal{G}}$ is well-defined by the uniqueness of the decomposition and $\varphi_{\mathcal{G}}$ preserves the *ia*-decomposition. The fact that $|\varphi_{\mathcal{G}}(\mathbf{w})| = |\mathbf{w}|$ holds and that the above diagram commutes follows from the fact that $\ell_{\mathcal{G}}, \pi_{\mathcal{G}}, \pi_{\mathcal{T}_{\mathcal{G}}}$ are all homomorphisms and the maps $\varphi_{\mathcal{G}_i}$ also satisfy these properties. So it remains to construct the inverse map.

In a similar way, we now define the inverse map $\psi_{\mathcal{G}}$ using the maps $\psi_{\mathcal{G}_i} = \varphi_{\mathcal{G}_i}^{-1}$. Let $\mathbf{w}' \in \mathcal{P}_{u'}(\mathcal{T}_{\mathcal{G}})$ be such that $u'(\mathbf{w}') = 1$. Again we have the *nbd*-decomposition $\mathbf{w}' = \mathbf{w}'_1 \cdots \mathbf{w}'_{d(u',\mathbf{w}')} u'$. We define

$$\psi_{\mathcal{G}}(\mathbf{w}') = \psi_{\mathcal{G}_1}(\mathbf{w}'_1)\psi_{\mathcal{G}_2}(\mathbf{w}'_2) \cdots \psi_{\mathcal{G}_{d(u',\mathbf{w}')}}(\mathbf{w}'_{d(u',\mathbf{w}')})u.$$

As before this is a well-defined map and preserves the *nbd*-decomposition. Using this, it is easy to see that $\varphi_{\mathcal{G}} \circ \psi_{\mathcal{G}}(\mathbf{w}) = \mathbf{w}$ and $\psi_{\mathcal{G}} \circ \varphi_{\mathcal{G}}(\mathbf{w}') = \mathbf{w}'$ for $\mathbf{w} \in \mathcal{P}_u(\mathcal{G})$, $\mathbf{w}' \in \mathcal{P}_{u'}(\mathcal{T}_{\mathcal{G}})$ with $u(\mathbf{w}) = u'(\mathbf{w}') = 1$.

If $\mathbf{w}' \in \mathcal{P}_{u'}(\mathcal{T}_{\mathcal{G}})$ with $u'(\mathbf{w}') > 1$, we extend the map using the *ia*-decomposition of $\mathbf{w}' = \mathbf{w}'_1 \cdots \mathbf{w}'_{u'(\mathbf{w}')}$, namely, we set

$$\psi_{\mathcal{G}}(\mathbf{w}') = \psi_{\mathcal{G}}(\mathbf{w}'_1) \cdots \psi_{\mathcal{G}}(\mathbf{w}'_{u'(\mathbf{w}')}). \tag{3.2}$$

As before, this is a well-defined map and preserves the *ia*-decomposition. Again we have $\varphi_{\mathcal{G}} \circ \psi_{\mathcal{G}} = \mathrm{Id}_{\mathcal{P}_{u'}(\mathcal{T}_{\mathcal{G}})}$ and $\psi_{\mathcal{G}} \circ \varphi_{\mathcal{G}} = \mathrm{Id}_{\mathcal{P}_u(\mathcal{G})}$, proving that $\varphi_{\mathcal{G}}$ is a bijection. □

3.3. The observation in Section 2.6 together with Theorem 1 immediately imply the multivariate Godsil identity

$$\frac{I(\mathcal{G} - u, \mathbf{x})}{I(\mathcal{G}, \mathbf{x})} = \frac{\ell_{\mathcal{G}}(I(\mathcal{T}_{\mathcal{G}} - u', \mathbf{x}))}{\ell_{\mathcal{G}}(I(\mathcal{T}_{\mathcal{G}}, \mathbf{x}))}.$$

We refer also to [11] for different generalizations of this identity.

4. Bipartite graphs and positive sum identities

4.1. Motivated by the Christoffel–Darboux type identities for the independence polynomial obtained in [2], we would like to achieve a similar type identity or a refined version of it without the alternating sign and in a multi-variate version. Our approach will be the same by observing the underlying indexing sets.

Let u, v be two distinct vertices of $\mathcal{G}$. Given a pair $(\mathbf{w}u, \mathbf{w}'v) \in \mathcal{P}_u(\mathcal{G}) \times \mathcal{P}_v(\mathcal{G})$ and a shortest path $\mathbf{p} = v_1 v_2 \cdots v_k$ connecting $u = v_1$ with $v = v_k$, we define a bipartite graph $\mathcal{H}$ whose vertices are given by $H = H_1 \cup H_2$, where

$$H_1 = \mathrm{IA}(\mathbf{w} \cdot v_2 \cdot v_4 \cdots), \quad H_2 = \mathrm{IA}(\mathbf{w}' \cdot v_1 \cdot v_3 \cdots).$$

Note that $v \in H_1$ and $u \in H_2$ if k is even and $u, v \in H_2$ otherwise. We consider the map

$$\mathcal{P}_u(\mathcal{G}) \times \mathcal{P}_v(\mathcal{G}) \to \dot{\bigsqcup}_H \mathcal{P}^{\emptyset}_{Z_1(\mathcal{H})}(\mathcal{G}) \times \mathcal{P}^{\emptyset}_{Z_2(\mathcal{H})}(\mathcal{G}) \tag{4.1}$$

$$(\mathbf{w}u, \mathbf{w}'v) \to \left(\frac{\mathbf{w} \cdot v_2 \cdot v_4 \cdots}{\prod_{y \in H_1} y}, \frac{\mathbf{w}' \cdot v_1 \cdot v_3 \cdots}{\prod_{y \in H_2} y} \right),$$

where the disjoint union runs over all connected bipartite subgraphs $\mathcal{H}$ of $\mathcal{G}$ containing the path $\mathbf{p}$ and satisfying

$$\begin{aligned} &H_1 \backslash \{v_2, v_4, \dots\} \subseteq N_{\mathcal{G}}[u], \quad H_2 \backslash \{v_1, v_3, \dots\} \subseteq N_{\mathcal{G}}[v], \\ &Z_1(\mathcal{H}) = N_{\mathcal{G}}[H_1 \backslash \{v_2, v_4, \dots\}] \cup (N_{\mathcal{G}}[H_1] \cap N_{\mathcal{G}}[u]), \\ &Z_2(\mathcal{H}) = N_{\mathcal{G}}[H_2 \backslash \{v_1, v_3, \dots\}] \cup (N_{\mathcal{G}}[H_2] \cap N_{\mathcal{G}}[v]). \end{aligned} \tag{4.2}$$

PROPOSITION

The map defined in (4.1) *is a bijection.*

Proof. We first show that the map is well-defined. Set $\mathbf{w}' = \frac{\mathbf{w} \cdot v_2 \cdot v_4 \cdots}{\prod_{z \in H_1} z}$, then we have

$$\mathbf{w} \cdot v_2 \cdot v_4 \cdots = \mathbf{w}' \prod_{z \in H_1} z \text{ and } \mathbf{w} = \mathbf{w}' \prod_{z \in H_1 \backslash \{v_2, v_4, \cdots\}} z.$$

Assume that a letter y is in the initial alphabet of the word $\mathbf{w}'$ which we assume to be non-empty. Suppose $y \in N_{\mathcal{G}}[H_1 \backslash \{v_2, v_4, \cdots\}]$, then we have $y \in Z_1(\mathcal{H})$. Otherwise $y \notin N_{\mathcal{G}}[H_1 \backslash \{v_2, v_4, \cdots\}]$ which implies $y \in \mathrm{IA}(\mathbf{w})$, hence $y \in N_{\mathcal{G}}[u]$. Suppose $y \in N_{\mathcal{G}}(H_1)$, then we have $y \in Z_1(\mathcal{H})$. Otherwise $y \notin N_{\mathcal{G}}(H_1)$, then $y \in \mathrm{IA}(\mathbf{w} \cdot v_2 \cdot v_4 \cdots) = H_1$. Again in this case we have $y \in Z_1(\mathcal{H})$. Similar calculation shows that the initial alphabet of the second component lies in $Z_2(\mathcal{H})$. This shows that the map is well-defined. For bijectivity, we construct the inverse map.

Given a bipartite connected graph $\mathcal{H}$ containing $\mathbf{p}$ (say $v_1, v_3, \dots \in H_2$) and satisfying (4.2), we define

$$\mathcal{P}^{\emptyset}_{Z_1(\mathcal{H})}(\mathcal{G}) \times \mathcal{P}^{\emptyset}_{Z_2(\mathcal{H})}(\mathcal{G}) \mapsto \mathcal{P}_u(\mathcal{G}) \times \mathcal{P}_v(\mathcal{G})$$

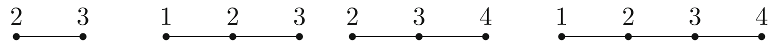
$$(\tilde{\mathbf{w}}, \tilde{\mathbf{w}}') \to \left(\tilde{\mathbf{w}} \prod_{y \in H_1 \backslash \{v_2, v_4, \dots\}} y \; u, \tilde{\mathbf{w}}' \prod_{y \in H_2 \backslash \{v_1, v_3, \dots\}} y \; v \right). \tag{4.3}$$

From (4.2) and the definition of $Z_i(\mathcal{H})$, $i = 1, 2$, we know that the above map is well defined. This map induces the inverse of (4.1) since

$$\mathrm{IA}(\tilde{\mathbf{w}} \prod_{y \in H_1} y) = H_1, \quad \mathrm{IA}(\tilde{\mathbf{w}}' \prod_{y \in H_2} y) = H_2.$$

□

Figure 2. Path graph P_4.



(a) (b) (c) (d)

Figure 3. Connected bipartite subgraphs of P_4 containing 2 and 3 for **(a)** H^1, **(b)** H^2, **(c)** H^3 and **(d)** H^4.

4.2. As an immediate consequence of Proposition 4.1, we obtain the following identity:

$$\left(\frac{I(\mathcal{G}-u,\mathbf{x})}{I(\mathcal{G},\mathbf{x})}-1\right)\left(\frac{I(\mathcal{G}-v,\mathbf{y})}{I(\mathcal{G},\mathbf{y})}-1\right)$$
$$=\sum_{H}\prod_{\substack{w\in H_1\setminus\{v_2,v_4,\dots\}\\ w'\in H_2\setminus\{v_1,v_3,\dots\}}} x_w y_{w'} x_u y_v \left(\frac{I(\mathcal{G}-Z_1(\mathcal{H}),\mathbf{x})}{I(\mathcal{G},\mathbf{x})}\right)\left(\frac{I(\mathcal{G}-Z_2(\mathcal{H}),\mathbf{y})}{I(\mathcal{G},\mathbf{y})}\right), \tag{4.4}$$

where the sum runs over all connected bipartite subgraphs $\mathcal{H}$ of $\mathcal{G}$ containing the path $\mathbf{p}$ and satisfying (4.2) (by convention, we denote always by H_2 the part which contains $v_1, v_3, \dots$).

Remark. If there is an edge between u and v, then the left-hand side of the above identity becomes (after evaluating $\mathbf{x}=\mathbf{y}$)

$$\frac{I(\mathcal{G}-u,\mathbf{x})}{I(\mathcal{G},\mathbf{x})}\frac{I(\mathcal{G}-v,\mathbf{x})}{I(\mathcal{G},\mathbf{x})}-\frac{I(\mathcal{G}-\{u,v\},\mathbf{x})}{I(\mathcal{G},\mathbf{x})}.$$

This part also appeared, for example, in Gutman's identity for trees (see [9]) and for general graphs in [2].

4.3. We will now see some examples that illustrate our results.

Example. Let us consider the path graph P_4 (see Fig. 2) and take $u=2$ and $v=3$. The connected bipartite subgraphs of P_4 containing u, v are given in Fig. 3.

In this case, we can rewrite equation (4.4) as follows:

$$(I(\mathcal{G}-u,\mathbf{x})-I(\mathcal{G},\mathbf{x}))\,(I(\mathcal{G}-v,\mathbf{y})-I(\mathcal{G},\mathbf{y}))$$
$$=\sum_{H}\prod_{\substack{w\in H_1\\ w'\in H_2}} x_w y_{w'} I(\mathcal{G}-Z_1(\mathcal{H}),\mathbf{x})I(\mathcal{G}-Z_2(\mathcal{H}),\mathbf{y}). \tag{4.5}$$

It is easy to see that

$$\begin{aligned}
I(\mathcal{G},\mathbf{x}) &= 1-x_1-x_2-x_3-x_4+x_1x_3+x_1x_4+x_2x_4\\
I(\mathcal{G}-u,\mathbf{x}) &= 1-x_1-x_3-x_4+x_1x_3+x_1x_4 \quad \text{and}\\
I(\mathcal{G}-v,\mathbf{x}) &= 1-y_1-y_2-y_4+y_1y_4+y_2y_4.
\end{aligned}$$

This gives

$$(I(\mathcal{G}-u,\mathbf{x})-I(\mathcal{G},\mathbf{x}))(I(\mathcal{G}-v,\mathbf{y})-I(\mathcal{G},\mathbf{y}))=x_2y_3(1-x_4)(1-y_1).$$

On the other hand, we have the parts arising from the bipartite parts which we list now:

(a) In this case, we have

$$H_1^1=\{3\},\ H_2^1=\{2\},\ Z_1(\mathcal{H}^1)=\{2,3\}=Z_2(\mathcal{H}^1)$$

and

$$I(\mathcal{G}-\{2,3\},\mathbf{x})=1-x_1-x_4+x_1x_4.$$

(b) In this case, we have

$$H_1^2=\{1,3\},\ H_2^2=\{2\},\ Z_1(\mathcal{H}^2)=\{1,2,3\},\ Z_2(\mathcal{H}^2)=\{2,3\}$$

and

$$I(\mathcal{G}-\{1,2,3\},\mathbf{x})=1-x_4,\quad I(\mathcal{G}-\{2,3\},\mathbf{y})=1-y_1-y_4+y_1y_4.$$

(c) In this case, we have

$$H_1^3=\{3\},\ H_2^3=\{2,4\},\ Z_1(\mathcal{H}^3)=\{2,3\},\ Z_2(\mathcal{H}^3)=\{2,3,4\}$$

and

$$I(\mathcal{G}-\{2,3\},\mathbf{x})=1-x_1-x_4+x_1x_4,\quad I(\mathcal{G}-\{2,3,4\},\mathbf{y})=1-y_1.$$

(d) In this case, we have

$$H_1^4=\{1,3\},\ H_2^4=\{2,4\},\ Z_1(\mathcal{H}^4)=\{1,2,3\},\ Z_2(\mathcal{H}^4)=\{2,3,4\}$$

and

$$I(\mathcal{G}-\{1,2,3\},\mathbf{x})=1-x_4,\quad I(\mathcal{G}-\{2,3,4\},\mathbf{y})=1-y_1.$$

If we simplify the RHS of equation (4.5) becomes $x_2y_3(1-x_4)(1-y_1)$ which is same as the LHS of equation (4.5).

Example. Let us consider the path graph P_4 (see Fig. 2) and take $u=1$ and $v=4$. The only connected bipartite subgraphs of P_4 containing u,v is P_4 itself. In this case, we have

$$I(\mathcal{G}-u,\mathbf{x})=1-x_2-x_3-x_4+x_2x_4$$

and

$$I(\mathcal{G}-v,\mathbf{y})=1-y_1-y_2-y_3+y_1y_3.$$

The LHS of equation (4.5) is equal to

$$x_1y_4(1-x_3-x_4)(1-y_1-y_2).$$

On the other hand, we have $H_1=\{2,4\}$, $H_2=\{1,3\}$, $Z_1(\mathcal{H})=\{1,2\}$ and $Z_2(\mathcal{H})=\{3,4\}$. This implies that the RHS of equation (4.5) is equal to

$$x_1y_4(1-x_3-x_4)(1-y_1-y_2),$$

which is same as the LHS of equation (4.5).

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