# Whittaker vectors in singular Whittaker modules

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**Abstract:** Let $\mathfrak{g}$ be a complex semisimple Lie algebra with Borel subalgebra $\mathfrak{b}$ and corresponding nilradical $\mathfrak{n}$. We show that singular Whittaker modules $M$ are simple if and only if the space $\mathrm{Wh}\, M$ of Whittaker vectors is 1-dimensional. For arbitrary locally $\mathfrak{n}$-finite $\mathfrak{g}$-modules $V$, an immediate corollary is that the dimension of $\mathrm{Wh}\, V$ is bounded by the composition length of $V$.

**Keywords:** Whittaker modules, Whittaker vectors, composition length

**MSC2020:** 17B10 (primary); 17B20 (secondary)

## 1 Introduction

Let $\mathfrak{g}$ be a complex semisimple Lie algebra with Cartan subalgebra $\mathfrak{h}$, root system $\Phi$, base $\Delta$ of simple roots, and corresponding set $\Phi_+$ of positive roots. The nilradical $\mathfrak{n} = \bigoplus_{\alpha \in \Phi_+} \mathfrak{g}_\alpha$ of the standard Borel acts trivially on the cyclic vectors of highest weight representations. More generally, we consider Whittaker modules $V$ of type $\psi$, cyclic modules generated by a vector $v \in V$ on which $\mathfrak{n}$ acts by a Whittaker character $\psi : \mathfrak{n} \to \mathbb{C}$. When $\psi(e_\alpha) \neq 0$ for all $\alpha \in \Delta$, the module $V$ and the character $\psi$ are said to be *non-singular*.

Whittaker modules were introduced by Kostant to address questions about primitive ideals, representations of semisimple Lie groups, and Toda integrable systems [2, 3]. They later played a prominent role in Block's classification [1] of all irreducible representations of $\mathfrak{sl}_2$. Recent interest has grown significantly due to equivalences between categories of generalized Whittaker modules and modules for finite $W$-algebras [7]. Under this correspondence, a generalized Whittaker module $M$ corresponds to its space $\mathrm{Wh}\, M$ of Whittaker vectors, viewed as a module over a finite $W$-algebra.

The purpose of this short paper is to show that a Whittaker module $M$ is simple if and only if its space of Whittaker vectors is 1-dimensional, even when $M$ is singular. This recovers a result of Kostant in the non-singular case [2, Theorem 3.6.1], and may be somewhat surprising, as the associated $W$-algebras are generally non-commutative.

For arbitrary locally $\mathfrak{n}$-finite $\mathfrak{g}$-modules $M$, an immediate corollary is that the dimension of $\mathrm{Wh}\, M$ is bounded by the composition length of $M$. In particular, the dimension of the space of Whittaker vectors in a Whittaker module with central character $\chi$ and Whittaker character $\psi$ is always bounded by the length of the universal module with these properties. The lengths of these universal modules are known, thanks to work by Miličić and Soergel [5] using Kazhdan-Lusztig theory.

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## 2 Simple Whittaker modules

We recall McDowell's classification of simple singular Whittaker modules [4]. Let $\mathfrak{g}$, $\mathfrak{h}$, $\Phi$, $\Phi_+$, and $\Delta$ be as in Section 1, and let $\mathfrak{n}$ be the nilradical of the standard Borel $\mathfrak{b}$ relative to $(\mathfrak{g}, \mathfrak{h}, \Delta)$. A Lie algebra homomorphism $\psi : \mathfrak{n} \to \mathbb{C}$ is called a *Whittaker character*; the character $\psi$ is *non-singular* if it is nonzero on the root spaces $\mathfrak{g}_\alpha$ for all simple roots $\alpha \in \Delta$. An element $v$ of a $\mathfrak{g}$-module $V$ is said to be a *Whittaker vector of type* $\psi$ if $xv = \psi(x)v$ for all $x \in \mathfrak{n}$. The space of Whittaker vectors in $V$ will be denoted by $\mathrm{Wh}\, V$. Cyclic modules generated by Whittaker vectors of type $\psi$ are called *Whittaker modules of type* $\psi$ and are *singular* or *non-singular*, depending on whether $\psi$ is singular or non-singular. Each Whittaker module admits a unique Whittaker character, so the notions of singular and non-singular are well defined.

Let $\psi$ be a Whittaker character, and let $\Delta_\psi$ be the set of simple roots $\alpha$ for which $\psi(\mathfrak{g}_\alpha) \neq 0$. We write $\mathfrak{l} = \mathfrak{l}_\psi$ for the reductive subalgebra generated by the sum of $\mathfrak{h}$ and the root spaces $\mathfrak{g}_\alpha$ with $\pm\alpha \in \Delta_\psi$. The Lie algebra $\mathfrak{l}$ has a triangular decomposition $\mathfrak{l} = \mathfrak{l}_- \oplus \mathfrak{h} \oplus \mathfrak{l}_+$, where $\mathfrak{l}_+$ and $\mathfrak{l}_-$ are the intersections of $\mathfrak{l}$ with the positive and negative root spaces of $\mathfrak{g}$, respectively. The centre $\mathfrak{z}$ of $\mathfrak{l}$ lies in $\mathfrak{h}$, and the corresponding parabolic subalgebra $\mathfrak{l} + \mathfrak{n}$ will be denoted by $\mathfrak{p}$. The subalgebra $\mathfrak{p}$ has a unique $\mathrm{ad}\,\mathfrak{h}$-stable decomposition $\mathfrak{p} = \mathfrak{l} \oplus \mathfrak{m}$, and $\mathfrak{g} = \overline{\mathfrak{m}} \oplus \mathfrak{l} \oplus \mathfrak{m}$, where $\overline{\mathfrak{m}}$ is the Lie subalgebra spanned by the root spaces $\mathfrak{g}_{-\alpha}$ for which $\mathfrak{g}_\alpha$ is contained in $\mathfrak{m}$. For any Lie algebra $\mathfrak{a}$, let $Z(\mathfrak{a})$ be the centre of its enveloping algebra $U(\mathfrak{a})$. For an arbitrary central character $\Omega : Z(\mathfrak{l}) \to \mathbb{C}$, consider the induced $\mathfrak{l}$-module

$$Y_{\Omega,\psi} = U(\mathfrak{l}) \otimes_{Z(\mathfrak{l})U(\mathfrak{l}_+)} \mathbb{C}v_{\Omega,\psi},$$

where $Z(\mathfrak{l})U(\mathfrak{l}_+) = Z(\mathfrak{l}) \otimes U(\mathfrak{l}_+)$ acts by the characters $\Omega$ and $\psi$ on the 1-dimensional space spanned by the vector $v_{\Omega,\psi}$. As the Whittaker character $\psi$ is nonzero on each root space $\mathfrak{g}_\alpha$ with $\alpha \in \Delta_\psi$, it is easy to show that $Y_{\Omega,\psi}$ is a simple left $U(\mathfrak{l})$-module. It remains irreducible when restricted to the (semisimple) derived subalgebra $\mathfrak{s} = [\mathfrak{l}, \mathfrak{l}]$ of $\mathfrak{l}$. We regard $Y_{\Omega,\psi}$ as a left $U(\mathfrak{p})$-module via the inflation map $\mathfrak{p} \to \mathfrak{p}/\mathfrak{m} = \mathfrak{l}$. Inducing to $U(\mathfrak{g})$ gives the Whittaker module

$$M_{\Omega,\psi} = U(\mathfrak{g}) \otimes_{U(\mathfrak{p})} Y_{\Omega,\psi}.$$

**Theorem 2.1** [4, Theorem 2.9] *The module $M_{\Omega,\psi}$ has a unique simple quotient $L_{\Omega,\psi}$, and every simple Whittaker module of type $\psi$ is isomorphic to a module $L_{\Omega,\psi}$ for some character $\Omega : Z(\mathfrak{l}) \to \mathbb{C}$.* □

## 3 Whittaker vectors

We maintain the notation of the previous section. Recall that the centre $\mathfrak{z}$ of the Lie algebra $\mathfrak{l} = \mathfrak{l}_\psi$ is contained in the Cartan subalgebra $\mathfrak{h}$ of $\mathfrak{g}$. Define a partial order

$\preceq$ on its algebraic dual $\mathfrak{z}^*$ by setting $\eta \preceq \mu$ if $\mu - \eta$ is the restriction of an element $\lambda \in \mathrm{Span}_{\mathbb{Z}_+}(\Delta \setminus \Delta_\psi)$ to $\mathfrak{z}$. We now state our main result.

**Theorem 3.1** *Let $V$ be a Whittaker module for $\mathfrak{g}$. Then $V$ is simple if and only if $\mathrm{Wh}\, V$ is 1-dimensional.*

**Proof** Let $\psi : \eta \to \mathbb{C}$ be the Whittaker character of $V$. The module $V$ has an obvious filtration $V = \bigcup_{k=0}^{\infty} V^{(k)}$, where

$$V^{(k)} = \{v \in V : \ x_0 \bullet x_1 \bullet \cdots \bullet x_k \bullet v = 0 \text{ for all } x_0, x_1, \dots, x_k \in \mathfrak{n}\},$$

and $x \bullet v$ is defined as $xv - \psi(x)v$ for all $x \in \mathfrak{n}$ and $v \in V$.

Let $v$ be a nonzero element of $V$. If $v \notin \mathrm{Wh}\, V$, then $x \bullet v \neq 0$ for some $x \in \mathfrak{n}$. But $x \bullet v \in V^{(k-1)}$ whenever $v \in V^{(k)}$, so $V^{(0)} \cap U(\mathfrak{g})v \neq 0$. If $\mathrm{Wh}\, V$ is 1-dimensional and spanned by a vector $w$, then by definition, $V^{(0)} = \mathrm{Wh}\, V = \mathbb{C}w$ and $V = U(\mathfrak{g})w$. In this case, we see that $\mathbb{C}w = V^{(0)} \subseteq U(\mathfrak{g})v$. Since $w$ generates $V$, the module $V$ is thus simple when $\mathrm{Wh}\, V$ is 1-dimensional.

Conversely, by McDowell's classification, any simple Whittaker module is of the form $L_{\Omega,\psi}$ for some Whittaker character $\psi : \mathfrak{n} \to \mathbb{C}$ and central character $\Omega \in Z(\mathfrak{l}_\psi)^*$. The centre $\mathfrak{z}$ of the Lie algebra $\mathfrak{l} = \mathfrak{l}_\psi$ acts semisimply on $Y_{\Omega,\psi}$. Since $\mathfrak{z} \subseteq \mathfrak{h}$, it also acts semisimply on the induced module $M_{\Omega,\psi} = U(\mathfrak{g}) \otimes_{U(\mathfrak{p})} Y_{\Omega,\psi}$ and on its simple quotient $L_{\Omega,\psi}$. That is,

$$M_{\Omega,\psi} = \bigoplus_{\mu \in \mathfrak{z}^*} M^{\mu}_{\Omega,\psi} \text{ and } L_{\Omega,\psi} = \bigoplus_{\mu \in \mathfrak{z}^*} L^{\mu}_{\Omega,\psi},$$

where $V^\mu = \{v \in V \ : \ hv = \mu(h)v \text{ for all } h \in \mathfrak{z}\}$ for any $\mathfrak{g}$-module $V$. Since $\mathfrak{z}$ acts on $Y_{\Omega,\psi}$ by the restriction $\overline{\Omega}$ of the character $\Omega$ to $\mathfrak{z}$, it acts on

$$M_{\Omega,\psi} = U(\overline{\mathfrak{m}}) \otimes_{\mathbb{C}} U(\mathfrak{p}) \otimes_{U(\mathfrak{p})} Y_{\Omega,\psi} = U(\overline{\mathfrak{m}}) \otimes_{\mathbb{C}} Y_{\Omega,\psi}$$

by weights bounded above by $\overline{\Omega}$. Similarly,

$$L_{\Omega,\psi} = \bigoplus_{\mu \preceq \overline{\Omega}} L^{\mu}_{\Omega,\psi}. \tag{3.2}$$

Let $v$ be a Whittaker vector in $L_{\Omega,\psi}$, with decomposition $v = \sum_{\mu \preceq \overline{\Omega}} v_\mu$ relative to (3.2). Let $x \in \mathfrak{n} = \mathfrak{l}_+ \oplus \mathfrak{m}$ be a root vector in a root space $\mathfrak{g}_\alpha$. If $x \in \mathfrak{l}_+$, then $\mathfrak{z}$ commutes with $x$ and $xL^{\mu}_{\Omega,\psi} \subseteq L^{\mu}_{\Omega,\psi}$. In particular, $\sum_\mu \psi(x)v_\mu = xv = \sum_\mu xv_\mu$, so by comparing graded components, $xv_\mu = \psi(x)v_\mu$ for all $\mu$. If $x \in \mathfrak{m}$, then

$$zx = x(z + \alpha(z)),$$

for all $z \in \mathfrak{z}$, so

$$xL^{\mu}_{\Omega,\psi} \subseteq L^{\mu+\overline{\alpha}}_{\Omega,\psi},$$

where $\overline{\alpha}$ is the restriction of $\alpha$ to $\mathfrak{z}$. But $\psi$ vanishes on $\mathfrak{m}$ by construction, so

$$0 = xv = \sum_{\mu \preceq \overline{\Omega}} xv_\mu.$$

Each $xv_\mu$ belongs to a distinct graded component, so $xv_\mu = 0$ for all $\mu$. The subalgebras $\mathfrak{l}_+$ and $\mathfrak{m}$ are spanned by root vectors, so $xv_\mu = \psi(x)v_\mu$ for all $x \in \mathfrak{n}$ and $\mu \preceq \overline{\Omega}$. That is, $v_\mu \in \mathrm{Wh}\, V$ for all $\mu$.

Let $\mu \prec \overline{\Omega}$. Then by degree considerations, the submodule

$$U(\mathfrak{g})v_\mu = U(\overline{\mathfrak{m}} \oplus \mathfrak{l})v_\mu \subseteq \sum_{\lambda \preceq \mu} L^\lambda_{\Omega,\psi}$$

is proper in $L_{\Omega,\psi}$. As $L_{\Omega,\psi}$ is simple, we see that $v_\mu = 0$ and $v = v_{\overline{\Omega}}$. Therefore, $v$ is a Whittaker vector in the simple $\mathfrak{s}$-Whittaker module $Y_{\Omega,\psi} = L^{\overline{\Omega}}_{\Omega,\psi}$ of type $\overline{\psi}$, where $\overline{\psi}$ is the restriction of $\psi$ to the positive part $\mathfrak{l}_+$ of $\mathfrak{s} = [\mathfrak{l}, \mathfrak{l}]$. As $Y_{\Omega,\psi}$ is non-singular, Kostant's criterion [2, Theorem 3.6.1] applies, and its space $W$ of Whittaker vectors is 1-dimensional. The space $\mathrm{Wh}\, L_{\Omega,\psi} \subseteq W$ is thus also 1-dimensional. □

Let $\mathcal{N}$ be the category of $\mathfrak{g}$-modules which are locally finite with respect to the action of $\mathfrak{n}$. An immediate consequence of Theorem 3.1 is that, for any module in $\mathcal{N}$, the dimension of its space of Whittaker vectors is bounded by its composition length.

**Corollary 3.3** *Let $M$ be a $\mathfrak{g}$-module in the category $\mathcal{N}$. Then* $\dim \mathit{Wh}\, M$ *is bounded by the length of $M$.*

**Proof** Without loss of generality, we may assume that $M$ is of finite length and has a composition series

$$0 = M_0 \subset M_1 \subset \cdots \subset M_\ell = M.$$

We induct on the length $\ell$ of $M$, and as the base case $\ell = 0$ is trivial, we may assume that $\ell > 0$.

For any $i > 0$ and nonzero $m \in M_i/M_{i-1}$, the Lie subalgebra $\mathfrak{n}$ acts on the finite dimensional space $U(\mathfrak{n})m$. By Lie's theorem, $U(\mathfrak{n})m \subseteq M_i/M_{i-1}$ contains a nonzero Whittaker vector $n$, and by simplicity $M_i/M_{i-1} = U(\mathfrak{g})n$ is thus a Whittaker module. The space $\mathrm{Wh}(M_i/M_{i-1})$ is 1-dimensional by Theorem 3.1.

If $v, w \in M$ are nonzero Whittaker vectors, then their images $v + M_{\ell-1}$ and $w + M_{\ell-1}$ lie in the 1-dimensional space $\mathrm{Wh}(M_\ell/M_{\ell-1})$. The vector $v$ is thus of the form $\lambda w + u$ for some $\lambda \in \mathbb{C}$ and $u \in M_{\ell-1}$. As $u = v - \lambda w$ is clearly also a Whittaker vector and $\dim \mathrm{Wh}(M_{\ell-1}) \leq \ell - 1$ by induction, it follows that $\dim \mathrm{Wh}\, M \leq 1 + \dim \mathrm{Wh}(M_{\ell-1}) \leq \ell$. □

**Remark 3.4** Any Whittaker module $V$ of type $\psi$ that admits a central character $\chi : Z(\mathfrak{g}) \to \mathbb{C}$ is a quotient of a universal module

$$\mathcal{Y}_{\chi,\psi} = U(\mathfrak{g}) \otimes_{Z(\mathfrak{g})U(\mathfrak{n})} \mathbb{C}w_{\chi,\psi},$$

where $Z(\mathfrak{g})U(\mathfrak{n}) = Z(\mathfrak{g}) \otimes U(\mathfrak{n})$ acts on the 1-dimensional space $\mathbb{C}w_{\chi,\psi}$ by $\chi \otimes \psi$. Such modules are always simple if $\psi$ is non-singular, but need not be simple for singular $\psi$. By the above corollary, the dimension of $\mathrm{Wh}\, V$ is bounded by the length of $\mathcal{Y}_{\chi,\psi}$. Such lengths have been computed by Miličić and Soergel [5] using Kazhdan-Lusztig theory.

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