



Sharp Adams Type Inequalities for the Fractional Laplace–Beltrami Operator on Noncompact Symmetric Spaces

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Abstract

We establish sharp Adams type inequalities on Sobolev spaces $W^{\alpha,n/\alpha}(X)$ of any fractional order $\alpha < n$ on Riemannian symmetric space X of noncompact type with dimension n and of arbitrary rank. We also establish sharp Hardy–Adams inequalities on the Sobolev spaces $W^{n/2,2}(X)$. We use Fourier analysis on the symmetric spaces to obtain these results.

Keywords Adams inequalities · Riemannian symmetric spaces · Fractional Laplacian · Sharp constants

Mathematics Subject Classification Primary 43A85 · Secondary 22E30 · 42B35

1 Introduction

The study and understanding of various kinds of sharp Sobolev inequalities which describe the embedding of Sobolev spaces into L^p spaces or Hölder spaces have been a matter of intensive research. They play an important role in calculus of variations, differential geometry, harmonic analysis, partial differential equations and other areas of modern mathematics. It is well-known that the Sobolev embedding holds for the case of compact Riemannian manifolds. To be precise, let (M, g) be a compact Riemannian manifold then the Sobolev embedding states that the Sobolev space $W^{k,p}(M)$ is continuously embedded into $L^{p^*}(M)$ where $p^* = \frac{np}{n-kp}$ provided $1 \leq p < \frac{n}{k}$. However, when M is a complete noncompact manifold then the Sobolev embedding is a non-trivial issue. In fact, there exists a complete noncompact Riemannian manifold M for which the Sobolev embedding $W^{k,p}(M) \hookrightarrow L^{p^*}(M)$ does not hold for any p satisfying $kp < n$. We refer to [14] for a detailed discussion on the topic.

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When M is compact and $p = n/k$, the space $W^{k,p}(M)$ is continuously embedded into $L^q(M)$ for all $q < \infty$ but not for $q = \infty$. When M is a bounded domain in \mathbb{R}^n with smooth boundary, Trudinger [34] established in the borderline case that $W_0^{1,n}(\Omega) \subset L_{\phi_n}(\Omega)$, where $L_{\phi_n}(\Omega)$ is the Orlicz space associated with the Young function $\phi_n(t) = \exp(\beta|t|^{n/(n-1)}) - 1$ for some $\beta > 0$. In 1971, Moser sharpened the Trudinger inequality in [27] by finding the optimal β . He showed that there exists a positive constant C depending only on n such that

$$\sup_{u \in C_c^\infty(\Omega), \int_\Omega |\nabla u|^n \leq 1} \int_\Omega e^{\beta|u(x)|^{n/(n-1)}} \, dx \leq C|\Omega|,$$

holds for every $\beta \leq \beta_n = n[\omega_{n-1}]^{1/(n-1)}$, where Ω is a bounded domain in \mathbb{R}^n , $|\Omega|$ denotes the volume of Ω and ω_{n-1} is the surface measure of the unit sphere in \mathbb{R}^n . Moreover, when $\beta > \beta_n$, the above supremum is infinite.

In 1988, D. Adams extended such an inequality on finite domain to higher order Sobolev spaces as follows.

Theorem 1.1 [1] *Let Ω be a domain in \mathbb{R}^n with finite Lebesgue measure and m be a positive integer less than n . There is a constant $c_0 = c_0(n, m)$ such that for all $u \in C^m(\mathbb{R}^n)$ with support contained in Ω and $\|\nabla^m u\|_{n/m} \leq 1$, the following uniform inequality holds*

$$\frac{1}{|\Omega|} \int_\Omega \exp\left(\beta(n, m)|u(x)|^{n/(n-m)}\right) \, dx \leq c_0, \tag{1.1}$$

where

$$\begin{aligned} \beta(n, m) &= \frac{n}{\omega_{n-1}} \left[\frac{\pi^{n/2} 2^m \Gamma((m+1)/2)}{\Gamma((n-m+1)/2)} \right]^{n/(n-m)}, \quad m \text{ is odd;} \\ &= \frac{n}{\omega_{n-1}} \left[\frac{\pi^{n/2} 2^m \Gamma(m/2)}{\Gamma((n-m)/2)} \right]^{n/(n-m)}, \quad m \text{ is even.} \end{aligned}$$

Furthermore, the constant $\beta(n, m)$ in (1.1) is sharp in the sense that if $\beta(n, m)$ is replaced by any larger number, then the integral in (1.1) cannot be bounded uniformly by any constant.

For $m = 1$, it recovers the Trudinger–Moser inequality. In [1], Theorem 1.1 was proved by representing functions by Riesz kernels and establishing inequalities for integral operators governed by these kernels.

There have been many extensions and generalizations of this result to different settings. For instance, L. Fontana in [10] obtained a sharp version of the inequality (1.1) on compact Riemannian manifolds. When Ω is a Euclidean ball, more refined results have been established. In dimension two, Wang and Ye [35] proved a Hardy–Trudinger–Moser inequality on a planar disk \mathbb{B}^2 by combining the Trudinger–Moser inequality with the Hardy inequality. Several variants of Trudinger–Moser and Adams type inequalities has been established in unbounded domains of Euclidean spaces. In

[21], N. Lam and G. Lu developed a new approach to establish these types of sharp inequalities in unbounded domains of Euclidean spaces without using the standard symmetrization. This approach can be applied in the context of Riemannian and sub-Riemannian manifolds where the symmetrization argument does not work (see e.g. [22, 37]). They proved the following Adams type inequalities on Sobolev space $W^{\alpha, n/\alpha}(\mathbb{R}^n)$ of fractional order α for $0 < \alpha < n$:

Theorem 1.2 *Let $0 < \alpha < n$ be an arbitrary real positive number, $p = n/\alpha$ and $\tau > 0$. There holds*

$$\sup_{u \in W^{\alpha, p}(\mathbb{R}^n), \|(\tau I - \Delta)^{\alpha/2} u\|_p \leq 1} \int_{\mathbb{R}^n} \Phi_p \left(\beta_0(n, \alpha) |u(x)|^{p'} \right) dx \leq C < \infty,$$

where $\beta_0(n, \alpha) = \frac{n}{\omega_{n-1}} \left[\frac{\pi^{n/2} 2^\alpha \Gamma(\alpha/2)}{\Gamma((n-\alpha)/2)} \right]^{p'}$, $1/p + 1/p' = 1$ and

$$\Phi_p(t) = e^t - \sum_{j=0}^{j_p-2} \frac{t^j}{j!}, \quad j_p = \min\{j \in \mathbb{N} : j \geq p\}.$$

Furthermore, this inequality is sharp in the sense that if $\beta_0(n, \alpha)$ is replaced by any $\beta > \beta_0(n, \alpha)$, then the supremum is infinite.

In the case of real hyperbolic spaces \mathbb{B}^n , Trudinger–Moser, Adams inequalities have been investigated in details. From a conformal point of view, an Adams inequality in the hyperbolic space was proved by Karmakar and Sandeep [18]. On the other hand, in a series of papers [23–25], using the Riesz kernel estimates and Fourier analysis on \mathbb{B}^n , Lu and Yang (also J. Li) proved sharp Adams and Hardy–Adams inequalities on \mathbb{B}^n . Precisely, in [24] they proved the results for any fractional order $\alpha < n$ on \mathbb{B}^n in all dimension n . Recently, in [9, 26], Lu and Yang (also in collaboration with Joshua Flynn), have established similar results on complex hyperbolic spaces and more generally on all rank one symmetric spaces of noncompact type. To prove their results, the authors derived the optimal bounds for the Green functions of the fractional Laplacians when the hyperbolic distance is small. To get these optimal bounds, they heavily used the explicit expression of the heat kernel available for the hyperbolic spaces. For the bounds corresponding to large hyperbolic distance, they used the results due to J. Anker and L. Ji [5].

Bertrand and Sandeep [6] established a Trudinger–Moser–Adams inequality on Cartan–Hadamard manifold with strictly negative sectional curvature. For other Trudinger–Moser–Adams inequalities on Riemannian manifolds, we refer to [20, 30, 32, 36, 37].

Our concern in this article is to establish these inequalities on Riemannian symmetric spaces X of noncompact type of all dimension $n \geq 3$ and of arbitrary rank. Precisely, we prove sharp local and global Adams inequalities (Theorems 1.3, 1.4) on the fractional order Sobolev spaces $W^{\alpha, n/\alpha}(X)$, $0 < \alpha < n$ and Hardy–Adams inequalities (Theorem 1.8) on $W^{n/2, 2}(X)$. Let Δ denote the Laplace–Beltrami operator

on X and let ρ denote the half-sum of all positive roots counted with their multiplicities (see (2.2) for the definition). We begin with the following sharp local and global Adams inequalities of fractional order on X .

Theorem 1.3 *Let $n \geq 3, 0 < \alpha < n$ be an arbitrary positive number, $p = n/\alpha$ and ζ satisfies $\zeta > 0$ if $1 < p < 2$ and $\zeta > 2|\rho|(1/2 - 1/p)$ if $p \geq 2$. Then for a measurable set E with finite volume in X , there exists $C = C(\zeta, n, \alpha, |E|)$ such that*

$$\frac{1}{|E|} \int_E \exp\left(\beta_0(n, \alpha)|u(x)|^{p'}\right) dx \leq C,$$

for any $u \in W^{\alpha,p}(X)$ with $\int_X |(-\Delta - |\rho|^2 + \zeta^2)^{\alpha/2}u(x)|^p dx \leq 1$. Furthermore, this inequality is sharp in the sense that if $\beta_0(n, \alpha)$ is replaced by any $\beta > \beta_0(n, \alpha)$, then the inequality can no longer hold with some C independent of u .

Theorem 1.4 *Let $n \geq 3, 0 < \alpha < n$ be an arbitrary positive number, $p = n/\alpha$ and ζ satisfies $\zeta > 2|\rho||1/2 - 1/p|$. Then there exists $C = C(\zeta, n, \alpha)$ such that*

$$\int_X \Phi_p\left(\beta_0(n, \alpha)|u(x)|^{p'}\right) dx \leq C,$$

for any $u \in W^{\alpha,p}(X)$ with $\int_X |(-\Delta - |\rho|^2 + \zeta^2)^{\alpha/2}u(x)|^p dx \leq 1$. Furthermore, this inequality is sharp in the sense that stated in Theorem 1.3.

We notice that $2|\rho||1/2 - 1/p| < |\rho|$ provided $p > 1$. Choosing $\zeta = |\rho|$ in Theorem 1.4, we have the following Adams inequality.

Theorem 1.5 *Let $n \geq 3, 0 < \alpha < n$ be an arbitrary positive number and $p = n/\alpha$. Then there exists $C = C(\zeta, n, \alpha)$ such that*

$$\int_X \Phi_p\left(\beta_0(n, \alpha)|u(x)|^{p'}\right) dx \leq C,$$

for any $u \in W^{\alpha,p}(X)$ with $\int_X |(-\Delta)^{\alpha/2}u(x)|^p dx \leq 1$.

In [7], the author in collaboration with S. Pusti have established a fractional Poincaré-Sobolev inequality on X (see Theorem 2.5). Using this we prove the following result in the special case $p = 2$.

Theorem 1.6 *Let $n \geq 3, \zeta > 0$ and s satisfies $0 < 2s < \min\{l + 2|\Sigma_0^+, n\}$. Then there exists $C = C(\zeta, n)$ such that*

$$\int_X \left[\exp\left(\beta_0(n, n/2)|u(x)|^2\right) - 1 - \beta_0(n, n/2)|u(x)|^2 \right] dx \leq C,$$

for any $u \in W^{n/2,2}(X)$ with

$$\int_X \left| (-\Delta - |\rho|^2)^{s/2} \left(-\Delta - |\rho|^2 + \zeta^2 \right)^{(n-2s)/4} u(x) \right|^2 dx \leq 1. \tag{1.2}$$

Furthermore, this inequality is sharp in the sense stated above.

Remark 1.7 The number $\nu = l + 2|\Sigma_0^+|$ is called ‘pseudo-dimension’ (see (2.1) for the definition). In the case of rank one symmetric spaces, in particular, for real hyperbolic space \mathbb{B}^n of dimension n , the pseudo-dimension $\nu = 3$.

Theorem 1.6 implies the following Hardy–Adams inequality.

Theorem 1.8 *Let $n \geq 3$, $\zeta > 0$ and s satisfies $2 \leq 2s < \min\{l + 2|\Sigma_0^+|, n\}$. Then there exists $C = C(\zeta, n)$ such that*

$$\int_X \left[\exp\left(\beta_0(n, n/2)|u(x)|^2\right) - 1 - \beta_0(n, n/2)|u(x)|^2 \right] dx \leq C,$$

for any $u \in W^{n/2,2}(X)$ with

$$\int_X |(-\Delta)^{s/2}(-\Delta - |\rho|^2 + \zeta^2)^{(n-2s)/4} u(x)|^2 dx - |\rho|^{2s} \zeta^{n-2s} \int_X |u(x)|^2 dx \leq 1.$$

Remark 1.9 For the real hyperbolic spaces, in contrast with [24, Theorem 1.14], our result improves the range of s with $1 \leq s < 3/2$ for all dimension n .

The article is organized as follows. In Sect. 2, we review some preliminaries of Riemannian symmetric spaces and Fourier analysis on them. Using Anker’s multiplier theorem on X , we derive a Sobolev embedding theorem on fractional Sobolev spaces (Corollary 2.4). Section 3 focuses on the optimal Bessel–Green–Riesz kernel estimates near the origin for the fractional operators. We also need to establish sharp estimates for the convolution of the fractional kernels. Section 4 devotes to the preparation of the proof of the important local Adams inequality (Theorem 1.3). In Sect. 5, we prove all the results using Fourier analysis on X .

2 Riemannian Symmetric Spaces of Noncompact Type

In this section, we describe the necessary preliminaries regarding semisimple Lie groups and harmonic analysis on Riemannian symmetric spaces. These are standard and can be found, for example, in [12, 15–17]. To make the article self-contained, we shall gather only those results which will be used throughout this paper.

2.1 Notations

Let G be a connected, noncompact, real semisimple Lie group with finite center and \mathfrak{g} its Lie algebra. We fix a Cartan involution θ of \mathfrak{g} and write $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$ where \mathfrak{k} and \mathfrak{p} are $+1$ and -1 eigenspaces of θ respectively. Then \mathfrak{k} is a maximal compact subalgebra of \mathfrak{g} and \mathfrak{p} is a linear subspace of \mathfrak{g} . The Cartan involution θ induces an automorphism Θ of the group G and $K = \{g \in G \mid \Theta(g) = g\}$ is a maximal compact subgroup of G . Let B denote the Cartan Killing form of \mathfrak{g} . It is known that $B|_{\mathfrak{p} \times \mathfrak{p}}$ is positive definite

and hence induces an inner product and a norm $|\cdot|$ on \mathfrak{p} . The homogeneous space $X = G/K$ is a Riemannian symmetric space of noncompact type. The tangent space of X at the point $o = eK$ can be naturally identified to \mathfrak{p} and the restriction of B on \mathfrak{p} then induces a G -invariant Riemannian metric \mathbf{d} on X . For $x \in X$, we denote $|x|$ is the distance of x from the origin o . For a given $x \in X$ and a positive number r we define

$$\mathbf{B}(x, r) = \{y \in X : \mathbf{d}(x, y) < r\},$$

to be the open ball with center x and radius r .

We fix a maximal abelian subspace \mathfrak{a} in \mathfrak{p} . The rank of X is the dimension l of \mathfrak{a} . We shall identify \mathfrak{a} endowed with the inner product induced from \mathfrak{p} with \mathbb{R}^l and let \mathfrak{a}^* be the real dual of \mathfrak{a} . The set of restricted roots of the pair $(\mathfrak{g}, \mathfrak{a})$ is denoted by Σ . It consists of all $\alpha \in \mathfrak{a}^*$ such that

$$\mathfrak{g}_\alpha = \{X \in \mathfrak{g} \mid [Y, X] = \alpha(Y)X, \text{ for all } Y \in \mathfrak{a}\},$$

is non-zero with $m_\alpha = \dim(\mathfrak{g}_\alpha)$. We choose a system of positive roots Σ_+ and with respect to Σ_+ , the positive Weyl chamber $\mathfrak{a}_+ = \{X \in \mathfrak{a} \mid \alpha(X) > 0, \text{ for all } \alpha \in \Sigma_+\}$. We also let Σ_0^+ be the set of positive indivisible roots, that is, $\Sigma_0^+ = \{\alpha \in \Sigma^+ \mid 2\alpha \notin \Sigma\}$. Let n be the dimension of X and ν be the pseudo-dimension:

$$n = l + \sum_{\alpha \in \Sigma_+} m_\alpha, \text{ and } \nu = l + 2|\Sigma_0^+|. \tag{2.1}$$

We notice that one cannot compare n and ν without specifying the geometric structure of X . For example, when G is complex, we have $n = \nu$; but when X has normal real form, we have $n = l + |\Sigma_0^+|$ which is strictly smaller than ν . Let $\rho \in \mathfrak{a}^*$ denote the half-sum of all positive roots counted with their multiplicities

$$\rho = \frac{1}{2} \sum_{\alpha \in \Sigma_+} m_\alpha \alpha. \tag{2.2}$$

It is known that the L^2 -spectrum of the Laplace-Beltrami operator Δ on X is the half-line $(-\infty, -|\rho|^2]$. Let \mathfrak{n} be the nilpotent Lie subalgebra of \mathfrak{g} associated to Σ_+ , that is, $\mathfrak{n} = \bigoplus_{\alpha \in \Sigma_+} \mathfrak{g}_\alpha$. If $N = \exp \mathfrak{n}$ and $A = \exp \mathfrak{a}$ then N is a nilpotent Lie subgroup and A normalizes N . For the group G , we now have the Iwasawa decomposition $G = KAN$, that is, every $g \in G$ can be uniquely written as

$$g = \kappa(g) \exp H(g) \eta(g), \quad \kappa(g) \in K, H(g) \in \mathfrak{a}, \eta(g) \in N,$$

and the map $(k, a, n) \mapsto kan$ is a global diffeomorphism of $K \times A \times N$ onto G . Let M' and M be the normalizer and centralizer of \mathfrak{a} in K respectively. Then M is a normal subgroup of M' and normalizes N . The quotient $W = M'/M$ is a finite group, called the Weyl group of the pair $(\mathfrak{g}, \mathfrak{k})$. W acts on \mathfrak{a} by the adjoint action. It is known that W acts as a group of orthogonal transformations (preserving the Cartan-Killing

form) on \mathfrak{a} . Each $w \in W$ permutes the Weyl chambers and the action of W on the Weyl chambers is simply transitive. Let $A_+ = \overline{\exp \mathfrak{a}_+}$. Since $\exp : \mathfrak{a} \rightarrow A$ is an isomorphism we can identify A with \mathbb{R}^l . Let $\overline{A_+}$ denote the closure of A_+ in G . One has the polar decomposition $G = KAK$, that is, each $g \in G$ can be written as

$$g = k_1(\exp Y)k_2, \quad k_1, k_2 \in K, Y \in \mathfrak{a}.$$

In the above decomposition, the A component of \mathfrak{g} is uniquely determined modulo W . In particular, it is well defined in $\overline{A_+}$. The map $(k_1, a, k_2) \mapsto k_1ak_2$ of $K \times A \times K$ into G induces a diffeomorphism of $K/M \times A_+ \times K$ onto an open dense subset of G . It follows that if $gK = k_1(\exp Y)K \in X$ then

$$|gK| = \mathbf{d}(o, gK) = |Y|.$$

We recall the following property of the Iwasawa projection map H [16, Lemma 1.14, p. 217]:

$$|H(\exp Yk)| \leq |Y|, \quad \text{for } Y \in \mathfrak{a}, k \in K. \tag{2.3}$$

We extend the inner product on \mathfrak{a} induced by B to \mathfrak{a}^* by duality, that is, set

$$\langle \lambda, \mu \rangle = B(Y_\lambda, Y_\mu), \quad \lambda, \mu \in \mathfrak{a}^*, Y_\lambda, Y_\mu \in \mathfrak{a},$$

where Y_λ is the unique element in \mathfrak{a} such that

$$\lambda(Y) = B(Y_\lambda, Y), \quad \text{for all } Y \in \mathfrak{a}.$$

This inner product induces a norm, again denoted by $|\cdot|$, on \mathfrak{a}^* ,

$$|\lambda| = \langle \lambda, \lambda \rangle^{\frac{1}{2}}, \quad \lambda \in \mathfrak{a}^*.$$

The elements of the Weyl group W act on \mathfrak{a}^* by the formula

$$sY_\lambda = Y_{s\lambda}, \quad s \in W, \lambda \in \mathfrak{a}^*.$$

Let $\mathfrak{a}_\mathbb{C}^*$ denote the complexification of \mathfrak{a}^* , that is, the set of all complex-valued real linear functionals on \mathfrak{a} . The inner products have complex bilinear extensions to the complexifications $\mathfrak{a}_\mathbb{C}$ and $\mathfrak{a}_\mathbb{C}^*$. All these bilinear forms are denoted by the same symbol $\langle \cdot, \cdot \rangle$.

Through the identification of A with \mathbb{R}^l , we use the Lebesgue measure on \mathbb{R}^l as the Haar measure da on A . As usual on the compact group K , we fix the normalized Haar measure dk and dn denotes a Haar measure on N . The following integral formulae describe the Haar measure of G corresponding to the Iwasawa and polar decompositions respectively.

$$\int_G f(g)g = \int_K \int_{\mathfrak{a}} \int_N f(k \exp Yn) e^{2\rho(Y)} dn dY dk, \quad f \in C_c(G);$$

$$= \int_K \int_{A_+} \int_K f(k_1 a k_2) J(a) dk_1 da dk_2,$$

where dY is the Lebesgue measure on \mathbb{R}^l and for $Y \in \overline{\mathfrak{a}_+}$

$$J(\exp Y) = c \prod_{\alpha \in \Sigma^+} (\sinh \alpha(Y))^{m_\alpha} \asymp \left\{ \prod_{\alpha \in \Sigma^+} \left(\frac{\alpha(Y)}{1 + \alpha(Y)} \right)^{m_\alpha} \right\} e^{2\rho(Y)}, \tag{2.4}$$

where c (in the equality above) is a normalizing constant. If f is a function on $X = G/K$ then f can be thought of as a function on G which is right invariant under the action of K . It follows that on X we have a G invariant measure dx such that

$$\int_X f(x) dx = \int_{K/M} \int_{\mathfrak{a}_+} f(k \exp Y) J(\exp Y) dY dk_M, \tag{2.5}$$

where dk_M is the K -invariant measure on K/M .

2.2 Fourier Analysis on X

For a sufficiently nice function f on X , its Fourier transform \tilde{f} is defined on $\mathfrak{a}_\mathbb{C}^* \times K$ by the formula

$$\tilde{f}(\lambda, k) = \int_G f(g) e^{(i\lambda - \rho)H(g^{-1}k)} dg, \quad \lambda \in \mathfrak{a}_\mathbb{C}^*, k \in K, \tag{2.6}$$

whenever the integral exists [16, P. 199]. As M normalizes N the function $k \mapsto \tilde{f}(\lambda, k)$ is right M -invariant. It is known that if $f \in L^1(X)$ then $\tilde{f}(\lambda, k)$ is a continuous function of $\lambda \in \mathfrak{a}^*$, for almost every $k \in K$ (in fact, holomorphic in λ on a domain containing \mathfrak{a}^*). If in addition, $\tilde{f} \in L^1(\mathfrak{a}^* \times K, |\mathbf{c}(\lambda)|^{-2} d\lambda dk)$ then the following Fourier inversion holds,

$$f(gK) = |W|^{-1} \int_{\mathfrak{a}^* \times K} \tilde{f}(\lambda, k) e^{-(i\lambda + \rho)H(g^{-1}k)} |\mathbf{c}(\lambda)|^{-2} d\lambda dk, \tag{2.7}$$

for almost every $gK \in X$ [16, Chapter III, Theorems 1.8, 1.9]. Here $\mathbf{c}(\lambda)$ denotes Harish Chandra’s c -function. Moreover, $f \mapsto \tilde{f}$ extends to an isometry of $L^2(X)$ onto $L^2(\mathfrak{a}_+^* \times K, |\mathbf{c}(\lambda)|^{-2} d\lambda dk)$ [16, Chapter III, Theorem 1.5], that is,

$$\int_X |f(x)|^2 dx = c_G \int_{\mathfrak{a}_+^* \times K} |\tilde{f}(\lambda, k)|^2 |\mathbf{c}(\lambda)|^{-2} d\lambda dk, \tag{2.8}$$

where c_G is a positive number that depends only on G .

We now specialize to the case of K -biinvariant functions f , that is, f satisfies $f(k_1 g k_2) = f(g)$, for all $k_1, k_2 \in K$ and $g \in G$. Using the polar decomposition of G we may view an integrable or a continuous K -biinvariant function f on G as a function

on A_+ , or by using the inverse exponential map we may also view f as a function on \mathfrak{a}_+ solely determined by its values on \mathfrak{a}_+ . Henceforth, we shall denote the set of K -biinvariant functions in $L^p(G)$ by $L^p(G//K)$, for $1 \leq p \leq \infty$; and K -biinvariant compactly supported smooth functions by $C_c^\infty(G//K)$. If $f \in L^1(G//K)$ then the Fourier transform \tilde{f} can also be written as

$$\tilde{f}(\lambda, k) = \hat{f}(\lambda) = \int_G f(g)\phi_{-\lambda}(g) dg, \tag{2.9}$$

where ϕ_λ is Harish Chandra’s elementary spherical function defined by

$$\phi_\lambda(g) = \int_K e^{-(i\lambda+\rho)(H(g^{-1}k))} dk, \quad \lambda \in \mathfrak{a}_\mathbb{C}^*. \tag{2.10}$$

We now list down some well-known properties of the elementary spherical functions which are important for us ([12, Prop 3.1.4], [5, Prop. 2.2.12], [16, Thm 1.1, p. 200; Lemma 1.18, p. 221]).

- Theorem 2.1** (1) $\phi_\lambda(g)$ is K -biinvariant in $g \in G$ and W -invariant in $\lambda \in \mathfrak{a}_\mathbb{C}^*$.
 (2) $\phi_\lambda(g)$ is C^∞ in $g \in G$ and holomorphic in $\lambda \in \mathfrak{a}_\mathbb{C}^*$.
 (3) For all $\lambda \in \overline{\mathfrak{a}_+^*}$ and $g \in G$ we have $|\phi_\lambda(g)| \leq \phi_0(g) \leq 1$.
 (4) The elementary spherical function ϕ_0 satisfies the following estimate:

$$\phi_0(\exp Y) \asymp \left\{ \prod_{\alpha \in \Sigma_0^+} (1 + \alpha(Y)) \right\} e^{-\rho(Y)}, \quad \text{for all } Y \in \overline{\mathfrak{a}^+}. \tag{2.11}$$

- (5) For $\lambda \in \mathfrak{a}^*$, there holds $\Delta\phi_\lambda = -(|\lambda|^2 + |\rho|^2)\phi_\lambda$.
 (6) For $\lambda \in \mathfrak{a}^*$, the function ϕ_λ satisfies the following

$$\phi_{-\lambda}(hg) = \int_K e^{(i\lambda-\rho)(H(g^{-1}k))} e^{-(i\lambda+\rho)(H(hk))} dk, \quad g, h \in G. \tag{2.12}$$

We now recall the following asymptotic estimates of the heat kernel $h_t(x)$ on X established by Anker and Ji [5, Theorem 3.7].

Theorem 2.2 Let κ be an arbitrary positive number. Then there exist positive constants C_1, C_2 (depending on κ) such that

$$C_1 \leq \frac{h_t(\exp Y)}{t^{-\frac{n}{2}}(1+t)^{\frac{n-l}{2}-|\Sigma_0^+|} \left\{ \prod_{\alpha \in \Sigma_0^+} (1 + \alpha(Y)) \right\} e^{-|\rho|^2 t - \rho(Y) - \frac{|Y|^2}{4t}}} \leq C_2,$$

for all $t > 0$, and $Y \in \overline{\mathfrak{a}^+}$, with $|Y| \leq \kappa(1+t)$.

Let $\alpha > 0$ and $1 < p < \infty$. We recall that the Sobolev space $W^{\alpha,p}(X)$ is the image of $L^p(X)$ under the operator $(-\Delta)^{-\alpha/2}$, equipped with the norm

$$\|f\|_{W^{\alpha,p}(X)} = \|(-\Delta)^{\alpha/2} f\|_{L^p(X)}.$$

If $\alpha = N$ is a non-negative integer, then $W^{\alpha,p}(X)$ coincides with the classical Sobolev space

$$W^{N,p}(X) = \{f \in L^p(X) : \nabla^j f \in L^p(X), \forall 1 \leq j \leq N\},$$

defined by means of covariant derivatives. We refer to [33] for more details about function spaces on Riemannian manifolds. For $p = 2$, the Sobolev space of order α on X is equivalently defined by

$$\begin{aligned} W^{\alpha,2}(X) &= \{f \in L^2(X) \mid \|f\|_{W^{\alpha,2}(X)}^2 \\ &:= \int_{\mathfrak{a}^* \times K} |\tilde{f}(\lambda, k)|^2 (|\lambda|^2 + |\rho|^2)^\sigma |\mathbf{c}(\lambda)|^{-2} \, d\lambda \, dk < \infty\}. \end{aligned}$$

In [2], J.-P. Anker proved the following Hörmander-Mikhlin type multiplier theorem in the context of Riemannian symmetric spaces of the noncompact type. Let k be a K -biinvariant tempered distribution on G and let m be its spherical Fourier transform.

Theorem 2.3 *Let $1 < p < \infty$, $v = |1/p - 1/2|$ and $N = [vn] + 1$. Then $Tf = f * k$ is a bounded operator on $L^p(X)$, provided that*

- (a) m extends to a holomorphic function inside the tube $\mathcal{I}^v = \mathfrak{a}^* + i \operatorname{co}(W \cdot 2v\rho)$,
- (b) $\nabla^i m$ (for $i = 0, \dots, N$) extends continuously to the whole of \mathcal{I}^v , with

$$\sup_{\lambda \in \mathcal{I}^v} (1 + |\lambda|)^{-i} |\nabla^i m(\lambda)| < \infty.$$

The multiplier of the operator $(-\Delta - |\rho|^2 + \zeta^2)^{-\alpha/2}$ is given by $(\langle \lambda, \lambda \rangle + \zeta^2)^{-\alpha/2}$ and this can be extended to a holomorphic function inside the tube $\{\lambda \in \mathfrak{a}_\mathbb{C}^* : |\Im \lambda| < \zeta\}$. Therefore, by using the theorem above we have the following Sobolev embedding theorem on the fractional order Sobolev spaces $W^{\alpha,p}(X)$.

Corollary 2.4 *Let $1 < p < \infty$, $\alpha > 0$ and $\zeta > 2|\rho||1/p - 1/2|$. Then there exists a positive constant $S_p > 0$ such that for all $f \in W^{\alpha,p}(X)$*

$$\|f\|_p \leq S_p \|(-\Delta - |\rho|^2 + \zeta^2)^{\frac{\alpha}{2}} f\|_p.$$

We recall the following analogue of the Poincaré-Sobolev inequality for the fractional Laplace-Beltrami operator on X . For proof, we refer the reader to [7, Theorem 1.11].

Theorem 2.5 *Let $\dim X = n \geq 3$ and $0 < \sigma < \min\{l + 2|\Sigma_0^+|, n\}$. Then for $2 < p \leq \frac{2n}{n-\sigma}$ there exists $S = S(n, \sigma, p)$ such that for all $u \in W^{\frac{\sigma}{2},2}(X)$,*

$$\|(-\Delta - |\rho|^2)^{\sigma/4} u\|_2^2 \geq S \|u\|_p^2.$$

3 Optimal Asymptotic Estimates of Bessel–Green–Riesz Kernels

In what follows, $a \lesssim b$ or $a = \mathcal{O}(b)$ will stand for $a \leq Cb$ with a positive constant C and $a \sim b$ stand for $C^{-1}b \leq a \leq Cb$.

We first set

$$\gamma(\alpha) = \frac{2^\alpha \pi^{n/2} \Gamma(\alpha/2)}{\Gamma((n-\alpha)/2)}, \quad \text{for } 0 < \alpha < n. \tag{3.1}$$

It is well-known that, on the Euclidean spaces, the following identity of convolution holds [31, Chapter V]: for $\alpha, \beta > 0$ with $\alpha + \beta < n$

$$\int_{\mathbb{R}^n} \|t\|^{\alpha-n} \|t-s\|^{\beta-n} dt = \frac{\gamma(\alpha)\gamma(\beta)}{\gamma(\alpha+\beta)} \|s\|^{\alpha+\beta-n}, \quad s \in \mathbb{R}^n; \tag{3.2}$$

where the function $\gamma(\eta)$ is defined in (3.1) and $\|\cdot\|$ is the Euclidean norm. In order to prove Adams inequality on compact Riemannian manifold (M, L) , Fontana proved an analogue of this formula on M [10, Lemma 2.1]. In this section, we need the following version of this formula valid on a compact subset of X . This is essentially proved in [10], but for the sake of completeness we sketch the proof.

Lemma 3.1 *Suppose $\alpha, \beta > 0$ satisfying $\alpha + \beta < n$ and $r > 0$. Then there exists ϵ satisfying $0 < \epsilon < \min\{1, n - \alpha - \beta\}$ such that for $x \in \mathbf{B}(o, r)$*

$$\int_{\mathbf{B}(o,r)} |y|^{\alpha-n} |y^{-1}x|^{\beta-n} dy \leq \frac{\gamma(\alpha)\gamma(\beta)}{\gamma(\alpha+\beta)} |x|^{\alpha+\beta-n} (1 + \mathcal{O}(|x|^\epsilon)).$$

Proof For Riemannian symmetric spaces of noncompact type, it is well-known that the sectional curvature is everywhere less than or equals zero [15, Theorem 3.1, p. 241]. On the other hand, on a compact subset of a Riemannian manifold, the sectional curvature is bounded [8, Corollary, p. 167]. Therefore, for $r > 0$ there exists $\mathcal{K}_r > 0$ such that for any plane section P at any point $x \in \mathbf{B}(o, r)$ the sectional curvature $\mathcal{K}(P)$ satisfies $-\mathcal{K}_r \leq \mathcal{K}(P) \leq 0$.

Let \mathbb{B}^n be the n -dimensional hyperbolic space of constant curvature $-\mathcal{K}_r$ and \exp' be the corresponding exponential map. Let $o' \in \mathbb{B}^n$ and $\mathcal{B}(o', r)$ be the geodesic ball centred at o' and of radius r in \mathbb{B}^n . We consider normal geodesic coordinates on $\mathbf{B}(o, r)$ and on $\mathcal{B}(o', r)$ in X and \mathbb{B}^n respectively. It is a feature of these coordinates that the tangent space at the center is isometric to the standard n -dimensional Euclidean space. So, by choosing orthonormal basis in $T_oX = \mathfrak{p}$ and $T_{o'}\mathbb{B}^n$, we can identify both the tangent spaces with standard \mathbb{R}^n .

If x and y are two points in $\mathbf{B}(o, r) \subset X$, we consider their normal geodesic coordinates s and t points in $\mathfrak{p} \cong \mathbb{R}^n$, uniquely determined by $x = \exp(s)$ and $y = \exp(t)$. We now construct two points x', y' in \mathbb{B}^n by $x' = \exp'(s)$ and $y' = \exp'(t)$. The Rauch Comparison Theorem [10, Theorem 2.3] implies that

$$\|s\| \leq |x| \leq d_{\mathbb{B}^n}(o', x'), \quad \|t\| \leq |y| \leq d_{\mathbb{B}^n}(o', y'), \quad \text{and} \quad \|s-t\| \leq |y^{-1}x| \leq d_{\mathbb{B}^n}(x', y'), \tag{3.3}$$

where $d_{\mathbb{B}^n}$ is the hyperbolic metric on \mathbb{B}^n . As in [10, eqn.(15)], we have that

$$d_{\mathbb{B}^n}(x', y') \leq \|s - t\| \left\{ 1 + \mathcal{O} \left((\|s\| + \|t\|)^2 \right) \right\}, \tag{3.4}$$

where the quantity $\mathcal{O} \left((\|s\| + \|t\|)^2 \right) \leq C_{r, \mathcal{K}_r} (\|s\| + \|t\|)^2$, for some $C_{r, \mathcal{K}_r} > 0$ depends only on the radius r and curvature \mathcal{K}_r . Also, we have $dy = (1 + \mathcal{O}(\|t\|)) dt$ on the compact set $\overline{\mathbf{B}(o, r)}$ [10, Prop. 2.2]. We now choose a small number ϵ such that $0 < \epsilon < \min\{1, n - \alpha - \beta\}$. Then, in normal geodesic coordinates around the origin o , using the notations introduced above, we obtain by (3.3) that

$$\begin{aligned} \int_{\mathbf{B}(o, r)} |y|^{\alpha-n} |y^{-1}x|^{\beta-n} dy &\leq \int_{B(0, r)} \|t\|^{\alpha-n} \|s - t\|^{\beta-n} (1 + \mathcal{O}(\|t\|)) dt \\ &\leq \int_{B(0, r)} \|t\|^{\alpha-n} \|s - t\|^{\beta-n} (1 + \mathcal{O}(\|t\|^\epsilon)) dt. \end{aligned}$$

Therefore, using the Euclidean relation (3.2), the estimates (3.3) and (3.4) we have

$$\begin{aligned} &\int_{\mathbf{B}(o, r)} |y|^{\alpha-n} |y^{-1}x|^{\beta-n} dy \\ &\leq \frac{\gamma(\alpha)\gamma(\beta)}{\gamma(\alpha + \beta)} \|s\|^{\alpha+\beta-n} (1 + \mathcal{O}(\|s\|^\epsilon)) \\ &\leq \frac{\gamma(\alpha)\gamma(\beta)}{\gamma(\alpha + \beta)} d_{\mathbb{B}^n}(o', x')^{\alpha+\beta-n} (1 + \mathcal{O}(d_{\mathbb{B}^n}(o', x')^\epsilon)) \left(1 + \mathcal{O}(\|s\|^2) \right)^\eta \\ &\leq \frac{\gamma(\alpha)\gamma(\beta)}{\gamma(\alpha + \beta)} |x|^{\alpha+\beta-n} (1 + \mathcal{O}(|x|^\epsilon)), \end{aligned}$$

where η is a positive number and the quantity $\mathcal{O}(\|s\|^2)$ is bounded on $B(0, r)$. □

Let $k_{\zeta, \alpha}$ be the Schwartz kernel of the operator $(-\Delta - |\rho|^2 + \zeta^2)^{-\alpha/2}$, for $\alpha \in \mathbb{R}$ and $\zeta > 0$. Also, let k_α be the kernel of $(-\Delta - |\rho|^2)^{-\alpha/2}$, for $0 < \alpha < l + 2|\Sigma_0^+|$. The following asymptotic estimates of the Bessel-Green-Riesz kernels at infinity is due to Anker and Ji [5, Theorem 4.2.2].

Theorem 3.2 (i) For $\zeta > 0$ and $\beta > 0$ there holds

$$k_{\zeta, \beta}(x) \sim |x|^{(\beta-l-1)/2-|\Sigma_0^+|} \phi_0(x) e^{-\zeta|x|}, \quad |x| \geq 1.$$

(ii) For $\zeta = 0$ and for $0 < \alpha < l + 2|\Sigma_0^+|$ there holds

$$k_\alpha(x) \sim |x|^{\alpha-l-2|\Sigma_0^+|} \phi_0(x), \quad |x| \geq 1.$$

In the remaining part of this section we first derive the optimal bounds for the kernels k_α , for $0 < \alpha < l + 2|\Sigma_0^+|$ and $k_{\zeta, \beta}$, for $\zeta > 0, \beta < n$ near the origin (Propositions 3.3 and 3.4). Then we establish sharp estimates for the convolutions

$k_\alpha * k_{\zeta, \beta}$, for $0 < \alpha + \beta < n$ near the origin and away from the origin using Fourier analysis on symmetric spaces (Propositions 3.5 and 3.6).

Proposition 3.3 *Let $0 < \alpha < \min\{n, l + 2|\Sigma_0^+\}|$. There holds*

$$k_\alpha(x) \leq \frac{1}{\gamma(\alpha)} \frac{1}{|x|^{n-\alpha}} + \mathcal{O}\left(\frac{1}{|x|^{n-\alpha-1}}\right), \quad 0 < |x| < 1;$$

where $\gamma(\alpha)$ is defined in (3.1)

Proof By the Mellin type expression

$$(-\Delta - |\rho|^2)^{-\alpha/2} = \frac{1}{\Gamma(\alpha/2)} \int_0^\infty t^{\alpha/2-1} e^{-t(-\Delta-|\rho|^2)} dt. \tag{3.5}$$

We will now use the following local expansion of the heat kernel $h_t(x)$

$$h_t(x) = e^{-|x|^2/4t} t^{-n/2} v_0(x) + \mathcal{O}\left(e^{-c|x|^2/t} t^{-n/2+1}\right),$$

where $v_0(x) = (4\pi)^{-n/2} + \mathcal{O}(|x|^2)$ and $0 < c < 1/4$ [3, eqn.(3.9), p. 278]. Using this it follows from (3.5) that on the kernel level

$$\begin{aligned} k_\alpha(x) &= \frac{1}{\Gamma(\alpha/2)} \int_0^\infty t^{\alpha/2-1} h_t(x) e^{|\rho|^2 t} dt \\ &= \frac{1}{\Gamma(\alpha/2)} \int_0^1 t^{\alpha/2-1} \left(e^{-|x|^2/4t} t^{-n/2} v_0(x) + \mathcal{O}\left(e^{-c|x|^2/t} t^{-n/2+1}\right) \right) e^{|\rho|^2 t} dt \\ &\quad + \frac{1}{\Gamma(\alpha/2)} \int_1^\infty t^{\alpha/2-1} h_t(x) e^{|\rho|^2 t} dt \\ &= \frac{1}{\Gamma(\alpha/2)} \int_0^1 t^{\alpha/2-1} \left(e^{-|x|^2/4t} t^{-n/2} v_0(x) + \mathcal{O}\left(e^{-c|x|^2/t} t^{-n/2+1}\right) \right) (1 + \mathcal{O}(t)) dt \\ &\quad + \frac{1}{\Gamma(\alpha/2)} \int_1^\infty t^{\alpha/2-1} h_t(x) e^{|\rho|^2 t} dt \\ &\leq \frac{1}{\Gamma(\alpha/2)} v_0(x) \int_0^1 e^{-|x|^2/4t} t^{\alpha/2-1-n/2} dt + \int_0^1 \mathcal{O}\left(e^{-c|x|^2/t} t^{\alpha/2-n/2}\right) dt \\ &\quad + \frac{1}{\Gamma(\alpha/2)} \int_1^\infty t^{\alpha/2-1} h_t(x) e^{|\rho|^2 t} dt. \end{aligned} \tag{3.6}$$

Now, we have that

$$\begin{aligned} \int_0^1 e^{-|x|^2/4t} t^{\alpha/2-1-n/2} dt &= \left(\frac{2}{|x|}\right)^{n-\alpha} \int_{|x|^2/4}^\infty e^{-s} s^{(n-\alpha)/2-1} ds \\ &\leq \left(\frac{2}{|x|}\right)^{n-\alpha} \Gamma\left(\frac{n-\alpha}{2}\right). \end{aligned} \tag{3.7}$$

Similarly, the second integral is of $\mathcal{O}(|x|^{\alpha-n+1})$. For the third term, we use the estimate on h_t [3, Theorem 3.1, ii)] that there exists $C > 0$ such that

$$h_t(x) \leq Ct^{-l/2-|\Sigma_0^+|} (1 + |x|^2)^{|\Sigma_0^+|/2} e^{-|\rho|^2 t - \rho(\log x) - |x|^2/(4t)}, \quad t \geq 1, \quad |x| \leq \sqrt{t}.$$

Using this estimate it follows that for all $0 < |x| < 1$ and $0 < \alpha < l + 2|\Sigma_0^+|$

$$\frac{1}{\Gamma(\alpha/2)} \int_1^\infty t^{\alpha/2-1} h_t(x) e^{|\rho|^2 t} dt \leq C \int_1^\infty t^{\alpha/2-1-l/2-|\Sigma_0^+|} dt < \infty. \quad (3.8)$$

Using $v_0(x) = (4\pi)^{-n/2} + \mathcal{O}(|x|^2)$ and the estimates (3.7) and (3.8), it follows from the equation (3.6) that for $0 < |x| < 1$

$$\begin{aligned} k_\alpha(x) &\leq \frac{1}{\Gamma(\alpha/2)} (4\pi)^{-n/2} \Gamma((n - \alpha)/2) 2^{n-\alpha} |x|^{\alpha-n} + \mathcal{O}(|x|^{\alpha-n+1}) + \mathcal{O}(1) \\ &\leq \frac{1}{\gamma(\alpha)} |x|^{\alpha-n} + \mathcal{O}(|x|^{\alpha-n+1}). \end{aligned}$$

This completes the proof. □

The following three results, that is, Propositions 3.4, 3.5 and Proposition 3.6 are extension of Lemmas 3.4, 3.5 and 3.6 proved in [24] respectively in the context of hyperbolic spaces.

Proposition 3.4 *Let $\zeta > 0$ and $0 < \beta < n$. Then there exists $\tilde{\epsilon}$ satisfying $0 < \tilde{\epsilon} < \min\{1, n - \beta\}$ such that*

$$k_{\zeta,\beta}(x) = \frac{1}{\gamma(\beta)} \frac{1}{|x|^{n-\beta}} + \mathcal{O}\left(\frac{1}{|x|^{n-\beta-\tilde{\epsilon}}}\right), \quad 0 < |x| < 1.$$

Proof We first prove the result when $\beta = m$ is an integer satisfying $1 \leq m < n - 1$. Precisely, we prove that there exists ϵ_0 satisfying $0 < \epsilon_0 < \min\{1, n - m\} = 1$ such that

$$k_{\zeta,m}(x) \leq \frac{1}{\gamma(m)} \frac{1}{|x|^{n-m}} + \mathcal{O}\left(\frac{1}{|x|^{n-m-\epsilon_0}}\right), \quad 0 < |x| < 1. \quad (3.9)$$

This will be done by induction. It follows by the Mellin type expressions that on the kernel level

$$\begin{aligned} k_{\zeta,1}(x) &= \frac{1}{\Gamma(1/2)} \int_0^\infty t^{-1/2} h_t(x) e^{\rho^2 t - \zeta^2 t} dt \\ &\leq \frac{1}{\Gamma(1/2)} \int_0^\infty t^{-1/2} h_t(x) e^{\rho^2 t} dt \\ &= k_1(x). \end{aligned}$$

Thus, by Proposition 3.3, the estimate (3.9) holds for $m = 1$. In fact, this is true as long as $m < \min\{n, l + 2|\Sigma_0^+|\}$ by Proposition 3.3. Now, suppose that the estimate (3.9) is

valid for $m \in \mathbb{Z}$ satisfying $1 \leq m < n - 2$ with $0 < \epsilon_0 < \min\{1, n - m\} = 1$. We prove this is also true for $m + 1$, that is, there exists $\tilde{\epsilon}$ satisfying $0 < \tilde{\epsilon} < \min\{1, n - m - 1\}$ such that

$$k_{\zeta, m+1}(x) \leq \frac{1}{\gamma(m+1)} \frac{1}{|x|^{n-m-1}} + \mathcal{O}\left(\frac{1}{|x|^{n-m-1-\tilde{\epsilon}}}\right), \quad 0 < |x| < 1. \quad (3.10)$$

Let $x \in X$ with $0 < |x| < 1$. Then

$$k_1 * k_{\zeta, m}(x) = \int_{\mathbf{B}(o, 2)} k_1(y) k_{\zeta, m}(y^{-1}x) \, dy + \int_{X \setminus \mathbf{B}(o, 2)} k_1(y) k_{\zeta, m}(y^{-1}x) \, dy := I_1 + I_2. \quad (3.11)$$

We first prove that the second integral I_2 on the right-hand side is uniformly bounded independent of x . Indeed, by Hölder’s inequality, Theorem 3.2, integral formula (2.5) with (2.4) and the estimate (2.11) of ϕ_0 , it follows that for $|x| < 1$

$$\begin{aligned} I_2 &= \int_{X \setminus \mathbf{B}(o, 2)} k_1(y) k_{\zeta, m}(y^{-1}x) \, dy \\ &\leq \left(\int_{X \setminus \mathbf{B}(o, 2)} |k_1(y)|^2 \, dy \right)^{1/2} \left(\int_{X \setminus \mathbf{B}(o, 1)} |k_{\zeta, m}(z)|^2 \, dz \right)^{1/2} \\ &\lesssim \left(\int_{X \setminus \mathbf{B}(o, 2)} |y|^{2-2l-4|\Sigma_0^+|} \phi_0(y)^2 \, dy \right)^{1/2} \\ &\quad \left(\int_{X \setminus \mathbf{B}(o, 1)} |z|^{m-l-1-2|\Sigma_0^+|} \phi_0(z)^2 e^{-2\zeta|z|} \, dz \right)^{1/2} \\ &\lesssim \left(\int_{\{Y \in \overline{\mathfrak{a}_+^+} : |Y| \geq 2\}} |Y|^{2-2l-4|\Sigma_0^+|} |Y|^{2|\Sigma_0^+|} e^{-2\rho(Y)} e^{2\rho(Y)} \, dY \right)^{1/2} \\ &\quad \left(\int_{\{Y \in \overline{\mathfrak{a}_+^+} : |Y| \geq 1\}} |Y|^{m-l-1-2|\Sigma_0^+|} |Y|^{2|\Sigma_0^+|} e^{-2\rho(Y)} e^{-2\zeta|Y|} e^{2\rho(Y)} \, dY \right)^{1/2} \\ &\lesssim \left(\int_2^\infty r^{2-2l-2|\Sigma_0^+|} r^{l-1} \, dr \right)^{1/2} \left(\int_1^\infty r^{m-l-1} e^{-2\zeta r} r^{l-1} \, dr \right)^{1/2} < \infty. \end{aligned}$$

Now, we estimate the first integral I_1 in (3.11). Since $m < n - 2$ and $\epsilon_0 < 1$, we can choose a $\epsilon_1 \in (0, 1)$ such that $m + \epsilon_0 + \epsilon_1 < n - 1$. We first observe that $|y|^{-(n-2)} \leq |y|^{-(n-1-\epsilon_1)}$, for small $|y|$. Using this fact, the estimate of k_α in Proposition 3.3 and the estimate (3.9) of $k_{\zeta, m}$ we get that

$$\begin{aligned} I_1 &= \int_{\mathbf{B}(o, 2)} k_1(y) k_{\zeta, m}(y^{-1}x) \, dy \\ &\leq \int_{\mathbf{B}(o, 2)} \left(\frac{1}{\gamma(1)} \frac{1}{|y|^{n-1}} + \frac{C_1}{|y|^{n-1-\epsilon_1}} \right) \left(\frac{1}{\gamma(m)} \frac{1}{|y^{-1}x|^{n-m}} + \frac{C_2}{|y^{-1}x|^{n-m-\epsilon_0}} \right) \, dy \\ &= \frac{1}{\gamma(1)\gamma(m)} \int_{\mathbf{B}(o, 2)} \frac{1}{|y|^{n-1}} \frac{1}{|y^{-1}x|^{n-m}} \, dy + \frac{C_2}{\gamma(1)} \int_{\mathbf{B}(o, 2)} \frac{1}{|y|^{n-1}} \frac{1}{|y^{-1}x|^{n-m-\epsilon_0}} \, dy \end{aligned}$$

$$+ \frac{C_1}{\gamma(m)} \int_{\mathbf{B}(o,2)} \frac{1}{|y|^{n-1-\epsilon_1}} \frac{1}{|y^{-1}x|^{n-m}} dy + C_1 C_2 \int_{\mathbf{B}(o,2)} \frac{1}{|y|^{n-1-\epsilon_1}} \frac{1}{|y^{-1}x|^{n-m-\epsilon_0}} dy.$$

Since $m + 1 + \epsilon_1 + \epsilon_0 < n$, using Lemma 3.1 we get that there exists $0 < \tilde{\epsilon} < \min\{1, n - (m + 1 + \epsilon_0 + \epsilon_1)\}$ such that for $|x| < 1$

$$I_1 \leq \frac{1}{\gamma(m+1)} \frac{1}{|x|^{n-m-1}} + \mathcal{O}\left(\frac{1}{|x|^{n-m-1-\tilde{\epsilon}}}\right).$$

Putting this in (3.11) we finally have for $|x| < 1$

$$k_{\zeta, m+1}(x) = k_{\zeta, 1} * k_{\zeta, m}(x) \leq k_1 * k_{\zeta, m}(x) \leq \frac{1}{\gamma(m+1)} \frac{1}{|x|^{n-m-1}} + \mathcal{O}\left(\frac{1}{|x|^{n-m-1-\tilde{\epsilon}}}\right).$$

By induction this completes the proof of the lemma when $\beta = m \in \mathbb{Z}$ with $1 \leq m < n - 1$.

Now, we prove the required estimate for arbitrary β (not necessarily integer) with $0 < \beta < n$. We choose $0 < \tilde{\beta} < 3$ and an integer m with $0 \leq m < n - 1$ such that $\beta = \tilde{\beta} + m < n$. Without loss of generality, we can assume $m \geq 1$. Thus

$$\begin{aligned} k_{\zeta, \beta}(x) &= k_{\zeta, \tilde{\beta}} * k_{\zeta, m} \leq k_{\tilde{\beta}} * k_{\zeta, m} \\ &= \int_{\mathbf{B}(o,2)} k_{\tilde{\beta}}(y) k_{\zeta, m}(y^{-1}x) dy + \int_{X \setminus \mathbf{B}(o,2)} k_{\tilde{\beta}}(y) k_{\zeta, m}(y^{-1}x) dy. \end{aligned} \tag{3.12}$$

The second integral is bounded as in the case of I_2 in (3.11). The first integral can be estimated as I_1 in (3.11). To see this, we first notice that we may choose ϵ_0 small enough satisfying $0 < \epsilon_0 < \min\{1, n - \beta\}$ such that (3.9) holds. Let us choose ϵ_2 such that $0 < \epsilon_2 < \min\{1, n - \beta - \epsilon_0\}$. By Proposition 3.3 it follows that

$$\begin{aligned} &\int_{\mathbf{B}(o,2)} k_{\tilde{\beta}}(y) k_{\zeta, m}(y^{-1}x) dy \\ &\leq \int_{\mathbf{B}(o,2)} \left(\frac{1}{\gamma(\tilde{\beta})} \frac{1}{|y|^{n-\tilde{\beta}}} + \frac{C_1}{|y|^{n-\tilde{\beta}-\epsilon_2}} \right) \left(\frac{1}{\gamma(m)} \frac{1}{|y^{-1}x|^{n-m}} + \frac{C_2}{|y^{-1}x|^{n-m-\epsilon_0}} \right) dy. \end{aligned}$$

Therefore, by Lemma 3.1 there exists $\tilde{\epsilon}$ satisfying $0 < \tilde{\epsilon} < \min\{1, n - \beta - \epsilon_0 - \epsilon_2\}$ such that

$$\int_{\mathbf{B}(o,2)} k_{\tilde{\beta}}(y) k_{\zeta, m}(y^{-1}x) dy \leq \frac{1}{\gamma(\tilde{\beta})} \frac{1}{|x|^{n-\beta}} + \mathcal{O}\left(\frac{1}{|x|^{n-\beta-\tilde{\epsilon}}}\right).$$

Putting this in (3.12) we complete the proof. □

Proposition 3.5 *Let $\zeta > 0, 0 < \alpha < l + 2|\Sigma_0^+|$ and $0 < \beta < n$ such that $0 < \alpha + \beta < n$. There exists ϵ' satisfying $0 < \epsilon' < \min\{1, n - \alpha - \beta\}$ such that*

$$k_\alpha * k_{\zeta, \beta}(x) \leq \frac{1}{\gamma(\alpha + \beta)} \frac{1}{|x|^{n-\alpha-\beta}} + \mathcal{O}\left(\frac{1}{|x|^{n-\alpha-\beta-\epsilon'}}\right), \quad 0 < |x| < 1.$$

Proof The proof is exactly the same as that of Proposition 3.4. □

Proposition 3.6 *Let $\zeta > 0, 0 < \alpha < l + 2|\Sigma_0^+|$ and $0 < \beta < n$ such that $0 < \alpha + \beta < n$. For $\zeta' \in (0, \zeta)$ we have*

$$k_\alpha * k_{\zeta, \beta}(x) \lesssim e^{-\zeta'|x|} \phi_0(x) + \left(\chi_{1/2}|\cdot|^{\alpha-l-2|\Sigma_0^+|} \phi_0(\cdot)\right) * k_{\zeta, \beta}(x), \text{ for } |x| \geq 1,$$

where $\chi_{1/2}$ is the cutoff function vanishing in $\mathbf{B}(o, 1/2)$ and identically equals 1 otherwise.

Proof By Theorem 3.2 (ii) we have

$$\begin{aligned} k_\alpha * k_{\zeta, \beta}(x) &= \int_{\mathbf{B}(o, 1/2)} k_\alpha(y)k_{\zeta, \beta}(y^{-1}x) \, dy + \int_{X \setminus \mathbf{B}(o, 1/2)} k_\alpha(y)k_{\zeta, \beta}(y^{-1}x) \, dy \\ &\lesssim \int_{\mathbf{B}(o, 1/2)} k_\alpha(y)k_{\zeta, \beta}(y^{-1}x) \, dy \\ &\quad + \int_{X \setminus \mathbf{B}(o, 1/2)} |y|^{\alpha-l-2|\Sigma_0^+|} \phi_0(y) k_{\zeta, \beta}(y^{-1}x) \, dy \\ &= \int_{\mathbf{B}(o, 1/2)} k_\alpha(y)k_{\zeta, \beta}(y^{-1}x) \, dy \\ &\quad + \left(\chi_{1/2}|\cdot|^{\alpha-l-2|\Sigma_0^+|} \phi_0(\cdot)\right) * k_{\zeta, \beta}(x). \end{aligned} \tag{3.13}$$

We notice that, if $|y| < 1/2$ and $|x| \geq 1$, then $|y^{-1}x| \geq |x| - |y| \geq 1/2$. Therefore, by the estimates of k_α (Proposition 3.3) and $k_{\zeta, \beta}$ (Theorem 3.2 (i)) we have for $|y| < 1/2$ and $|x| \geq 1$ that

$$k_\alpha(y) \lesssim |y|^{\alpha-n}, \text{ and } k_{\zeta, \beta}(y^{-1}x) \lesssim e^{-\zeta'|y^{-1}x|} \phi_0(y^{-1}x),$$

where $\zeta' \in (0, \zeta)$ and the constant depends on ζ' . Using the above estimates, the property (2.12), the fact $|H(y^{-1}k)| \leq |y|$ (see equation (2.3)) and the integral formula (2.5), it follows that

$$\begin{aligned} &\int_{\mathbf{B}(o, 1/2)} k_\alpha(y)k_{\zeta, \beta}(y^{-1}x) \, dy \\ &\lesssim \int_{\mathbf{B}(o, 1/2)} \frac{1}{|y|^{n-\alpha}} e^{-\zeta'|y^{-1}x|} \phi_0(y^{-1}x) \, dy \\ &\leq e^{-\zeta'|x|} \int_{\mathbf{B}(o, 1/2)} \frac{1}{|y|^{n-\alpha}} e^{\zeta'|y|} \int_K e^{-\rho(H(y^{-1}k))} e^{-\rho(H(x^{-1}k))} \, dk \, dy \\ &\leq e^{-\zeta'|x|} \int_K e^{-\rho H(x^{-1}k)} \, dk \int_{\mathbf{B}(o, 1/2)} \frac{1}{|y|^{n-\alpha}} e^{\zeta'|y|} e^{|\rho||y|} \, dy \\ &\lesssim e^{-\zeta'|x|} \phi_0(x) \int_{\{Y \in \mathfrak{a}; |Y| < 1/2\}} \frac{1}{|Y|^{n-\alpha}} J(\exp Y) \, dY \\ &\lesssim e^{-\zeta'|x|} \phi_0(x). \end{aligned}$$

Putting this in (3.13) we get the required result. □

Remark 3.7 There exists $C > 0$ such that $k_\alpha * k_{\zeta,\beta}(x) \leq C$, for all $|x| \geq 1$. Indeed, by Proposition 3.6 it is enough to show that $(\chi_{1/2} \cdot |\cdot|^{\alpha-l-2|\Sigma_0^+|} \phi_0(\cdot)) * k_{\zeta,\beta}(x) \leq C$, for $|x| \geq 1$. To see this we first observe that if $|x| \geq 1$ and $|y| < 1/2$, then $|y^{-1}x| \geq 1/2$. Using this, the property (2.12), the estimate of $k_{\zeta,\beta}$ (Theorem 3.2 and Proposition 3.4) it follows by Cauchy-Schwarz inequality that

$$\begin{aligned} & (\chi_{1/2} \cdot |\cdot|^{\alpha-l-2|\Sigma_0^+|} \phi_0(\cdot)) * k_{\zeta,\beta}(x) \\ & \lesssim \int_{X \setminus \mathbf{B}(o,1/2)} |y|^{\alpha-l-2|\Sigma_0^+|} \phi_0(y) k_{\zeta,\beta}(y^{-1}x) \, dy \\ & \lesssim \int_{\{y \in X \setminus \mathbf{B}(o,1/2) : |y^{-1}x| < 1/2\}} |y|^{\alpha-l-2|\Sigma_0^+|} \phi_0(y) |y^{-1}x|^{\beta-n} \, dy \\ & \quad + \int_{\{y \in X \setminus \mathbf{B}(o,1/2) : |y^{-1}x| \geq 1/2\}} |y|^{\alpha-l-2|\Sigma_0^+|} \phi_0(y) k_{\zeta,\beta}(y^{-1}x) \, dy \\ & \lesssim \int_{\{y \in X : |y^{-1}x| < 1/2\}} |y^{-1}x|^{\beta-n} \, dy \\ & \quad + \int_{\{y \in X \setminus \mathbf{B}(o,1/2) : |y^{-1}x| \geq 1/2\}} |y|^{\alpha-l-2|\Sigma_0^+|} \phi_0(y) k_{\zeta,\beta}(y^{-1}x) \, dy \\ & \lesssim C + \left(\int_{X \setminus \mathbf{B}(o,1/2)} |y|^{2\alpha-2l-4|\Sigma_0^+|} (\phi_0(y))^2 \, dy \right)^{1/2} \\ & \left(\int_{X \setminus \mathbf{B}(o,1/2)} e^{-2\zeta'|z|} (\phi_0(z))^2 \, dz \right)^{1/2} < \infty. \end{aligned}$$

4 Asymptotic Estimates of Non-increasing Rearrangement of the Bessel–Green–Riesz Kernels

For a real valued function f on X , the non-increasing rearrangement of f is defined by

$$f^*(t) = \inf\{s > 0 : \lambda_f(s) \leq t\},$$

where the distribution function λ_f of f is given by

$$\lambda_f(s) = |\{x \in X : |f(x)| > s\}| = \int_{\{x \in X : |f(x)| > s\}} \, dx.$$

Here we use the notation $|E|$ for the measure of a subset E of X . We need the following two properties of the non-increasing rearrangement [13, Prop. 1.4.5, p. 46]:

- (i) By definition $\lambda_f(f^*(t)) \leq t$. If $|f| \leq |g|$ almost everywhere, then $f^* \leq g^*$.
- (ii) If there exists $c > 0$ such that $|\{x \in X : |f(x)| \geq f^*(t) - c\}| < \infty$, then $t \leq |\{x \in X : |f(x)| \geq f^*(t)\}|$.

For the convenience of the reader we summarize the results of Sect. 3 here. By Theorem 3.2, Propositions 3.3, 3.4, Propositions 3.5 and 3.6 we have

a) For $\zeta = 0$ and $0 < \alpha < \min\{n, l + 2|\Sigma_0^+\}|$

$$k_\alpha(x) \leq \frac{1}{\gamma(\alpha)} \frac{1}{|x|^{n-\alpha}} + \mathcal{O}\left(\frac{1}{|x|^{n-\alpha-1}}\right), \quad 0 < |x| < 1;$$

$$\lesssim |x|^{\alpha-l-2|\Sigma_0^+|} \phi_0(x), \quad |x| \geq 1.$$

b) Let $\zeta > 0$ and $0 < \beta < n$. There exists $\tilde{\epsilon}$ satisfying $0 < \tilde{\epsilon} < \min\{1, n - \beta\}$ such that

$$k_{\zeta,\beta}(x) \leq \frac{1}{\gamma(\beta)} \frac{1}{|x|^{n-\beta}} + \mathcal{O}\left(\frac{1}{|x|^{n-\beta-\tilde{\epsilon}}}\right), \quad 0 < |x| < 1;$$

$$\lesssim |x|^{(\beta-l-1)/2-|\Sigma_0^+|} e^{-\zeta|x|} \phi_0(x), \quad |x| \geq 1. \tag{4.1}$$

c) Let $\zeta > 0, 0 < \alpha < l + 2|\Sigma_0^+|$ and $0 < \beta < n$ such that $0 < \alpha + \beta < n$. There exists ϵ' satisfying $0 < \epsilon' < \min\{1, n - \alpha - \beta\}$ such that

$$k_\alpha * k_{\zeta,\beta}(x) \leq \frac{1}{\gamma(\alpha + \beta)} \frac{1}{|x|^{n-\alpha-\beta}} + \mathcal{O}\left(\frac{1}{|x|^{n-\alpha-\beta-\epsilon'}}\right), \quad 0 < |x| < 1.$$

$$\lesssim e^{-\zeta'|x|} \phi_0(x) + \left(\chi_{1/2} \cdot |x|^{\alpha-l-2|\Sigma_0^+|} \phi_0(\cdot)\right) * k_{\zeta,\beta}(x), \quad \text{for } |x| \geq 1, \tag{4.2}$$

where $0 < \zeta' < \zeta$.

The behaviour of the volume of a small geodesic ball around the origin in X can be expressed as follows (see [11, Theorem 3.98], [20, equation (8)]):

$$|\mathbf{B}(o, r)| = \frac{\omega_{n-1}}{n} r^n + \mathcal{O}(r^{n+1}), \quad 0 < r < 1. \tag{4.3}$$

Lemma 4.1 *Let $\zeta, \alpha, \beta, \epsilon'$ be as in Proposition 3.5. Then for $0 < t < 2$*

$$[k_\alpha * k_{\zeta,\beta}]^*(t) \leq \frac{1}{\gamma(\alpha + \beta)} \left(\frac{nt}{\omega_{n-1}}\right)^{(\alpha+\beta-n)/n} + \mathcal{O}\left(t^{(\alpha+\beta+\epsilon'-n)/n}\right).$$

Proof By (4.2) there exists $C > 0$ such that for $0 < |x| < 1, k_\alpha * k_{\zeta,\beta}(x) \leq h(x)$, where the function h is defined by

$$h(x) = \frac{1}{\gamma(\alpha + \beta)} \frac{1}{|x|^{n-\alpha-\beta}} + \frac{C}{|x|^{n-\alpha-\beta-\epsilon'}}, \quad x \in X.$$

By Remark 3.7 there exists $C_0 > 0$ such that $k_\alpha * k_{\zeta,\beta}(x) \leq C_0$ for $|x| \geq 1$. Therefore, $k_\alpha * k_{\zeta,\beta}(x) \leq C_0 + h(x)$ for all $x \in X$ and hence it is enough to prove the required estimate for the function h^* .

First, we note that if $f(t) = At^{-a}(1+Bt^b)$, $t > 0$, for positive constants A, B, a, b then there exists a $C > 0$ such that

$$f^{-1}(t) \leq [At^{-1}]^{1/a}[1 + Ct^{-b/a}], \text{ for } t > 1.$$

Applying this to the function $f(|x|) = h(x)$, we get that

$$h^{-1}(t) \leq \left[\frac{t^{-1}}{\gamma(\alpha + \beta)} \right]^{1/(n-\alpha-\beta)} \left[1 + Ct^{-\epsilon'/(n-\alpha-\beta)} \right], \quad t > 1.$$

Using the above inequality and (4.3) we get that for $t > 1$

$$\begin{aligned} |\{x \in X : h(x) \geq t\}| &\leq |\mathbf{B}(o, h^{-1}(t))| \leq \frac{\omega_{n-1}}{n} (h^{-1}(t))^n + C(h^{-1}(t))^{n+1} \\ &\leq \frac{\omega_{n-1}}{n} \left(\frac{t^{-1}}{\gamma(\alpha + \beta)} \right)^{n/(n-\alpha-\beta)} \left[1 + C't^{-\epsilon'/(n-\alpha-\beta)} \right]. \end{aligned}$$

Again, if $g(t) = At^{-a}[1+Bt^{-b}]$, $t > 0$, for positive constants A, B, a, b , then there exists a $C > 0$ such that

$$g^{-1}(t) \leq [At^{-1}]^{1/a}[1 + Ct^{b/a}], \text{ for } 0 < t \leq 2.$$

Using this we get that for $0 < t \leq 2$

$$h^*(t) \leq \frac{1}{\gamma(\alpha + \beta)} \left(\frac{\omega_{n-1}}{nt} \right)^{(n-\alpha-\beta)/n} [1 + \mathcal{O}(t^{\epsilon'/n})].$$

This completes the proof. □

Lemma 4.2 *Let $\zeta, \beta, \tilde{\epsilon}$ be as in Proposition 3.4. Then there holds*

$$[k_{\zeta, \beta}]^*(t) \leq \frac{1}{\gamma(\beta)} \left(\frac{nt}{\omega_{n-1}} \right)^{(\beta-n)/n} + \mathcal{O}(t^{(\beta+\tilde{\epsilon}-n)/n}), \quad 0 < t < 2.$$

Proof The proof is similar to that of Lemma 4.1. □

Lemma 4.3 *Let $\zeta > 0, 0 < \beta < n$ and $\zeta' \in (0, \zeta)$. There holds*

$$[k_{\zeta, \beta}]^*(t) \lesssim t^{-1/2-\zeta'/2|\rho|} (\log t)^{2|\rho|/(\zeta'+|\rho|)}, \quad t \geq 2.$$

Proof Let us choose $r > 0$ such that $|\mathbf{B}(o, r)| < 1$ and set

$$\begin{aligned} h(x) &= \frac{1}{|x|^{n-\beta}}, \quad 0 < |x| < r; \\ &= 0, \quad |x| \geq 1, \end{aligned}$$

and $f(x) = e^{-\rho(\log x)} e^{-\zeta'|x|}$, for $x \in X$. Using the estimate (2.11) of ϕ_0 , it follows from (4.1) that there exists $C > 0$ such that

$$k_{\zeta,\beta}(x) \leq C(h(x) + f(x)), \quad x \in X. \tag{4.4}$$

We now observe that for $s > 0$

$$|\{x \in X : h(x) > s\}| \leq |\mathbf{B}(o, r)|.$$

This inequality yields

$$\begin{aligned} |\{x \in X : h(x) + f(x) > s\}| &\leq |\{x \in X : h(x) > s/2\}| + |\{x \in X : f(x) > s/2\}| \\ &\leq |\mathbf{B}(o, r)| + |\{x \in X : f(x) > s/2\}|. \end{aligned}$$

Therefore, for $t > |\mathbf{B}(o, r)|$ we have

$$\begin{aligned} (h + f)^*(t) &= \inf_s \{s > 0 : |\{x \in X : h(x) + f(x) > s\}| \leq t\} \\ &\leq \inf_s \{s > 0 : |\{x \in X : f(x) > s/2\}| \leq t - |\mathbf{B}(o, r)|\} \\ &= 2f^*(t - |\mathbf{B}(o, r)|). \end{aligned} \tag{4.5}$$

Integral formula (2.5) yields

$$\begin{aligned} \lambda_f(f^*(t)) &= \int_{\{Y \in \overline{\mathfrak{a}}_+ : f(Y) > f^*(t)\}} c \prod_{\alpha \in \Sigma_+} (\sinh \alpha(Y))^{m_\alpha} dY \\ &\lesssim \int_{\{Y \in \overline{\mathfrak{a}}_+ : e^{\rho(Y) + \zeta'|Y|} < \frac{1}{f^*(t)}\}} e^{2\rho(Y)} dY. \end{aligned} \tag{4.6}$$

Let us fix a basis of \mathfrak{a}^* as $\{\epsilon_1, \dots, \epsilon_{l-1}, \rho/|\rho|\}$, where $\rho^\perp = \text{span}\{\epsilon_1, \dots, \epsilon_{l-1}\}$. If $Y \in \mathfrak{a}$, we write $Y = (Y_1, \dots, Y_l)$ with its coordinates with respect to the corresponding dual basis. We observe that $\rho(Y) = |\rho|Y_l$. Since $\rho(Y) \geq 0$ for $Y \in \overline{\mathfrak{a}}_+$, it follows that

$$\begin{aligned} \left\{ Y \in \overline{\mathfrak{a}}_+ : e^{\rho(Y) + \zeta'|Y|} < \frac{1}{f^*(t)} \right\} &\subset \left\{ Y \in \overline{\mathfrak{a}}_+ : e^{\rho(Y) + \zeta'Y_l} < \frac{1}{f^*(t)}, e^{\zeta'|Y|} < \frac{1}{f^*(t)} \right\} \\ &\subset \left\{ Y \in \overline{\mathfrak{a}}_+ : e^{|\rho|Y_l + \zeta'Y_l} < \frac{1}{f^*(t)}, |Y| < \frac{1}{\zeta'} \log\left(\frac{1}{f^*(t)}\right) \right\}. \end{aligned}$$

Therefore, by (4.6) we get that

$$\begin{aligned} \lambda_f(f^*(t)) &\lesssim \int_{\{Y \in \overline{\mathfrak{a}}_+ : e^{|\rho|Y_l + \zeta'Y_l} < \frac{1}{f^*(t)}, |Y| < \frac{1}{\zeta'} \log\left(\frac{1}{f^*(t)}\right)\}} e^{2|\rho|Y_l} dY \\ &\lesssim \left(\frac{1}{f^*(t)}\right)^{2|\rho|/(\zeta' + |\rho|)} \left(\log\left(\frac{1}{f^*(t)}\right)\right)^l. \end{aligned}$$

Since $f^*(t)$ is non-zero it follows from the above estimate that $\lambda_f(f^*(t - c)) < \infty$ for $c \in (0, f^*(t))$. Therefore,

$$t = \lambda_f(f^*(t)) \lesssim \left(\frac{1}{f^*(t)}\right)^{2|\rho|/(\zeta'+|\rho|)} \left(\log\left(\frac{1}{f^*(t)}\right)\right)^l.$$

Using the lemma below (Lemma 4.4) we get that for all $t \geq 1$

$$f^*(t) \lesssim t^{-(\zeta'+|\rho|)/2|\rho|} (\log t)^{l(\zeta'+|\rho|)/2|\rho|}.$$

This fact together with (4.4) and (4.5) completes the proof. □

Lemma 4.4 *Let h be a non-increasing function on the positive real axis. Suppose there exist two positive numbers a, b such that for $t \geq 1$*

$$t \leq \left(\frac{1}{h(t)}\right)^a \log\left(\frac{1}{h(t)}\right)^b.$$

Then there holds

$$h(t) \leq t^{-1/a} (\log t)^{b/a}, \text{ for } t \geq 1.$$

Proof Let $h(t) = s$. By the hypothesis we have for all $t \geq 1$

$$h^{-1}(s) = t \lesssim \frac{1}{s^a} \left(\log \frac{1}{s}\right)^b.$$

Therefore, for $s \geq 1$

$$\begin{aligned} h^{-1}\left(\left(\frac{(\log s)^b}{s}\right)^{1/a}\right) &\lesssim \frac{s}{(\log s)^b} \left(\log\left(\frac{s^{1/a}}{(\log s)^{b/a}}\right)\right)^b \\ &= s \left(\frac{\log s^{1/a} - \log (\log s)^{b/a}}{\log s}\right)^b \end{aligned}$$

Since the term inside the bracket on the right-hand side goes to a finite positive limit as s goes to infinity, we get that

$$h^{-1}\left(\left(\frac{(\log s)^b}{s}\right)^{1/a}\right) \lesssim s, \text{ for } s \geq 1.$$

Since h is non-increasing, it follows that

$$h(s) \lesssim \frac{1}{s^{1/a}} (\log s)^{b/a}.$$

□

The following result is analogous to Lemma 4.1 in [24].

Lemma 4.5 *Let $0 < \alpha < 3/2$, $\zeta > 0$ and $0 < \beta < n - \alpha$. Then for each $a > 0$, we have $\int_a^\infty ([k_\alpha * k_{\zeta,\beta}]^*(t))^2 dt < \infty$.*

Proof By Lemma 4.1, it is enough to show that there exists $c_0 > 0$ such that the integral $\int_{c_0}^\infty ([k_\alpha * k_{\zeta,\beta}]^*(t))^2 dt < \infty$. We set

$$f_1(x) = \frac{1}{|x|^{n-\alpha-\beta}}, \quad |x| < 1; \\ = 0, \quad |x| \geq 1,$$

and $f_2(x) = (\chi_{1/2} \cdot |\alpha-l-2|\Sigma_0^+| \phi_0(\cdot)) * k_{\zeta,\beta}(x)$. Let $\zeta' \in (0, \zeta)$. By (4.2) there exists $C > 0$ such that

$$k_\alpha * k_{\zeta,\beta}(x) \leq C \left(f_1(x) + e^{-\zeta'|x|} \phi_0(x) + f_2(x) \right), \quad x \in X. \tag{4.7}$$

Proceeding as before we get by (4.5) that for $t > |\mathbf{B}(o, 1)|$

$$(f_1 + e^{-\zeta'|t|} \phi_0 + f_2)^*(t) = 2 \left(e^{-\zeta'|t|} \phi_0 + f_2 \right)^*(t - c_0),$$

where $c_0 = |\mathbf{B}(o, 1)|$. By (4.7) and equation above it follows that

$$\int_{c_0}^\infty [(k_\alpha * k_{\zeta,\beta})^*(t)]^2 dt \leq 4C^2 \int_{c_0}^\infty \left[\left(e^{-\zeta'|t|} \phi_0 + f_2 \right)^*(t - c_0) \right]^2 dt \\ = 4C^2 \int_X \left| e^{-\zeta'|x|} \phi_0(x) + f_2(x) \right|^2 dx \\ \leq 4C^2 \left\{ \left(\int_X e^{-2\zeta'|x|} \phi_0(x)^2 dx \right)^{1/2} + \left(\int_X (f_2(x))^2 dx \right)^{1/2} \right\}^2.$$

The integral formula (2.5) and the estimate (2.11) yield

$$\int_X e^{-2\zeta'|x|} \phi_0(x)^2 dx < \infty.$$

On the other hand, by Plancherel formula (2.8) for K -biinvariant functions we have for $0 < \alpha < 3/2$ that

$$\int_X (f_2(x))^2 dx = c_G \int_{\mathfrak{a}^*} |\widehat{(k_{\zeta,\beta})}(\lambda)|^2 \left| (\chi_{1/2} \cdot |\alpha-l-2|\Sigma_0^+| \phi_0)^\wedge(\lambda) \right|^2 |\mathbf{c}(\lambda)|^{-2} d\lambda \\ = c_G \int_{\mathfrak{a}^*} (|\lambda|^2 + \zeta^2)^{-\beta} \left| (\chi_{1/2} \cdot |\alpha-l-2|\Sigma_0^+| \phi_0)^\wedge(\lambda) \right|^2 |\mathbf{c}(\lambda)|^{-2} d\lambda \\ \leq C \int_{\mathfrak{a}^*} \left| (\chi_{1/2} \cdot |\alpha-l-2|\Sigma_0^+| \phi_0)^\wedge(\lambda) \right|^2 |\mathbf{c}(\lambda)|^{-2} d\lambda$$

$$= C \int_{\{x \in X: |x| \geq 1/2\}} |x|^{2\alpha - 2l - 4|\Sigma_0^+|} (\phi_0(x))^2 \, dx < \infty.$$

This completes the proof. □

5 Proof of the Theorems

Proof of Theorem 1.3 Let $u \in W^{\alpha,p}(X)$ and we write $f = (-\Delta - |\rho|^2 + \zeta^2)^{\alpha/2}u$. Then clearly $u = f * k_{\zeta,\alpha}$ and by the hypothesis $\|f\|_p \leq 1$. Applying O’Neil’s lemma [28, Lemma 1.5] for the rearrangement of convolution, we have for $t > 0$

$$u^*(t) \leq \frac{1}{t} \int_0^t f^*(s) \, ds \int_0^t k_{\zeta,\alpha}^*(s) \, ds + \int_t^\infty f^*(s) k_{\zeta,\alpha}^*(s) \, ds.$$

Therefore,

$$\begin{aligned} \frac{1}{|E|} \int_E \exp(\beta_0(n, \alpha) |u(x)|^{p'}) \, dx &\leq \frac{1}{|E|} \int_0^{|E|} \exp(\beta_0(n, \alpha) |u^*(t)|^{p'}) \, dt \\ &\leq \frac{1}{|E|} \int_0^{|E|} \exp\left(\beta_0(n, \alpha) \left| \frac{1}{t} \int_0^t f^*(s) \, ds \int_0^t k_{\zeta,\alpha}^*(s) \, ds + \int_t^\infty f^*(s) k_{\zeta,\alpha}^*(s) \, ds \right|^{p'}\right) \, dt \\ &\leq \int_0^\infty \exp\left(-t + \beta_0(n, \alpha) \left| \frac{1}{|E|e^{-t}} \int_0^{|E|e^{-t}} f^*(s) \, ds \int_0^{|E|e^{-t}} k_{\zeta,\alpha}^*(s) \, ds \right. \right. \\ &\quad \left. \left. + \int_{|E|e^{-t}}^\infty f^*(s) k_{\zeta,\alpha}^*(s) \, ds \right|^{p'}\right) \, dt. \end{aligned} \tag{5.1}$$

To get the last equation, we use the substitution $t \mapsto |E|e^{-t}$. Next, we change the variables

$$\phi(t) = (|E|e^{-t})^{1/p} f^*(|E|e^{-t}); \tag{5.2}$$

$$\psi(t) = \beta_0(n, \alpha)^{1/p'} (|E|e^{-t})^{1/p'} k_{\zeta,\alpha}^*(|E|e^{-t}). \tag{5.3}$$

It is now easy to check that

$$\begin{aligned} \int_0^{|E|e^{-t}} f^*(s) \, ds \int_0^{|E|e^{-t}} k_{\zeta,\alpha}^*(s) \, ds &= \frac{|E|}{\beta_0(n, \alpha)^{1/p'}} \int_t^\infty e^{-s/p'} \phi(s) \, ds \int_t^\infty e^{-s/p} \psi(s) \, ds; \\ \int_{|E|e^{-t}}^\infty f^*(s) k_{\zeta,\alpha}^*(s) \, ds &= \frac{1}{\beta_0(n, \alpha)^{1/p'}} \int_{-\infty}^t \phi(s) \psi(s) \, ds. \end{aligned}$$

Putting the above quantities in (5.1) we get that

$$\frac{1}{|E|} \int_E \exp(\beta_0(n, \alpha) |u(x)|^{p'}) \, dx$$

$$\begin{aligned} &\leq \int_0^\infty \exp\left(-t + \left|e^t \int_t^\infty e^{-s/p'} \phi(s) ds \int_t^\infty e^{-s/p} \psi(s) ds + \int_{-\infty}^t \phi(s) \psi(s) ds\right|^{p'}\right) dt \\ &= \int_0^\infty e^{-F(t)} dt, \end{aligned} \tag{5.4}$$

where

$$F(t) = t - \left(e^t \int_t^\infty e^{-s/p'} \phi(s) ds \int_t^\infty e^{-s/p} \psi(s) ds + \int_{-\infty}^t \phi(s) \psi(s) ds\right)^{p'}.$$

We now set

$$\begin{aligned} a(s, t) &= \psi(s), & s < t; \\ &= e^t \left(\int_t^\infty e^{-r/p} \psi(r) dr\right) e^{-s/p'}, & s > t. \end{aligned} \tag{5.5}$$

Then, by (5.4) we have

$$\frac{1}{|E|} \int_E \exp\left(\beta_0(n, \alpha)|u(x)|^{p'}\right) dx \leq \int_0^\infty e^{-F(t)} dt,$$

where

$$F(t) = t - \left(\int_{\mathbb{R}} a(s, t) \phi(s) ds\right)^{p'}. \tag{5.6}$$

Now, we prove that there exists C independent of u such that $\int_0^\infty e^{-F(t)} dt \leq C$. The proof is inspired by similar ideas used by Adams [1, Lemma 1] and has been carried out in details in [24]. For the sake of completeness, we sketch the proof. First, notice that

$$\int_0^\infty e^{-F(t)} dt = \int_{\mathbb{R}} |E_\lambda| e^{-\lambda} d\lambda,$$

where $E_\lambda = \{t \geq 0 : F(t) \leq \lambda\}$ and $|E_\lambda|$ is the Lebesgue measure of E_λ . It is enough to show the following two facts:

- (i) There exists a constant $c \geq 0$ which is independent of ϕ such that $\inf_{t \geq 0} F(t) \geq -c$.
- (ii) There exist constants B_1 and B_2 which are both independent of ϕ and λ such that $|E_\lambda| \leq B_1|\lambda| + B_2$.

We first prove (i). We set $L(t) = \left(\int_t^\infty \phi(s)^p ds\right)^{1/p}$. By the definition (5.2) of ϕ , it follows that

$$\int_{-\infty}^t \phi(s)^p ds = \int_{\mathbb{R}} \phi(s)^p ds - L(t)^p = \|f\|_p^p - L(t)^p \leq 1 - L(t)^p.$$

By using the above estimate and Hölder’s inequality, it follows from (5.6) that if $t \in E_\lambda$,

$$\begin{aligned}
 t - \lambda &\leq \left[\int_{\mathbb{R}} a(s, t) \phi(s) ds \right]^{p'} \\
 &= \left[\int_{-\infty}^t a(s, t) \phi(s) ds + \int_t^{\infty} a(s, t) \phi(s) ds \right]^{p'} \\
 &\leq \left[\left(\int_{-\infty}^t a(s, t)^{p'} ds \right)^{1/p'} (1 - L(t)^p)^{1/p} + \left(\int_t^{\infty} a(s, t)^{p'} ds \right)^{1/p'} L(t) \right]^{p'} \\
 &= \left[\left(\int_{-\infty}^t \psi(s)^{p'} ds \right)^{1/p'} (1 - L(t)^p)^{1/p} \right. \\
 &\quad \left. + e^t \left(\int_t^{\infty} e^{-r/p} \psi(r) dr \right) \left(\int_t^{\infty} e^{-s} ds \right)^{1/p'} L(t) \right]^{p'}. \tag{5.7}
 \end{aligned}$$

By the definition (5.3) of ψ , the estimate given in Lemma 4.2 and the fact that $p = n/\alpha$ we have

$$\psi(t) = \beta_0(n, \alpha)^{1/p'} (|E|e^{-t})^{1/p'} k_{\zeta, \alpha}^* (|E|e^{-t}) \leq 1 + \mathcal{O}\left(e^{-\frac{\zeta t}{n}}\right), \text{ for all } t > 0. \tag{5.8}$$

Let $\zeta > 0$ if $1 < p < 2$ and $\zeta > 2|\rho| (\frac{1}{2} - \frac{1}{p})$ if $p \geq 2$. We choose $\zeta' \in (0, \zeta)$ with $\zeta - \zeta'$ small enough such that ζ' satisfies the same properties as ζ , that is, $\zeta' > 0$ if $1 < p < 2$ and $\zeta' > 2|\rho| (\frac{1}{2} - \frac{1}{p})$ if $p \geq 2$. Then by Lemma 4.3 we have

$$\begin{aligned}
 \int_{-\infty}^0 \psi(s)^{p'} ds &= \beta_0(n, \alpha) \int_{|E|}^{\infty} \left(k_{\zeta, \alpha}^*(t)\right)^{p'} dt \\
 &\lesssim \int_{|E|}^{\infty} \left(t^{-1/2-\zeta'/2|\rho|} (\log t)^{2|\rho|/(\zeta'+|\rho|)}\right)^{p'} dt < \infty.
 \end{aligned}$$

Therefore, using (5.8) and the above estimate we have

$$\begin{aligned}
 \int_{-\infty}^t \psi(s)^{p'} ds &= \int_{-\infty}^0 \psi(s)^{p'} ds + \int_0^t \psi(s)^{p'} ds \\
 &\leq b_1 + \int_0^t \left(1 + \mathcal{O}(e^{-\tilde{\epsilon}s/n})\right)^{p'} ds \leq b_2 + t, \tag{5.9}
 \end{aligned}$$

and

$$\begin{aligned}
 e^t \left(\int_t^{\infty} e^{-r/p} \psi(r) dr \right) \left(\int_t^{\infty} e^{-s} ds \right)^{1/p'} &\leq e^t \int_t^{\infty} e^{-r/p} \left(1 + \mathcal{O}(e^{-\tilde{\epsilon}r/n})\right) dr e^{-t/p'} \\
 &\leq C \int_t^{\infty} e^{-(r-t)/p} dr = b_3 < \infty, \tag{5.10}
 \end{aligned}$$

where the constants b_1, b_2 and b_3 are independent of ϕ . Using the estimates (5.9), (5.10) it follows from (5.7) that

$$t - \lambda \leq \left[(b_2 + t)^{1/p'} (1 - L(t)^p)^{1/p} + b_3 L(t) \right]^{p'}$$

The rest of the proof of (a) is similar to that in [1] (see the proof after eqn.(16) in [1]). Since the proof of (b) is the same as that in [24] we will omit here.

The sharpness of the constant $\beta_0(n, \alpha)$ can be verified by the process similar to that in [1, 19, 24, 29] and thus the proof of Theorem 1.3 is completed. \square

Proof of Theorem 1.4 Let $u \in W^{\alpha,p}$ with $\int_X |(-\Delta - |\rho|^2 + \zeta^2)^{\alpha/2} u(x)|^p dx \leq 1$. By Corollary 2.4, we have

$$\int_X |u(x)|^p dx \leq S_p \int_X |(-\Delta - |\rho|^2 + \zeta^2)^{\alpha/2} u(x)|^p dx \leq S_p,$$

provided $\zeta > 2|\rho||1/2 - 1/p|$. We now set $\Omega(u) = \{x \in X : |u(x)| \geq 1\}$. Then from the above inequality it follows that

$$|\Omega(u)| = \int_{\Omega(u)} dx \leq \int_X |u(x)|^p dx \leq S_p.$$

Therefore, we have $|\Omega(u)| \leq S_p$, which is independent of u satisfying $\|(-\Delta - |\rho|^2 + \zeta^2)^{\alpha/2} u\|_p \leq 1$ provided $\zeta > 2|\rho||1/2 - 1/p|$. We now write

$$\begin{aligned} & \int_X \Phi_p \left(\beta_0(n, \alpha) |u(x)|^{p'} \right) dx \\ &= \int_{\Omega(u)} \Phi_p \left(\beta_0(n, \alpha) |u(x)|^{p'} \right) dx + \int_{X \setminus \Omega(u)} \Phi_p \left(\beta_0(n, \alpha) |u(x)|^{p'} \right) dx. \end{aligned}$$

We now notice that $j_p = p$ if p is an integer and $j_p = [p] + 1$ if p is not an integer. Therefore, $(j_p - 1)p' \geq p$ for all $p > 1$. We also notice that on the domain $X \setminus \Omega(u)$, $|u(x)| < 1$. Thus

$$\begin{aligned} \int_{X \setminus \Omega(u)} \Phi_p \left(\beta_0(n, \alpha) |u(x)|^{p'} \right) dx &\leq \sum_{k=j_p-1}^{\infty} \frac{\beta_0(n, \alpha)^k}{k!} \int_{X \setminus \Omega(u)} |u(x)|^{p'k} dx \\ &\leq \sum_{k=j_p-1}^{\infty} \frac{\beta_0(n, \alpha)^k}{k!} \int_{X \setminus \Omega(u)} |u(x)|^p dx \\ &\leq \sum_{k=j_p-1}^{\infty} \frac{\beta_0(n, \alpha)^k}{k!} \|u\|_p^p \leq C_1. \end{aligned} \tag{5.11}$$

Since $\zeta > 2|\rho||1/p - 1/2|$, by Theorem 1.3 there exists $C_2 > 0$ independent of u such that

$$\int_{\Omega(u)} \Phi_p \left(\beta_0(n, \alpha) |u(x)|^{p'} \right) dx \leq \int_{\Omega(u)} \exp \left(\beta_0(n, \alpha) |u(x)|^{p'} \right) dx \leq C_2. \tag{5.12}$$

Combining equations (5.11) and (5.12) it follows that

$$\int_X \Phi_p \left(\beta_0(n, \alpha) |u(x)|^{p'} \right) dx \leq C_1 + C_2 = C,$$

for all u satisfying $\|(-\Delta - |\rho|^2 + \zeta^2)^{\alpha/2}\|_p \leq 1$ provided $\zeta > 2|\rho||1/2 - 1/p|$. The sharpness of the constant $\beta_0(n, \alpha)$ can be verified by the process similar to that in the proof of Theorem 1.3. □

Lemma 5.1 *Let $n \geq 3$, $\zeta > 0$ and $0 < 2s < \min\{l + 2|\Sigma_0^+|, n\}$. Then for $2 < q \leq \frac{2n}{n-2s}$ there exists $C = C(n, s, q, \zeta)$ such that for all $u \in W^{n/2,2}(X)$*

$$\int_X |(-\Delta - |\rho|^2)^{s/2} (-\Delta - |\rho|^2 + \zeta^2)^{(n-2s)/4} u(x)|^2 dx \geq C \|u\|_q^2.$$

Proof By the Plancherel formula (2.8) and the Poincaré-Sobolev inequality (Theorem 2.5), it follows that

$$\begin{aligned} & \int_X |(-\Delta - |\rho|^2)^{s/2} (-\Delta - |\rho|^2 + \zeta^2)^{(n-2s)/4} u(x)|^2 dx \\ &= c_G \int_{\mathfrak{a}_+^* \times K} |\lambda|^{2s} (|\lambda|^2 + \zeta^2)^{(n-2s)/2} |\tilde{u}(\lambda, k)|^2 |\mathfrak{c}(\lambda)|^{-2} d\lambda dk \\ &\geq \zeta^{n-2s} c_G \int_{\mathfrak{a}_+^* \times K} |\lambda|^{2s} |\tilde{u}(\lambda, k)|^2 |\mathfrak{c}(\lambda)|^{-2} d\lambda dk \\ &= \zeta^{n-2s} \int_X |(-\Delta - |\rho|^2)^{s/2} u(x)|^2 dx \geq C \|u\|_q^2, \quad 2 < q \leq \frac{2n}{n-2s}. \end{aligned}$$

This completes the proof. □

Proof of Theorem 1.6 Let $u \in W^{n/2,2}(X)$ satisfying (1.2). We choose some q_0 satisfying $2 < q_0 \leq \min\{2n/(n-2s), 4\}$. Then by Lemma 5.1 we have

$$\|u\|_{q_0}^2 \leq C_0 \int_X |(-\Delta - |\rho|^2)^{s/2} (-\Delta - |\rho|^2 + \zeta^2)^{(n-2s)/4} u(x)|^2 dx \leq C.$$

We now set $\Omega(u) = \{x \in S : |u(x)| \geq 1\}$, then

$$|\Omega(u)| = \int_{\Omega(u)} dx \leq \int_X |u(x)|^{q_0} dx \leq C^{q_0/2},$$

where the constant $C^{q_0/2}$ is independent of u . Since $q_0 \leq 4$, it follows that

$$\begin{aligned} & \int_{X \setminus \Omega(u)} \left[\exp \left(\beta_0(n, n/2) u(x)^2 \right) - 1 - \beta_0(n, n/2) u(x)^2 \right] dx \\ &= \sum_{k=2}^{\infty} \frac{\beta_0(n, n/2)^k}{k!} \int_{X \setminus \Omega(u)} u(x)^{2k} dx \end{aligned}$$

$$\leq \sum_{k=2}^{\infty} \frac{\beta_0(n, n/2)^k}{k!} \int_{X \setminus \Omega(u)} |u(x)|^{q_0} dx < \infty. \tag{5.13}$$

Next, we show that $\int_{\Omega(u)} \exp(\beta_0(n, n/2)u(x)^2) dx$ is bounded by some constant independent of u . We rewrite

$$v = (-\Delta - |\rho|^2)^{s/2}(-\Delta - |\rho|^2 + \zeta^2)^{(n-2s)/4} u.$$

Then $\|v\|_2 \leq 1$ and $u = v * (k_s * k_{\zeta, (n-2s)/2})$. By Lemma 4.1, the kernel $k_s * k_{\zeta, (n-2s)/2}$ satisfies

$$[k_s * k_{\zeta, (n-2s)/2}]^*(t) \leq \frac{1}{\gamma(n/2)} \cdot \left(\frac{nt}{\omega_{n-1}}\right)^{-1/2} + O(t^{-1/2+\epsilon'/n}), \text{ for } 0 < t < 1,$$

and by Lemma 4.5, for each $a > 0$

$$\int_a^{\infty} |[k_s * k_{\zeta, (n-2s)/s}]^*(t)|^2 dt < \infty.$$

Following the proof of Theorem 1.3, we can find a constant C independent of u such that

$$\begin{aligned} & \int_{\Omega(u)} \exp(\beta_0(n, n/2)u(x)^2) dx \\ &= \int_{\Omega(u)} \exp(\beta_0(n, n/2)[v * (k_s * k_{\zeta, (n-2s)/2})]^2) dx \leq C. \end{aligned}$$

Combining equation (5.13) with the above inequality we complete the proof.

The sharpness of the constant $\beta_0(n, \alpha)$ can be verified by the process similar to that in the proof of Theorem 1.3 □

Proof of Theorem 1.8 Let $u \in C_c^\infty(X)$ with

$$\int_X |(-\Delta)^{s/2}(-\Delta - |\rho|^2 + \zeta^2)^{(n-2s)/4} u(x)|^2 dx - |\rho|^{2s} \zeta^{n-2s} \int_X |u(x)|^2 dx \leq 1.$$

The Plancherel formula (2.8) yields

$$\begin{aligned} & \int_X |(-\Delta - |\rho|^2)^{s/2} (-\Delta - |\rho|^2 + \zeta^2)^{(n-2s)/4} u(x)|^2 dx \\ &= c_G \int_{\mathfrak{a}_+^* \times K} |\lambda|^{2s} (|\lambda|^2 + \zeta^2)^{(n-2s)/2} |\tilde{u}(\lambda, k)|^2 |\mathbf{c}(\lambda)|^{-2} d\lambda dk \\ &\leq c_G \int_{\mathfrak{a}_+^* \times K} \left[(|\lambda|^2 + |\rho|^2)^s (|\lambda|^2 + \zeta^2)^{(n-2s)/2} - \zeta^{n-2s} |\rho|^{2s} \right] |\tilde{u}(\lambda, k)|^2 |\mathbf{c}(\lambda)|^{-2} d\lambda dk \\ &= \int_X |(-\Delta)^{s/2}(-\Delta - |\rho|^2 + \zeta^2)^{(n-2s)/4} u(x)|^2 dx - \zeta^{n-2s} |\rho|^{2s} \int_X |u(x)|^2 dx \leq 1. \end{aligned}$$

Therefore, by Theorem 1.6 we complete the proof. \square

We conclude the paper with the following remark.

Remark 5.2 We recall that Damek–Ricci spaces are non-symmetric generalization of rank one Riemannian symmetric spaces. Though symmetric spaces are the most important prototypes, they form a very small subclass of the set of all Damek–Ricci spaces (see [4]). A Damek–Ricci space is a Riemannian manifold and a solvable Lie group but in general not a symmetric space, i.e. cannot be realized as a quotient space G/K , for a semisimple Lie group G . It will be interesting to see whether Adams-type inequalities can be proved in the context of Damek–Ricci spaces.

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