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Eccentric graph of trees and their Cartesian products

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ABSTRACT

Let G be an undirected simple connected graph. We say a vertex u is eccentric to a vertex v in G if $d(u,v) = \max\{d(v,w) : w \in V(G)\}$. The eccentric graph of G, say $Ec(G)$, is a graph defined on the same vertex set as of G and two vertices are adjacent if one is eccentric to the other. We find the structure and the girth of the eccentric graph of trees and see that the girth of the eccentric graph of a tree can either be zero, three, or four. Further, we study the structure of the eccentric graph of the Cartesian product of graphs and prove that the girth of the eccentric graph of the Cartesian product of trees can only be zero, three, four or six. Furthermore, we provide a comprehensive classification when the eccentric girth assumes these values. We also give the structure of the eccentric graph of the grid graphs and the Cartesian product of two cycles. Finally, we determine the conditions under which the eccentricity matrix of the Cartesian product of trees becomes invertible.

1. Introduction

Let G be a simple undirected graph on n vertices with m edges and $V(G)$ denote the set of vertices in G. If two vertices $v, w \in V(G)$ are adjacent, we will write $v \sim_G w$. The *neighbourhood* of a vertex v in G is defined as $N_G(v) = \{w : v \sim_G w\}$. If the graph G is connected, the *distance* $d_G(v,w)$, between two vertices v and w is the length of the shortest path in G connecting them. The *distance matrix* of a connected graph G, denoted as $D(G)$, is the $n \times n$ matrix indexed by $V(G)$ whose (v,w)th-entry is equal to $d_G(v,w)$. We will only consider simple, undirected graphs on *at least* two vertices in this paper.

The *eccentricity*, $e_G(v)$, of a vertex $v \in V(G)$ is defined as

$$e_G(v) = \max\{d(u,v) : u \in V(G)\},$$

we will use $e(v)$ instead of $e_G(v)$ whenever there is no confusion about the underlying graph. If $d(u,v) = e(v)$, then we will say u is *eccentric* to v and a shortest path between u and v is called an *eccentric path* (starting from v). The *diameter* of G, $diam(G)$, is the maximum of eccentricities of the vertices in G. A *diametrical path* is a longest path among all eccentric paths in the graph G.

The *eccentricity matrix* of a connected graph G, denoted by $\mathcal{E}_G$, is constructed from the distance matrix $D(G)$, retaining the largest distances in each row and each column, while other elements of the distance matrix are set to zero. In other words,

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$$(\mathcal{E}_G)_{ij} = \begin{cases} d(u_i, u_j) & \text{if } d(u_i, u_j) = \min\{e(u_i), e(u_j)\}, \\ 0 & \text{otherwise.} \end{cases}$$

Definition 1. The *eccentric graph* [1] of a connected graph G is the simple graph with the vertex set same as that of G and uv is an edge in $Ec(G)$ if either v is eccentric to u or u is eccentric to v. In that case, we call u and v are adjacent in $Ec(G)$ and denote it as $u \sim_{Ec(G)} v$.

Note that the adjacency matrix of the eccentric graph $Ec(G)$ is obtained by replacing the non-zero entries in the eccentricity matrix $\mathcal{E}_G$, by 1.

Recall the girth of a graph G is the length of the shortest cycle present in G. If a graph G has no cycles, we will say that G has girth 0. We will call the girth of the eccentric graph as *eccentric girth* and denote it as $g(Ec(G))$. Girth is the dual concept to edge connectivity, in the sense that the girth of a planar graph is the edge connectivity of its dual graph, and vice versa. Calculating the girth of a graph is an important task in graph theory, as it helps us understand the graph's structure and properties.

The notion of eccentricity matrix was first introduced by Randić as the D_{max}-matrix in 2013 [6] and subsequently, Wang et al. renamed it as the eccentricity matrix in 2018 [8]. The eccentricity matrix of a graph is also called as anti-adjacency matrix in the following sense. The eccentricity matrix is obtained from the distance matrix by preserving only the largest distances in each row and column; on the other hand, the adjacency matrix is obtained from the distance matrix by preserving only the smallest non-zero distances in each row and column. Unlike the adjacency matrix and the distance matrix, the eccentricity matrix of a connected graph need not be irreducible. The eccentricity matrix of a complete bipartite graph is reducible and the eccentricity matrix of a tree is irreducible [8,4].

Spectra of the eccentricity matrix for some graphs are studied by Mahato et al. [4] and Wang et al. [8], the lower and upper bounds for the $\mathcal{E}$-spectral radius of graphs are also discussed in [8]. J. Wang et al. studied the non-isomorphic co-spectral graphs with respect to the eccentricity matrix [9]. Eccentricity matrix has interesting applications, its main application is in the field of chemical graph theory [6,7]. In other direction, some eccentricity based indices have also been studied. Xu et al. [11] has obtained bounds on the non-self-centrality number (NSC number) of a graph G. The bounds for the difference of the eccentric connectivity index (ECI) and the connective eccentricity index (CEI) of a tree have been studied and the corresponding extremal trees have also been classified [10].

A necessary and sufficient condition for $Ec(G)$ to be isomorphic to G or the complement of G is given by Akiyama et al. [1]. Kaspar et al. gave complete structure of the eccentric graph for some well-known graphs like paths and cycles [2]. A *star graph* S_n on $(n+1)$ vertices is a graph with n vertices of degree 1 and one vertex, called the center, of degree n. A *double star* $S_{s,t}$ is a graph obtained by adding an edge between the center vertices of two stars S_s and S_t. Let P_n denote the *path* graph on n vertices with the natural labelling $1, 2, \ldots, n$. Then,

$$Ec(P_n) = \begin{cases} K_n, \text{ the complete graph} & \text{if } n \leq 3, \\ S_{\frac{n-2}{2}, \frac{n-2}{2}}, \text{ a double star} & \text{if } n > 3 \text{ is even}, \\ H_{\frac{n-3}{2}} & \text{if } n > 3 \text{ is odd}, \end{cases}$$

where K_t denotes the complete graph on t vertices and H_t is a graph obtained by adding t pendant vertices to each of any two of the vertices of a triangle (see Fig. 1).

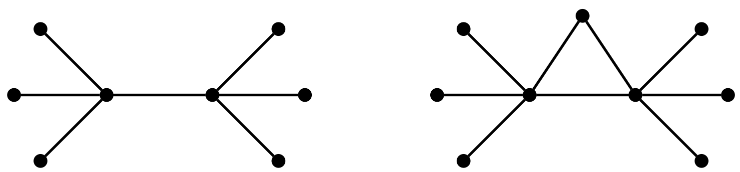


Fig. 1. Eccentric graphs of the path graphs P_8 and P_9 ($S_{3,3}$ and H_3).

Let C_n denote the cycle graph on n vertices and the vertices are labeled as $1, 2, \ldots, n$. Then,

$$Ec(C_n) = \begin{cases} \frac{n}{2} K_2 & \text{if } n \text{ is even}, \\ C_n & \text{if } n \text{ is odd}. \end{cases} \tag{1}$$

Also, $Ec(K_n) = K_n$ and $Ec(K_{s,t}) = K_s \cup K_t$ for $s, t > 1$ [2]. Throughout the paper, we will use the notation P_n and C_n to denote the path graph and the cycle graph on n vertices.

Numerous interesting properties of the eccentricity matrix of a tree have been established so far. For instance, Mahato showed that the eccentricity matrix of a tree is invertible only if the tree is a star [3]. Additionally, the diameter of the tree is odd if and only if eigenvalues of its eccentricity matrix are symmetric about the origin [5].

In Section 2, we will give a complete structure of the eccentric graph of a general tree and point out one more structural information in Proposition 1. In Section 3, we will prove that the eccentric girth of a tree can either be zero, three or four. In Section 4, we will present some structural properties of the eccentric graph of the Cartesian product of graphs and classify all the possible values of the eccentric girth of the Cartesian product of trees.

Lastly, in section 5, generalising the result of Mahato [3, Theorem 2.1], we will analyze and classify the conditions under which the eccentricity matrix of the Cartesian product of trees becomes invertible.

2. Structure of eccentric graph of a tree

In this section, we will focus on the structure of the eccentric graph of a tree. Recall that a *tree* is a connected graph with no cycles and the *degree* of a vertex v in a simple graph G is the number of vertices adjacent to it. A vertex of degree 1 is called a *leaf* or a *pendant* vertex. The *union* of two graphs G_1 and G_2 is the simple graph whose vertex set and edge set are formed by taking the union of the vertex sets of G_1 and G_2 and the edge sets of G_1 and G_2, respectively.

Definition 2. Let T be a tree and v be a leaf in T. We define the path from v to the nearest vertex of degree greater than two as the *stem* at v and the *branching vertex* is an endpoint of the stem which has degree greater than two in T.

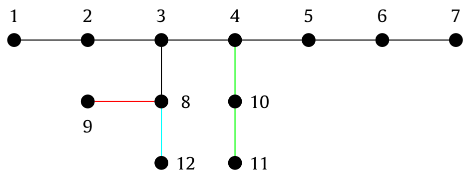


Fig. 2. A tree T on 12 vertices with different coloured stems at vertices 9, 11 and 12.

Note that a path graph P_n has no stems.

Definition 3. Let P be a diametrical path in a tree T. We define the *tree induced from the path* P as the subtree of T obtained by removing stems except branching vertices at those leaves (except endpoints of P), which are an endpoint of some diametrical path other than P.

Consider the tree T shown in Fig. 2. T has three diametrical paths and the subtrees induced by these are shown in Fig. 3.

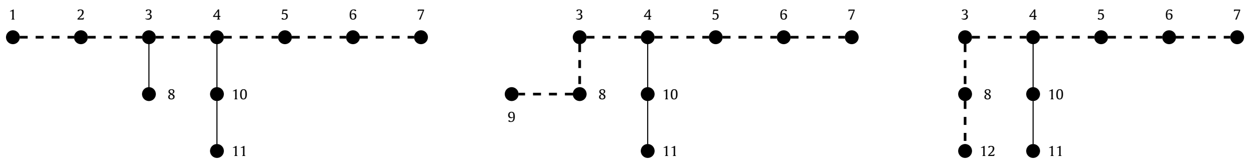


Fig. 3. Subtrees induced by different diametrical paths (dashed) of the tree in Fig. 2.

Note that the structure of the eccentric graph of a subtree induced from a diametrical path in T depends on the diameter of T. In case of an even diameter, it looks as shown in the left of Fig. 4 and in case of odd diameter, it looks as shown in the right of Fig. 4.

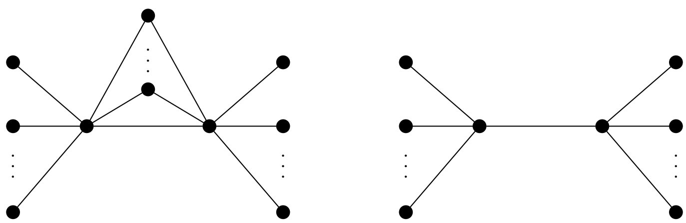


Fig. 4. Eccentric graphs of subtrees induced by diametrical paths.

For example, the eccentric graphs of the subtrees in Fig. 3 have been shown in Fig. 5.

The following result shows that the graphs shown in Fig. 4 are the building blocks for the eccentric graph of a tree.

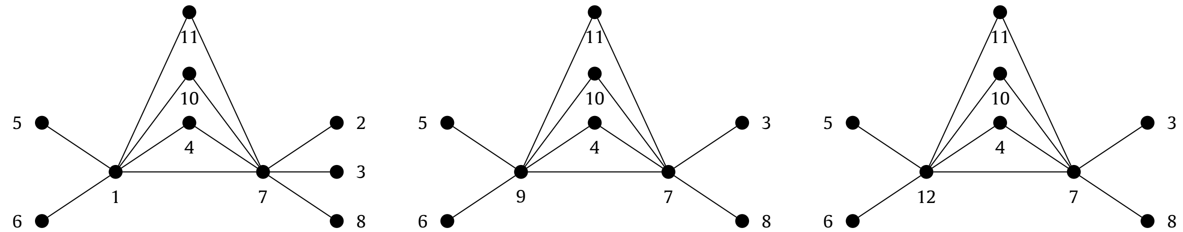


Fig. 5. Eccentric graphs of the three subtrees in Fig. 3 of the tree in Fig. 2.

Theorem 1. *Let $Q_1, \cdots, Q_k$ be possible diametrical paths in T with starting point $v_0^1, \dots, v_0^k$ and ending point $v_n^1, \dots, v_n^k$, respectively. Let $T_1, \dots, T_k$ be induced trees from $Q_1, \dots, Q_k$, respectively. Then, $Ec(T) = \cup_{i=1}^k Ec(T_i)$.*

Proof. It is clear that each vertex of T lies in at least one tree induced from a diametrical path.

For $i \in [k]$, let e be an edge in the eccentric graph $Ec(T_i)$. As Q_i is the unique diametrical path in T_i, it follows that one of the endpoints of e is either v_0^i or v_n^i, assume that $e = vv_n^i$. Thus, $e_{T_i}(v) = d_{T_i}(v, v_n^i) = d_T(v, v_n^i) = e_T(v)$. Thus, $Ec(T_i)$ is a subgraph of $Ec(T)$.

Now, let $v \sim_{Ec(T)} w$ which implies that one of v or w (say v) is an endpoint of a diametrical path say Q_j $(1 \leq j \leq k)$ in T. It is enough to show that v and w both lie on the same tree T_s for some $s \in [k]$. If $w \notin Ec(T_j)$, eccentric graph of the tree induced from Q_j, then w lies on a stem at some leaf z in T. In that case, the path joining from v to z is a diametrical path and the tree induced by this diametrical path T_s contains both v and w. $\square$

The following example illustrates Theorem 1.

Example 2. Let T be the tree shown in Fig. 2. The eccentric graph of T (see Fig. 6) is the union of the eccentric graphs (shown in Fig. 5) of the subtrees (shown in Fig. 3) induced from the three diametrical paths of T.

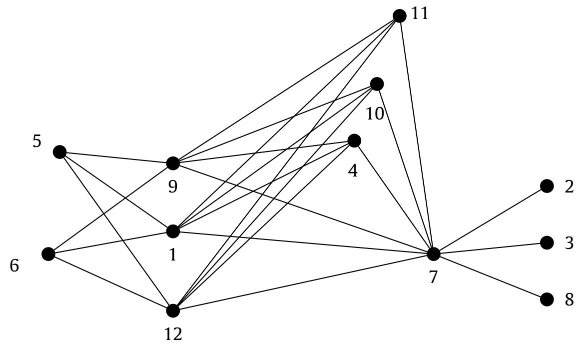


Fig. 6. Eccentric graph of the tree in Fig. 2 which is the union of the graphs in Fig. 5.

In the remaining part of this section, we will highlight more structural information about the eccentric graph of a tree.

Proposition 1. *Let T be a tree. There does not exist $v_1, v_2, v_3 \in V(T)$ such that $v_1 \sim_{Ec(T)} v_2$, $v_2 \sim_{Ec(T)} v_3$ and $e_T(v_1) < e_T(v_2) < e_T(v_3)$.*

Proof. On the contrary, assume that such $v_1, v_2, v_3 \in V(T)$ exist. Then $d_T(v_1, v_2) = \min\{e_T(v_1), e_T(v_2)\} = e_T(v_1)$, i.e., v_2 is eccentric to v_1. Therefore, v_2 is an endpoint of a diametrical path. Again, $d_T(v_2, v_3) = \min\{e_T(v_2), e_T(v_3)\} = e_T(v_2)$, which implies that the path from v_2 to v_3 is a diametrical path and therefore $e_T(v_2) = e_T(v_3)$, a contradiction. $\square$

The essence of Proposition 1 can be summarized as the eccentricity of a vertex $v \in V(T)$ is either the smallest or the largest among the eccentricities of its neighbours in the eccentric graph of T.

3. Eccentric girth of a tree

In this section, we will determine the eccentric girth of a tree and its potential values. In addition, we will classify the instances in which these possible values of the eccentric girth can be achieved. It is well-known that two paths of maximum length must pass through a common point. Thus, it is evident that two diametrical paths in a tree must intersect. But this is not true in general, the graph in Fig. 7 has two diametrical paths (dashed) but they do not intersect.

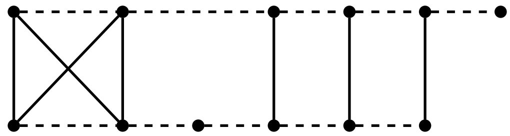


Fig. 7. A graph having two non-intersecting diametrical paths.

Now, we will present the main result of this section which classifies the eccentric girth of a tree.

Theorem 3. *Let T be a tree. Then the girth of the eccentric graph of T is either zero, three, or four. Moreover,*

$$g(Ec(T)) = \begin{cases} 3 & \text{if the diameter of } T \text{ is even,} \\ 0 & \text{if the diameter of } T \text{ is odd with unique diametrical path,} \\ 4 & \text{otherwise.} \end{cases}$$

Proof. The proof is divided into the following cases depending on the parity of the diameter of T.

First, let the diameter of T be even and $P = v_0\, v_1 \dots v_k\, v_{k+1} \dots v_{2k}$ be a diametrical path. Note that $e(v_0) = 2k = e(v_{2k})$ and $d(v_0, v_{2k}) = 2k$, therefore $v_0 \sim_{Ec(T)} v_{2k}$. If $e(v_k) > k$, then one of $e(v_0)$ or $e(v_{2k})$ will be greater than $2k$, which is not possible. Also, $d(v_0, v_k) = k = d(v_k, v_{2k})$, therefore $e(v_k) = k$ and $v_k \sim_{Ec(T)} v_0$, $v_k \sim_{Ec(T)} v_{2k}$. Thus, v_0, v_k, and v_{2k} form a triangle in $Ec(T)$.

Second, if the diameter of T is odd and $P = v_0\, v_1 \dots v_k\, v_{k+1} \dots v_{2k+1}$ is the unique diametrical path in T. It is sufficient to show that for any vertex $i \in V(T)$ exactly one of v_0 or v_{2k+1} is eccentric to i and no other vertex is eccentric to i. Note that, in a tree, if a vertex j is eccentric to some vertex, then j must be a pendant vertex.

Let $i \in V(P)$, if possible, there exists a vertex $j \in V(T)$ other than v_0 and v_{2k+1} which is eccentric to i, that is, $d(i, j) = e(i)$, then j is a leaf of a branch emerging from some vertex $p \in V(P)$. Assume that p is on the left of i in P, then $d(i, j) \geq d(i, v_0)$, which implies $d(v_{2k+1}, j) = d(v_{2k+1}, i) + d(i, j) \geq d(v_{2k+1}, i) + d(i, v_0) = 2k + 1$, which contradicts the fact that P is the only diametrical path. A similar argument can be given when p is on the right of i.

Now suppose that $i \in V(T) \setminus V(P)$ lies on some branch emerging from a vertex $i' \in V(P)$. Again let there exists $j \in V(T)$ other than v_0 and v_{2k+1} which is eccentric to i. Note that j cannot lie on the same branch; otherwise, the eccentricity of one of v_0 or v_{2k+1} will increase. Thus, j must be eccentric to i' which cannot happen as proved in the preceding paragraph. Moreover, because of odd diameter, exactly one of v_0 or v_{2k+1} can be eccentric to i. For illustration, $Ec(T)$ in this scenario is shown in Fig. 8.

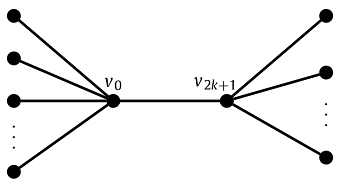


Fig. 8. Eccentric graph of a tree (of odd diameter) with unique diametrical path.

Third, let the diameter of T be odd and $P = v_0\, v_1 \dots v_k\, v_{k+1} \dots v_{2k+1}$, $P' = w_0\, w_1 \dots w_k\, w_{k+1} \dots w_{2k+1}$ be two diametrical paths in T. As mentioned at the start of Section 3, they must intersect. Therefore, it is reasonable to assume that P and P' have one common endpoint say $v_0 = w_0$, otherwise one of the paths joining from v_0 to w_0 or w_{2k+1} (say w_{2k+1}) is a diametrical path and we can create two such diametrical paths by replacing P' with the diametrical path from v_0 to w_{2k+1}. Hence, $(v_0, v_{2k+1}, v_1, w_{2k+1})$ forms a 4-cycle in $Ec(T)$. Now, if there is a triangle (z_1, z_2, z_3) in $Ec(T)$ and $e(z_1) \leq e(z_2) \leq e(z_3)$. Without loss of generality, assume that z_1 is a vertex on some branch emerging from w_p, $1 \leq p \leq k$ (Note that z_1 can be w_0). If z is any vertex eccentric to z_1, then z must be a vertex on some branch emerging from w_i, for some $k + 1 \leq i \leq 2k$; if not, then $d(z, w_{2k+1})$ is greater than the diameter $2k + 1$. Now z_2 being eccentric to z_1 must lie on some branch emerging from w_q, $k + 1 \leq q \leq 2k$ (Note that z_2 can be w_{2k+1}). Again, as z_3 is eccentric to z_2, z_3 is a vertex on some branch emerging from w_r, $1 \leq r \leq k$, but then z_3 cannot be eccentric to z_1. Hence $Ec(T)$ cannot have a triangle. $\square$

4. Eccentric graph of the Cartesian product of graphs

In this section, we will examine some properties of the eccentric graph of the Cartesian product of general graphs and calculate the eccentric girth of the Cartesian product of trees in Section 4.1. We begin by recalling the definition of the *Cartesian product* and the *Kronecker product* of two graphs.

Definition 4. Let G_1 and G_2 be two simple connected graphs. The *Cartesian product* of G_1 and G_2 denoted as $G_1 \Box G_2$ is a graph with vertex set $V(G_1) \times V(G_2)$, and two vertices (u_1, u_2) and (v_1, v_2) are adjacent if and only if either $u_1 = v_1$ and $u_2 \sim_{G_2} v_2$ or $u_1 \sim_{G_1} v_1$ and $u_2 = v_2$.

The following equations follow directly from Definition 4.

$$d_{G_1 \Box G_2}\big((u_1, u_2), (v_1, v_2)\big) = d_{G_1}(u_1, v_1) + d_{G_2}(u_2, v_2), \tag{2}$$

and

$$e_{G_1 \Box G_2}\big((u_1, u_2)\big) = e_{G_1}(u_1) + e_{G_2}(u_2). \tag{3}$$

The above definition and the equations can be generalised to the Cartesian product of k graphs $G_1, \ldots, G_k$ denoted as $G_1 \Box \cdots \Box G_k$.

Definition 5. Let G_1 and G_2 be two simple connected graphs. The *Kronecker product* of G_1 and G_2 denoted as $G_1 \times G_2$ is a graph with vertex set $V(G_1) \times V(G_2)$, and two vertices (u_1, u_2) and (v_1, v_2) are adjacent if and only if $u_1 \sim_{G_1} v_1$ and $u_2 \sim_{G_2} v_2$.

Lemma 1. *Let $G_1, \ldots, G_k$ be simple connected graphs and $G = G_1 \Box \cdots \Box G_k$ be their Cartesian product. Let $u = (u_1, \ldots, u_k)$, $v = (v_1, \ldots, v_k) \in V(G)$ where $u_i, v_i \in V(G_i)$ for $i \in [k]$. Then, v is eccentric to u if and only if v_i is eccentric to u_i for all $i \in [k]$.*

Proof. Let v be eccentric to u, i.e., $d_G\big(u, v\big) = \max\{d_G\big(u, x\big) : x \in V(G)\}$. Then by (2) we can express this as:

$$\sum_{i=1}^{k} d_{G_i}\big(u_i, v_i\big) = \max\left\{\sum_{i=1}^{k} d_{G_i}\big(u_i, x_i\big) : x_i \in V(G_i)\right\}.$$

Which holds only if

$$d_{G_i}\big(u_i, v_i\big) = \max\{d_{G_i}\big(u_i, x_i\big) : x_i \in V(G_i)\} \text{ for all } i \in [k]\}.$$

Thus, v_i is eccentric to u_i for all $i \in [k]$. Furthermore, we can reverse the steps of this argument to establish the converse part. □

Note that if $(u_1, \ldots, u_k) \sim_{Ec(G_1 \Box \cdots \Box G_k)} (v_1, \ldots, v_k)$, then $u_i \neq v_i$ for all $i \in [k]$. Also, it is clear from Lemma 1 that if $u \sim_{Ec(G)} v$, then $u_i \sim_{Ec(G_i)} v_i$ for all $i \in [k]$, but the converse is not true. For example, $1 \sim_{Ec(P_4)} 3$ and $2 \sim_{Ec(P_4)} 4$, but $(1, 2) \nsim_{Ec(P_4 \Box P_4)} (3, 4)$ (see Fig. 9).

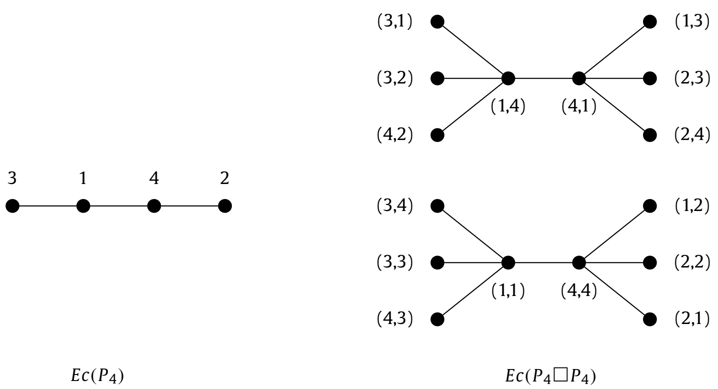


Fig. 9. Eccentric graphs of naturally labelled P_4 and $P_4 \Box P_4$.

Corollary 1. *Let G_1 and G_2 be simple connected graphs such that all the vertices in both G_1 and G_2 have the same eccentricities. Then $Ec(G_1 \square G_2)$ is isomorphic to $Ec(G_1) \times Ec(G_2)$, the Kronecker product of $Ec(G_1)$ and $Ec(G_2)$.*

Lemma 2. *Let $G_1, \ldots, G_k$ be simple connected graphs and $G = G_1 \square \cdots \square G_k$. If for some $s, t \in [k]$ there exists $u_s, v_s, w_s \in V(G_s)$ such that $u_s \sim_{Ec(G_s)} v_s$, $v_s \sim_{Ec(G_s)} w_s$ and $e_{G_s}(v_s) \geq \max\{e_{G_s}(u_s), e_{G_s}(w_s)\}$, and there exists $u_t, v_t, w_t \in V(G_t)$ such that $u_t \sim_{Ec(G_t)} v_t$, $v_t \sim_{Ec(G_t)} w_t$ and $e_{G_t}(v_t) \leq \min\{e_{G_t}(u_t), e_{G_t}(w_t)\}$, then there exists a 4-cycle in $Ec(G)$.*

Proof. Without loss of generality, assume that $s = 1$ and $t = 2$ and for $i = 3, \ldots, k$, let $\{u_i, v_i\}$ be an edge in $Ec(G_i)$ such that $e_{G_i}(u_i) \geq e_{G_i}(v_i)$, i.e., u_i is eccentric to v_i for $i = 3, \ldots, k$. By the inequalities in the hypothesis, v_1 is eccentric to both u_1 and w_1, u_2 is eccentric to v_2 and w_2 is eccentric to v_2. Thus by Lemma 1, $a = (u_1, v_2, v_3, \ldots v_k)$, $b = (v_1, w_2, u_3, \ldots, u_k)$, $c = (w_1, v_2, v_3, \ldots, v_k)$ and $d = (v_1, u_2, u_3, \ldots, u_k)$ form a 4-cycle in $Ec(G)$ (see Fig. 10). □

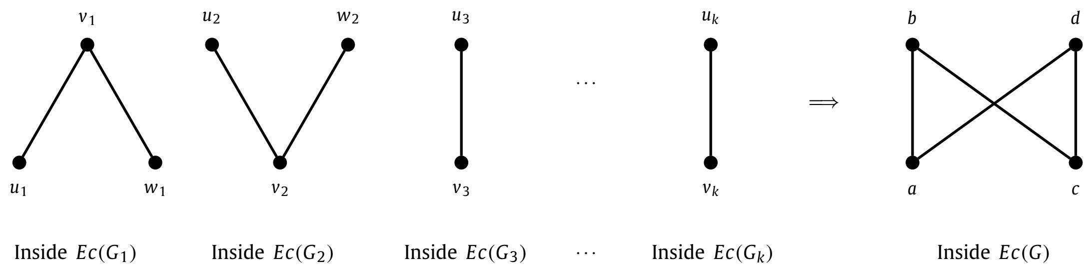


Fig. 10. Formation of 4-cycle in $Ec(G)$.

We will now prove that there is a triangle in the eccentric graph of the Cartesian product of k graphs if and only if there is a triangle in the eccentric graph of each of the individual graphs.

Theorem 4. *Let $G_1, \ldots, G_k$ be simple connected graphs and G be their Cartesian product. Then the girth of $Ec(G)$ is 3 if and only if the girth of $Ec(G_i)$ is 3 for all $i \in [k]$.*

Proof. First, suppose that there is a triangle in $Ec(G_i)$ for all $i \in [k]$. Let $\{u_i, v_i, w_i\}$ be a triangle in $Ec(G_i)$ such that $e_{G_i}(u_i) \leq e_{G_i}(v_i) \leq e_{G_i}(w_i)$ for all $i \in [k]$. In other words, v_i is eccentric to u_i and w_i is eccentric to both u_i and v_i for all i. Therefore by Lemma 1, $(u_1, \ldots, u_k), (v_1, \ldots, v_k)$ and $(w_1, \ldots, w_k)$ form a triangle in $Ec(G)$. Conversely, suppose $(u_1, \ldots, u_k), (v_1, \ldots, v_k)$ and $(w_1, \ldots, w_k)$ form a triangle in $Ec(G)$, then again by Lemma 1, $\{u_i, v_i, w_i\}$ forms a triangle in $Ec(G_i)$ for all $i \in [k]$. □

Theorem 5. *Let $G_1, \ldots, G_k$ be simple connected graphs such that the eccentric girths of at least two of them are greater than two. Let $G = G_1 \square \cdots \square G_k$, then the girth of $Ec(G)$ is four except when the girth of $Ec(G_i)$ is exactly three for all $i \in [k]$.*

Proof. Suppose that $Ec(G_1)$ and $Ec(G_2)$ have girths greater than two and C_1 and C_2 are cycles in $Ec(G_1)$ and $Ec(G_2)$, respectively. Let v_1 be a vertex of the largest eccentricity on C_1 and v_2 be a vertex of the smallest eccentricity on C_2. In particular, if u_1, w_1 are neighbours of v_1 in C_1 and u_2, w_2 are neighbours of v_2 in C_2, then

$$e_{G_1}(v_1) \geq \max\{e_{G_1}(u_1), e_{G_1}(w_1)\} \text{ and } e_{G_2}(v_2) \leq \min\{e_{G_2}(u_2), e_{G_2}(w_2)\}.$$

Hence, the result follows from Theorem 4 and Lemma 2. □

Based on the above-stated theorems, it can be concluded that the eccentric girth of the Cartesian product of graphs in which at least two have non-zero eccentric girth is either three or four.

4.1. *Eccentric girth of the Cartesian product of trees*

Recall that in section 3, we observed that the eccentric girth of a tree could either be zero, three or four. Now, we will prove that for the Cartesian product of trees, it can also be six in addition to the above values. We will now characterize completely the eccentric girth of the Cartesian product of trees and present an analogous result to Theorem 3.

Theorem 6. *Let $T_1, \dots, T_k$ be trees and $G = T_1 \square \cdots \square T_k$. Then,*

$$g(Ec(G)) = \begin{cases} 0 & \text{if the girth of } Ec(T_i) = 0 \text{ for all } i \in [k], \\ 3 & \text{if the girth of } Ec(T_i) = 3 \text{ for all } i \in [k], \\ 6 & \text{if } G = T_1 \square P_2 \square \cdots \square P_2 \text{ and } Ec(T_1) \text{ is } C_4\text{-free with girth three}, \\ 4 & \text{otherwise.} \end{cases}$$

Proof. First, assume that $T_1, \dots, T_k$ are trees with eccentric girth 0. By Theorem 3, there exists a unique diametrical path of odd length in T_i with endpoints u_i and v_i for all $i \in [k]$. Consider the set of vertices $S = \{(x_1, \dots, x_k) : x_i \in \{u_i, v_i\}, i \in [k]\}$ in $V(G)$. Any vertex $u \in V(G) \setminus S$ cannot be eccentric to anyone in G and the vertices eccentric to u lie in S. Moreover, exactly one vertex in S is eccentric to u because each of the $T_i's$ has a unique diametrical path. Consequently, u is adjacent to exactly one vertex in the eccentric graph $Ec(G)$. Also, note that any two vertices in S are adjacent if and only if they differ at each component, therefore $Ec(G)$ is an acyclic graph with 2^{k-1} connected components.

Second, only one of $T_i's$ say T_1 has non-zero eccentric girth. Now there are two cases, one is when at least one of T_i, $i = 2, \dots, k$, is not P_2 and the other is $T_i = P_2$ for all $i = 2, \dots, k$.

If suppose that $T_2 \neq P_2$, and since $Ec(T_2)$ has girth zero, by Theorem 3 there exists a unique diametrical path with endpoints u_2 and v_2 and $u_2 \sim_{Ec(T_2)} v_2$. Now, as $T_2 \neq P_2$ and $Ec(T_2)$ is connected [8], there is a vertex w_2, adjacent to either u_2 or v_2, let's say $v_2 \sim_{Ec(T_2)} w_2$. Clearly, $e_{T_2}(v_2) \geq \max\{e_{T_2}(u_2), e_{T_2}(w_2)\}$. Additionally, as the girth of $Ec(T_1)$ is nonzero (it is either 3 or 4 by Theorem 3), it is possible to choose $u_1, v_1, w_1 \in V(T_1)$ such that $u_1 \sim_{Ec(T_1)} v_1$, $v_1 \sim_{Ec(T_1)} w_1$ and $e_{T_1}(v_1) \leq \min\{e_{T_1}(u_1), e_{T_1}(w_1)\}$. Therefore by Lemma 2 and Theorem 4, the girth of $Ec(G)$ is four.

Let $T_i = P_2$ with endpoints $\{u_i, v_i\}$ for $i = 2, \dots, k$. If $Ec(T_1)$ contains a 4-cycle, $\{u_1, v_1, w_1, x_1\}$, then $\{(u_1, u_2, \dots, u_k), (v_1, v_2, \dots, v_k), (w_1, u_2, \dots, u_k), (x_1, v_2, \dots, v_k)\}$ forms a 4-cycle in $Ec(G)$. Therefore the girth of $Ec(G)$ is four as $Ec(G)$ can not contain any odd cycle (because $T_2 = P_2$). If $Ec(T_1)$ doesn't contain a 4-cycle, then by Theorem 3, girth of $Ec(T_1)$ is 3. Let $\{u_1, v_1, w_1\}$ be a 3-cycle in $Ec(T_1)$ then $\{(u_1, u_2, \dots, u_k), (v_1, v_2, \dots, v_k), (w_1, u_2, \dots, u_k), (u_1, v_2, \dots, v_k), (v_1, u_2, \dots, u_k), (w_1, v_2, \dots, v_k)\}$ forms a 6-cycle in $Ec(G)$. If $Ec(G)$ contains a 4-cycle, then so is $Ec(T_1)$ as $T_i = P_2$ for all $i = 2, \dots, k$.

Finally, the rest of the cases follow from Theorems 4 and 5. $\square$

As an illustration, we will now discuss the structure and the girth of the eccentric graph of the graphs obtained as the Cartesian product of two path graphs and two cycle graphs.

4.2. Cartesian product of two path graphs

An $m \times n$ *grid graph* is the Cartesian product of the path graphs P_m and P_n, denoted as $P_m \square P_n$. Let the vertices of $P_m \square P_n$ be $\{(i, j) : 1 \leq i \leq m,\ 1 \leq j \leq n\}$. For the sake of simplicity in figures, we label a vertex (i, j) by $(i-1)n + j$. Fig. 11 shows the mentioned labelling for the grid graph $P_3 \square P_5$.

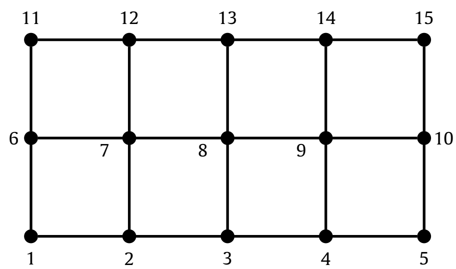


Fig. 11. The grid graph, $P_3 \square P_5$.

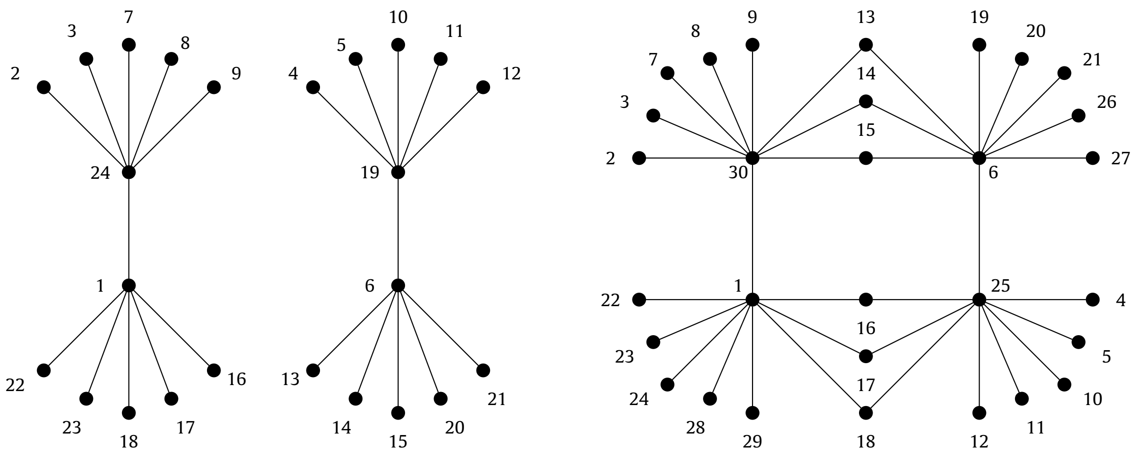
Let $G = P_m \square P_n$ be a grid. Then the eccentricity of each vertex is given by

$$e\big((i,j)\big) = \begin{cases} d\big((i,j),(m,n)\big) & \text{if } 1 \leq i \leq \lceil \frac{m}{2} \rceil,\ 1 \leq j \leq \lceil \frac{n}{2} \rceil, \\ d\big((i,j),(1,1)\big) & \text{if } \lfloor \frac{m}{2} \rfloor < i \leq m,\ \lfloor \frac{n}{2} \rfloor < j \leq n, \\ d\big((i,j),(m,1)\big) & \text{if } 1 \leq i \leq \lceil \frac{m}{2} \rceil,\ \lfloor \frac{n}{2} \rfloor < j \leq n, \\ d\big((i,j),(1,n)\big) & \text{if } \lfloor \frac{m}{2} \rfloor < i \leq m,\ 1 \leq j \leq \lceil \frac{n}{2} \rceil. \end{cases}$$

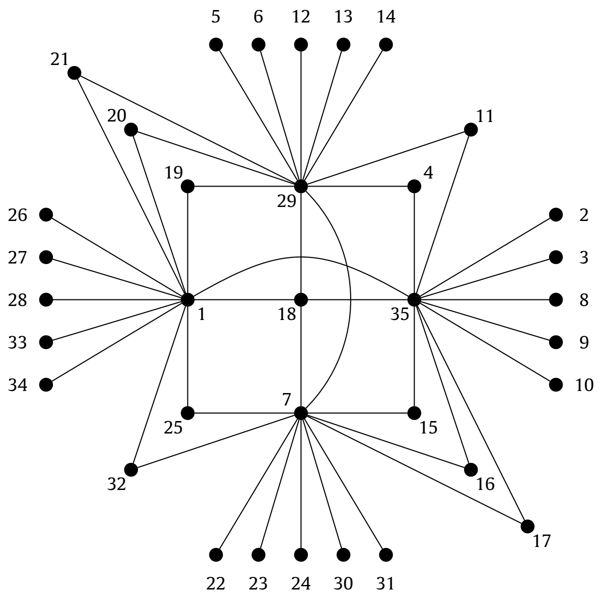
Note that $(1,1)$, $(1,n)$, $(m,1)$ and (m,n) have the maximum eccentricity, which is $m+n$. Therefore,

$$(i,j) \sim_{Ec(G)} \begin{cases} (m,n) & \text{if } 1 \le i \le \lceil \frac{m}{2} \rceil,\ 1 \le j \le \lceil \frac{n}{2} \rceil, \\ (1,1) & \text{if } \lfloor \frac{m}{2} \rfloor < i \le m,\ \lfloor \frac{n}{2} \rfloor < j \le n, \\ (m,1) & \text{if } 1 \le i \le \lceil \frac{m}{2} \rceil,\ \lfloor \frac{n}{2} \rfloor < j \le n, \\ (1,n) & \text{if } \lfloor \frac{m}{2} \rfloor < i \le m,\ 1 \le j \le \lceil \frac{n}{2} \rceil. \end{cases}$$

From the above adjacency relations, it is clear that the eccentric graph of $P_m \Box P_n$ has a specific structure depending on the parity of m and n. Example for each of the three cases, depending on whether both m and n are even, both are odd, or one is even and the other is odd, are presented in Fig. 12. *Further, note that the girth of the eccentric graph $Ec(P_m \Box P_n)$ is zero if both m and n are even, four if exactly one of m and n is even and greater than two, six if exactly one of m and n is two and the other is odd, and three if both m and n are odd.*



(a) $Ec(P_4 \Box P_6)$ (b) $Ec(P_5 \Box P_6)$



(c) $Ec(P_5 \Box P_7)$

Fig. 12. Eccentric graphs of different grid graphs.

Moreover, if n is odd and $m > 2$ is even, then

$$\left((1,1)\left(\frac{n+1}{2},m-1\right)(n,1)\left(\frac{n+1}{2},m\right)\right)$$

form a 4-cycle in $Ec(P_n \square P_m)$. If n is odd and $m = 2$, then

$$\left((1,1)\left(\frac{n+1}{2},2\right)(n,1)(1,2)\left(\frac{n+1}{2},1\right)(n,2)\right)$$

form a 6-cycle in $Ec(P_n \square P_m)$. If both n and m are odd, then

$$\left((1,1)\left(\frac{n+1}{2},\frac{m+1}{2}\right)(n,m)\right)$$

form a 3-cycle in $Ec(P_n \square P_m)$.

4.3. Cartesian product of two cycle graphs

As discussed in Section 1, $Ec(C_n)$ is isomorphic to the $\frac{n}{2}$ copies of K_2 for an even n. Thus when n and m both are even, each vertex in $Ec(C_n \square C_m)$ has degree 1. In other words, $Ec(C_n \square C_m)$ is isomorphic to a graph containing $\frac{nm}{2}$ copies of K_2.

For an even n and an odd m, each vertex in $Ec(C_n)$ and $Ec(C_m)$ has degree 1 and 2 respectively. Therefore, $Ec(C_n \square C_m)$ is a 2-regular graph. Consequently, $Ec(C_n \square C_m)$ is either a cycle or a union of cycles. Moreover, $Ec(C_n \square C_m)$ consists $\frac{n}{2}$ cycles of length $2m$, namely

$$\left((i,1)\left(\frac{n}{2}+i,2\right)\dots(i,m)\left(\frac{n}{2}+i,1\right)(i,2)\dots\left(\frac{n}{2}+i,m\right)\right)$$

for $i \in [\frac{n}{2}]$. Fig. 13 shows the eccentric graph of the Cartesian product of C_4 and C_3.

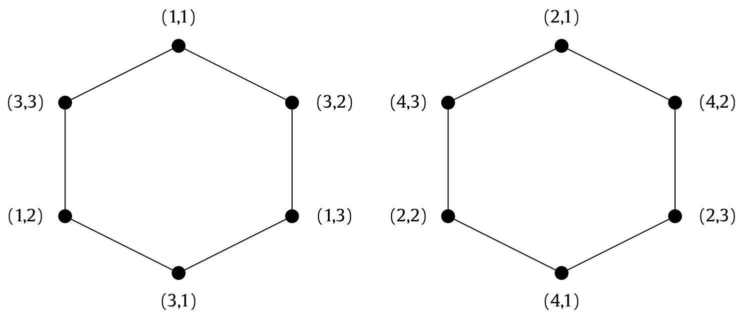


Fig. 13. Eccentric graph of the Cartesian product of C_4 and C_3.

When m and n both are 3, the eccentric graph of $C_n \square C_m$ is shown in Fig. 14 and its girth is 3 by Theorem 4, which can be seen in the figure as well.

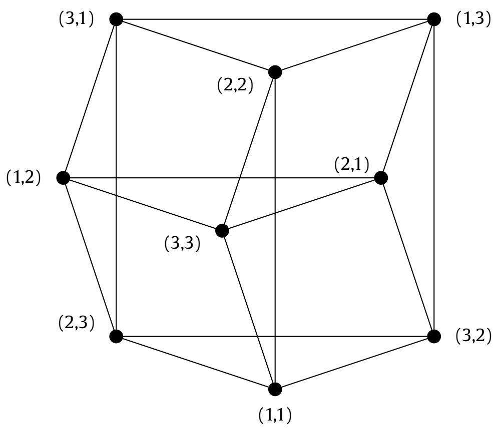


Fig. 14. Eccentric graph of the Cartesian product of a 3-cycle with itself.

Finally, for the remaining case, it follows from Theorem 5 that the eccentric girth of $Ec(C_n \square C_m)$ is four and

$$\left(\left(1,\frac{m+3}{2}\right)\left(\frac{n+1}{2},1\right)\left(1,\frac{m+1}{2}\right)\left(\frac{n+3}{2},1\right)\right)$$

form a 4-cycle in $Ec(C_n \square C_m)$.

The following statement summarizes the above discussion: The eccentric girth of the Cartesian product of two cycle graphs is even except when both cycles are triangles. Moreover,

$$g(Ec(C_n \square C_m)) = \begin{cases} 0 & \text{if both } n \text{ and } m \text{ are even,} \\ 3 & \text{if } n = m = 3, \\ 2m & \text{if } n \text{ is even and } m \text{ is odd,} \\ 4 & \text{otherwise.} \end{cases}$$

We will end this section with the following observation.

Proposition 2. *For odd n, $Ec(C_n \square C_n)$ is isomorphic to $C_n \square C_n$.*

Proof. By Corollary 1, it is enough to show that $C_n \square\, C_n$ is isomorphic to $C_n \times C_n$ for an odd n. We assume the natural labelling on the vertices of C_n. Now, we define an isomorphism f from $C_n \square\, C_n$ to $C_n \times C_n$ as follows

$$\begin{aligned} f\big((1,1)\big) &= (1,1), \\ f\big((i,1)\big) &= (n+2-i, n+2-i) \text{ for } i = 2,\dots,n, \\ f\big((i,j)\big) &= \big[f\big((i,1)\big) + (j-1, 1-j)\big] \,(\text{mod } n). \end{aligned}$$

We will write 0 as n in the computation of f. To see f is a bijection, first note that $f\big((i,1)\big) \neq f\big((j,1)\big)$ for $i \neq j$. Now assume that $(i,j) \neq (k,l)$, this happens in either of three cases, (a) $i \neq k$ and $j = l$, (b) $i = k$ and $j \neq l$, or (c) $i \neq k$ and $j \neq l$.

Consider the first case $i \neq k$ and $j = l$ and let $f\big((i,1)\big) = (s,s)$ and $f\big((k,1)\big) = (t,t)$, clearly $s \neq t$. Now, if $f\big((i,j)\big) = f\big((k,l)\big)$, then $s + j - 1 \equiv t + j - 1 (\text{mod } n)$, which leads to $s = t$, a contradiction. Therefore $f\big((i,j)\big) \neq f\big((k,l)\big)$. Similarly, we can show for the second case. Now consider the third case $i \neq k$ and $j \neq l$, and again let $f\big((i,1)\big) = (s,s)$ and $f\big((k,1)\big) = (t,t)$, clearly $s \neq t$. Now, if $f\big((i,j)\big) = f\big((k,l)\big)$, then $s + j - 1 \equiv t + l - 1 (\text{mod } n)$ and $s + 1 - j \equiv t + 1 - l (\text{mod } n)$, compatibility with addition of congruence leads to again $s = t$ (because n is odd), a contradiction. Therefore, f is a bijection.

Now, let $(i,j) \in V(C_n \square C_n)$ and $f\big((i,j)\big) = (s,t)$. Then $f\big((i \pm 1, j)\big) = (s \pm 1, t \pm 1)(\text{mod } n)$ and $f\big((i, j \pm 1)\big) = (s \pm 1, t \mp 1)(\text{mod } n)$. This proves that f preserves the adjacency. □

5. Invertibilty of eccentricity matrix of the Cartesian product of trees

In this section, we will focus on the invertibility of the eccentricity matrix for the Cartesian product of trees. First, recall the definition of the *Kronecker product* of two matrices.

Definition 6. Let $A = (a_{i,j})$ be an $m \times n$ matrix and $B = (b_{i,j})$ be a $p \times q$ matrix, then the *Kronecker product*, $A \otimes B$, is an $mp \times nq$ block matrix defined as

$$\begin{pmatrix} a_{11}B & \cdots & a_{1n}B \\ \vdots & \ddots & \vdots \\ a_{m1}B & \cdots & a_{mn}B \end{pmatrix}.$$

Kronecker product of two matrices is non-commutative in general. If A and B are square matrices of order n and p, respectively, then

$$\det A \otimes B = (\det A)^p (\det B)^n.$$

Lemma 3. *Let T_1 be a tree which is not a star or P_4, then the eccentricity matrix of $T_1 \square \underbrace{P_2 \square \cdots \square P_2}_{k-1}$ is not invertible.*

Proof. Let $G = T_1 \square P_2 \square \cdots \square P_2$ and the i^{th} graph in this product be the path P_2 with endpoints $\{u_i, v_i\}$ for $i = 2,\dots,k$. Note that a vertex $(x_1, x_2 \dots, x_k)$ is adjacent to $(u_1, u_2, \dots, u_k)$ in $Ec(G)$ if and only if $x_i = v_i$ for $i = 2,\dots,k$ and either x_1 is eccentric to u_1 in T_1 or u_1 is eccentric to x_1 in T_1. In other words, adjacency with $(u_1, u_2, \dots, u_k)$ in $Ec(G)$ solely depends on the adjacency of u_1 in $Ec(T_1)$. Now we consider three cases.

Case 1: $diam(T_1) = 3$.

Let $P = a_1\, b_1\, c_1\, d_1$ be a diametrical path in T_1. As $T_1 \neq P_4$, there must be a leaf vertex, say e_1, adjacent to either b_1 or c_1. Let us assume that e_1 is adjacent to b_1. Now we claim that $N_{Ec(G)}\big((a_1, u_2, \dots, u_k)\big) = N_{Ec(G)}\big((e_1, u_2, \dots, u_k)\big)$. If a vertex f_1 is eccentric to a_1 then f_1 is also eccentric to e_1 because $d_{T_1}(a_1, f_1) = d_{T_1}(e_1, f_1)$, and if a_1 is eccentric to some vertex f_1 then so is e_1 because $d_{T_1}(a_1, f_1) = d_{T_1}(e_1, f_1)$. This proves our claim and hence the rows corresponding to these two vertices in $\mathcal{E}_G$ are exactly the same and therefore $\det(\mathcal{E}_G) = 0$.

Case 2: $diam(T_1) = 4$.

Let $P = a_1\, b_1\, c_1\, d_1\, e_1$ be a diametrical path in T_1. Let $\{b_1, d_1, p_1, \dots, p_\ell\}$ be the set of neighbours of c_1. Note that if a vertex x is eccentric to a neighbour of c_1 then it is also eccentric to c_1. Further, note that none of c_1 or its neighbours can be eccentric to any vertex in T_1. Therefore, the row corresponding to $(c_1, u_2, \dots, u_k)$ in the matrix $\mathcal{E}_G$ is a constant multiple of the sum of the rows corresponding to $(b_1, u_2, \dots, u_k), (d_1, u_2, \dots, u_k), (p_1, u_2, \dots, u_k), \dots (p_\ell, u_2, \dots, u_k)$.

Case 3: $diam(T_1) > 4$.

Let $P = a_1\, b_1\, c_1\, d_1 \dots z_1$ be a diametrical path in T_1. A vertex eccentric to b_1 in T_1 is also eccentric to c_1 in T_1 and vice versa. Also, b_1 and c_1 cannot be eccentric to any vertex in T_1 as they are not leaves. Therefore, b_1 and c_1 have same neighbourhood in $Ec(T_1)$. As a result, the rows corresponding to $(b_1, u_2, \dots, u_k)$ and $(c_1, u_2, \dots, u_k)$ in $\mathcal{E}_G$ are constant multiple of each other and hence $\det(\mathcal{E}_G) = 0$. $\square$

Now, we will present the main result of this section.

Theorem 7. *Let $T_1, \dots, T_k$ be trees and $G\ (= T_1 \square \cdots \square T_k)$ be their Cartesian product. Then the eccentricity matrix of G, $\mathcal{E}_G$, is invertible if and only if one of them is a star or P_4 and the rest are P_2.*

Proof. Let $T_1, \dots, T_k$ be trees with at least two vertices and $G = T_1 \square \cdots \square T_k$. Assume that T_1 is a star on $n+1$ vertices and $T_2 = \cdots = T_k = P_2$. Then the eccentricity matrix of G is

$$\mathcal{E}_G = \begin{pmatrix} 0 & k & k & \cdots & k \\ k & 0 & k+1 & \cdots & k+1 \\ \vdots & \vdots & \ddots & & \\ k & k+1 & \cdots & 0 & k+1 \\ k & k+1 & \cdots & k+1 & 0 \end{pmatrix} \otimes J_{2^{k-1}},$$

where, J_s is a $s \times s$ antidiagonal matrix with all antidiagonal entries as 1.

Note that $\det \begin{pmatrix} 0 & k & k & \cdots & k \\ k & 0 & k+1 & \cdots & k+1 \\ \vdots & \vdots & \ddots & & \\ k & k+1 & \cdots & 0 & k+1 \\ k & k+1 & \cdots & k+1 & 0 \end{pmatrix}$ is $(-1)^n nk^2(k+1)^{n-1}$, also $\det J_{2^{k-1}} \neq 0$. Therefore $\det \mathcal{E}_G \neq 0$.

Now if $T_1 = P_4$, then the eccentricity matrix of G is

$$\mathcal{E}_G = \begin{pmatrix} 0 & 0 & k+1 & k+2 \\ 0 & 0 & 0 & k+1 \\ k+1 & 0 & 0 & 0 \\ k+2 & k+1 & 0 & 0 \end{pmatrix} \otimes J_{2^{k-1}}.$$

Again, $\det \mathcal{E}_G \neq 0$, as $\det \begin{pmatrix} 0 & 0 & k+1 & k+2 \\ 0 & 0 & 0 & k+1 \\ k+1 & 0 & 0 & 0 \\ k+2 & k+1 & 0 & 0 \end{pmatrix} = (k+1)^4$.

For the converse part, let T_1 be neither a star nor P_4. Thus the diameter of $T_1 > 2$ and let $P = u_1 u_2 \dots u_s$ be a diametrical path in T_1. If each of $T_2, \dots, T_k$ contains only pendant vertices, then the conclusion follows from Lemma 3. Therefore, we can assume without loss of generality that T_2 has a non-pendant vertex v. Now we want to show that $\det \mathcal{E}_G$ is zero. This assertion holds if we can show in general $\det \mathcal{E}_K$ is zero, where K is the Cartesian product of T_1, T_2 and a simple connected graph H. Let $(u_i, v, x) \in V(K)$. Note that (u_i, v, x) cannot be farthest from (and hence, eccentric to) any vertex in K because v is a non-pendant. Consequently, only those vertices are adjacent to (u_i, v, x) (in $Ec(K)$) which are eccentric to (u_i, v, x). Thus by Lemma 1,

$$N_{Ec(K)}(u_i, v, x) = \{(w_i, w, y) : w_i, w, y \text{ are eccentric to } u_i, v, x \text{ respectively}\}. \quad (4)$$

Now if any vertex is eccentric to u_1 in T_1 then the same vertex is eccentric to u_2 as well in T_1 leading to

$$N_{Ec(K)}(u_1, v, x) = N_{Ec(K)}(u_2, v, x).$$

Thus, the row corresponding to (u_1, v, x) in $\mathcal{E}_K$ is a constant multiple of that of (u_2, v, x), proving the non-invertibility of $\mathcal{E}_K$. $\square$

Declaration of competing interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

Data availability

No data was used for the research described in the article.

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