

Multiple Estimation Models for Discrete-time Adaptive Iterative Learning Control

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Abstract

This article focuses on making discrete-time Adaptive Iterative Learning Control more effective using multiple estimation models. Existing strategies use the tracking error to adjust the parametric estimates. Our strategy uses the last component of the identification error to tune these estimates of the model parameters. We prove that this strategy results in bounded estimates of the parameters, and bounded and convergent identification and tracking errors. We emphasize that the proof does not use the Key Technical Lemma. Rather, it uses the properties of square-summable sequences. We extend this strategy to include multiple estimation models and show that all the signals are bounded, and the errors converge. It is also shown that this works whether we switch between the models at every instant and every iteration or at the end of every iteration. Simulation results demonstrate the efficacy of the proposed method with a faster convergence using multiple estimation models.

1 Introduction

Many practical, modern engineering systems require that a reference trajectory be tracked for a specific finite interval, and this task is then repeated for multiple iterations. Extensive research has been dedicated to using Iterative Learning Control (ILC) for such tasks. In the last two decades, ILC has evolved into a highly popular strategy to achieve requirements of finite-interval, high precision tracking control, yet simultaneously maintaining acceptable levels of control energy [1–5]. These requirements cannot be achieved satisfactorily by standard feedback control techniques. In particular, feedback control laws do not update over iterations, and the error profile is identical in every iteration. Further, feedback control guarantees only asymptotic convergence of error, which is unsuitable when considering finite-interval tracking. High-precision tracking using feedback control requires prohibitively significant control energy. Each of these issues is addressed by ILC.

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The primary notion in ILC is that performance of a system can be improved by learning from error and control signals of previous iterations. Such a notion led to the design of numerous laws that constructed the control input in iteration $(k + 1)$ based directly on the control input and error signal in iteration k . This eventually developed into a contraction-mapping (CM), operator-theoretic framework for ILC [3]. Many popular ILC strategies were designed based on this framework [4–9], and it continues to be popular, with numerous applications in large-scale industrial manufacturing [10, 11], chemical batch processes [12, 13], hybrid actuation systems [14] and robotics [15–18]. This framework has also been extensively analyzed, with established convergence and robustness results [19–23].

An alternative framework for ILC is based on Composite Energy Functions (CEFs), which are Lyapunov-like energy functions over iterations. This framework is particularly useful when system parameters are unknown, and ILC design must incorporate parameter estimation. The approach closely follows adaptive control strategies, with minor differences in parameter update laws [24, 25]. CEF-based ILC has numerous advantages over the contraction-mapping approach to ILC. First, the restrictive requirement of globally Lipschitz nonlinearities can be relaxed. French and Rogers [26] proposed one of the earliest CEF-based techniques to achieve this in continuous-time Adaptive ILC. Further, it provides a unified method to address nonlinear systems, systems subjected to disturbances, and systems with time-varying parameters. This framework can also handle iteration-varying reference trajectories and random initial conditions on system states. The CEF approach to continuous-time Adaptive ILC was formalized in a series of papers [24, 25, 27–29] in the early 2000s. In contrast to adaptive control, a prominent feature of the proposed strategies was the *discrete* update of parameter estimates over iterations. Further, monotonicity of energy functions was demonstrated, resulting in pointwise convergence of tracking error. Continuous-time Adaptive ILC has also been extensively applied to robot manipulators [5, 30–33], high-speed trains [34–36] and vibration control [37, 38].

Using the analogy between the iteration axis and discrete-time axis, Chi *et al.* [39] proposed a discrete-time Adaptive ILC strategy using Composite Energy Functions for a nonlinear system subjected to disturbances. The primary features of this strategy included the ability to deal with iteration-varying reference trajectories, random initial conditions on the system state and time-varying system parameters. Applying the Key Technical Lemma (KTL) [40] over iterations, the convergence of tracking error was demonstrated. This technique was further formalized in [41–43]. In [44–46], the problem that arises when the sign of the input coefficient is unknown was addressed. In [47, 48] time- and iteration-varying parameters were both in the problem setup, and a novel dead-zone approach was proposed to tackle the additional complexity. Learning control for a system with binary-valued observations was achieved in [49]. A dynamic linearization framework was used in [50] for Adaptive ILC on MIMO systems. In [51], a predictive ILC scheme was used for learning control of nonaffine, nonlinear systems.

The design and analysis of discrete-time Adaptive ILC closely follow discrete-time adaptive control. Poor transient response is a significant issue in adaptive control and arises from using a single model for parameter estimation. (In what follows, a model is a mathematical representation of the given dynamical system.) A poor initial estimate can contribute to large initial tracking and identification errors. (In this paper, the tracking error is the deviation of the system’s state from the reference model state and the identification error is the

deviation of the system’s state from the state of the estimation model. The latter is defined explicitly in Section 3.) The Multiple Models, Switching and Tuning (MMST) methodology [52–54] was proposed to combat this problem. By initializing many estimation models in the parameter space, one of these models is likely sufficiently close to the actual parameter, resulting in improved identification and tracking performance. Most Adaptive ILC schemes use the tracking error to update parameter estimates, similar to certain discrete-time adaptive control strategies. However, such an estimation law does not lend itself well to the extension to multiple estimation models. Adaptive control strategies which use the identification error in place of tracking error for updating parameters have been explored [55, 56], and these strategies result in improved convergence with multiple models. The objective of this article is to present the MMST methodology in the context of discrete-time Adaptive ILC, by modifying the control and identification laws.

There is very little existing research on using multiple models for Adaptive ILC, particularly in the CEF framework. In [57, 58], Li *et al.* present a strategy with multiple fuzzy neural networks estimating part of the system’s parameters. In [59], Freeman and French use multiple estimation models in the contraction-mapping setting to present robust stability and performance-bounds results. In [60, 61], the authors present MMST for Adaptive ILC in a contraction-mapping framework and Multiple Models with Second-Level Adaptation (MM-SLA) (see [62, 63]) for Adaptive ILC to achieve lower computational complexity. In all the above cases, however, iteration-varying references cannot be tracked, and the standard estimation and certainty-equivalent control procedure in Adaptive ILC are not used. Further, systems subjected to disturbances are not addressed, and convergence is not demonstrated based on composite energy functions.

In this article, we present a complete framework for using multiple estimation models in Adaptive ILC, addressing the above drawbacks. The primary contributions of this article are as follows:

- A new control with a single model identification scheme are presented for discrete-time Adaptive ILC, and the identification error (rather than tracking error) is used to update parameter estimates.
- Convergence is proved using CEFs and the properties of square-summable sequences rather than using the KTL.
- Next, a strategy with multiple estimation models based on MMST is proposed. A complete overview of the control and identification laws is provided, two switching schemes are outlined, and convergence is proved in a unified manner for both schemes, using the properties of square-summable sequences.
- Simulation results indicate that both the single model and multiple model estimation schemes demonstrate satisfactory tracking performance. The multiple model scheme results in faster convergence of tracking errors for linear time-invariant and time-varying systems, as well as nonlinear, discrete-time systems subjected to disturbances.

The remainder of this article is organized as follows. In Section 2, the general discrete-time Adaptive ILC problem is introduced, with standard assumptions and remarks. Section

3 presents a new single estimation model solution to this problem. This is extended to a strategy with multiple estimation models in Section 4. We present simulation results for the proposed strategies in Section 5, and concluding remarks in Section 6.

Throughout this article, \mathbb{N} denotes the set of natural numbers $\{1, 2, \dots\}$, and \mathbb{R}^n denotes the vector space of all n -tuples of real numbers. For a vector $y = [y_1, \dots, y_n]^T \in \mathbb{R}^n$, $\|y\|$ denotes the Euclidean norm, defined as

$$\|y\| \triangleq \sqrt{\sum_{i=1}^n y_i^2}.$$

ℓ_2 denotes the Hilbert space of all square-summable sequences, i.e. all sequences $\{x_n\}_{n \in \mathbb{N}}$ such that

$$\sum_n |x_n|^2 < \infty,$$

and ℓ_∞ denotes the Banach space of all bounded sequences, i.e. all sequences $\{x_n\}_{n \in \mathbb{N}}$ such that

$$\sup_n |x_n| < \infty.$$

2 Problem Formulation

In this section, we formulate the general discrete-time Adaptive ILC problem. Consider the following discrete-time, nonlinear, uncertain n th order system with matched, time-varying uncertainty:

$$\begin{aligned} x_{i,k}(t+1) &= x_{i+1,k}(t), \quad i = 1, \dots, (n-1), \\ x_{n,k}(t+1) &= \theta_1^T(t) \xi(x_k(t)) + b(t)u_k(t) + d(t), \end{aligned} \quad (1)$$

where $k \in \mathbb{N}$ denotes the index for iterations and each iteration consists of samples indexed $\{0, 1, \dots, T\}$. The time index t is within the set $\mathcal{I}_T = \{0, 1, \dots, T-1\}$. Note that this set does not include the final sample T . $x_{i,k}(t)$ denotes state i in iteration k at sample t . $x_k(t) = [x_{1,k}(t), \dots, x_{n,k}(t)]^T \in \mathbb{R}^n$ is the measurable state vector. $\theta_1(t) \in \mathbb{R}^p$ is an unknown parameter vector, $b(t) \in \mathbb{R}$ is the unknown input coefficient, and $d(t) \in \mathbb{R}$ is an unknown exogenous disturbance. Each of these quantities is iteration invariant. $\xi(x_k(t)) \equiv \xi_k(t) \in \mathbb{R}^p$, called the *regression vector*, is a known, bounded nonlinear vector function of the state $x_k(t)$, and $u_k(t) \in \mathbb{R}$ is the input to the system in iteration k and at sample t . Equation (1) can be rewritten as follows:

$$\begin{aligned} x_{i,k}(t+1) &= x_{i+1,k}(t), \quad i = 1, \dots, (n-1), \\ x_{n,k}(t+1) &= \theta^T(t) \phi_k(t), \end{aligned} \quad (2)$$

where $\theta(t) \triangleq [\theta_1^T(t), b(t), d(t)]^T \in \mathbb{R}^{p+2}$ is the overall unknown parameter vector, and $\phi_k(t) \triangleq [\xi_k^T(t), u_k(t), 1]^T \in \mathbb{R}^{p+2}$ is the overall known regression vector.

The objective of the Adaptive ILC problem is to design an appropriate control input $u_k(t)$ such that the system state $x_k(t)$ tracks the state $x_{m,k}(t)$ of the following stable, iteration-varying reference model:

$$\begin{aligned} x_{i,m,k}(t+1) &= x_{i+1,m,k}(t) \quad i = 1, \dots, (n-1), \\ x_{n,m,k}(t+1) &= \rho_k(t), \end{aligned} \quad (3)$$

for some known $\rho_k(t)$, with asymptotic tracking over iterations k . Defining the state tracking error $e_k(t) \triangleq x_k(t) - x_{m,k}(t) = [e_{1,k}(t), \dots, e_{n,k}(t)]^T \in \mathbb{R}^n$, asymptotic tracking over iterations implies:

$$\lim_{k \rightarrow \infty} e_k(t) = 0 \quad (4)$$

for each sample $t \in \mathcal{I}_T$.

The following assumptions are made:

Assumption 1. The unknown quantities $\theta_1(t)$, $b(t)$ and $d(t)$ are bounded, and hence the parameter vector $\theta(t)$ is bounded.

Assumption 2. The sign of $b(t)$ is known and invariant, i.e. $b(t)$ is either positive or negative for all time t , and $b(t)$ is non-singular. Without loss of generality, assume $b(t) \geq b_{\min} > 0$. This assumption implies that the control direction is known.

Remark 1. Throughout this article, the case with iteration invariant parameters $\theta_1(t)$, $b(t)$ and $d(t)$, and hence iteration invariant $\theta(t)$ is considered. This can be extended to the case with time- and iteration-varying parameters, which was addressed in [48] using a novel dead-zone approach. Applying this to the proposed techniques is an interesting avenue for future work on this topic.

Remark 2. Throughout this article, we do not assume identical initial conditions on the plant and reference model. However, it has been observed in [39] that with random, non-zero initial conditions on the plant (2), random non-zero initial errors will propagate, and the state errors $e_{i,k}(t)$ for $i = 1, \dots, (n-1)$ and $t = 0, \dots, (n-i)$ cannot be ‘learned’, as these errors are not affected by the input $u_k(t)$. The remaining errors are dependent on $u_k(t)$ and hence can be driven to zero. If the plant and reference model have identical initial conditions, it can be shown that each component of the identification and tracking error vectors converge to zero.

Remark 3. Assumption 2 can be relaxed by employing the technique of discrete Nussbaum gain [64], as explored in [44, 45, 65]. An alternative approach without using the Nussbaum gain was also explored in [46] by fully exploiting the convergence properties of parameter estimates and incorporating two modifications in the control and parameter update laws. Extending the techniques proposed here by incorporating these approaches when the control direction is unknown is a promising avenue for future work.

Remark 4. In numerous works on discrete-time Adaptive ILC, an additional assumption — usually called the linear growth rate or sector-bounded condition — is made. This assumption states that the nonlinearity $\xi_k(t)$ satisfies:

$$\|\xi_k(t)\| \leq c_1 + c_2 \|x_k(t)\|,$$

for some positive constants c_1 and c_2 . This assumption plays a key role in analyzing the convergence of the tracking error over iterations as part of the assumptions for the KTL [40]. In contrast, our analysis does not involve the KTL, and hence we do not make this assumption.

3 A New Solution for Discrete-time Adaptive ILC

In this section, we formulate a new control law and parameter update law for the problem formulated in Section 2. Existing solutions mainly incorporate the principle of certainty equivalence for control design and use the tracking error for estimating and updating parameters. The disadvantage of using the tracking error is that this strategy cannot be extended to the use of multiple estimation models. Using the identification error in place of tracking error has been explored for discrete-time adaptive control in [55, 56], with results demonstrating improved convergence with multiple estimation models. Further, the stability proofs do not invoke the KTL and instead use Lyapunov theory and the properties of square-summable, or ℓ_2 sequences. Here, we use the analogy between the discrete-time and iteration axes to formulate a corresponding Adaptive ILC strategy.

3.1 Control and Identification Laws

Construct an identification model with state $\hat{x}_k(t)$:

$$\begin{aligned}\hat{x}_{i,k}(t+1) &= \hat{x}_{i+1,k}(t), \quad i = 1, \dots, (n-1), \\ \hat{x}_{n,k}(t+1) &= \hat{\theta}_{1,k}^T(t)\xi(x_k(t)) + \hat{b}_k(t)u_k(t) + \hat{d}_k(t) \\ &= \hat{\theta}_k^T(t)\phi_k(t).\end{aligned}\tag{5}$$

The purpose of establishing this identification model is to construct an identification error that is used to estimate the unknown parameter vector $\theta(t)$. $\hat{\theta}_{1,k}(t)$, $\hat{b}_k(t)$ and $\hat{d}_k(t)$ denote the estimates of the quantities $\theta_1(t)$, $b(t)$ and $d(t)$ in iteration k and sample t . Correspondingly, $\hat{\theta}_k(t)$ denotes the estimate of the parameter vector $\theta(t)$. Define $\tilde{\theta}_k(t) \triangleq \theta(t) - \hat{\theta}_k(t)$. Further, define the state identification error $\hat{e}_k(t) \triangleq x_k(t) - \hat{x}_k(t)$. Then, from (2) and (5),

$$\begin{aligned}\hat{e}_{i,k}(t+1) &= \hat{e}_{i+1,k}(t), \quad i = 1, \dots, (n-1), \\ \hat{e}_{n,k}(t+1) &= \tilde{\theta}_k^T(t)\phi_k(t).\end{aligned}\tag{6}$$

Finally, the tracking error $e_k(t) = x_k(t) - x_{m,k}(t)$ can be described by:

$$\begin{aligned}e_{i,k}(t+1) &= e_{i+1,k}(t), \quad i = 1, \dots, (n-1), \\ e_{n,k}(t+1) &= \theta_1^T(t)\xi_k(t) + b(t)u_k(t) + d(t) - \rho_k(t).\end{aligned}\tag{7}$$

Using (7), the following control law is generated:

$$u_k(t) = \frac{1}{\hat{b}_k(t)} \left[\beta e_{n,k-1}(t+1) + \rho_k(t) - \hat{\theta}_{1,k}^T(t)\xi_k(t) - \hat{d}_k(t) \right],\tag{8}$$

where $0 < \beta < 1$. Add and subtract $\hat{b}_k(t)u_k(t)$ in (7), substitute (8) in (7) and use (6):

$$\begin{aligned} e_{n,k}(t+1) &= \tilde{\theta}_k^T(t)\phi_k(t) + \beta e_{n,k-1}(t+1) \\ &= \hat{e}_{n,k}(t+1) + \beta e_{n,k-1}(t+1). \end{aligned} \quad (9)$$

The presence of β in the control law is to provide some damping by incorporating previous iteration errors. Existing Adaptive ILC schemes set $\beta = 0$, with no previous iteration tracking error term, resulting in a deadbeat-like law.

The parameter vector estimate is updated according to the following adaptive law:

$$\hat{\theta}_{k+1}(t) = \text{Proj} \left[\hat{\theta}_k(t) + \frac{\phi_k(t)}{1 + \|\phi_k(t)\|^2} \hat{e}_{n,k}(t+1) \right]. \quad (10)$$

Note that this law uses the identification error, in contrast to existing Adaptive ILC strategies that use the tracking error for updating parameters. This law is similar to the projection algorithm widely used in adaptive control [40], except with the update over iterations rather than time. The error $\hat{e}_{n,k}(t+1)$ is available as the update is performed offline at the end of iteration k . The operator $\text{Proj}[\cdot]$ is defined below. Define a vector m as follows:

$$m \triangleq \left[\hat{\theta}_k(t) + \frac{\phi_k(t)}{1 + \|\phi_k(t)\|^2} \hat{e}_{n,k}(t+1) \right] = [m_1^T, m_2, m_3]^T,$$

where $m_1 \in \mathbb{R}^p$, $m_2 \in \mathbb{R}$ and $m_3 \in \mathbb{R}$ denote the estimates of $\theta_1(t)$, $b(t)$ and $d(t)$ prior to projection. Then,

$$\text{Proj}[m] \triangleq \begin{cases} [m_1^T, m_2, m_3]^T & \text{if } m_2 \geq b_{\min} \\ [m_1^T, b_{\min}, m_3]^T & \text{if } m_2 < b_{\min} \end{cases}. \quad (11)$$

The use of the projection operator defined above ensures that division by zero is avoided in the control law (8).

Remark 5. Throughout this article, the time index t is always in the set \mathcal{I}_T . The control (8) and adaptive (10) laws are defined on this time horizon. However, note that all state variables and errors are formulated on the time horizon $\{1, \dots, T\}$, apart from their initial conditions. Hence, state variables and errors use the index $(t+1)$ throughout, as evident from the control and adaptive laws above. Further, note that neither the control law nor the adaptive law is defined at the final sample T . However, they affect the state variables and errors corresponding to this sample.

3.2 Convergence Analysis

We have the following result for convergence of the proposed Adaptive ILC law:

Theorem 1. *For the system (2) with the objective of tracking the reference model (3), the control law (8) along with the adaptive law (10) guarantees the following:*

1. $\tilde{\theta}(t), \hat{\theta}(t) \in \ell_\infty$ for each $t \in \mathcal{I}_T$, i.e. the sequence of parametric errors $\tilde{\theta}_k(t)$ over iterations — and hence the sequence of parameter estimates $\hat{\theta}_k(t)$ over iterations — is bounded for each sample t .

2. $\hat{e}_n(t+1) \in \ell_2 \cap \ell_\infty$ for each $t \in \mathcal{I}_T$, i.e. the sequence of the n th component of the identification error over iterations is square-summable and bounded for each sample t .
3. With identical initial conditions on the plant and reference model, $\lim_{k \rightarrow \infty} \hat{e}_k(t+1) = 0$ for each $t \in \mathcal{I}_T$, i.e. each component of the identification error vector tends to zero with iterations, for each sample t .
4. With identical initial conditions on the plant and reference model, $\lim_{k \rightarrow \infty} e_k(t+1) = 0$ for each $t \in \mathcal{I}_T$, i.e. each component of the tracking error vector tends to zero with iterations, for each sample t .
5. $\lim_{k \rightarrow \infty} \left\| \hat{\theta}_k(t) - \hat{\theta}_{k-p}(t) \right\|^2 = 0$, for each $t \in \mathcal{I}_T$, for any $p \in \mathbb{N}$, i.e. the parameter vector estimates converge over iterations for each sample t .

Proof. The proof is organized into three parts. Part 1 derives the boundedness of $\tilde{\theta}(t)$, Part 2 demonstrates that all errors converge to zero over iterations, and Part 3 shows that parameter vector estimates converge over iterations. Thus, statement 1 of the theorem is proved in Part 1, statements 2, 3 and 4 are proved in Part 2, and statement 5 is proved in Part 3.

Part 1: Boundedness of Parametric Error:

Define a composite energy function (CEF) $V_k(t)$:

$$V_k(t) \triangleq \tilde{\theta}_k^T(t) \tilde{\theta}_k(t) = \left\| \tilde{\theta}_k(t) \right\|^2. \quad (12)$$

Let $\Delta V_k(t) \triangleq V_{k+1}(t) - V_k(t)$. Then, from (10),

$$\Delta V_k(t) = \left\| \theta(t) - \text{Proj}[m] \right\|^2 - \left\| \tilde{\theta}_k(t) \right\|^2. \quad (13)$$

Consider the scalar $|b(t) - \text{Proj}[m_2]|$, and note that $b(t) \geq b_{\min}$.

- When $m_2 \geq b_{\min}$, $\text{Proj}[m_2] = m_2$. Then, $|b(t) - \text{Proj}[m_2]| = |b(t) - m_2|$.
- When $m_2 < b_{\min} \leq b(t)$, $\text{Proj}[m_2] = b_{\min}$. Then, $|b(t) - \text{Proj}[m_2]| = |b(t) - b_{\min}| < |b(t) - m_2|$.

Thus, the relation $|b(t) - \text{Proj}[m_2]| \leq |b(t) - m_2|$ always holds. As $b(t)$ is simply part of the parameter vector $\theta(t)$, the relation $\|\theta(t) - \text{Proj}[m]\| \leq \|\theta(t) - m\|$ always holds. Hence, the parametric error magnitude does not increase using the projection operator. Using this in (13),

$$\Delta V_k(t) \leq \left\| \theta(t) - m \right\|^2 - \left\| \tilde{\theta}_k(t) \right\|^2. \quad (14)$$

Substitute $m = \left[\hat{\theta}_k(t) + \frac{\phi_k(t)}{1 + \|\phi_k(t)\|^2} \hat{e}_{n,k}(t+1) \right]$. Then,

$$\Delta V_k(t) \leq \left\| \tilde{\theta}_k(t) - \frac{\phi_k(t)}{1 + \|\phi_k(t)\|^2} \hat{e}_{n,k}(t+1) \right\|^2 - \left\| \tilde{\theta}_k(t) \right\|^2. \quad (15)$$

On simplification by expanding the norm and using (6), this reduces to:

$$\Delta V_k(t) \leq - \left(\frac{2 + \|\phi_k(t)\|^2}{(1 + \|\phi_k(t)\|^2)^2} \right) \hat{e}_{n,k}^2(t+1), \quad (16)$$

or,

$$\Delta V_k(t) \leq -\alpha_k^2(t) \hat{e}_{n,k}^2(t+1) \leq 0, \quad (17)$$

where $\alpha_k^2(t)$ denotes the positive quantity in parentheses in (16). Thus, the function $V_k(t)$ is non-increasing. From this and the construction of $V_k(t)$ (12), it is evident that $\tilde{\theta}_k(t)$ is a bounded sequence over iterations k , for every $t \in \mathcal{I}_T$, i.e. $\tilde{\theta}(t) \in \ell_\infty$. Subsequently, as $\theta(t)$ is bounded, the sequence of parameter estimates $\hat{\theta}_k(t)$ over iterations k is bounded, i.e. $\hat{\theta}(t) \in \ell_\infty$. This concludes Part 1 of the proof.

Part 2: Convergence of Errors:

From (17), note that $\lim_{N \rightarrow \infty} |V_{N+1}(t) - V_1(t)| < \infty$ for each $t \in \mathcal{I}_T$. This can be written as:

$$\lim_{N \rightarrow \infty} \left| \sum_{k=1}^N \Delta V_k(t) \right| \leq \lim_{N \rightarrow \infty} \sum_{k=1}^N \alpha_k^2(t) \hat{e}_{n,k}^2(t+1) < \infty.$$

Then, the sequence $\alpha_k(t) \hat{e}_{n,k}(t+1)$ is square-summable over iterations k , for each $t \in \mathcal{I}_T$. By the properties of ℓ_2 sequences, $\alpha(t) \hat{e}_n(t+1) \in \ell_2 \cap \ell_\infty$. Next, note that since $\xi_k(t)$ and $u_k(t)$ are bounded, so is $\|\phi_k(t)\|$ by definition. Hence, $\alpha_k(t)$ can never converge to 0. Further, $0 < \alpha_k(t) < \sqrt{2}$ and we have $\hat{e}_n(t+1) \in \ell_2 \cap \ell_\infty$, and hence,

$$\lim_{k \rightarrow \infty} \hat{e}_{n,k}(t+1) = 0 \quad (18)$$

for each $t \in \mathcal{I}_T$. Further, consider eq. (9). This is an iteration-domain difference equation, with a forcing function $\hat{e}_{n,k}(t+1) \rightarrow 0$ as $k \rightarrow \infty$. As $0 < \beta < 1$, it is evident that:

$$\lim_{k \rightarrow \infty} e_{n,k}(t+1) = 0 \quad (19)$$

for each $t \in \mathcal{I}_T$. Finally, under the assumption of identical initial conditions, (18) and (19) imply that:

$$\lim_{k \rightarrow \infty} \hat{e}_k(t+1) = 0 \quad (20)$$

and

$$\lim_{k \rightarrow \infty} e_k(t+1) = 0 \quad (21)$$

for each $t \in \mathcal{I}_T$, i.e. the identification and tracking error *vectors* converge to 0 as $k \rightarrow \infty$. This concludes Part 2 of the proof.

Part 3: Convergence of Parameter Estimates:

Consider the update law (10), and consider the scalar $\left| \text{Proj}[m_2] - \hat{b}_k(t) \right|$. From the preceding update of parameter estimates, $\hat{b}_k(t) \geq b_{\min}$, by the use of projection.

- When $m_2 \geq b_{\min}$, $\text{Proj}[m_2] = m_2$, and $\left| \text{Proj}[m_2] - \hat{b}_k(t) \right| = \left| m_2 - \hat{b}_k(t) \right|$.

- When $m_2 < b_{\min} \leq \hat{b}_k(t)$, $\text{Proj}[m_2] = b_{\min}$. Then, $\left| \text{Proj}[m_2] - \hat{b}_k(t) \right| = \left| b_{\min} - \hat{b}_k(t) \right| < \left| m_2 - \hat{b}_k(t) \right|$.

Thus, the relation $\left| \text{Proj}[m_2] - \hat{b}_k(t) \right| \leq \left| m_2 - \hat{b}_k(t) \right|$ always holds. By extension, the relation

$$\left\| \text{Proj}[m] - \hat{\theta}_k(t) \right\| \leq \left\| m - \hat{\theta}_k(t) \right\|$$

always holds. Using (10) and substituting m ,

$$\left\| \hat{\theta}_{k+1}(t) - \hat{\theta}_k(t) \right\|^2 \leq \left\| \frac{\phi_k(t)}{1 + \|\phi_k(t)\|^2} \hat{e}_{n,k}(t+1) \right\|^2 \leq \hat{e}_{n,k}^2(t+1).$$

Thus,

$$\lim_{N \rightarrow \infty} \sum_{k=1}^N \left\| \hat{\theta}_{k+1}(t) - \hat{\theta}_k(t) \right\|^2 \leq \lim_{N \rightarrow \infty} \sum_{k=1}^N \hat{e}_{n,k}^2(t+1) < \infty,$$

or,

$$\lim_{k \rightarrow \infty} \left\| \hat{\theta}_{k+1}(t) - \hat{\theta}_k(t) \right\| = 0 \quad (22)$$

for each $t \in \mathcal{I}_T$. Thus, parameter estimates one iteration apart converge. This result can easily be extended to the difference between parameter estimates p iterations apart, as follows:

$$\begin{aligned} & \left\| \hat{\theta}_k(t) - \hat{\theta}_{k-p}(t) \right\| \\ &= \left\| \hat{\theta}_k(t) - \hat{\theta}_{k-1}(t) + \hat{\theta}_{k-1}(t) - \hat{\theta}_{k-2}(t) + \dots + \hat{\theta}_{k-p+1}(t) - \hat{\theta}_{k-p}(t) \right\| \\ &\leq \left\| \hat{\theta}_k(t) - \hat{\theta}_{k-1}(t) \right\| + \left\| \hat{\theta}_{k-1}(t) - \hat{\theta}_{k-2}(t) \right\| + \dots + \left\| \hat{\theta}_{k-p+1}(t) - \hat{\theta}_{k-p}(t) \right\|. \end{aligned}$$

Thus, taking the limit as $k \rightarrow \infty$ on both sides,

$$\lim_{k \rightarrow \infty} \left\| \hat{\theta}_k(t) - \hat{\theta}_{k-p}(t) \right\| \leq \lim_{k \rightarrow \infty} \left\| \hat{\theta}_k(t) - \hat{\theta}_{k-1}(t) \right\| + \dots + \lim_{k \rightarrow \infty} \left\| \hat{\theta}_{k-p+1}(t) - \hat{\theta}_{k-p}(t) \right\|.$$

As each limit on the right is zero from (22), and the norm is always non-negative,

$$\lim_{k \rightarrow \infty} \left\| \hat{\theta}_k(t) - \hat{\theta}_{k-p}(t) \right\| = 0, \quad (23)$$

for each $t \in \mathcal{I}_T$, i.e. parameter estimates converge over iterations. This concludes the proof of Theorem 1. \blacksquare

In summary, this section has presented a new approach to solving the discrete-time Adaptive ILC problem. A new control law with an additional scaled tracking error term is formulated, and parameter estimates are updated using the identification error rather than the tracking error. It is then proved that each component of the identification and tracking error vectors converges to 0 with iterations k . The proof of convergence does not involve the KTL. Instead, simple inferences from the non-increasing nature of $V_k(t)$ are used concurrently with properties of ℓ_2 sequences. The approach presented in this section also enables the extension to the multiple estimation models case, as described in the following section.

4 Multiple Estimation Models for Adaptive ILC

Adaptive control strategies can suffer from the poor transient performance of identification, tracking and parametric errors when a single model is used for parameter estimation. In particular, the initial parametric uncertainty is likely large, contributing significantly to poor transient response. The methodology of Multiple Models, Switching and Tuning (MMST) was proposed to address this issue [52–54]. The methodology in discrete-time adaptive control is as follows. A number of models (say M) are initialized in the parameter space with different initial conditions. Each of these is updated according to standard parameter estimation algorithms [40] every sample. At each sample, one model is chosen according to a criterion, and the parameter estimates corresponding to that model are used for control design. The most common criterion used is a minimum identification error criterion, stated as follows. At each sample, pick the model j^* that satisfies $j^* = \arg \min_{j=1,\dots,M} |\hat{e}_j(t)|$, where $\hat{e}_j(t)$ denotes the identification error corresponding to model j at time instant t .

As mentioned in Section 1, there is very little existing research on the use of the MMST methodology in Adaptive ILC. This section presents the main results of this article, designing a general approach to using multiple models in Adaptive ILC, proposing two strategies for switching between models and proving convergence in both cases. Section 4.1 describes the formulation of control and identification laws for the proposed strategies, and Section 4.2 presents the proof of convergence of the identification and tracking errors.

4.1 Control and Identification Laws

The basic formulation of the problem remains the same as described in Section 2. However, instead of a single identification model (5) as in Section 3, we construct M identification models. Let $\mathcal{M} = \{1, \dots, M\}$ denote the set of model indices. Then, each model has a state $\hat{x}_{j,k}(t)$, $j \in \mathcal{M}$, that evolves as follows:

$$\begin{aligned} \hat{x}_{i,j,k}(t+1) &= \hat{x}_{i+1,j,k}(t), \quad i = 1, \dots, (n-1), \\ \hat{x}_{n,j,k}(t+1) &= \hat{\theta}_{1,j,k}^T(t) \xi(x_k(t)) + \hat{b}_{j,k}(t) u_k(t) + \hat{d}_{j,k}(t) \\ &= \hat{\theta}_{j,k}^T(t) \phi_k(t). \end{aligned} \tag{24}$$

$\hat{x}_{i,j,k}(t)$ denotes state i of identification model j in iteration k and sample t , and $\hat{\theta}_{j,k}(t)$ denotes the parameter estimate of model j in iteration k and sample t . Define the M parametric errors $\tilde{\theta}_{j,k}(t) \triangleq \theta(t) - \hat{\theta}_{j,k}(t)$, and the M identification errors $\hat{e}_{j,k}(t) \triangleq x_k(t) - \hat{x}_{j,k}(t)$. Using (2) and (24),

$$\begin{aligned} \hat{e}_{i,j,k}(t+1) &= \hat{e}_{i+1,j,k}(t), \quad i = 1, \dots, (n-1), \\ \hat{e}_{n,j,k}(t+1) &= \tilde{\theta}_{j,k}^T(t) \phi_k(t). \end{aligned} \tag{25}$$

As before, the tracking error is described by eq. (7).

We are now presented with two options:

4.1.1 Case 1

Continue using the analogy between the discrete-time axis in adaptive control and the iteration axis in Adaptive ILC, and switch between models only once every iteration, at the end. The criterion for switching is then chosen as:

$$j_k^* = \arg \min_{j \in \mathcal{M}} \left[\sum_{t \in \mathcal{I}_T} |\hat{e}_{n,j,k-1}(t+1)|^2 \right], \quad (26)$$

i.e. the model producing minimum energy in the n th component of the identification error (and hence minimum energy in the identification error vector) in iteration $(k-1)$ is chosen for control design in iteration k .

4.1.2 Case 2

Switch between models at every sample t in every iteration k . The criterion for switching is then chosen as:

$$j_k^*(t) = \arg \min_{j \in \mathcal{M}} |\hat{e}_{n,j,k}(t)|, \quad (27)$$

i.e. at every sample, a new model is chosen based on the minimum identification error at that sample, and is used for control design at that sample.

Remark 6. In Case 2, as identification error is on the time horizon $\{1, \dots, T\}$, so is the sequence of models $j_k^*(t)$. However, the control design is on the horizon $t \in \mathcal{I}_T$. Hence, the final model chosen, $j_k^*(T)$, is used for designing $u_{k+1}(0)$, the initial control input of the next iteration.

Note how the best model j^* depends only on iteration k in Case 1, but depends on both iteration k and time t in Case 2. The control law can then be formulated as follows:

$$u_k(t) = \frac{1}{\hat{b}_{j^*,k}(t)} \left[\beta e_{n,k-1}(t+1) + \rho_k(t) - \hat{\theta}_{1,j^*,k}^T(t) \xi_k(t) - \hat{d}_{j^*,k}(t) \right], \quad (28)$$

where $0 < \beta < 1$, and j^* denotes either j_k^* or $j_k^*(t)$, depending on whether criterion (26) or (27) is being used. Evidently, the control law uses parameter estimates corresponding to the model with minimum identification error in the sense of either criterion. In iteration 1, model 1 is chosen for control design without loss of generality. Note that all models continue to be updated irrespective of which model is chosen in (28). Substituting (28) in (7), and using (25),

$$\begin{aligned} e_{n,k}(t+1) &= \tilde{\theta}_{j^*,k}^T(t) \phi_k(t) + \beta e_{n,k-1}(t+1) \\ &= \hat{e}_{n,j^*,k}(t+1) + \beta e_{n,k-1}(t+1). \end{aligned} \quad (29)$$

Each model $j \in \mathcal{M}$ is updated according to the following law, similar to (10):

$$\hat{\theta}_{j,k+1}(t) = \text{Proj}[m] = \text{Proj} \left[\hat{\theta}_{j,k}(t) + \frac{\phi_k(t)}{1 + \|\phi_k(t)\|^2} \hat{e}_{n,j,k}(t+1) \right], \quad (30)$$

where $\text{Proj}[\cdot]$ is defined in (11). The algorithms 1 and 2 summarize the above procedure for both Case 1 and 2.

Algorithm 1 Computational flow with Multiple Models Case 1.

Initialisation: $\hat{\theta}_{j,0}(t) \leftarrow \text{random}, j \in \mathcal{M}, t \in \mathcal{I}_T; j_1^* = 1.$
for $k = 1, 2, \dots$ **do**
 for $t = 0, 1, \dots$ **do**
 Determine $\hat{x}_{j,k}(t+1)$ with $\phi_k(t)$ and $\hat{\theta}_{j,k}(t)$. {(24)}
 Determine $\hat{e}_{n,j,k}(t), j \in \mathcal{M}$ with $x_{n,k}(t), \hat{x}_{n,j,k}(t)$. {(25)}
 Determine $e_{n,k}(t)$, with $x_{n,k}(t), x_{m,n,k}(t)$. {(7)}
 Compute $u_k(t)$ with $\rho_k(t), \hat{\theta}_{j^*,k}(t)$ and $e_{n,k-1}(t+1)$. {(28)}
 end for
 Compute $\hat{\theta}_{j,k+1}(t)$ with $\hat{\theta}_{j,k}(t), \phi_k(t)$ and $\hat{e}_{n,j,k}(t+1)$. {(30)}
 Determine j_{k+1}^* using $\hat{e}_{n,j,k}(t+1)$. {(26)}
end for

Algorithm 2 Computational flow with Multiple Models Case 2.

Initialisation: $\hat{\theta}_{j,0}(t) \leftarrow \text{random}, j \in \mathcal{M}; j_1^*(t) = 1, t \in \mathcal{I}_T$
for $k = 1, 2, \dots$ **do**
 for $t = 0, 1, \dots$ **do**
 Determine $\hat{x}_{j,k}(t+1)$ with $\phi_k(t)$ and $\hat{\theta}_{j,k}(t)$. {(24)}
 Determine $\hat{e}_{n,j,k}(t), j \in \mathcal{M}$ with $x_{n,k}(t), \hat{x}_{n,j,k}(t)$. {(25)}
 Determine $e_{n,k}(t)$, with $x_{n,k}(t), x_{m,n,k}(t)$. {(7)}
 Determine $j_k^*(t)$ using $\hat{e}_{n,j,k}(t)$. {(27)}
 Compute $u_k(t)$ with $\rho_k(t), \hat{\theta}_{j^*,k}(t)$ and $e_{n,k-1}(t+1)$. {(28)}
 end for
 Compute $\hat{\theta}_{j,k+1}(t)$ with $\hat{\theta}_{j,k}(t), \phi_k(t)$ and $\hat{e}_{n,j,k}(t+1)$. {(30)}
end for

4.2 Convergence Analysis

We now present the primary result of this article for convergence of Adaptive ILC using multiple models:

Theorem 2. *For the system (2) with to track the reference model (3), the control law (28) along with the adaptive law (30) guarantees the following:*

1. $\tilde{\theta}_j(t), \hat{\theta}_j(t) \in \ell_\infty$ for each $t \in \mathcal{I}_T$, for each $j \in \mathcal{M}$, i.e. the sequence of parametric errors $\tilde{\theta}_{j,k}(t)$ over iterations — and hence the sequence of parameter estimates $\hat{\theta}_{j,k}(t)$ over iterations — is bounded for each sample t and model j .
2. $\hat{e}_{n,j}(t+1) \in \ell_2 \cap \ell_\infty$ for each $t \in \mathcal{I}_T$, for each $j \in \mathcal{M}$, i.e. the sequence of the n th component of the identification error over iterations is square-summable and bounded for each sample t and model j .
3. With identical initial conditions on the plant and reference model, $\lim_{k \rightarrow \infty} \hat{e}_{j,k}(t+1) = 0$ for each $t \in \mathcal{I}_T$, for each $j \in \mathcal{M}$, i.e. each component of the identification error vector

tends to zero with iterations, for each sample t and model j .

4. With identical initial conditions on the plant and reference model, $\lim_{k \rightarrow \infty} e_k(t+1) = 0$ for each $t \in \mathcal{I}_T$, i.e. each component of the tracking error vector tends to zero with iterations, for each sample t .
5. $\lim_{k \rightarrow \infty} \left\| \hat{\theta}_{j,k}(t) - \hat{\theta}_{j,k-p}(t) \right\|^2 = 0$ for each $t \in \mathcal{I}_T$, for each $j \in \mathcal{M}$, for any $p \in \mathbb{N}$, i.e. the parameter vector estimates converge over iterations for each sample t and model j .

Proof. As with the proof of Theorem 1, the proof of Theorem 2 is organized in three parts, with the statement 1 proved in Part 1, statements 2, 3 and 4 proved in Part 2 and statement 5 proved in Part 3.

Part 1: Boundedness of Parametric Error:

Define a composite energy function (CEF) $V_k(t)$ as:

$$V_k(t) \triangleq \sum_{j \in \mathcal{M}} V_{j,k}(t), \quad (31a)$$

$$V_{j,k}(t) \triangleq \tilde{\theta}_{j,k}^T(t) \tilde{\theta}_{j,k}(t) = \left\| \tilde{\theta}_{j,k}(t) \right\|^2, \quad (31b)$$

and let $\Delta V_{j,k}(t) \triangleq V_{j,k+1}(t) - V_{j,k}(t)$. By arguments similar to the ones made in the proof of Theorem 1,

$$\Delta V_{j,k}(t) \leq - \left(\frac{2 + \|\phi_k(t)\|^2}{(1 + \|\phi_k(t)\|^2)^2} \right) \hat{e}_{n,j,k}^2(t+1), \quad (32)$$

or,

$$\Delta V_{j,k}(t) \leq -\alpha_k^2(t) \hat{e}_{n,j,k}^2(t+1) \leq 0, \quad (33)$$

where, as before, $\alpha_k^2(t)$ denotes the positive quantity within parentheses in (32). Hence, $V_{j,k}(t)$ is non-increasing for each j , and thus,

$$\Delta V_k(t) \triangleq V_{k+1}(t) - V_k(t) = \sum_{j \in \mathcal{M}} \Delta V_{j,k}(t) \leq 0, \quad (34)$$

or, $V_k(t)$ is a non-increasing function. From the construction of $V_{j,k}(t)$, it is evident that the sequence of parametric errors $\tilde{\theta}_{j,k}(t)$ is a bounded sequence over iterations k , for each $t \in \mathcal{I}_T$, i.e. $\tilde{\theta}_j(t) \in \ell_\infty$. As $\theta(t)$ is bounded, we conclude that $\hat{\theta}_j(t) \in \ell_\infty$, i.e. the sequence of parameter estimates $\hat{\theta}_{j,k}(t)$ is bounded over iterations k , for each $t \in \mathcal{I}_T$.

Part 2: Convergence of Errors:

From (33), $\lim_{N \rightarrow \infty} |V_{j,N+1}(t) - V_{j,1}(t)| < \infty$ for each $t \in \mathcal{I}_T$. Rewriting this,

$$\lim_{N \rightarrow \infty} \left| \sum_{k=1}^N \Delta V_{j,k}(t) \right| \leq \lim_{N \rightarrow \infty} \sum_{k=1}^N \alpha_k^2(t) \hat{e}_{n,j,k}^2(t+1) < \infty.$$

Using the same arguments as earlier, $\hat{e}_{n,j}(t+1) \in \ell_2 \cap \ell_\infty$, and

$$\lim_{k \rightarrow \infty} \hat{e}_{n,j,k}(t+1) = 0 \quad (35)$$

for each $t \in \mathcal{I}_T$, for each model $j \in \mathcal{M}$. Under the assumption of identical initial conditions, this implies that:

$$\lim_{k \rightarrow \infty} \hat{e}_{j,k}(t+1) = 0. \quad (36)$$

Now consider eq. (29). While we know that $\hat{e}_{n,j,k}(t+1) \rightarrow 0$ as $k \rightarrow \infty$, the actual sequence $\hat{e}_{n,j^*,k}(t+1)$ that acts as a forcing function here depends on the switching criterion considered, either (26) or (27). We now show that $\hat{e}_{n,j^*,k}(t+1) \rightarrow 0$ as $k \rightarrow \infty$, where j^* denotes j_k^* or $j_k^*(t)$, depending on whether criterion (26) or (27) is used. For notational simplicity, let $\hat{e}_{j,k}$ denote $\hat{e}_{n,j,k}(t+1)$. Construct the following sequence at each sample t :

$$S = \underbrace{\hat{e}_{1,1}, \dots, \hat{e}_{M,1}}_{j \in \mathcal{M}, k=1}, \underbrace{\hat{e}_{1,2}, \dots, \hat{e}_{M,2}}_{j \in \mathcal{M}, k=2}, \dots, \underbrace{\hat{e}_{1,k}, \dots, \hat{e}_{M,k}}_{j \in \mathcal{M}, k=k}, \dots \quad (37)$$

This is a sequence of identification errors of each model, considered one iteration after another. Since $\hat{e}_{j,k} \rightarrow 0$ as $k \rightarrow \infty$, the above sequence $S \rightarrow 0$ as $k \rightarrow \infty$. Then, if S^* denotes any subsequence of S , $S^* \rightarrow 0$ as $k \rightarrow \infty$. We exploit this fact to show that using either criterion (26) or (27), the forcing function $\hat{e}_{n,j^*,k}(t+1)$ in (29) converges to 0 as $k \rightarrow \infty$.

With criterion (26), switching takes place only once every iteration, at the end. The optimal model j_k^* does not depend on the sample t . Then, the forcing function sequence in (29) can be written as $S^* = \hat{e}_{j_1^*,1}, \hat{e}_{j_2^*,2}, \dots$, with each $j_k^* \in \mathcal{M}$. S^* is evidently a subsequence of S , and as $S \rightarrow 0$ as $k \rightarrow \infty$, $S^* \rightarrow 0$ as $k \rightarrow \infty$, and hence $\hat{e}_{n,j^*,k}(t+1) \rightarrow 0$ as $k \rightarrow \infty$, for each t .

The arguments for criterion (27) are very similar. The optimal model $j_k^*(t)$ is now dependent on the sample t . For a given t , the forcing function sequence in (29) can be written as $S^*(t) = \hat{e}_{j_1^*(t),1}, \hat{e}_{j_2^*(t),2}, \dots$, with each $j_k^*(t) \in \mathcal{M}$. $S^*(t)$ is a subsequence of S for each t , and by the above arguments, $S^*(t) \rightarrow 0$ as $k \rightarrow \infty$, and hence $\hat{e}_{n,j^*,k}(t+1) \rightarrow 0$ as $k \rightarrow \infty$, for each t .

The minor difference between the two arguments lies in the fact that the subsequence constructed depends on the sample t in the second case. For both criteria (26) and (27),

$$\lim_{k \rightarrow \infty} \hat{e}_{n,j^*,k}(t+1) = 0 \quad (38)$$

for each $t \in \mathcal{I}_T$. Then, eq. (29) is an iteration-domain difference equation with a forcing function that converges to 0. As $0 < \beta < 1$, it is evident that:

$$\lim_{k \rightarrow \infty} e_{n,k}(t+1) = 0 \quad (39)$$

for each $t \in \mathcal{I}_T$. Under the assumption of identical initial conditions,

$$\lim_{k \rightarrow \infty} e_k(t+1) = 0. \quad (40)$$

Part 3: Convergence of Parameter Estimates:

The final part of the proof is straightforward and simply extends the arguments made in the corresponding part of the proof of Theorem 1 to the multiple model case. It can first

be shown that the relation $\left\| \text{Proj}[m] - \hat{\theta}_{j,k}(t) \right\| \leq \left\| m - \hat{\theta}_{j,k}(t) \right\|$ always holds, for each model $j \in \mathcal{M}$. Then, using (30),

$$\left\| \hat{\theta}_{j,k+1}(t) - \hat{\theta}_{j,k}(t) \right\|^2 \leq \hat{e}_{n,j,k}^2(t+1).$$

Then, summing the above inequality over iterations k and using the properties of the ℓ_2 sequence $\hat{e}_{n,j}(t+1)$,

$$\lim_{k \rightarrow \infty} \left\| \hat{\theta}_{j,k+1}(t) - \hat{\theta}_{j,k}(t) \right\| = 0 \quad (41)$$

for each $t \in \mathcal{I}_T$. Thus, for each model $j \in \mathcal{M}$, parameter estimates one iteration apart converge for each sample t . For parameter estimates p iterations apart, $\left\| \hat{\theta}_{j,k}(t) - \hat{\theta}_{j,k-p}(t) \right\|$ is written as a telescoping series, as shown earlier. By the same arguments,

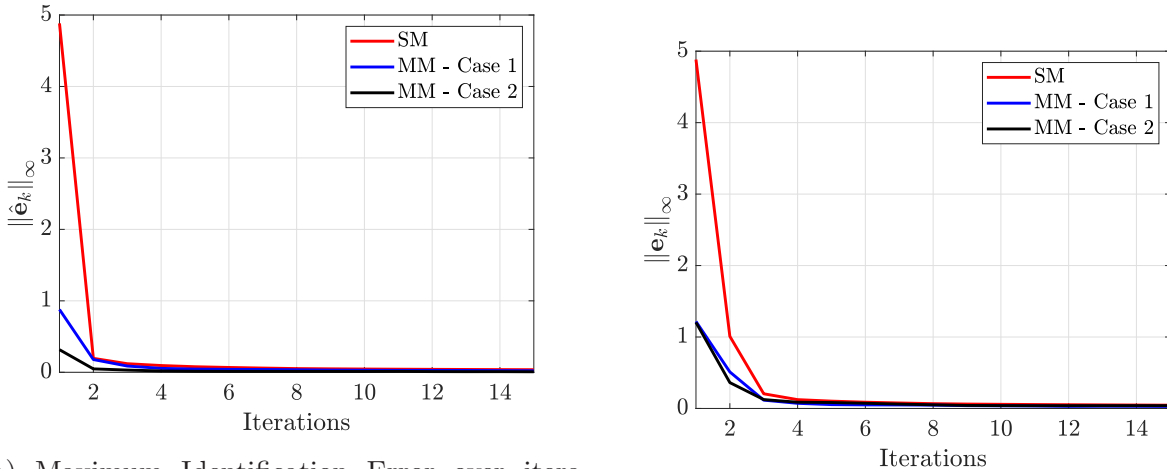
$$\lim_{k \rightarrow \infty} \left\| \hat{\theta}_{j,k}(t) - \hat{\theta}_{j,k-p}(t) \right\| = 0 \quad (42)$$

for each $t \in \mathcal{I}_T$, for each model $j \in \mathcal{M}$. This concludes the proof of Theorem 2. \blacksquare

Summarizing the results of this section, we have presented an approach using multiple estimation models to solve the discrete-time Adaptive ILC problem. This approach is enabled by using each model's identification error in updating the corresponding parameter estimates. The control law is formulated based on the optimal model at sample t , in iteration k . We provide two options for switching between models — either once in an iteration or once every sample — and each option has its own criterion. Using either criterion, we prove that each component of the identification and tracking error vectors converge to 0 with iterations k . A key step in this proof is to show that the sequence of identification errors corresponding to the best model — $\hat{e}_{n,j^*,k}(t+1)$ — converges to 0 as $k \rightarrow \infty$, using either criterion. As with the strategy in Section 3, the proof of convergence does not involve the KTL and the properties of $V_k(t)$ and ℓ_2 sequences are used instead.

5 Simulation Examples

In this section, we present simulation examples to demonstrate the efficacy of the single-model strategy proposed in Section 3, and the two switching strategies with multiple models in Section 4. Four different first-order systems are considered: a linear, time-invariant system not subjected to disturbances (LTI), a linear, time-varying system subjected to disturbances (LTV-D), a nonlinear system not subjected to disturbances (NL) and a nonlinear system subjected to disturbances (NL-D). In each example, the time interval for each iteration is $\{0, 1, \dots, 100\}$, and hence the time index t is in the set $\mathcal{I}_T = \{0, 1, \dots, 99\}$. The parameter β in the control laws (8) and (28) is set to 0.2. Zero initial conditions on the plant and reference are assumed in all examples. For the multiple-model cases, the number of models is set to $M = 10$, and parameters are initialized randomly in the parameter space. The strategy that applies the single model control law (8) is designated “SM”, and the strategies that use the multiple-model control law (28) with criterion (26) or (27) are designated “MM



(a) Maximum Identification Error over iterations.

(b) Maximum Tracking Error over iterations.

Figure 1: Example 1: Error profiles over iterations for an LTI system without disturbances.

- Case 1” and “MM - Case 2” respectively. For each example, the objective is to track the following iteration-invariant reference:

$$x_m(t) = \pi^2 (2 - 3 \sin^3(2\pi t/100)) \sin(2\pi t/100)/10, \quad (43)$$

which is similar to the trajectory considered in [39]. The efficacy of each strategy is measured based on the peak identification and tracking errors over iterations, which ideally converge to zero. This is the same as considering the ∞ -norm of both errors, defined below for the identification error:

$$\|\hat{e}_k\|_\infty = \max_t |x_k(t+1) - \hat{x}_k(t+1)| = \max_t |\hat{e}_k(t+1)|, \quad (44)$$

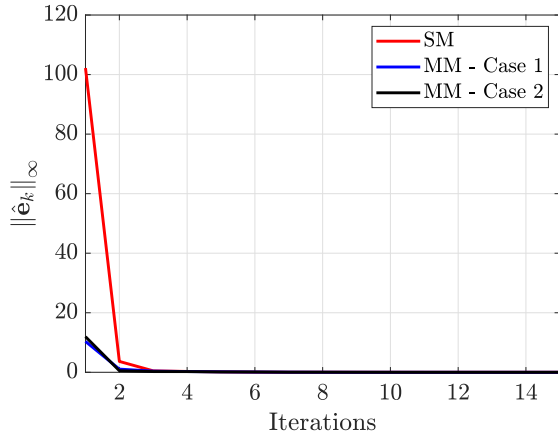
and defined similarly for the tracking error. An iteration-invariant trajectory is considered for simplicity, to highlight the advantages of faster convergence in multiple models. The final example in this section presents results for tracking an iteration-varying trajectory.

5.1 Example 1: LTI System without Disturbances

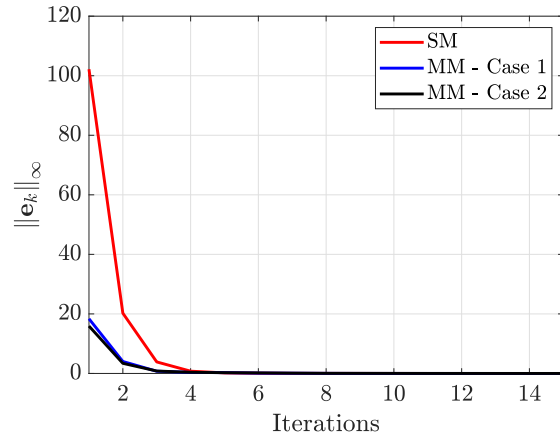
Consider the system:

$$x_k(t+1) = 0.5x_k(t) + u_k(t), \quad (45)$$

a simple, stable LTI system without disturbances. The objective is for $x_k(t)$ to track the reference $x_m(t)$ in (43). The results for identification and tracking performance are shown in Fig. 1, in terms of the peak amplitude of errors over iterations for each strategy. It is evident that both multiple-model strategies converge faster than the single-model strategy, mainly because the initialization of multiple estimation models leads to better estimates in earlier iterations, hence improving transient performance. Further, the multiple model strategy with criterion (27), i.e. MM - Case 2 converges marginally faster than MM - Case 1, due to models switching more frequently.

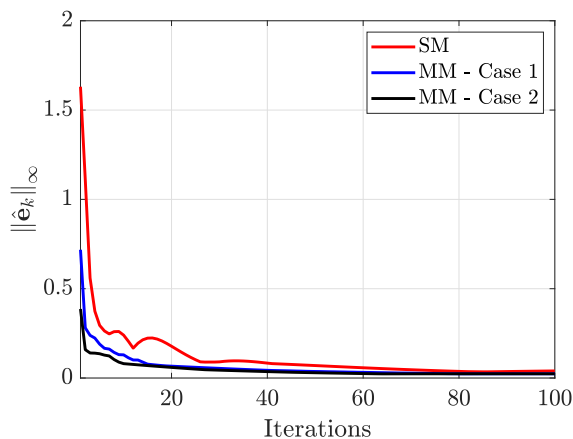


(a) Maximum Identification Error over iterations.

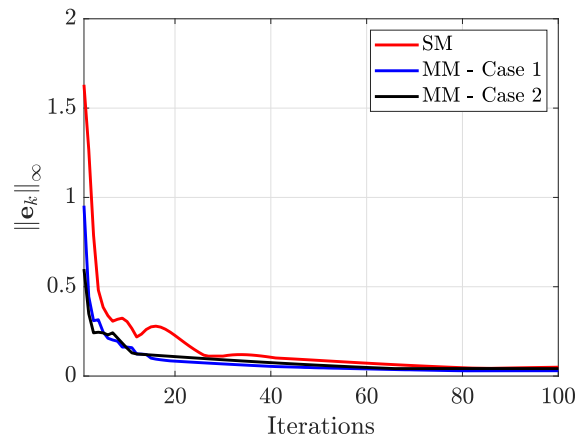


(b) Maximum Tracking Error over iterations.

Figure 2: Example 2: Error profiles over iterations for an LTV system with disturbances.



(a) Maximum Identification Error over iterations.



(b) Maximum Tracking Error over iterations.

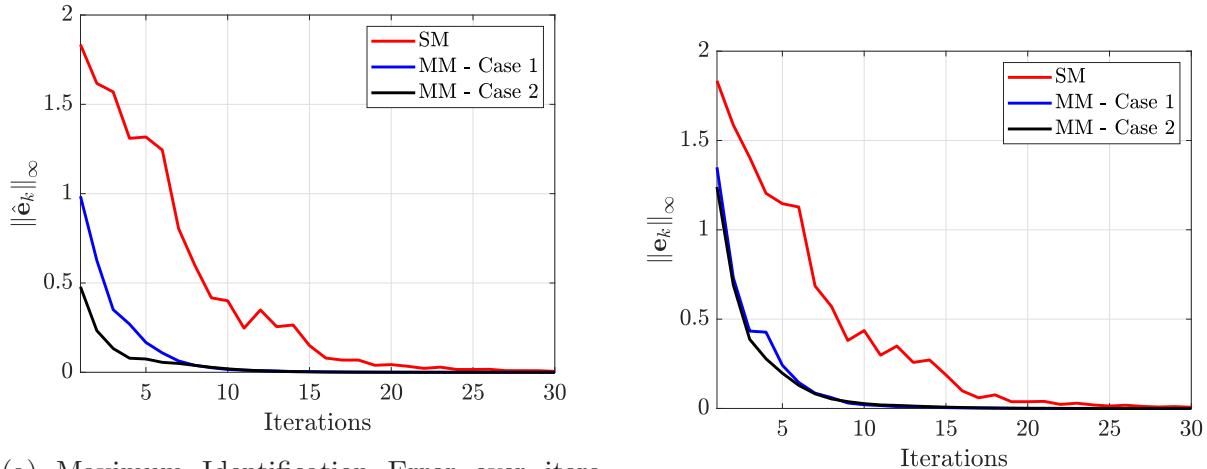
Figure 3: Example 3: Error profiles over iterations for a nonlinear system without disturbances.

5.2 Example 2: LTV System with Disturbances

In this example, the following LTV system with disturbances is considered:

$$x_k(t+1) = \theta_1(t)x_k(t) + b(t)u_k(t) + d(t), \quad (46)$$

where $\theta_1(t) = 1 + 0.5\sin(t)$, $b(t) = 3 + 0.5\sin(2\pi t)$ and $d(t) = \sin^3(2\pi t)$, an external disturbance. The results for achieving the tracking objective are shown in Fig. 2. As before, the two multiple-model cases achieve faster convergence due to improved transient response, whereas the single-model case has very poor transient response due to large initial parametric



(a) Maximum Identification Error over iterations.

(b) Maximum Tracking Error over iterations.

Figure 4: Example 4: Error profiles over iterations for a nonlinear system with disturbances.

errors. This example also demonstrates the first instance of time-varying parameters being successfully identified over iterations.

5.3 Example 3: Nonlinear System without Disturbances

The following nonlinear system is considered:

$$x_k(t+1) = \theta_1(t) \sin^2(x_k(t)) + b(t)u_k(t), \quad (47)$$

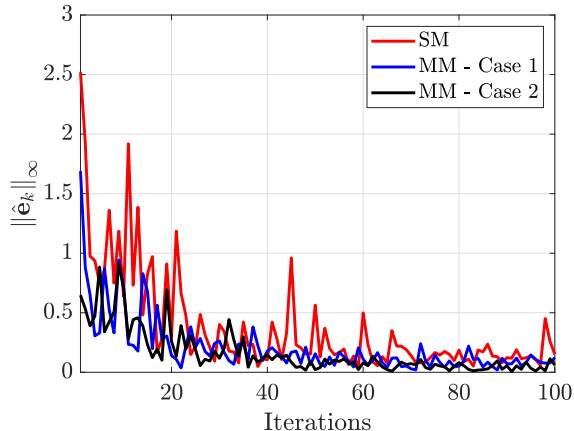
where $\theta_1(t) = 1 + 0.5 \sin(t)$ and $b(t) = 3 + 0.5 \sin(2\pi t)$, as before. Note the nonlinearity in the regression vector, in contrast to the previous examples. Fig. 3 shows the results for identification and tracking performance for this system, in terms of the peak amplitude of the errors. The two multiple-model cases are seen to achieve faster convergence, and in particular, the strategy “MM - Case 2” converges marginally faster due to higher frequency of switching.

5.4 Example 4: Nonlinear System with Disturbances

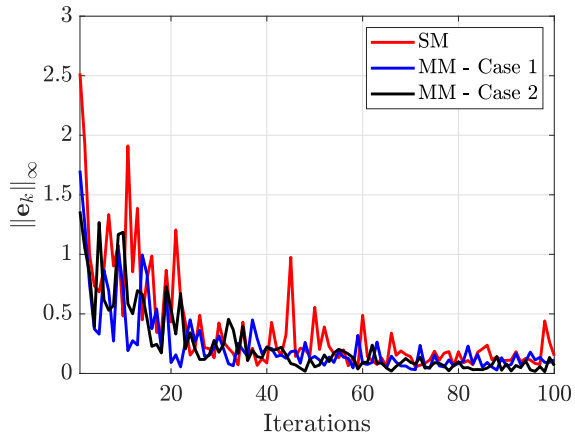
Finally, the most general system is considered:

$$x_k(t+1) = \theta_1(t) \sin^2(x_k(t)) + b(t)u_k(t) + d(t), \quad (48)$$

where $\theta_1(t) = 1.5 + 0.5 \sin(t)$, $b(t) = 3 + 0.5 \sin(2\pi t)$ and $d(t) = \sin^3(2\pi t)$, the same disturbance considered in (46). This is very similar to the system considered in [39]. Fig. 4 shows the performance for tracking the reference (43) over iterations, in terms of peak identification and tracking errors. It is evident that the convergence for both multiple-model cases is significantly faster than the single-model case, with MM - Case 2 providing the fastest convergence. Interestingly, the errors for this system also converge faster than the errors for the system (47), which was not affected by disturbances. This is due to the presence of $d(t)$ and its estimate, resulting in a persistently exciting control law.



(a) Maximum Identification Error over iterations.



(b) Maximum Tracking Error over iterations.

Figure 5: Example 5: Error profiles over iterations for a nonlinear system with disturbances, with an iteration-varying reference trajectory.

5.5 Example 5: Iteration-varying Reference Trajectory

We also present results for the system (48) tracking an iteration-varying reference trajectory:

$$x_{m,k}(t) = \vartheta(k)\pi^2 (2 - 3 \sin^3(2\pi t/100)) \sin(2\pi t/100)/10, \quad (49)$$

where $\vartheta(k) \sim \mathcal{U}[-0.5, 0.5]$, i.e. a uniformly distributed random variable between -0.5 and 0.5 , in each iteration k . This is similar to the trajectory considered in [39]. The results for this example are shown in Fig. 5. It is easily seen that all errors are decreasing and are quite close to 0. Further, the two multiple-model strategies are seen to perform better than the single-model strategy. To reinforce this, Table 1 presents the root-mean-square value of peak error amplitudes over iterations for identification and tracking errors for all three strategies. This metric is defined below for the identification error:

$$\text{Metric} = \sqrt{\frac{1}{K} \sum_{k=1}^K \|\hat{\mathbf{e}}_k\|_\infty^2}, \quad (50)$$

and defined similarly for the tracking error. K denotes the total number of iterations in the simulation. Lower values of this metric indicate better performance. From Table 1, it is evident that both multiple-model strategies have significantly smaller values of this metric, indicating smaller identification and tracking errors and faster convergence. The smallest values are in MM - Case 2, indicating that this strategy achieves the best possible performance. Further, Table 2 presents the number of iterations k^* taken for tracking convergence. In particular, we consider the number of iterations taken for the peak tracking error $\|\mathbf{e}_k\|_\infty$ to fall below 2% of the first iteration peak tracking error $\|\mathbf{e}_1\|_\infty$ of the single model strategy. It is once again evident that both multiple model strategies converge faster, with MM - Case 2 converging fastest. As mentioned earlier, this is partly because the initialization of

Table 1: Root Mean-Square Maximum Errors

Strategy	Identification Error	Tracking Error
Single Model	6.3288	6.2884
Multiple Models – Case 1	3.0544	3.4948
Multiple Models – Case 2	2.0728	3.1338

Table 2: Iterations for Tracking Convergence

Strategy	Number of Iterations k^*
Single Model	79
Multiple Models – Case 1	58
Multiple Models – Case 2	47

multiple estimation models leads to better estimates in earlier iterations, leading to faster convergence.

In conclusion, all simulation examples demonstrate that the proposed strategies result in convergence of identification and tracking errors to zero. The multiple model strategies converge much faster than the single model strategy, and the multiple model strategy with criterion (27) converges faster than that with criterion (26). Note how the initial error magnitudes were larger for the linear systems compared to the nonlinear systems. This is due to the presence of the nonlinearity $\sin^2(x_k(t))$, which is always bounded between 0 and 1 irrespective of the value of $x_k(t)$. The nonlinear system subjected to disturbances also shows faster error convergence than the system without disturbances.

6 Concluding Remarks

In this article, we have proposed a complete framework for using the Multiple Models, Switching and Tuning (MMST) methodology in the context of discrete-time Adaptive Iterative Learning Control (ILC). First, the single estimation model case is considered, a new control and identification scheme is presented, and convergence is proved using the properties of square-summable, or ℓ_2 sequences. The update law for parameter estimates uses the identification error rather than the tracking error, in contrast to existing Adaptive ILC schemes. This enables the extension to multiple estimation models. In the case of multiple models, we have described two criteria for switching between models — either at the end of each iteration or at each sample. In both options, convergence is proved in a unified manner using the properties of square-summable sequences. An extensive set of simulation results are presented for four different types of systems. In all cases, it is seen that the identification and tracking errors converge to zero. In particular, it is seen that the second switching criterion for multiple-models outperforms the first, which in turn outperforms the single model case.

A drawback of the strategies presented here is their high computational complexity,

particularly for the second switching criterion (27) in multiple models. It is known that the Multiple Models with Second-Level Adaptation (MM-SLA) scheme [62, 63] has lower computational complexity compared to MMST, as a much smaller number of estimation models, is required. This was explored for Adaptive ILC in a contraction-mapping setting in [61], and an interesting avenue for future work is to explore this in the context of CEF-based Adaptive ILC. Further, as mentioned in Section 1, the techniques proposed here can be extended to the case with time- and iteration-varying parameters, and also to the case when the control direction is unknown.

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