CRITERIA FOR THE AMPLENESS OF CERTAIN VECTOR BUNDLES

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Abstract. We prove that certain vector bundles over surfaces are ample if they are so when restricted to divisors, certain numerical criteria hold, and they are semistable (with respect to $\det(E)$). This result is a higher-rank version of a theorem of Schneider and Tancredi for vector bundles of rank two over surfaces. We also provide counterexamples indicating that our theorem is sharp.

1. Introduction

A holomorphic vector bundle E over a compact complex manifold X is said to be ample if $\mathcal{O}_{\mathbb{P}(E^*)}(1)$ is ample over $\mathbb{P}(E^*)$ (the projective bundle over X parametrising the lines in the fibres of E^*). Ample vector bundles play an important role in algebraic geometry because of the various vanishing theorems that ensue from ampleness. Many of these vanishing theorems have numerous geometric consequences. It is therefore of interest to find criteria for ampleness.

For any line bundle L over a projective manifold X, the Nakai-Moizeshon criterion gives a numerical condition to decide the ampleness of L [4]. It says that L is ample if for all $1 \leq k \leq \dim X$, we have $c_1(L)^k.Y > 0$ for every closed subvariety Y of dimension k of X. However, no numerical criterion can exist for deciding the ampleness of vector bundles [2]. Notwithstanding this negative result, Schneider and Tancredi proved the following criterion (that is not purely numerical) for rank two vector bundles over surfaces.

Theorem 1.1 ([7, p. 134])**.** *Let E be a holomorphic vector bundle of rank two over a compact complex surface X. Assume that $c_1(E) > 0$ and that E is semistable with respect to $\det(E)$. Suppose $E\big|_C$ is ample for every closed curve $C \subset X$, and*

$$(c_1(E)^2 - 2c_2(E)).X > 0, \quad c_2(E).X > 0.$$

Then E is ample.

This result and its improvements carried out in [8] are useful in studying the ampleness of cotangent bundles [6]. They also find use in an approach of Demailly to the Green-Griffiths-Lang conjecture [1]. In this paper we aim to

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generalise this theorem to vector bundles of other ranks for further possible applications.

We were motivated by the following result of Lübke [5].

Theorem 1.2 ([5, p. 313, Theorem 2.1])**.** *Let $(E,\, h)$ be a holomorphic Hermitian rank-r vector bundle over a compact Kähler manifold $(X,\, \omega)$ of dimension n. Suppose $F_h \wedge \omega^{n-1} \,=\, -\sqrt{-1}\lambda\omega^n$, where F_h is the curvature of the Chern connection of h and $\lambda \,>\, 0$ is a constant. Assume that*

$$c_1(E,\, h) \,=\, \frac{r\lambda}{2\pi}\omega.$$

Also, suppose there exists a positive function ψ such that either of the following holds:

(1) $n \,=\, 2$ *and* $c_1^2(E,h) - \frac{2r(r-1)}{r^2-2r+2}c_2(E,\, h) \,=\, \psi\omega^2$, *or*
(2) $r \,=\, 2$ *and* $c_1^2(E,h) - \frac{4(n-1)^2}{n^2-2n+2}c_2(E,\, h) \,=\, \psi\omega^2$.

Then h is Griffiths-positively curved, i.e., $\langle v,\, \sqrt{-1}F_h v\rangle$ is a Kähler form whenever $v \,\neq\, 0$ is a vector in E.

The following is the main result of this paper.

Theorem 1.3. *Let E be a holomorphic vector bundle of rank r over a compact complex manifold X of dimension two. Suppose $c_1(E) \,>\, 0$ and E is semistable with respect to* $\det(E)$. *Also assume that E restricted to every curve is ample, and that that $(c_1^2 - c_2)(E).X \,>\, 0$. Then E is ample if*

$$\left(c_1^2(E) - \frac{2r(r-1)}{r^2-2r+2}c_2(E)\right).X \,>\, 0. \tag{1.1}$$

The proof of Theorem 1.3 (carried out in Section 2) uses the existence of approximately Hermitian-Einstein metrics on semistable vector bundles [3].

In Section 3 we provide examples to indicate that Lübke's Chern class inequality in Theorem 1.2 cannot be dispensed with for $n \,=\, 2$ (and arbitrary r).

2. Proof of Theorem 1.3

In this section, we prove Theorem 1.3. The Nakai-Moishezon criterion will be used [4]. Our aim is to show that $(c_1(\mathcal{O}_{\mathbb{P}(E^*)}(1)))^d.Y \,>\, 0$ for every subvariety Y of $\mathbb{P}(E^*)$ of dimension d. Let

$$\pi \,:\, \mathbb{P}(E^*) \,\longrightarrow\, X$$

be the natural projection. If $\pi(Y)$ is a point, we are done trivially. If $\pi(Y)$ is a curve, then since E restricted to $\pi(Y)$ is ample, we are done. So assume that $\pi(Y) \,=\, X$.

In this case we shall compute the intersection number by choosing an appropriate smooth metric h on E and considering the Chern-Weil representative of the induced metric $\widetilde{h}$ on $\mathcal{O}_{\mathbb{P}(E^*)}(1)$. Firstly, we fix a Kähler form ω on X. Let

$$S \,\subset\, X$$

be the smallest Zariski closed proper subset such that $Y \setminus \pi^{-1}(S)$ consists of regular points of the projection map over $X \setminus S$, i.e., $Y \setminus \pi^{-1}(S)$ is a smooth fibre bundle over $X \setminus S$.

For every ϵ, there exists an approximate Hermitian-Einstein metric h_ϵ (with curvature F_ϵ) satisfying $c_1(h_\epsilon) = \omega$ and (2.3) [3]. We shall choose ϵ later. Let $\Theta_\epsilon = \frac{\sqrt{-1}}{2\pi} F_\epsilon$. Let $(p, [v]) \in Y$. The key point is that if we choose a holomorphic normal trivialisation of E near p, then

$$c_1(\mathcal{O}_{\mathbb{P}(E^*)}(1), \widetilde{h}_\epsilon)(p, [v]) = \frac{\langle v, \pi^* \Theta_\epsilon v\rangle}{\langle v, v\rangle} + \omega_{FS}, \tag{2.1}$$

where ω_{FS} is the Fubini-Study metric on the fibres of π (restricted to Y). Therefore,

$$c_1(\mathcal{O}_{\mathbb{P}(E^*)}(1), \widetilde{h}_\epsilon)^d(p, [v]) =$$

$$\frac{d(d-1)}{2}\left(\pi^* \frac{\langle v, \Theta_\epsilon, v\rangle}{\langle v, v\rangle}\right)^2 \omega_{FS}^{d-2} + \omega_{FS}^d + d\pi^* \frac{\langle v, \Theta_\epsilon, v\rangle}{\langle v, v\rangle}\omega_{FS}^{d-1}$$

$$= \frac{d(d-1)}{2}\left(\pi^* \frac{\langle v, \Theta_\epsilon, v\rangle}{\langle v, v\rangle}\right)^2 c_1(\mathcal{O}_{\mathbb{P}(E^*)}(1), \widetilde{h}_\epsilon)^{d-2}(p, [v]), \tag{2.2}$$

where we noted that on a surface, at most two powers of $\frac{\langle v, \Theta_\epsilon, v\rangle}{\langle v, v\rangle}$ are non-zero, and since the dimension of the fibre is $d-2$, we have $\omega_{FS}^{d-1} = 0$ on $Y \cap \pi^{-1}(p)$ for any $p \in S$. Note that the expression

$$\frac{d(d-1)}{2}\left(\pi^* \frac{\langle v, \Theta_\epsilon, v\rangle}{\langle v, v\rangle}\right)^2 c_1(\mathcal{O}_{\mathbb{P}(E^*)}(1), \widetilde{h}_\epsilon)^{d-2}(p, [v])$$

is independent of the choice of the local trivialisation of E.

At this juncture, the following lemma, which is a pointwise assertion, will be used to lower bound the right-hand-side of (2.2).

Lemma 2.1. *For every $\epsilon > 0$, let Θ_ϵ be a (normalized) Chern curvature endomorphism of a Hermitian holomorphic vector bundle (E, h_ϵ), of rank r, at a point p on a surface X. Let $v \in E_p$. Suppose ω is a Kähler form at p and $c_1(h_\epsilon) = \omega$. Moreover, assume that for every $\epsilon > 0$, there exists a trace-free endomorphism B_ϵ of E at p satisfying $\left|(B_\epsilon)_i^j\right| \leq \epsilon$ for all i, j, and*

$$\Theta_\epsilon \wedge \omega = \frac{1}{r}\omega^2 + \frac{B_\epsilon}{2}\omega^2. \tag{2.3}$$

Then

$$c_1^2(h_\epsilon) - \frac{2r(r-1)}{r^2-2r+2}c_2(h_\epsilon) \leq \frac{r^2}{r^2-2r+2}\left(\frac{\langle v, \Theta_\epsilon v\rangle_{h_\epsilon}}{\langle v, v\rangle_{h_\epsilon}}\right)^2 + \frac{4r + r(r^2-1)\epsilon}{4(r^2-2r+2)}\epsilon\omega^2. \tag{2.4}$$

Proof. Since the statement does not depend on the choice of trivialisation as well as the choice of coordinates, we can assume that h_ϵ is given by the identity matrix at this point p, in other words, h_ϵ is the trivial Hermitian structure. Moreover, without loss of generality, we assume that $v = (1, 0, 0)$. For ease of notation, we drop the ϵ subscript on Θ_ϵ. Lastly, we can choose coordinates so that

$$\omega = \sum_{i=1}^{r} \Theta_i^i = \sqrt{-1}dz^1 \wedge d\overline{z}^1 + \sqrt{-1}dz^2 \wedge d\overline{z}^2,$$

and $\Theta_1^1 = \sqrt{-1}\mu_1 dz^1 \wedge d\overline{z}^1 + \sqrt{-1}\mu_2 dz^2 \wedge d\overline{z}^2$. By the approximate Hermitian-Einstein condition (2.3) we see that

$$\begin{aligned}
\mu_1 + \mu_2 &= \frac{2}{r} + (B_\epsilon)_1^1 \\
\Theta_2^2 + \Theta_3^3 + \ldots &= (1-\mu_1)\sqrt{-1}dz^1 \wedge d\overline{z}^1 + \left(1 - \frac{2}{r} + \mu_1 - (B_\epsilon)_1^1\right)\sqrt{-1}dz^2 \wedge d\overline{z}^2, \\
\Theta_i^i \wedge \omega &= \frac{1}{r}\omega^2 + \frac{1}{2}(B_\epsilon)_i^i \omega^2, \\
(\Theta_i^j)_{1\overline{1}} + (\Theta_i^j)_{2\overline{2}} &= (B_\epsilon)_i^j. \qquad (2.5)
\end{aligned}$$

Now,

$$\begin{aligned}
\frac{c_1(h)^2 - \frac{2r(r-1)}{r^2-2r+2}c_2(h)}{\sqrt{-1}dz^1 \wedge d\overline{z}^1 \wedge \sqrt{-1}dz^2 \wedge d\overline{z}^2} &= 2 - \frac{r(r-1)}{r^2-2r+2}\frac{\sum_{i\neq j}\left(-\Theta_i^j\overline{\Theta_i^j} + \Theta_i^i\Theta_j^j\right)}{\sqrt{-1}dz^1 \wedge d\overline{z}^1 \wedge \sqrt{-1}dz^2 \wedge d\overline{z}^2} \\
&\leq 2 - \frac{r(r-1)}{r^2-2r+2}\times \\
\frac{\sum_{i\neq j}\left(-((\Theta_i^j)_{1\overline{1}}\overline{(\Theta_i^j)_{2\overline{2}}} + (\Theta_i^j)_{2\overline{2}}\overline{(\Theta_i^j)_{1\overline{1}}})\sqrt{-1}dz^1 \wedge d\overline{z}^1 \wedge \sqrt{-1}dz^2 \wedge d\overline{z}^2 + \Theta_i^i\Theta_j^j\right)}{\sqrt{-1}dz^1 \wedge d\overline{z}^1 \wedge \sqrt{-1}dz^2 \wedge d\overline{z}^2}&.
\end{aligned}$$

Using (2.5) we see that

$$
\begin{aligned}
&\frac{c_1(h)^2 - \frac{2r(r-1)}{r^2-2r+2}c_2(h)}{\sqrt{-1}dz^1\wedge d\overline{z}^1\wedge\sqrt{-1}dz^2\wedge d\overline{z}^2} \leq \\
&2+\frac{r(r-1)\left|(B_\epsilon)_i^j\right|^2}{r^2-2r+2} - \frac{r(r-1)\sum_{i\neq j}\left(\Theta_i^i\Theta_j^j\right)}{(r^2-2r+2)\sqrt{-1}dz^1\wedge d\overline{z}^1\wedge\sqrt{-1}dz^2\wedge d\overline{z}^2} \\
&= 2+\frac{r(r-1)\left|(B_\epsilon)_i^j\right|^2}{r^2-2r+2} - \frac{r(r-1)\left(\omega^2-\sum_i(\Theta_i^i)^2\right)}{(r^2-2r+2)\sqrt{-1}dz^1\wedge d\overline{z}^1\wedge\sqrt{-1}dz^2\wedge d\overline{z}^2} \\
&\leq 2+\frac{r(r-1)\epsilon^2}{r^2-2r+2} - \frac{2r(r-1)}{r^2-2r+2} + \frac{r(r-1)\sum_i(\Theta_i^i)^2}{(r^2-2r+2)\sqrt{-1}dz^1\wedge d\overline{z}^1\wedge\sqrt{-1}dz^2\wedge d\overline{z}^2} \\
&= \frac{4-2r+r(r-1)\epsilon^2}{r^2-2r+2} + \frac{r(r-1)\sum_i(\Theta_i^i)^2}{(r^2-2r+2)\sqrt{-1}dz^1\wedge d\overline{z}^1\wedge\sqrt{-1}dz^2\wedge d\overline{z}^2} \\
&\leq \frac{4-2r+r(r-1)\epsilon^2}{r^2-2r+2} + \frac{r(r-1)(\Theta_1^1)^2}{(r^2-2r+2)\sqrt{-1}dz^1\wedge d\overline{z}^1\wedge\sqrt{-1}dz^2\wedge d\overline{z}^2} \\
&\quad + \frac{2r(r-1)\sum_{i\geq 2}(\Theta_i^i)_{1\overline{1}}(\Theta_i^i)_{2\overline{2}}}{r^2-2r+2} \\
&= \frac{4-2r+r(r-1)\epsilon^2}{r^2-2r+2} + \frac{r(r-1)(\Theta_1^1)^2}{(r^2-2r+2)\sqrt{-1}dz^1\wedge d\overline{z}^1\wedge\sqrt{-1}dz^2\wedge d\overline{z}^2} \\
&\quad + \frac{2r(r-1)\sum_{i\geq 2}(\Theta_i^i)_{1\overline{1}}\left(\frac{2}{r}+(B_\epsilon)_i^i-(\Theta_i^i)_{1\overline{1}}\right)}{r^2-2r+2} \\
&\Rightarrow \frac{(r^2-2r+2)c_1(h)^2-2r(r-1)c_2(h)}{\sqrt{-1}dz^1\wedge d\overline{z}^1\wedge\sqrt{-1}dz^2\wedge d\overline{z}^2} \leq \\
&4-2r+r(r-1)\epsilon^2 + \frac{r(r-1)(\Theta_1^1)^2}{\sqrt{-1}dz^1\wedge d\overline{z}^1\wedge\sqrt{-1}dz^2\wedge d\overline{z}^2} \\
&+2r(r-1)\sum_{i\geq 2}(\Theta_i^i)_{1\overline{1}}\left(\frac{2}{r}+(B_\epsilon)_i^i-(\Theta_i^i)_{1\overline{1}}\right). \qquad (2.6)
\end{aligned}
$$

Now we want to maximize $f = \sum_{i\geq 2}(\Theta_i^i)_{1\overline{1}}\left(\frac{2}{r}+(B_\epsilon)_i^i-(\Theta_i^i)_{1\overline{1}}\right)$ subject to the condition $\sum_{i\geq 2}(\Theta_i^i)_{1\overline{1}} = 1-\mu_1$. Clearly, f tends to $-\infty$ at infinity. Therefore, using Lagrange's multipliers we conclude that the maximum of f is as follows (we replace μ_1 by μ for the remainder of this section):

(2.7)

$$
\sum_{i\geq 2}\left(\left(\frac{1-\mu}{r-1}+\frac{(B_\epsilon)_1^1}{2(r-1)}+\frac{(B_\epsilon)_i^i}{2}\right)\left(\frac{2}{r}+(B_\epsilon)_i^i\right)-\left(\frac{1-\mu}{r-1}+\frac{(B_\epsilon)_1^1}{2(r-1)}+\frac{(B_\epsilon)_i^i}{2}\right)^2\right).
$$

Thus we have

$$\frac{(r^2-2r+2)c_1(h)^2-2r(r-1)c_2(h)}{\sqrt{-1}dz^1\wedge d\overline{z}^1\wedge\sqrt{-1}dz^2\wedge d\overline{z}^2}$$
$$\leq\ 4-2r+r(r-1)\epsilon^2+\frac{r(r-1)(\Theta_1^1)^2}{\sqrt{-1}dz^1\wedge d\overline{z}^1\wedge\sqrt{-1}dz^2\wedge d\overline{z}^2}$$
$$+2r-4-2r(B_\epsilon)_1^1-\frac{r}{2}((B_\epsilon)_1^1)^2+\frac{r(r-1)\sum_{i\geq 2}((B_\epsilon)_i^i)^2}{2}+2r\mu\left(\frac{2}{r}+(B_\epsilon)_1^1-\mu\right)$$
$$\leq\ \frac{r(r^2-1)\epsilon^2}{2}+2r\epsilon+\frac{r^2(\Theta_1^1)^2}{\sqrt{-1}dz^1\wedge d\overline{z}^1\wedge\sqrt{-1}dz^2\wedge d\overline{z}^2}. \tag{2.8}$$

This completes the proof of the lemma. □

Returning to the situation at hand, choose

$$\epsilon\ =\ \min\left(1,\ \frac{2\int((r^2-2r+2)c_1(h)^2-2r(r-1)c_2(h))}{r(r^2+1)\int\omega^2}\right)$$

and let $h\ =\ h_\epsilon$. Since $c_1(\mathcal{O}_{\mathbb{P}(E^*)}(1),\,\widetilde{h}_\epsilon)$ is a closed form representing the cohomology class $c_1(\mathcal{O}_{\mathbb{P}(E^*)}(1))$, we have

$$(c_1(\mathcal{O}_{\mathbb{P}(E^*)}(1)))^d.Y\ =\ \int_Y c_1(\mathcal{O}_{\mathbb{P}(E^*)}(1),\widetilde{h}_\epsilon)^d.$$

Therefore, Lemma 2.1 shows that

$$(c_1(\mathcal{O}_{\mathbb{P}(E^*)}(1)))^d.Y\ \geq$$
$$\frac{d(d-1)}{4r^2}\int_S((r^2-2r+2)c_1^2-2r(r-1)c_2)(p)\left(\int_{\pi^{-1}(p)\cap Y}\omega_{FS}^{d-2}\right).$$

Now $\int_{\pi^{-1}(p)\cap Y}\omega_{FS}^{d-2}$ is the degree of $\pi^{-1}(p)\bigcap Y\ \subset\ \pi^{-1}(p)\ =\ \mathbb{P}(E_p^*)$, and hence the given condition (1.1) in Theorem 1.3 implies that

$$(c_1(\mathcal{O}_{\mathbb{P}(E^*)}(1)))^d.Y\ >\ 0.$$

This completes the proof.

3. Counterexamples

Consider vector bundles of rank r on surfaces. In this section we provide counterexamples to show that if Lübke's condition

$$c_1^2(E).X\ >\ \frac{2r(r-1)}{r^2-2r+2}c_2(E).X$$

is not met, then the conclusion of the theorem cannot hold in general. Likewise for semistability with respect to $\det(E)$. For $r\ =\ 2$, the counterexample for semistability was provided in [7] and the sharpness of the Chern class inequality was shown in [9]. For the remainder of this section we assume that $r\ \geq\ 3$.

Indeed, just as in [7], let M be a Riemann surface of genus at least two, and let F be a stable vector bundle of rank two on M such that $c_1(F) = 0$ and the symmetric product $S^m F$ is stable for all $m \geq 1$. Set $X = \mathbb{P}(F)$, and let $L = \mathcal{O}_{\mathbb{P}(F)}(1)$. (Then $c_1(L)^2 = 0$ and $L\big|_C$ is positive for all curves C on X.) Let H be an ample line bundle on X, and set

$$E = L \oplus H \oplus H \oplus \ldots \oplus H.$$

Note that E is not ample because its quotient L is not ample. However, E is ample when restricted to curves. Then we have

$$c_1(E) = c_1(L) + (r-1)c_1(H),$$
$$c_2(E) = (r-1)c_1(L)c_1(H) + \frac{(r-1)(r-2)}{2}c_1^2(H).$$

The slope of L is $c_1(L).c_1(E) = (r-1)c_1(L)c_1(H)$ and that of H is

$$c_1(H).c_1(E) = c_1(H).c_1(L) + (r-1)c_1(H)^2.$$

Therefore, the semistability of E (with respect to $\det(E)$) holds if and only if

$$(r-2)c_1(L).c_1(H) = (r-1)c_1(H)^2.$$

Certainly this inequality cannot be met for an arbitrary H (and thus ampleness can fail if semistability does not hold). Suppose that semistability is met. Then we have the following:

$$c_1(E)^2.X = (r-1)^2 c_1(H)^2.X + 2(r-1)c_1(L)c_1(H).X = (r-1)rc_1(L)c_1(H).X,$$
$$\frac{2r(r-1)}{r^2-2r+2}c_2(E).X = \frac{2r(r-1)}{r^2-2r+2}\frac{2(r-1)+(r-2)^2}{2}c_1(L)c_1(H) = c_1(E)^2.$$

Therefore, if Lübke's Chern class inequality is not met, ampleness cannot hold in general.

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