J. Math. Anal. Appl. 537 (2024) 128382

Contents lists available at ScienceDirect

Journal of Mathematical Analysis and Applications

journal homepage: www.elsevier.com/locate/jmaa

Regular Articles

Local continuous extension of proper holomorphic maps: Low-regularity and infinite-type boundaries

Annapurna Banik

Department of Mathematics, Indian Institute of Science, Bangalore 560012, India

A R T I C L E I N F O

Article history:
Received 7 July 2023
Available online 3 April 2024
Submitted by E.J. Straube

Keywords:
Proper holomorphic maps
Local extension
Low boundary regularity
Kobayashi metric

A B S T R A C T

We prove a couple of results on local continuous extension of proper holomorphic maps $F : D \to \Omega$, $D, \Omega \subsetneq \mathbb{C}^n$, making local assumptions on ∂D and $\partial\Omega$. The first result allows us to have much lower regularity, for the patches of $\partial D, \partial\Omega$ that are relevant, than in earlier results. The second result (and a result closely related to it) is in the spirit of a result by Forstnerič–Rosay. However, our assumptions allow $\partial\Omega$ to contain boundary points of infinite type.

1. Introduction

In this paper, we present some results on local continuous extension of proper holomorphic maps $F : D \to \Omega$, D, $\Omega \subsetneq \mathbb{C}^n$, given local assumptions on ∂D and $\partial\Omega$. Our results are motivated by the well-known work of Forstnerič–Rosay [10]. There is also extensive literature on the problem of *global* extension of proper holomorphic maps (see [9] and the references therein), from which too we take a few cues.

With F, D, and Ω as above, let $p \in \partial D$ and let $C(F, p)$ be the *cluster set* of F at p, defined as:

$$C(F,p) := \Big\{ w \in \mathbb{C}^n : \exists \text{ a sequence } \{z_\nu\} \subset D \text{ such that } \lim_{\nu\to\infty} z_\nu = p \text{ and } \lim_{\nu\to\infty} F(z_\nu) = w \Big\}.$$

If Ω is bounded, then $C(F,p) \neq \emptyset$. F being proper, $C(F,p) \subset \partial\Omega$. In [10], the "local assumptions" alluded to above are imposed on $\partial D \cap U$ and $\partial\Omega \cap W$, where U and W are neighbourhoods of p and q respectively, where $q \in C(F,p)$. Since we will have occasion to mention the main result in [10] let us state it (also see [1,23,17] for results of a similar flavour as the following).

Result 1.1 *([10, Theorem 1.1]). Let D and Ω be domains in $\mathbb{C}^n$, Ω bounded, and let $F : D \to \Omega$ be a proper holomorphic map. Let $p \in \partial D$. Assume that there is a continuous, negative plurisubharmonic function ρ*

E-mail address: annapurnab@iisc.ac.in.

https://doi.org/10.1016/j.jmaa.2024.128382

on D and a neighbourhood U of p such that $\partial D \cap U$ is a $\mathcal{C}^{1+\varepsilon}$ submanifold of U for some $\varepsilon > 0$, and $\rho(z) \geq -\delta_D(z)$ for all $z \in U \cap D$. If the cluster set $C(F,p)$ contains a point q at which $\partial\Omega$ is strongly pseudoconvex, then F extends continuously to p.

With W and q as above, Result 1.1 assumes $\partial\Omega \cap W$ to be strongly pseudoconvex. Later results — [1,23], for instance — weaken this requirement to admit certain families of domains Ω such that $\partial\Omega$ is of finite type at each point of $\partial\Omega \cap W$. In this paper, we wish to extend this analysis and, in contrast, focus on the situation when, among other things, $\partial\Omega \cap W$ is not even $\mathcal{C}^1$-smooth or, if it is smooth, then q is of infinite type; see Examples 2.2 and 2.4, respectively.

Concerning the notation δ_D in Result 1.1: given an open set $D \subsetneq \mathbb{C}^n$ and $z \in D$, we write $\delta_D(z) := \text{dist}(z, \mathbb{C}^n \setminus D)$. The idea of the proof in [10] (although not the details thereof) goes back to Vormoor [24]. The technique used in [10] relies on the classical Hopf Lemma for plurisubharmonic functions, which imposes constraints on the regularity of $\partial\Omega \cap W$. For F, D, Ω, and p as above, and given a sequence $\{z_\nu\} \subset D$ such that $z_\nu \to p$, some form of a Hopf-type lemma is the simplest tool to control $\{F'(z_\nu)\}$ — provided $\partial\Omega$ is at least $\mathcal{C}^2$ near $\{F(z_\nu)\}$ (see [13,21,7], for instance). We explore a different paradigm from the one in [10] that allows us to greatly lower the regularity of $\partial D \cap U$ and $\partial\Omega \cap W$. Loosely speaking, instead of picking a point in $C(F,p)$ and imposing conditions on $\partial\Omega$ near it, we consider an *interior* condition (i.e., on a suitable open set in Ω) which is notably less restrictive; see Theorem 1.5. Before we present this result, a few definitions:

Definition 1.2. A function $\omega : ([0,\infty),0) \to ([0,\infty),0)$ is said to satisfy a *Dini condition* if ω is monotone increasing and $\int_0^\varepsilon r^{-1}\omega(r)dr < \infty$ for some (hence for any) $\varepsilon > 0$.

Definition 1.3. Let $D \subset \mathbb{C}^n$ be a domain, $p \in \partial D$, and U be a neighbourhood of p. Let $S := \partial D \cap U$.

(1) We say that S is a *Lipschitz submanifold of U* if for each $\xi \in S$, there exists a neighbourhood $\mathcal{N}_\xi$ of ξ, $\mathcal{N}_\xi \subset U$, a unitary map U_ξ, a constant $r_\xi > 0$, and a Lipschitz function $\varphi_\xi : B^{n-1}(0,r_\xi) \times (-r_\xi, r_\xi) \to \mathbb{R}$ such that, writing the affine map $z \mapsto \mathsf{U}_\xi(z-\xi)$ as U^ξ, we have

$$\mathsf{U}^\xi(\mathcal{N}_\xi \cap D) \subset \{(Z', Z_n) \in \mathbb{C}^{n-1} \times \mathbb{C} : \mathsf{Im}(Z_n) > \varphi_\xi(Z', \mathsf{Re}(Z_n)),\ \|Z'\| < r_\xi \text{ and } |\mathsf{Re}(Z_n)| < r_\xi\}, \tag{1.1}$$

$$\mathsf{U}^\xi(\mathcal{N}_\xi \cap \partial D) = \{(Z', Z_n) \in \mathbb{C}^{n-1} \times \mathbb{C} : \mathsf{Im}(Z_n) = \varphi_\xi(Z', \mathsf{Re}(Z_n)),\ \|Z'\| < r_\xi \text{ and } |\mathsf{Re}(Z_n)| < r_\xi\}. \tag{1.2}$$

(2) (Cf. Kukavica–Nyström [15], for instance) We say that S is a $\mathcal{C}^{1,\text{Dini}}$ *submanifold of U* if there exists a function $\omega : ([0,\infty),0) \to ([0,\infty),0)$ satisfying a Dini condition, and for each $\xi \in S$, there exist $\mathcal{N}_\xi$, U^ξ, and $r_\xi > 0$ as described by (1) above, and a $\mathcal{C}^1$ function $\varphi_\xi : B^{n-1}(0,r_\xi) \times (-r_\xi, r_\xi) \to \mathbb{R}$ such that

$$\|\nabla\varphi_\xi(x) - \nabla\varphi_\xi(y)\| \leq \omega(\|x-y\|) \quad \forall x, y \in B^{n-1}(0,r_\xi) \times (-r_\xi, r_\xi),$$

and $\mathsf{U}^\xi(\mathcal{N}_\xi \cap D)$ and $\mathsf{U}^\xi(\mathcal{N}_\xi \cap \partial D)$ are described by (1.1) and (1.2), respectively.

When $n = 1$, the expressions $B^{n-1}(0,r_\xi) \times (-r_\xi, r_\xi)$ and $\mathbb{C}^{n-1} \times \mathbb{C}$ must be read as $(-r_\xi, r_\xi)$ and $\mathbb{C}$, respectively. The conditions (1.1) and (1.2) must then be read *mutatis mutandis*.

Before we give the next definition, let us fix some notations. An *open right circular cone with aperture θ* is the following open set

$$\Gamma(v,\theta) := \{z \in \mathbb{C}^n : \mathsf{Re}\,\langle z, v\rangle > \cos(\theta/2)\,\|z\|\},$$

where $v \in \mathbb{C}^n$ is a unit vector and $\theta \in (0,\pi)$ (here $\langle\cdot\,,\cdot\rangle$ denotes the standard Hermitian inner product on $\mathbb{C}^n$). Given $z_0 \in \mathbb{C}^n$, the *axis* of the (translated) cone $z_0 + \Gamma(v,\theta)$ is the ray $\{z_0 + tv : t > 0\}$.

Definition 1.4. Let $D \subset \mathbb{C}^n$ be a domain and let W be an open set such that $\partial D \cap W$ is a non-empty ∂D-open set. We say that D satisfies a *uniform interior cone condition in* W if, given any open set $\mathcal{U} \subset D$ such that $\mathcal{U} \Subset W$, there exist constants $r_{\mathcal{U}} > 0$ and $\theta_{\mathcal{U}} \in (0,\pi)$ such that for each $w \in \mathcal{U}$, there exists $\xi_w \in \partial D \cap W$ and a unit vector v_w such that

- w lies on the axis of $\xi_w + \Gamma(v_w, \theta_{\mathcal{U}})$.
- $w \in (\xi_w + \Gamma(v_w, \theta_{\mathcal{U}})) \cap B^n(\xi_w, r_{\mathcal{U}}) \subset W \cap D$.

Given a domain $D \subset \mathbb{C}^n$, $k_D : D \times \mathbb{C}^n \to [0,\infty)$ will denote the Kobayashi pseudometric for D. We are now in a position to state our first theorem.

Theorem 1.5. *Let D and Ω be domains in $\mathbb{C}^n$ and let $F : D \to \Omega$ be a proper holomorphic map. Let $p \in \partial D$. Assume that there is a continuous, negative plurisubharmonic function ρ on D, a neighbourhood U of p, and a constant $s \in (0,1]$ such that $\partial D \cap U$ is a Lipschitz submanifold of U, and $\rho(z) \geq -(\delta_D(z))^s$ for all $z \in U \cap D$. Suppose there exists a neighbourhood U^* of p, $U^* \Subset U$, and an open set W such that $\partial\Omega \cap W \neq \emptyset$, and such that*

- *$F(U^* \cap D) \Subset W$, and*
- *Ω satisfies a uniform interior cone condition in W.*

Suppose there exists a function $M : ([0,\infty),0) \to ([0,\infty),0)$ satisfying a Dini condition so that

$$k_\Omega(w; v) \geq \|v\|/M(\delta_\Omega(w)) \quad \forall (w,v) \in (W \cap \Omega) \times \mathbb{C}^n. \tag{1.3}$$

Then, there exists a ∂D-neighbourhood $\mathscr{V}$ of p such that F extends to $D \cup \mathscr{V}$ as a continuous map.

One may ask whether, given that Ω in Theorem 1.5 is assumed to satisfy a uniform interior cone condition in W, one also requires the condition (1.3). There is a vital point related to this, which is best discussed after we state our next theorem and prove Theorem 1.5; see Remark 6.1.

The proof of our next result follows many of the techniques used in [10]. Even so, with D, p, and U as above, one is able to admit $\partial D \cap U$ having lower regularity than in [10]. The argument used in [10] is *very delicate*, and is necessitated by an aspect of the hypothesis of Forstnerič–Rosay: i.e., picking a point in $C(F,p)$ and imposing conditions on $\partial\Omega$ near it (compare the hypotheses of Result 1.1 and Theorem 1.5). Given such a hypothesis, the proof relies on certain intrinsic constants matching up precisely. This is why, as hinted at earlier, one needs to assume $\mathcal{C}^2$ regularity near a point $q \in C(F,p)$: which is done both in Result 1.1 and in Theorem 1.7. Our greatest departure from [10] involves a concept — namely, local log-type convexity — introduced by Liu–Wang in [16]. The assumption of Ω being log-type convex near q (which substitutes the assumption of strong pseudoconvexity in Result 1.1) is broad enough to admit, on the one hand, domains Ω such that $\partial\Omega \cap W$ is a $\mathcal{C}^2$ submanifold and, on the other hand, Ω such that q is of infinite type. Quantitatively, this condition admits useful lower bounds for the Kobayashi distance for Ω, which is the delicate part of proving Theorem 1.7.

Before we present Theorem 1.7, we need to define log-type convexity. But first, we must introduce a quantity that is very natural in the context of bounded convex domains. For instance, by a theorem of

Graham [11,12] (see Result 5.1 below), this quantity provides an optimum estimate for the Kobayashi metric for such a domain. Let D be a bounded convex domain in $\mathbb{C}^n$. For each $z \in D$ and $v \in \mathbb{C}^n \setminus \{0\}$, define

$$\delta_D(z; v) := \sup\left\{r > 0 : \left(z + (r\mathbb{D})\frac{v}{\|v\|}\right) \subset D\right\}.$$

Definition 1.6 *(Liu–Wang, [16, Definition 1.1]).* A bounded convex domain $D \subset \mathbb{C}^n$, $n \geq 2$, is called *log-type convex* if there are constants $C,\ \nu > 0$ such that

$$\delta_D(z; v) \leq \frac{C}{\left|\log \delta_D(z)\right|^{1+\nu}} \quad \forall z \in D \text{ and } \forall v \in \mathbb{C}^n \setminus \{0\}. \tag{1.4}$$

Theorem 1.7. *Let D and Ω be domains in $\mathbb{C}^n$, $n \geq 2$, Ω bounded, and let $F : D \to \Omega$ be a proper holomorphic map. Let $p \in \partial D$ and $q \in C(F, p)$. Assume that there is a continuous, negative plurisubharmonic function ρ on D and a neighbourhood U of p such that $\partial D \cap U$ is a $\mathcal{C}^{1,\,\mathrm{Dini}}$ submanifold of U, and $\rho(z) \geq -\delta_D(z)$ for all $z \in U \cap D$. Suppose there exists a neighbourhood $\mathscr{O}$ of q such that*

- *$\partial\Omega \cap \mathscr{O}$ is a $\mathcal{C}^2$ submanifold of $\mathscr{O}$, and*
- *$\mathscr{O} \cap \Omega$ is log-type convex.*

Then, there exists a ∂D-neighbourhood $\mathscr{V}$ of p such that F extends to $D \cup \mathscr{V}$ as a continuous map.

Remark 1.8. The reason Theorem 1.7 has been stated for domains in $\mathbb{C}^n$ with $n \geq 2$ is because its proof relies crucially on results about log-type convex domains (see Section 5), which are defined as domains in $\mathbb{C}^n$, $n \geq 2$. Note, though, that *any* bounded convex domain in $\mathbb{C}$ satisfies the defining inequality (1.4) for a log-type convex domain. However, the proofs of the above-mentioned results have been written with domains in higher dimensions in mind — additional arguments would be needed in the planar case. For this reason, we restrict Theorem 1.7 to $n \geq 2$. Furthermore, one suspects that Theorem 1.7, suitably restated, is already known for planar domains.

We must mention a connection between Theorem 1.5 and the proof of Theorem 1.7. Note that the conclusion of Theorem 1.7 is stronger than that of Result 1.1. Now, the former conclusion can, in principle, be deduced under the assumptions in Result 1.1 too. This requires an auxiliary argument alluded to in [10] (see page 241) — by which one may deduce Hölder continuity of the extension with exponent 1/2. That specific argument does not, in general, work in the context of Theorem 1.7. Instead, it turns out that once we get a continuous extension of F to p (for Theorem 1.7), we are able to use Theorem 1.5 and get the stronger conclusion. We can also estimate the modulus of continuity of the extension given by Theorems 1.5 and 1.7. As both theorems admit domains Ω such that $\partial\Omega \cap W$ contains points of infinite type (see Remark 6.1 and Example 2.4), one **cannot** expect these extensions, in general, to be Hölder. This leads to the much more technical discussion of the modulus of continuity of the extension, which we omit.

We should also emphasise that — with D, U, and $p \in \partial D$ as in all the results above — the observations just made hold true for $\partial D \cap U$ having much lower regularity than in any earlier result on the present theme.

Readers familiar with the essence of the argument of Forstnerič–Rosay in [10] and with the results in [5] by Bracci et al. might ask whether the conclusions of Theorem 1.7 may be obtainable under weaker conditions on $\mathscr{O} \cap \Omega$. We shall address this in Section 7: see Remark 7.1 and Theorem 7.2.

Given any two domains $D, \Omega \subseteq \mathbb{C}^n$, $n \geq 2$, it is rare for the pair (D, Ω) to admit a proper holomorphic map $F : D \to \Omega$. Since the domains D, Ω in Theorems 1.5 and 1.7 must satisfy several conditions, the question arises: are there any domains $D, \Omega \subseteq \mathbb{C}^n$ that satisfy these conditions **and** admit a non-trivial proper holomorphic map $F : D \to \Omega$ when $n \geq 2$? We provide examples of such domains in Section 2. Recall:

the conditions in Theorem 1.7 allow $\partial\Omega$ to be of infinite type at $C(F, p)$. Section 2 provides an example of (D, Ω) where Ω has the latter property and there exists a non-trivial proper holomorphic $D \to \Omega$ map. As for the proofs of Theorems 1.5 and 1.7: they are presented in Sections 6 and 7, respectively.

2. Two examples

In this section, we discuss the examples mentioned above. But we first explain the notation used below and in later sections (some of which has also been used without clarification in Section 1).

2.1. Common notations

(1) For $v \in \mathbb{R}^N$, $\|v\|$ denotes the Euclidean norm. For any $a \in \mathbb{R}^N$ and $B \subset \mathbb{R}^N$, we write $\text{dist}(a, B) := \inf\{\|a - b\| : b \in B\}$.
(2) Given a point $z \in \mathbb{C}^n$ and $r > 0$, $B^n(z, r)$ denotes the open Eulidean ball with radius r and centre z. For simplicity, we write $\mathbb{D} := B^1(0, 1)$.
(3) Given a domain $D \subset \mathbb{C}^n$, $K_D : D \times D \to [0, \infty)$ denotes the Kobayashi distance for D.

2.2. Examples

We are now in a position to present the examples referred to several times in Section 1. To this end, we need the following result by Sibony:

Result 2.1 *(paraphrasing [22, Proposition 6]). Let $D \subset \mathbb{C}^n$ be a domain and $z \in D$. There exists a uniform constant $\alpha > 0$ (i.e., it does not depend on z) such that if u is a negative plurisubharmonic function on D that is of class $\mathcal{C}^2$ in a neighbourhood of z and satisfies*

$$\langle v, (\mathfrak{H}_{\mathbb{C}} u)(z) v\rangle \geq c\|v\|^2 \quad \forall v \in \mathbb{C}^n,$$

where c is some positive constant, then

$$k_D(z; v) \geq \left(\frac{c}{\alpha}\right)^{1/2} \frac{\|v\|}{|u(z)|^{1/2}}.$$

Here $\langle \cdot, \cdot \rangle$ denotes the standard Hermitian inner product and $\mathfrak{H}_{\mathbb{C}}$ denotes the complex Hessian.

Example 2.2. An example demonstrating that there exist domains D, $\Omega \subset \mathbb{C}^2$, and a proper holomorphic map $F : D \to \Omega$ such that D, Ω, and F satisfy the conditions stated in Theorem 1.5.

Let us define

$$D := \{(z, w) \in \mathbb{C}^2 : |z|^2 + |w| < 1\},$$
$$\Omega := \{(z, w) \in \mathbb{C}^2 : |z| + |w| < 1\},$$

and $F(z, w) := (z^2, w)$ for $(z, w) \in D$. Then $F : D \to \Omega$ and it is clear that F is proper. Finally, let $p = (1, 0) \in \partial D$. That the stated conditions are satisfied will be discussed in two steps.

Step 1. We shall show that there exists a constant $\widetilde{C} > 0$ such that, writing

$$\rho(z, w) := \widetilde{C}\left(|z|^2 + |w| - 1\right),\ (z, w) \in D,$$

ρ satisfies the desired estimate in $U \cap D$, where $U := \{(z, w) \in \mathbb{C}^2 : 9/10 < |z| < 11/10,\ |w| < \ 1\}$. Since ∂D is **not** smooth around p, this task will be slightly involved. Note that ρ is a continuous, negative, plurisubharmonic function on D.

Since D is Reinhardt, it is elementary to show that if $(z, w) \in D,\ \theta_z,\ \theta_w \in \mathbb{R}$ are such that $z = |z|e^{i\theta_z},\ w = |w|e^{i\theta_w}$, and $(\zeta, \eta) \in \partial D$ are such that $\|(z, w) - (\zeta, \eta)\| = \delta_D(z, w)$, then

- $\exists(\zeta, \eta) \in \partial D$ with the above property such that $\zeta = |\zeta|e^{i\theta_z}$ and $\eta = |\eta|e^{i\theta_w}$.
- For (ζ, η) as described by the above bullet point, $\big\|\big(|z|, |w|\big) - \big(|\zeta|, |\eta|\big)\big\| = \delta_D\big(|z|, |w|\big)$.

Due to the above, it suffices to show that

$$\rho\big(|z|, |w|\big) \geq -\delta_D\big(|z|, |w|\big) \quad \forall(z, w) : |z|^2 + |w| < 1,\ 9/10 < |z| < 1,\ |w| < 1/10. \tag{2.1}$$

Clearly, to establish (2.1), we need to estimate the infimum of the set $\mathcal{S}(x_0, y_0) := \{\|(x, y) - (x_0, y_0)\| : x^2 + |y| = 1\}$, having fixed (x_0, y_0) such that $9/10 < x_0 < 1,\ 0 \leq y_0 < 1/10$, and ${x_0}^2 + y_0 < 1$. It is elementary to show that, as the curve $x^2 + |y| = 1$ has a non-smooth point at $(1, 0)$ and bounds a convex region, $\mathcal{S}(x_0, y_0)$ must attain its minimum in the set $\{(x, y) \in \mathbb{R} \times (0, \infty) : x^2 + y = 1\}$, which is a smooth curve $\mathscr{C}$. So, we can apply the method of Lagrange multipliers to deduce that if $\mathcal{S}(x_0, y_0)$ attains its minimum at (X, Y), then (X, Y) satisfies

$$2X^3 + (2y_0 - 1)X - x_0 = 0, \tag{2.2}$$

$$\frac{X - x_0}{Y - y_0} = 2X. \tag{2.3}$$

To estimate $\min \mathcal{S}(x_0, y_0)$, we define an auxiliary function $\phi : \mathbb{R} \to \mathbb{R},\ \phi(x) := 2x^3 + (2y_0 - 1)x - x_0$. Independent of the choice of y_0, ϕ is strictly increasing on $[1/\sqrt{6}, \infty)$. By the nature of the curve $\mathscr{C}$ and the choice of x_0, it follows that $9/10 < x_0 < X < 1$. Thus, in order to locate X, it suffices to examine $\phi|_{[9/10,1]}$. Let $C := \sup\{6x^2 + (2y - 1) : x \in [9/10, 1],\ y \in [0, 1/10]\}$. Since $\sup_{x\in[9/10,1]} \phi'(x) \leq C$, writing $A = \phi(9/10),\ B = \phi(1)$ we have

$$\begin{aligned} |\phi(x) - \phi(x')| \leq C|x - x'| \quad \forall x,\ x' \in [9/10, 1] \\ \Rightarrow \big|\phi^{-1}(a) - \phi^{-1}(a')\big| \geq C^{-1}|a - a'| \quad \forall a,\ a' \in [A, B]. \end{aligned} \tag{2.4}$$

Thus, by (2.2) and (2.4), since X is a root of (2.2),

$$|X - x_0| = \big|\phi^{-1}(0) - \phi^{-1}(\phi(x_0))\big| \geq C^{-1}|\phi(x_0)| = \frac{2x_0}{C}\,\big|x_0^2 + y_0 - 1\big|.$$

From the above inequality, we deduce

$$\min \mathcal{S}(x_0, y_0) = \sqrt{(X - x_0)^2 + (Y - y_0)^2} \geq |X - x_0| \geq \frac{9}{5C}\big|x_0^2 + y_0 - 1\big|. \tag{2.5}$$

We set $\widetilde{C} = \frac{9}{5C}$. Then, (2.5) implies $\rho(x_0, y_0) \geq -\min \mathcal{S}(x_0, y_0)$. Recalling the purpose of the set $\mathcal{S}(x_0, y_0)$, $\min \mathcal{S}(|z|, |w|) = \delta_D(|z|, |w|)$ for the above mentioned constraints. Thus, the last inequality gives us (2.1). By the discussion preceding (2.1),

$$\rho(z, w) \geq -\delta_D(z, w) \quad \forall(z, w) \in U \cap D.$$

Step 2. We shall now show that k_Ω satisfies the desired estimate in $(W \cap \Omega) \times \mathbb{C}^2$ where $W := B^2((1, 0), \delta),\ \delta > 0$ is small enough — with $M(t) := \tilde{c}\,t^{1/2},\ t \geq 0$, for some $\tilde{c} > 0$.

Clearly, $u(z, w) := |z| + |w| - 1,\ (z, w) \in \Omega$, is negative, plurisubharmonic on Ω. Note that u is smooth at a given point $(z, w) \in W \cap \Omega$ if and only if $w \neq 0$. To establish the required lower bound on k_Ω at smooth points of u, we will use Result 2.1. For the non-smooth points, we will use the convexity of Ω and apply the estimate by Graham as given by Result 5.1.

Since Ω is Reinhardt, it satisfies the same properties as described by the bullet points (prior to (2.1)) in Step 1. Thus, using coordinate geometry, it is elementary to see that

$$\delta_\Omega(z, w) = \frac{1 - |z| - |w|}{\sqrt{2}} = \frac{|u(z, w)|}{\sqrt{2}}. \tag{2.6}$$

Let $(z, w) \in W \cap \Omega$ with $w \neq 0$. It is easy to compute that

$$\langle v, (\mathfrak{H}_{\mathbb{C}} u)(z, w)v\rangle = \frac{1}{4}\left(\frac{|v_1|^2}{|z|} + \frac{|v_2|^2}{|w|}\right) \geq \frac{\|v\|^2}{4} \quad \forall v = (v_1, v_2) \in \mathbb{C}^2.$$

Then, by Result 2.1 and (2.6), there exists $\beta > 0$ such that

$$k_\Omega((z, w); v) \geq \frac{\beta\|v\|}{(\delta_\Omega(z, w))^{1/2}} \quad \forall (z, w) \in W \cap \Omega \text{ with } w \neq 0 \text{ and } \forall v \in \mathbb{C}^2. \tag{2.7}$$

Now, let $(z, w) \in W \cap \Omega$ with $w = 0$ and let $v = (v_1, v_2) \in \mathbb{C}^2$. Write $z = |z|e^{i\theta_z}$, $v_k = |v_k|e^{i\theta_k}$ where $\theta_z,\ \theta_k \in \mathbb{R}$, $k = 1, 2$. By the invariance of k_Ω under the automorphism $\Omega \ni (z, w) \mapsto \big(e^{-i\theta_z} z, e^{-i(-\theta_z+\theta_1-\theta_2)} w\big)$, we get

$$\begin{aligned} k_\Omega((z, 0); v) &= k_\Omega\big((|z|, 0); e^{i(\theta_1-\theta_z)}\big(|v_1|, |v_2|\big)\big) \geq \frac{\|v\|}{2\,\delta_\Omega\big((|z|, 0); e^{i(\theta_1-\theta_z)}\big(|v_1|, |v_2|\big)\big)} \\ &= \frac{\|v\|}{2\,\delta_\Omega\big((|z|, 0); \big(|v_1|, |v_2|\big)\big)}, \end{aligned} \tag{2.8}$$

where the inequality is due to Result 5.1. It is easy to see that

$$\delta_\Omega\big((|z|, 0); \big(|v_1|, |v_2|\big)\big) \leq \big\|(|z|, 0) - (1, 0)\big\| = \sqrt{2}\,\delta_\Omega(z, 0).$$

Hence, (2.8) implies

$$k_\Omega((z, 0); v) \geq \frac{\|v\|}{2\sqrt{2}\,\delta_\Omega(z, 0)} \quad \forall z : (z, 0) \in W \cap \Omega \text{ and } \forall v \in \mathbb{C}^2. \tag{2.9}$$

If we set $\tilde{c} = \max\big(1/\beta, 2\sqrt{2}\big)$, then from (2.7) and (2.9) we get the estimate

$$k_\Omega((z, w); v) \geq \|v\|/M(\delta_\Omega(z, w)) \quad \forall (z, w) \in W \cap \Omega \text{ and } \forall v \in \mathbb{C}^2$$

for the M introduced above.

All that remains is to produce a neighbourhood U^* of $(1, 0)$ such that $(U^* \cap D)$ is mapped by F as desired. Now, the main point of this example is — given that proper holomorphic maps between a given pair of domains in $\mathbb{C}^n$ are rare when $n \geq 2$ — to show that there exist domains D and Ω satisfying the respective *geometric* conditions imposed by Theorem 1.5 that admit a proper non-trivial holomorphic map $F : D \to \Omega$. In the present example, F is holomorphic on $\mathbb{C}^2$. Clearly, it extends continuously to a ∂D-neighbourhood, say $\mathscr{V}$, of $(1, 0)$. The existence of the desired U^* easily follows from the continuity of F at $p = (1, 0)$. ◀

Recall the discussion in Section 1 on how the assumptions stated in Theorem 1.7 admit continuous extension of $F : D \to \Omega$ even if $q \in \partial\Omega$ — D, Ω, F, and q as in the statement of Theorem 1.7 — is a point of infinite type. This is very different from the assumptions encountered in previous results on local extension of proper holomorphic maps. For this reason, an example, showing that the result implicit in the first sentence of this paragraph isn't vacuously true, is desirable. In discussing such an example, the following well-known result (see, for instance, [14, Chapter 3]) will be useful.

Result 2.3. *If D is a domain in $\mathbb{C}^n$, $p \in \partial D$, and ∂D is a $\mathcal{C}^2$ submanifold of some neighbourhood $\mathcal{N}$ of p, then there exists an $\varepsilon > 0$ such that $B^n(p,\varepsilon) \Subset \mathcal{N}$ and:*

- *$\delta_D(z) = \delta_{\mathcal{N}\cap D}(z)$ for all $z \in B^n(p,\varepsilon) \cap D$.*
- *If we define*

$$\widetilde{\rho}(z) := \begin{cases} -\delta_D(z), & \text{if } z \in \overline{D}, \\ \delta_{(\mathbb{C}^n \setminus \overline{D})}(z), & \text{if } z \in \mathbb{C}^n \setminus \overline{D}, \end{cases}$$

then $\widetilde{\rho} \in \mathcal{C}^2\big(B^n(p,\varepsilon)\big)$. Moreover, $\widetilde{\rho}$ is a local defining function for D on $B^n(p,\varepsilon)$.

Example 2.4. An example demonstrating that there exist domains $D,\ \Omega \subset \mathbb{C}^2$, and a proper holomorphic map $F : D \to \Omega$ such that D, Ω, and F satisfy the conditions stated in Theorem 1.7 and such that, with $p \in \partial D$ and for $q \in C(F,p)$, $\partial\Omega$ is of infinite type at q.

Let us define the function $\varphi : [0,\infty) \to \mathbb{R}$ as

$$\varphi(x) := \begin{cases} \exp\big(-\frac{1}{x^{1/2}}\big), & \text{if } x \neq 0, \\ 0, & \text{otherwise.} \end{cases}$$

Now, set

$$\begin{aligned} D &:= \{(z,w) \in \mathbb{C}^2 : \mathsf{Re} z > \varphi(|w|^2)\} \cap \{(z,w) \in \mathbb{C}^2 : |z|^2 + |w|^4 < 1\}, \\ \Omega &:= \{(z,w) \in \mathbb{C}^2 : \mathsf{Re} z > \varphi(|w|)\} \cap B^2((0,0),1), \end{aligned}$$

and $F(z,w) := (z,w^2)$ for $(z,w) \in D$. Then $F : D \to \Omega$ and it is clear that F is proper. Let $p = (0,0) \in \partial D$ and $q = (0,0) \in C(F,p)$. Take $\mathscr{O} := B^2(q,r)$, where $r \in (0,1)$ is such that $\mathscr{O} \cap \Omega$ is log-type convex (see [16, Example 1.3]). Consider the function $\rho : D \to \mathbb{R}$ defined as

$$\rho(z,w) := \varphi(|w|^2) - \mathsf{Re} z, \ (z,w) \in D.$$

By definition, ρ is smooth and negative in D. A computation gives

$$\frac{\partial^2 \rho}{\partial w \partial \overline{w}}(z,w) = \begin{cases} 4^{-1}|w|^{-3}\exp\big(-\frac{1}{|w|}\big)\Big(\frac{1}{|w|} - 1\Big), & \text{if } w \neq 0, \\ 0, & \text{otherwise.} \end{cases}$$

This implies that ρ is a plurisubharmonic function on D.

We will now show that there exist a constant $C > 0$ and a neighbourhood U of p such that the plurisubharmonic function $D \ni (z,w) \mapsto C\rho(z,w)$ satisfies the desired estimate in $U \cap D$. Since ∂D is smooth near p, there exist a neighbourhood $\mathcal{N}$ of p and $\varepsilon > 0$ with $B^2(p,\varepsilon) \Subset \mathcal{N} \Subset \{(z,w) \in \mathbb{C}^2 : |z|^2 + |w|^4 < 1\}$ such that the conclusion of Result 2.3 holds and such that $\rho|_{\mathcal{N}}$ is a local defining function for D at $p \in \partial D$. Now, since the function $\widetilde{\rho}$ as in Result 2.3 is a local defining function of D on $B^2(p,\varepsilon)$, there exists a neighbourhood $\mathcal{N}'$ of $\partial D \cap B^2(p,\varepsilon)$ and a positive function $H \in \mathcal{C}^2(\mathcal{N}')$ satisfying

$$\widetilde{\rho}(z,w) = \rho(z,w)H(z,w) \quad \forall (z,w) \in \mathcal{N}'.$$

So, fixing a neighbourhood U of p such that $U \Subset \mathcal{N}'$, write $C := \inf_U H > 0$. Then,

$$C\rho(z,w) \geq -\delta_D(z,w) \quad \forall (z,w) \in U \cap D.$$

Finally, note that ∂D is of infinite type at $q = (0,0)$. Hence, the triple (D,Ω,F) has the desired properties. ◄

3. Preliminary analytic lemmas

This short section is devoted to a few facts from analysis that would be needed in the proofs of the theorems stated in Section 1.

Lemma 3.1. *Let $p \in \partial D$ and U, U^* be as in Theorem 1.5. Then (in the notation of Definition 1.3) there exist a neighbourhood V of p, $V \Subset \mathcal{N}_p \cap U^*$ and a constant $C > 1$ such that*

$$\delta_D(z) \leq Y(\mathsf{U}^p(z)) \leq C\delta_D(z) \quad \forall z \in V \cap D$$

where we define $Y(Z',Z_n) := \mathsf{Im}(Z_n) - \varphi_p(Z',\mathsf{Re}(Z_n))$ for $(Z',Z_n) \in B^n(0,r_p)$.

Proof. The first inequality is immediate. To prove the second inequality, first we choose a neighbourhood V of p, $V \Subset \mathcal{N}_p \cap U^*$ such that $\operatorname{diam}(V) < \operatorname{dist}(\overline{V}, \mathbb{C}^n \setminus \mathcal{N}_p)$. Then

$$\delta_D(z) = \delta_{\mathcal{N}_p \cap D}(z) \quad \forall z \in V \cap D.$$

The choice of V ensures that if $z \in V$, there exists $w_z \in \partial D \cap \mathcal{N}_p$ such that $\delta_D(z) = \|z - w_z\|$.

Now, clearly the function Y is Lipschitz with some Lipschitz constant $C > 1$ and it vanishes on $\mathsf{U}^p(\partial D \cap \mathcal{N}_p)$. Thus, denoting $Z = \mathsf{U}^p(z)$, we get

$$\begin{aligned} Y(Z) &= \|Y(\mathsf{U}^p(z)) - Y(\mathsf{U}^p(w_z))\| \leq C\|\mathsf{U}^p(z) - \mathsf{U}^p(w_z)\| = C\|z - w_z\| \\ &= C\delta_D(z) \quad \forall z \in V \cap D. \end{aligned}$$

Hence the result. □

The next result is the first step on the path — via the distance decreasing property for the Kobayashi metric — to get an integrable bound on the norm of the total derivative of the map F in the proof of Theorem 1.5. This will allow us to use a type of "Hardy–Littlewood trick" to establish local continuous extension of proper holomorphic maps. The latter idea, which we will adapt to our low-regularity setting, is inspired by the proof of [7, Lemma 8] (where the relevant estimates are absent, but the idea for obtaining them is hinted at).

Result 3.2 *(paraphrasing [19, Proposition 1.4]). Let $D \subset \mathbb{C}^n$ be a domain and W be an open set such that $\partial D \cap W$ is a non-empty ∂D-open set. Suppose D satisfies a uniform interior cone condition in W. Let $\varphi : D \to [-\infty, 0)$ be a plurisubharmonic function. Then, given any open set $\mathcal{U} \subset D$ with $\mathcal{U} \Subset W$, there exist constants $C_{\mathcal{U}} > 0$ and $\alpha_{\mathcal{U}} > 1$ such that*

$$\varphi(w) \le -C_{\mathcal{U}}(\delta_D(w))^{\alpha_{\mathcal{U}}} \quad \forall w \in \mathcal{U}. \tag{3.1}$$

Remark 3.3. We remark here that the above is a local version of the Hopf Lemma due to Mercer given by [19, Proposition 1.4]. The proof of the local version is routine and we shall skip it. (Indeed, a lemma of Hopf-type is typically a local statement; Mercer's result was formulated as a global statement presumably because, then, its proof is similar to that of the global statement given by [8, Proposition 12.2].)

The next lemma is an application of the above version of a Hopf-type lemma — suited for the geometry of $W \cap \Omega$ — applied to the "pushforward" of the plurisubharmonic function ρ on D. To be precise, (using classical results for proper holomorphic maps) the function

$$\tau : \Omega \to \mathbb{R},\ \tau(w) := \max\{\rho(z) : z \in F^{-1}\{w\}\},\ w \in \Omega,$$

is a continuous, negative, plurisubharmonic function on Ω. Having this we now prove

Lemma 3.4. *Let U^* be the open set and $F : D \to \Omega$ be the map occurring in Theorem 1.5. Let τ be as defined above. Then, there exist constants (which depend on U^*) $\alpha_* > 1$ and $C_* > 0$ such that*

$$\tau(w) \le -C_*(\delta_\Omega(w))^{\alpha_*} \quad \forall w \in F(U^* \cap D).$$

Proof. Let W^* be an open set such that $F(U^* \cap D) \Subset W^* \Subset W$. Since Ω satisfies a uniform interior cone condition in W, by Result 3.2, there exist $C_* := C_{W^* \cap \Omega} > 0$ and $\alpha_* := \alpha_{W^* \cap \Omega} > 1$ such that

$$\tau(w) \le -C_*(\delta_\Omega(w))^{\alpha_*} \quad \forall w \in W^* \cap \Omega.$$

This proves the lemma. □

4. Some geometric lemmas

In this section we wish to obtain an estimate on K_D, where D is as in Theorem 1.7, similar to the estimate provided by Forstnerič–Rosay in [10, Proposition 2.5]. Their estimate is obtained by embedding a specific simply-connected bounded planar domain $\mathcal{D}$ into the domain they considered in [10, Proposition 2.5]. The boundary of the domain that they considered is of class $\mathcal{C}^{1,\epsilon}$ near a given boundary point. However, in our situation the boundary near $p \in \partial D$ has lower (namely, $\mathcal{C}^{1,\,\mathrm{Dini}}$) regularity, so we need to modify their construction. For this reason, we introduce a class of domains $\mathcal{D}(\beta, \varepsilon)$, $\beta, \varepsilon > 0$ (see the definition below) — analogous to $\mathcal{D}$ in [10, Proposition 2.5] — that can be embedded into D for a suitable choice of (β, ε). These domains appeared in the work of Maitra [18]. In fact, we shall adapt some of the arguments in [18, Proposition 4.2] to the present case.

In this discussion we need some definitions. Let ω, φ_p, and r_p be as introduced by Definition 1.3-(2). Here $n \ge 2$. Let us define $\omega_p : [0, 2\sqrt{2}r_p) \to [0, \infty)$ as

$$\omega_p(r) := \sup\left\{\|\nabla\varphi_p(x) - \nabla\varphi_p(y)\| : x,\ y \in B^{n-1}(0, r_p) \times (-r_p, r_p),\ \|x - y\| \le r\right\}. \tag{4.1}$$

One can check that ω_p satisfies the following properties:

- ω_p is monotone increasing.
- For any $r \in [0, 2\sqrt{2}r_p)$, $\omega_p(r) \le \omega(r)$. In particular, ω_p satisfies a Dini condition.
- ω_p is sub-additive, i.e., given $\sigma,\ \tau \ge 0,\ \sigma + \tau < 2\sqrt{2}r_p,\ \omega_p(\sigma + \tau) \le \omega_p(\sigma) + \omega_p(\tau)$.

Now we define $h : (-2\sqrt{2}r_p, 2\sqrt{2}r_p) \to [0, \infty)$ as

$$h(t) := \begin{cases} \int_0^t \omega_p(r)dr, & \text{if } t \geq 0, \\ \int_t^0 \omega_p(-r)dr, & \text{if } t < 0. \end{cases} \tag{4.2}$$

It is easy to see that h is strictly increasing on $[0, 2\sqrt{2}r_p)$, strictly decreasing on $(-2\sqrt{2}r_p, 0]$, and $h(0) = h'(0) = 0$. Now, given $\beta,\ \varepsilon > 0$, define the domain

$$\mathcal{D}(\beta, \varepsilon) := \{\zeta = s + it \in \mathbb{C} : |t| < \varepsilon,\ \beta h(t) < s < \varepsilon\}.$$

With these definitions we now prove

Proposition 4.1. *Let D, U be as in Theorem 1.7. Let $p \in \partial D \cap U$ and $\mathcal{N}_p$ be as given by Definition 1.3-(2). For $\xi \in \partial D \cap U$, let $\Psi_\xi : \mathbb{C} \to \mathbb{C}^n$ denote the $\mathbb{C}$-affine map $\zeta \mapsto \xi + \zeta\eta_\xi$ (where η_ξ denotes the unit inward normal vector at ξ). Then, for any neighbourhood V of p, $V \Subset \mathcal{N}_p$, there exist constants (that depend on V) $\beta,\ \varepsilon > 0$ such that $\Psi_\xi(\mathcal{D}(\beta, \varepsilon)) \subset U \cap D$ for all $\xi \in \partial D \cap V$.*

Proof. Let $\mathsf{U}^p,\ \varphi_p$ be as in Definition 1.3-(2). Let us denote $\widetilde{S} = \mathsf{U}^p(S)$ for any subset $S \subset \mathbb{C}^n$. We shall indicate that we are working in the coordinate system given by U^p by using $Z_1, \dots, Z_n$, where $(Z_1, \dots, Z_n) = Z = \mathsf{U}^p(z)$. Clearly, $\rho(Z) := \varphi_p(Z', \mathsf{Re}Z_n) - \mathsf{Im}Z_n,\ Z = (Z', Z_n) \in \widetilde{\mathcal{N}_p}$, is a local defining function for $\widetilde{D}$ near $0 \in \partial\widetilde{D}$. Since ω_p as defined in (4.1) is increasing, we have

$$\|\nabla\rho(Z) - \nabla\rho(W)\| \leq \omega_p\big(\|Z - W\|\big) \quad \forall Z,\ W \in \widetilde{\mathcal{N}_p}. \tag{4.3}$$

Since $V \Subset \mathcal{N}_p$, $m \equiv m_V := \inf\big\{\|D\rho\big(\widetilde{\xi}\big)\| : \widetilde{\xi} \in \partial\widetilde{D} \cap \widetilde{V}\big\} > 0$. Now choose $\beta \equiv \beta_V > 1$ such that $1/\beta \leq m/4\sqrt{2}$. The need for such a choice for β will become evident later. Take $r_V > 0$ with $r_V < r_p$ satisfying $\big(\bigcup_{\xi \in \partial D \cap V} B^n(\xi, r_V)\big) \cap D \subset \mathcal{N}_p \cap D$. Also, recall that the function h as defined in (4.2) satisfies $h'(0) = 0$. So, we can choose a sufficiently small $\varepsilon \equiv \varepsilon_V > 0$ such that

$$\sqrt{2}\varepsilon < r_V \quad \text{and} \quad x/h^{-1}(x) < 1/\beta \ \ \forall x \in (0, \varepsilon). \tag{4.4}$$

It is clear that with the choice of $\beta,\ \varepsilon$ above, $\mathcal{D}(\beta, \varepsilon) \subset D(0, r_V)$. It suffices to show that $\mathsf{U}^p\big(\Psi_\xi(\mathcal{D}(\beta, \varepsilon))\big) \subset \widetilde{\mathcal{N}_p} \cap \widetilde{D}$ for all $\xi \in \partial D \cap V$. Fix $\xi \in \partial D \cap V$, $\zeta = s + it \in \mathcal{D}(\beta, \varepsilon)$. Now, $\mathsf{U}^p(\xi + \zeta\eta_\xi) = \widetilde{\xi} + \zeta\widetilde{\eta_\xi}$ where we write $\widetilde{\xi} := \mathsf{U}^p(\xi)$, $\widetilde{\eta_\xi} := \mathsf{U}_p(\eta_\xi)$. Then, by the Fundamental Theorem of Calculus,

$$\begin{aligned} \rho\big(\widetilde{\xi} + \zeta\widetilde{\eta_\xi}\big) &= \rho\big(\widetilde{\xi}\big) + D\rho\big(\widetilde{\xi}\big)\big(\zeta\widetilde{\eta_\xi}\big) + \int_0^1 \big(D\rho\big(\widetilde{\xi} + x\zeta\widetilde{\eta_\xi}\big) - D\rho\big(\widetilde{\xi}\big)\big)\big(\zeta\widetilde{\eta_\xi}\big)dx \\ &= s\big\langle\nabla\rho\big(\widetilde{\xi}\big) \mid \widetilde{\eta_\xi}\big\rangle + t\big\langle\nabla\rho\big(\widetilde{\xi}\big) \mid \mathbb{J}\big(\widetilde{\eta_\xi}\big)\big\rangle + \int_0^1 \big\langle\nabla\rho\big(\widetilde{\xi} + x\zeta\widetilde{\eta_\xi}\big) - \nabla\rho\big(\widetilde{\xi}\big) \mid \zeta\widetilde{\eta_\xi}\big\rangle\, dx, \end{aligned}$$

where $\langle\,\cdot \mid \cdot\,\rangle$ denotes the standard inner product on $\mathbb{R}^{2n}$ and $\mathbb{J} : \mathbb{R}^{2n} \to \mathbb{R}^{2n}$ is the standard almost complex structure. Now, U_p being unitary, $\widetilde{\eta_\xi} = -\nabla\rho\big(\widetilde{\xi}\big)/\|\nabla\rho\big(\widetilde{\xi}\big)\|$. Thus, using (4.3), the last equation gives

$$\rho\big(\widetilde{\xi}+\zeta\widetilde{\eta_\xi}\big) \le -sm + |\zeta| \int_0^1 \omega_p(x|\zeta|)\,dx. \tag{4.5}$$

Since $\zeta = s+it \in \mathcal{D}(\beta,\varepsilon)$, in view of (4.4)

$$|\zeta|^2 \le s^2 + \big(h^{-1}(s/\beta)\big)^2 = \big(h^{-1}(s/\beta)\big)^2\left(\beta^2\Big(\frac{(s/\beta)}{h^{-1}(s/\beta)}\Big)^2 + 1\right) \le 2\big(h^{-1}(s/\beta)\big)^2.$$

In view of this estimate, (4.5) implies

$$\begin{aligned}\rho\big(\widetilde{\xi}+\zeta\widetilde{\eta_\xi}\big) &\le -sm + \sqrt{2}h^{-1}(s/\beta)\int_0^1 \omega_p\big(2xh^{-1}(s/\beta)\big)\,dx\\ &\le -sm + 2\sqrt{2}h^{-1}(s/\beta)\int_0^1 \omega_p\big(xh^{-1}(s/\beta)\big)\,dx\\ &= -sm + 2\sqrt{2}\int_0^{h^{-1}(s/\beta)} \omega_p(u)\,du = -sm + 2\sqrt{2}s/\beta \le -sm + sm/2 < 0,\end{aligned}$$

where the second inequality is due to the sub-additivity of ω_p and the last inequality is due to our choice of β. This shows that $\widetilde{\xi}+\zeta\widetilde{\eta_\xi} \in \widetilde{\mathcal{N}_p}\cap\widetilde{D}$ for all $\zeta\in\mathcal{D}(\beta,\varepsilon)$. Hence the result. □

In order to present the next two results, we require a definition and some related remarks.

Definition 4.2. A bounded domain $\mathcal{D} \subsetneq \mathbb{C}$ is called a *model domain* if $\mathcal{D}$ is an open subset of $\{\zeta\in\mathbb{C} : \mathsf{Re}\,\zeta > 0\}$ that is symmetric about $\mathbb{R}$, whose boundary is a Jordan curve with $0 \in \partial\mathcal{D}$, and such that, if g denotes the unique biholomorphic mapping of $\mathcal{D}$ onto $\mathbb{D}$ such that $g(\mathcal{D}\cap\mathbb{R}) = (-1,1)$ and $g(0) = 1$, then the limit

$$\lim_{\mathcal{D}\ni\zeta\to 0}\frac{g(\zeta)-1}{\zeta} \tag{4.6}$$

exists and is non-zero.

Remark 4.3. Since $\mathcal{D}$ in Definition 4.2 is enclosed by a Jordan curve, it follows from Carathéodory's theorem that any biholomorphic map G of $\mathcal{D}$ onto $\mathbb{D}$ extends to a homeomorphism from $\overline{\mathcal{D}}$ to $\overline{\mathbb{D}}$. Thus, for such a map, $G(0)$ makes sense. Now, it is classical that a biholomorphic map $g : \mathcal{D} \xrightarrow{\text{onto}} \mathbb{D}$ such that $g(\mathcal{D}\cap\mathbb{R}) = (-1,1)$ and $g(0) = 1$ exists and is unique.

Remark 4.4. The domains $\mathcal{D}(\beta,\varepsilon)$, $\beta,\ \varepsilon > 0$, introduced above are examples of model domains. It is obvious from its construction that $\mathcal{D}(\beta,\varepsilon)$ has the geometric properties that a model domain must have. As for the existence of the limit (4.6) for the biholomorphic map $g_{\beta,\varepsilon} : \mathcal{D}(\beta,\varepsilon)\to\mathbb{D}$ that maps $\mathcal{D}(\beta,\varepsilon)\cap\mathbb{R}$ onto $(-1,1)$ and such that $g_{\beta,\varepsilon}(0) = 1$ and the condition on it: the desired condition is given by [18, Lemma 4.3] by Maitra combined with [25, Theorem 1] by Warschawski.

The next proposition will lead to our desired estimate on K_D as discussed in the beginning of this section (for D as in Theorem 1.7). Although this proposition is stated for domains that satisfy very general conditions near a given boundary point, we shall apply it to domains that satisfy a much more geometrical hypothesis. To be explicit, the domains that are considered in Theorem 1.7 will be shown to satisfy the hypothesis of the following proposition. Here η_ξ will denote the unit inward normal vector to ∂D at $\xi\in\partial D\cap V$.

Proposition 4.5. *Let D be a domain in $\mathbb{C}^n$ and $p \in \partial D$. Suppose D admits a pair of balls $B^n(p, 4\varrho) =: V$ and $B^n(p, \varrho) =: V'$ such that $\partial D \cap V$ is a connected $\mathcal{C}^1$-submanifold of V, and such that*

- $\|\eta_\xi - \eta_p\| < 1/8$ *for every* $\xi \in \partial D \cap V$.
- *For each $z \in V' \cap D$, there exists a point $\xi \in \partial D \cap V$ such that $\delta_D(z) = \|z - \xi\|$.*
- *There exists a constant $c \in (5/8, 1)$ such that*

$$z + t\eta_\xi \in D \quad \text{and} \quad \delta_D(z + t\eta_\xi) > ct$$

for every $t \in [0, 2\varrho]$, for every $z \in V' \cap D$, and every $\xi \in \partial D \cap V$.

For $\xi \in \partial D \cap V$, let $\Psi_\xi : \mathbb{C} \to \mathbb{C}^n$ denote the $\mathbb{C}$-affine map $\zeta \mapsto \xi + \zeta\eta_\xi$. Assume that there exists a model domain $\mathcal{D} \subset \mathbb{C}$ such that $\Psi_\xi(\mathcal{D}) \subset D$ for each $\xi \in \partial D \cap V$. Then, there exists a constant $C > 0$ such that for each $z_1, z_2 \in V' \cap D$,

$$K_D(z_1, z_2) \leq \sum_{j=1}^{2} \frac{1}{2} \log \frac{1}{\delta_D(z_j)} - \sum_{j=1}^{2} \frac{1}{2} \log \left(\frac{1}{\delta_D(z_j) + \|z_2 - z_1\|} \right) + C.$$

The above result is, in essence, [10, Proposition 2.5] by Forstnerič–Rosay with the conditions that make the proof of the latter work emphasised in the hypothesis of Proposition 4.5 in place of the local $\mathcal{C}^{1,\epsilon}$ condition of Forstnerič–Rosay. Thus, the proof of Proposition 4.5 is nearly verbatim the proof of [10, Proposition 2.5] (keeping careful track of where U and $\widetilde{U}$ are required, which correspond to V' and V, respectively, in our case). Therefore, we shall omit the proof of the above proposition.

The following is the estimate on K_D alluded to at the beginning of this section.

Corollary 4.6. *Let $D \subset \mathbb{C}^n$ be a domain, $p \in \partial D$, and U be a neighbourhood of p such that $\partial D \cap U$ is a $\mathcal{C}^{1,\,\mathrm{Dini}}$ submanifold of U. Then, there exists a neighbourhood U' of p and a constant $C > 0$ such that for each $z_1, z_2 \in U' \cap D$,*

$$K_D(z_1, z_2) \leq \sum_{j=1}^{2} \frac{1}{2} \log \frac{1}{\delta_D(z_j)} - \sum_{j=1}^{2} \frac{1}{2} \log \left(\frac{1}{\delta_D(z_j) + \|z_2 - z_1\|} \right) + C.$$

Proof. The set $\mathcal{N}_p$ below will be as given by Definition 1.3-(2). Since $\partial D \cap U$ is a $\mathcal{C}^{1,\,\mathrm{Dini}}$ submanifold of U, we can choose a sufficiently small $\varrho > 0$ such that $B^n(p, 4\varrho) \Subset \mathcal{N}_p$ and such that — writing $V := B^n(p, 4\varrho)$, $V' := B^n(p, \varrho)$ — all but the last condition stated in Proposition 4.5 are satisfied. Now, by Proposition 4.1, there exist constants $\beta, \varepsilon > 0$ such that $\Psi_\xi(\mathcal{D}(\beta, \varepsilon)) \subset U \cap D$ for all $\xi \in \partial D \cap V$. Thus, in view of Remark 4.4, the result follows immediately from Proposition 4.5 (by taking $U' = V'$). □

5. Estimates near a locally log-type convex boundary

This section is devoted to obtaining a useful lower bound for K_Ω, where Ω is as in Theorem 1.7. Such a lower bound is the delicate part of the proof of Theorem 1.7. To this end, we first state a couple of results from the literature.

Result 5.1 *(Graham, [11,12]). Let Ω be a bounded convex domain in $\mathbb{C}^n$. Then:*

$$\frac{\|v\|}{2\delta_\Omega(z; v)} \leq k_\Omega(z; v) \leq \frac{\|v\|}{\delta_\Omega(z; v)} \quad \forall z \in \Omega \text{ and } \forall v \in \mathbb{C}^n \setminus \{0\}.$$

Here, the quantity $\delta_\Omega(z;v)$ is as introduced in Section 1.

Result 5.2 *(paraphrasing [16, Theorem 3.2] by Liu–Wang). Let Ω be a bounded domain, $q \in \partial\Omega$, and suppose there exists a neighbourhood $\mathscr{O}$ of q such that $\mathscr{O} \cap \Omega$ is log-type convex. Then, there exists a neighbourhood W of q, $W \Subset \mathscr{O}$, and a constant $C_q > 1$ such that*

$$k_{\mathscr{O}\cap\Omega}(w;v) \leq C_q k_\Omega(w;v) \quad \forall w \in W \cap \Omega \text{ and } \forall v \in \mathbb{C}^n.$$

Remark 5.3. [16, Theorem 3.2] has a seemingly more technical statement than the above paraphrasing. We get the above — in the language of [16, Theorem 3.2] — by focusing attention to $\xi = q \in \partial\Omega \cap \mathscr{O}$. Then, W is the ball $B^n(q,\varepsilon)$, where $\varepsilon > 0$ is as given by the latter theorem. The constant $C_q > 1$, then, is

$$\exp\left(c\left(\log \frac{1}{\sup_{x\in W\cap\Omega} \delta_\Omega(x)}\right)^{-(1+\nu)}\right),$$

where $c, \nu > 0$ are as given by [16, Theorem 3.2].

Lemma 5.4. *Let Ω, $q \in \partial\Omega$, and $\mathscr{O} \subsetneq_{\text{open}} \mathbb{C}^n$ be as in the statement of Result 5.2. Then, there exists a neighbourhood W of q, $W \Subset \mathscr{O}$, and constants c, $\nu > 0$ such that*

$$k_\Omega(w;v) \geq c\|v\|\left(\log \frac{1}{\delta_\Omega(w)}\right)^{1+\nu} \quad \forall w \in W \cap \Omega \text{ and } \forall v \in \mathbb{C}^n.$$

Proof. Since $\mathscr{O} \cap \Omega$ is log-type convex, there are constants $C, \nu > 0$ such that

$$\delta_{\mathscr{O}\cap\Omega}(w;v) \leq C\left|\log \delta_{\mathscr{O}\cap\Omega}(w)\right|^{-(1+\nu)} \quad \forall w \in \mathscr{O}\cap\Omega \text{ and } \forall v \in \mathbb{C}^n.$$

We now choose a sufficiently small neighbourhood W of q with $W \Subset \mathscr{O}$ such that the conclusion of Result 5.2 holds and such that $\delta_{\mathscr{O}\cap\Omega}(w) = \delta_\Omega(w) < 1$ for all $w \in W \cap \Omega$. Then, applying Result 5.1 to $k_{\mathscr{O}\cap\Omega}$ and using the inequality above, the result follows. □

The next proposition relies on the convex domain $\mathscr{O}\cap\Omega$, where $\mathscr{O}$ and Ω are as introduced above, having a geometry that is favourable for the estimates that we require. At this stage, we need to introduce a new notion; we wish to have a brief discussion about Goldilocks domains, which were introduced by Bharali–Zimmer [4, Definition 1.1]. In the proposition below, a key step is to show that $\mathscr{O}\cap\Omega$ is a Goldilocks domain. This fact and the convexity of $\mathscr{O} \cap \Omega$ together will allow us to prove this result. First, we need a couple of definitions:

Let $\Omega \subset \mathbb{C}^n$ be a bounded domain and $r > 0$. Define

$$M_\Omega(r) := \sup\left\{\frac{1}{k_\Omega(w;v)} : \delta_\Omega(w) \leq r,\ \|v\| = 1\right\}.$$

We say that Ω is a *Goldilocks domain* if

(1) for some (hence any) $\varepsilon > 0$ we have

$$\int_0^\varepsilon \frac{1}{r} M_\Omega(r) dr < \infty, \text{ and}$$

(2) for each $w_0 \in \Omega$ there exist constants $\alpha,\ \beta > 0$ (that depend on w_0) such that

$$K_\Omega(w, w_0) \leq \alpha + \beta \log \frac{1}{\delta_\Omega(w)} \quad \forall w \in \Omega.$$

See [4, Remark 1.3] for an explanation of the geometric significance of the two conditions above.

Proposition 5.5. *Let Ω, $q \in \partial\Omega$, and $\mathscr{O} \subsetneq_{\text{open}} \mathbb{C}^n$ be as in the statement of Result 5.2. Then, for $\xi \in (\partial\Omega \cap \mathscr{O}) \setminus \{q\}$, there exist constants $\varepsilon,\ K > 0$ such that $B^n(q, \varepsilon),\ B^n(\xi, \varepsilon) \subset \mathscr{O}$ and*

$$K_{\mathscr{O}\cap\Omega}(w_1, w_2) \geq \frac{1}{2} \log \frac{1}{\delta_\Omega(w_1)} + \frac{1}{2} \log \frac{1}{\delta_\Omega(w_2)} - K \tag{5.1}$$

for all $w_1 \in B^n(q, \varepsilon) \cap \Omega$, for all $w_2 \in B^n(\xi, \varepsilon) \cap \Omega$.

Proof. This result will be proved in the following two steps.

Step 1. *Showing that $\mathscr{O} \cap \Omega$ is a Goldilocks domain*

First, we shall prove that

$$\int_0^\varepsilon \frac{1}{r} M_{\mathscr{O}\cap\Omega}(r)dr < \infty$$

for some (hence for any) $\varepsilon > 0$. Fix $0 < \varepsilon < 1$, $r \in (0, \varepsilon]$, and $v \in \mathbb{C}^n$ with $\|v\| = 1$. Let $w \in \mathscr{O} \cap \Omega$ be such that $\delta_{\mathscr{O}\cap\Omega}(w) \leq r$. By Result 5.1 we get

$$\frac{1}{k_{\mathscr{O}\cap\Omega}(w; v)} \leq 2\delta_{\mathscr{O}\cap\Omega}(w; v) \leq 2C\left(\log \frac{1}{\delta_{\mathscr{O}\cap\Omega}(w)}\right)^{-(1+\nu)} \leq 2C\left(\log \frac{1}{r}\right)^{-(1+\nu)},$$

where $C,\ \nu > 0$ are as given by Definition 1.6. Since $\int_0^\varepsilon r^{-1}\big(\log(1/r)\big)^{-(1+\nu)} dr < \infty$, our desired integral is also convergent.

Now, $\mathscr{O} \cap \Omega$ being convex, by [4, Lemma 2.3], for each $w_0 \in \mathscr{O} \cap \Omega$ there are constants $\alpha,\ \beta > 0$ (that depend on w_0) such that

$$K_{\mathscr{O}\cap\Omega}(w_0, w) \leq \alpha + \beta \log \frac{1}{\delta_{\mathscr{O}\cap\Omega}(w)} \quad \forall w \in \mathscr{O} \cap \Omega.$$

The above argument shows that $\mathscr{O} \cap \Omega$ satisfies both of the conditions in [4, Definition 1.1] for being a Goldilocks domain.

Step 2. *Proving the estimate* (5.1)

Since $\mathscr{O} \cap \Omega$ is convex, it follows from [5, (2.4)] that

$$K_{\mathscr{O}\cap\Omega}(w, w') \geq \frac{1}{2}\left|\log \frac{\delta_{\mathscr{O}\cap\Omega}(w)}{\delta_{\mathscr{O}\cap\Omega}(w')}\right| \quad \forall w,\ w' \in \mathscr{O} \cap \Omega. \tag{5.2}$$

Now, fix a point $o \in \mathscr{O} \cap \Omega$ and let $\xi \in (\partial\Omega \cap \mathscr{O}) \setminus \{q\}$. So, q and ξ are two distinct boundary points of the Goldilocks domain $\mathscr{O} \cap \Omega$. Let $\varepsilon > 0$ be such that $B^n(q, \varepsilon),\ B^n(\xi, \varepsilon) \subset \mathscr{O}$, such that — denoting $V_q := B^n(q, \varepsilon) \cap \Omega,\ V_\xi := B^n(\xi, \varepsilon) \cap \Omega$ — $\overline{V_q} \cap \overline{V_\xi} = \emptyset$, and such that $\delta_{\mathscr{O}\cap\Omega}(w) = \delta_\Omega(w) < \delta_{\mathscr{O}\cap\Omega}(o)$ for all $w \in V_q \cup V_\xi$. Then, it follows from the proof of [4, Proposition 6.8] that there exists a constant $K' > 0$ such that

$$K_{\mathscr{O}\cap\Omega}(w_1,o)+K_{\mathscr{O}\cap\Omega}(o,w_2)-K_{\mathscr{O}\cap\Omega}(w_1,w_2)\le K' \quad \forall w_1\in V_q \text{ and } \forall w_2\in V_\xi. \tag{5.3}$$

Therefore, (5.2) and (5.3) together imply

$$K_{\mathscr{O}\cap\Omega}(w_1,w_2)\ge \frac{1}{2}\log\frac{1}{\delta_\Omega(w_1)}+\frac{1}{2}\log\frac{1}{\delta_\Omega(w_2)}+\log\delta_{\mathscr{O}\cap\Omega}(o)-K' \quad \forall w_1\in V_q \text{ and } \forall w_2\in V_\xi.$$

Thus, choosing $K>\max(0,K'-\log\delta_{\mathscr{O}\cap\Omega}(o))$, we get the desired conclusion. □

The following will play a key role in the proof of the main result of this section.

Result 5.6 *(Liu–Wang, [16, Theorem 1.4]). Let Ω be a bounded domain in $\mathbb{C}^n$. Suppose that there exists a connected open set $\mathscr{O}$ with $\partial\Omega\cap\mathscr{O}\ne\emptyset$ and $\mathscr{O}\cap\Omega$ is log-type convex. Then, for any open set W with $W\Subset\mathscr{O}$, there exists $K>0$ such that the Kobayashi distance satisfies*

$$K_\Omega(w_1,w_2)\le K_{\mathscr{O}\cap\Omega}(w_1,w_2)\le K_\Omega(w_1,w_2)+K \quad \forall w_1,\ w_2\in W\cap\Omega.$$

We now have all the ingredients to establish the lower bound for K_Ω that we need.

Proposition 5.7. *Let Ω, $q\in\partial\Omega$, and $\mathscr{O}\subsetneq_{\text{open}}\mathbb{C}^n$ be as in the statement of Result 5.2. Then, for $\xi\in(\partial\Omega\cap\mathscr{O})\setminus\{q\}$, there exist constants $\varepsilon,\ K>0$ such that $B^n(q,\varepsilon),\ B^n(\xi,\varepsilon)\subset\mathscr{O}$ and*

$$K_\Omega(w_1,w_2)\ge\frac{1}{2}\log\frac{1}{\delta_\Omega(w_1)}+\frac{1}{2}\log\frac{1}{\delta_\Omega(w_2)}-K$$

for all $w_1\in B^n(q,\varepsilon)\cap\Omega$, for all $w_2\in B^n(\xi,\varepsilon)\cap\Omega$.

Proof. Let us choose $\varepsilon>0$ such that $B^n(q,\varepsilon)\cup B^n(\xi,\varepsilon)\Subset\mathscr{O}$, such that (5.1) holds for all $w_1\in B^n(q,\varepsilon)\cap\Omega$ and for all $w_2\in B^n(\xi,\varepsilon)\cap\Omega$. By Result 5.6, there exists $K>0$ such that

$$K_{\mathscr{O}\cap\Omega}(w_1,w_2)\le K_\Omega(w_1,w_2)+K \quad \forall w_1,\ w_2\in\big(B^n(q,\varepsilon)\cup B^n(\xi,\varepsilon)\big)\cap\Omega.$$

Thus, using the above estimate, the result follows immediately from Proposition 5.5. □

6. The proof of Theorem 1.5

Before we present the proof, we fix some notations that will be used over the course of the proof. We will write $\widetilde{F}=\big(\widetilde{F}_1,\dots,\widetilde{F}_n\big)=F\circ(\mathsf{U}^p)^{-1}$, where U^p is as introduced in Definition 1.3. For any subset $S\subset\mathbb{C}^n$, we will write $\widetilde{S}=\mathsf{U}^p(S)$, and we will indicate that we are working in the coordinate system given by U^p by using $Z_1,\dots,Z_n$, as in the proof of Proposition 4.1.

Further, in the proof below C will denote a positive constant that depends *only* on the data in the hypothesis of Theorem 1.5. However, the magnitude of C may change from line to line.

Proof of Theorem 1.5. Let V be the neighbourhood of p as given by Lemma 3.1. Then, since $F(V\cap D)\subset W\cap\Omega$, for all $z\in V\cap D$ and $v\in\mathbb{C}^n$, we have

$$\frac{\|F'(z)v\|}{M\big(\delta_\Omega(F(z))\big)}\le k_\Omega(F(z);F'(z)v)\le k_D(z;v)\le\frac{\|v\|}{\delta_D(z)}.$$

The last inequality follows by the distance decreasing property for the Kobayashi metric under the inclusion map $B^n(z,\delta_D(z))\hookrightarrow D$. Now, using Lemma 3.4, for all $z\in V\cap D$

$$-(\delta_D(z))^s \le \rho(z) \le \tau(F(z)) \le -C_*\big(\delta_\Omega(F(z))\big)^{\alpha_*}, \tag{6.1}$$

which gives (since M is increasing), for all $z \in V \cap D$

$$M\big(\delta_\Omega(F(z))\big) \le M\big(C(\delta_D(z))^{s/\alpha_*}\big).$$

Combining the above inequalities we get

$$\|F'(z)v\| \le \frac{\|v\|}{\delta_D(z)} M\big(C(\delta_D(z))^{s/\alpha_*}\big) \quad \forall z \in V \cap D,\, \forall v \in \mathbb{C}^n. \tag{6.2}$$

We now express the estimate (6.2) in the new coordinate system $\mathbb{C}^n \ni z \mapsto Z = \mathsf{U}^p(z)$. Let us write $A := \mathsf{U}_p^{-1}$, which is a unitary transformation. Applying chain rule, (6.2) gives

$$\begin{aligned}\|\widetilde{F}'(Z)v\| = \|F'(z)(Av)\| \le \frac{\|Av\|}{\delta_D(z)} M\big(C(\delta_D(z))^{s/\alpha_*}\big) &= \frac{\|v\|}{\delta_D(z)} M\big(C(\delta_D(z))^{s/\alpha_*}\big)\\ &\le \frac{C\,\|v\|}{Y(Z)} M\big(C(Y(Z))^{s/\alpha_*}\big),\end{aligned} \tag{6.3}$$

where the last inequality holds by Lemma 3.1. Recall that M satisfies a Dini condition. It is elementary to see (using change of variables) that for fixed κ, $m > 0$, the composite function $t \mapsto M(\kappa\, t^m)$ also satisfies a Dini condition. Hence, we can rewrite (6.3) as:

$$\|\widetilde{F}'(Z)v\| \le \psi\big(Y(Z)\big)\|v\| \quad \forall Z \in \widetilde{V} \cap \widetilde{D},\, \forall v \in \mathbb{C}^n, \tag{6.4}$$

where ψ is a non-negative Lebesgue integrable function on range(Y).

Now we choose a neighbourhood $\widetilde{\mathcal{U}}$ of $\mathbf{0} \in \mathbb{C}^n$, $\widetilde{\mathcal{U}} \Subset \widetilde{V}$ and $\delta > 0$ such that

$$\Big(\bigcup_{\xi \in \partial\widetilde{D} \cap \widetilde{\mathcal{U}}} \overline{B^n(\xi, \delta)}\Big) \cap \widetilde{D} \subset \widetilde{V} \cap \widetilde{D}. \tag{6.5}$$

Pick $1 \le j \le n$, fix $\xi = (\xi', \zeta + i\,\eta) = (\xi', \zeta + i\,\varphi_p(\xi', \zeta)) \in \partial\widetilde{D} \cap \widetilde{\mathcal{U}}$, and $0 < t < t' < \delta$. Then,

$$\widetilde{F}_j(\xi + t'\boldsymbol{\epsilon}) - \widetilde{F}_j(\xi + t\boldsymbol{\epsilon}) = \int_t^{t'} i\,\frac{\partial \widetilde{F}_j}{\partial Z_n}(\xi + x\boldsymbol{\epsilon})\,dx \quad \text{where } \boldsymbol{\epsilon} = (0, \dots, 0, i) \in \mathbb{C}^n. \tag{6.6}$$

By (6.4),

$$\left|\int_t^{t'} i\,\frac{\partial \widetilde{F}_j}{\partial Z_n}(\xi + x\boldsymbol{\epsilon})\,dx\right| \le \int_t^{t'} \psi\big(Y(\xi + x\boldsymbol{\epsilon})\big)\,dx = \int_t^{t'} \psi(x)\,dx.$$

Hence, by integrability of ψ and the fact that $0 \in$ domain (ψ), the limit

$$\widetilde{F}_j^{\bullet}(\xi) := \widetilde{F}_j(\xi + t'\boldsymbol{\epsilon}) - \lim_{t \to 0^+} \int_t^{t'} i\,\frac{\partial \widetilde{F}_j}{\partial Z_n}(\xi + x\boldsymbol{\epsilon})\,dx$$

exists. (Note that, although the right hand side of the above expression involves the parameter t', in view of (6.6), it does not depend on t'.) Also, we get the estimate (which is uniform in ξ)

$$\left|\widetilde{F}_j^{\bullet}(\xi) - \widetilde{F}_j(\xi + t'\boldsymbol{\epsilon})\right| \leq \int_0^{t'} \psi(x)\, dx \quad \forall \xi \in \partial\widetilde{D} \cap \widetilde{\mathcal{U}} \text{ and } \forall t' \in (0, \delta). \tag{6.7}$$

We can now define $\widehat{F} = \big(\widehat{F}_1, \ldots, \widehat{F}_n\big) : \widetilde{D} \cup \big(\partial\widetilde{D} \cap \widetilde{\mathcal{U}}\big) \longrightarrow \overline{\Omega}$

$$\widehat{F}(Z) := \begin{cases} \widetilde{F}^{\bullet}(Z) = \big(\widetilde{F}_1^{\bullet}(Z), \ldots, \widetilde{F}_n^{\bullet}(Z)\big), & \text{if } Z \in \partial\widetilde{D} \cap \widetilde{\mathcal{U}}, \\ \\ \widetilde{F}(Z), & \text{otherwise.} \end{cases} \tag{6.8}$$

Our goal is to show that $\widehat{F}$ is continuous. It is enough to show its continuity on $\partial\widetilde{D} \cap \widetilde{\mathcal{U}}$. We will adapt a Hardy–Littlewood trick to complete the proof.

Let $\varepsilon > 0$. As ψ is integrable near 0, there exists $r(\varepsilon) \in (0, \delta)$ such that (6.7) gives

$$\left|\widetilde{F}_j^{\bullet}(\xi) - \widetilde{F}_j(\xi + r(\varepsilon)\boldsymbol{\epsilon})\right| \leq \int_0^{r(\varepsilon)} \psi(x)\, dx < \varepsilon/3 \quad \text{for every } \xi \in \partial\widetilde{D} \cap \widetilde{\mathcal{U}}. \tag{6.9}$$

Define $S(\varepsilon) := \Big\{\xi + r(\varepsilon)\boldsymbol{\epsilon} : \xi \in \partial\widetilde{D} \cap \widetilde{\mathcal{U}}\Big\}$. By the choice of $\widetilde{\mathcal{U}}$ as in (6.5), we have $S(\varepsilon) \Subset \widetilde{V} \cap \widetilde{D} \subset \widetilde{U} \cap \widetilde{D}$. Hence $\widetilde{F}_j\big|_{S(\varepsilon)}$ is uniformly continuous. So, there exists $\sigma \equiv \sigma(\varepsilon) > 0$ such that

$$\left|\widetilde{F}_j(Z) - \widetilde{F}_j(W)\right| < \varepsilon/3 \quad \text{whenever } Z, W \in S(\varepsilon) \text{ with } \|Z - W\| < \sigma. \tag{6.10}$$

Now, let $\xi_1, \xi_2 \in \partial\widetilde{D} \cap \widetilde{\mathcal{U}}$ such that $\|\xi_1 - \xi_2\| < \sigma$. Let $W_1, W_2 \in S(\varepsilon)$ such that $W_k := \xi_k + r(\varepsilon)\boldsymbol{\epsilon},\ k = 1, 2$. Since $\|W_1 - W_2\| = \|\xi_1 - \xi_2\| < \sigma$, (6.9) and (6.10) together imply

$$\left|\widetilde{F}_j^{\bullet}(\xi_1) - \widetilde{F}_j^{\bullet}(\xi_2)\right| \leq \sum_{k=1}^{2} \left|\widetilde{F}_j^{\bullet}(\xi_k) - \widetilde{F}_j(W_k)\right| + \left|\widetilde{F}_j(W_1) - \widetilde{F}_j(W_2)\right| < \varepsilon.$$

This shows that $\widehat{F}\big|_{\partial\widetilde{D} \cap \widetilde{\mathcal{U}}}$ is continuous.

Let us now fix $\xi = (\xi', \zeta + i\,\eta) = (\xi', \zeta + i\,\varphi_p(\xi', \zeta)) \in \partial\widetilde{D} \cap \widetilde{\mathcal{U}}$. It suffices to show that for any sequence $\{Z_\nu\} \subset \Big(\widetilde{\mathcal{U}} \cap \overline{\widetilde{D}}\Big) \setminus \{\xi\}$, converging to ξ, we have $\widehat{F}_j(Z_\nu) \to \widetilde{F}_j^{\bullet}(\xi)$ as $\nu \to \infty$. Define an auxiliary sequence $\{\widetilde{Z_\nu}\}$ as

$$\widetilde{Z_\nu} := \begin{cases} Z_\nu, & \text{if } Z_\nu \in \partial\widetilde{D}, \\ Z_\nu, & \text{if } Z_\nu = \big(\xi', \zeta + i\,(x + \varphi_p(\xi', \zeta))\big) \text{ for some } x > 0, \\ \pi(Z_\nu), & \text{otherwise,} \end{cases}$$

where $\pi(Z) := (Z', X_n + i\,\varphi_p(Z', X_n))$ and $Z = (Z', X_n + i\,Y_n) \in \widetilde{\mathcal{U}} \cap \widetilde{D}$ — i.e., the projection along $\mathbb{R}\boldsymbol{\epsilon}$ into the boundary of $\widetilde{D}$. Since π is continuous, $\widetilde{Z_\nu} \to \xi$ as $\nu \to \infty$. Now, using the estimates (6.7) and (6.9) together with the fact that $\widehat{F}\big|_{\partial\widetilde{D} \cap \widetilde{\mathcal{U}}}$ is continuous, we get

$$\lim_{\nu \to \infty} \Big(\widehat{F}_j(\widetilde{Z_\nu}) - \widehat{F}_j(Z_\nu)\Big) = 0 \quad \text{and} \quad \lim_{\nu \to \infty} \widehat{F}_j(\widetilde{Z_\nu}) = \widetilde{F}_j^{\bullet}(\xi) = \widehat{F}_j(\xi).$$

This proves that $\widehat{F}$ as defined in (6.8) is continuous. Since U^p is an automorphism of $\mathbb{C}^n$, this completes the proof. $\square$

Before we end this section, we must elaborate upon a point that was deferred in Section 1.

Remark 6.1. The curious reader might ask whether, since Ω in Theorem 1.5 is assumed to satisfy a uniform interior cone condition in W, one also requires the condition (1.3). The question might arise as the assumption of a uniform interior cone condition in W suggests the existence of a local plurisubharmonic barrier. Then, by Result 2.1, one may hope to deduce (1.3). In fact, the last two ingredients summarise the approach to the proof in [23] (also see [2]). However, such an approach actually requires a local plurisubharmonic barrier with *"good" estimates* — obtaining such estimates is quite hard, and a uniform interior cone condition is not sufficient for such an estimate. Now, note that the hypothesis of Theorem 1.5 imposes no regularity condition on $\partial\Omega$. In its absence, our uniform interior cone condition in W functions as a very mild (local) boundary-regularity condition on Ω. The latter enables us to use a version of the Hopf Lemma to obtain the inequality (6.1). In contrast, in [23] — and in most of the articles on the present theme cited above — the relevant patch of the boundary of the target domain is required to be $\mathcal{C}^2$-smooth (in order to deduce the analogue of (6.1)). That brings us to the condition (1.3): a careful perusal of [23,2] will reveal that (1.3) is **less restrictive** than the plurisubharmonic-barrier condition. Our function M is just a way to express quantitatively the requirement that $k_\Omega(w;\boldsymbol{\cdot})$ must grow as w approaches $\partial\Omega$ via $W\cap\Omega$ but may do so **relatively slowly**. Classically, (1.3) is true if $\partial\Omega\cap W$ is strongly pseudoconvex (with $M(r)=\sqrt{r}$); it is also true if $\partial\Omega\cap W$ is real analytic and all points in $\partial\Omega\cap W$ are of finite type (in which case M is some fractional power); see [7]. However, recall that we want Theorem 1.5 to be able to address the case when $\partial\Omega\cap W$ contains infinite-type points. This is what M (which is more general than a power) enables; see [3,16].

7. The proof of Theorem 1.7

We will present the proof in two steps. The method that we will use in the first step is the method of the proof of [10, Theorem 1.1]. In this step, we will show that F extends as a continuous map on $D\cup\{p\}$. In the second step, we will apply Theorem 1.5 to get the desired conclusion.

Step 1. *Proving that F extends continuously to $p\in\partial D$*

First, we note that the cluster set $C(F,p)$ is connected. This is so because there is a basis of neighbourhoods $\{\mathcal{N}_\nu\}$ of p such $\mathcal{N}_\nu\cap D$ is connected for each ν (see [6, Chapter 1, Section 1] for more details). Also, F being proper, $C(F,p)\subseteq\partial\Omega$.

We shall establish our goal by contradiction. Assume that F does not extend continuously to $p\in\partial D$. Then, $C(F,p)$ is not a singleton. Since $C(F,p)$ is connected, we can find a point $\xi\in(\partial\Omega\cap\mathscr{O})\cap C(F,p)$ with $\xi\neq q$. Consider a pair of sequences $\{z^1_\nu\}$ and $\{z^2_\nu\}$ in $U\cap D$ such that, writing $w^j_\nu := F\big(z^j_\nu\big)$ for $j=1,\ 2$, we have

$$z^j_\nu\to p \text{ for } j=1,\ 2, \text{ and } w^1_\nu\to q,\ w^2_\nu\to\xi \text{ as } \nu\to\infty.$$

So, we can find a non-negative integer $N>1$ such that for all $\nu\geq N$

$$K_D\big(z^1_\nu,z^2_\nu\big)\leq\frac{1}{2}\log\frac{1}{\delta_D\big(z^1_\nu\big)}+\frac{1}{2}\log\frac{1}{\delta_D\big(z^2_\nu\big)}-l(\nu)+C, \tag{7.1}$$

$$K_\Omega\big(w^1_\nu,w^2_\nu\big)\geq\frac{1}{2}\log\frac{1}{\delta_\Omega\big(w^1_\nu\big)}+\frac{1}{2}\log\frac{1}{\delta_\Omega\big(w^2_\nu\big)}-K, \tag{7.2}$$

where $C>0$ is given by Corollary 4.6, $K>0$ is given by Proposition 5.7, and

$$l(\nu):=\sum_{j=1}^{2}\frac{1}{2}\log\left(\frac{1}{\delta_D\big(z^j_\nu\big)+\big\|z^1_\nu-z^2_\nu\big\|}\right)\to\infty. \tag{7.3}$$

Recall that, the function $\tau : \Omega \to \mathbb{R},\ w \mapsto \max\{\rho(z) : z \in F^{-1}\{w\}\}$, is continuous, negative, and plurisubharmonic on Ω (see the discussion prior to Lemma 3.4). Since $\partial\Omega$ is of class $\mathcal{C}^2$ near q, we appeal to the classical Hopf Lemma for plurisubharmonic functions which, essentially, is the inequality (3.1) wherein — due to (local) $\mathcal{C}^2$-regularity — the exponent $\alpha_{\mathcal{U}} = 1$ independent of the open set $\mathcal{U}$. This lemma gives us, raising the value of the N above if needed, a constant $C_0 > 0$ such that for $j = 1,\ 2$, and for all $\nu \geq N$,

$$\tau\big(w_\nu^j\big) \leq -C_0\delta_\Omega\big(w_\nu^j\big).$$

This gives, for $j = 1,\ 2$, and for all $\nu \geq N$,

$$-\delta_D\big(z_\nu^j\big) \leq \rho\big(z_\nu^j\big) \leq \tau\big(w_\nu^j\big) \leq -C_0\delta_\Omega\big(w_\nu^j\big). \tag{7.4}$$

Then, (7.1), (7.2), and (7.4) together imply

$$\begin{aligned} K_D\big(z_\nu^1, z_\nu^2\big) &\leq \frac{1}{2}\log\frac{1}{\delta_\Omega\big(w_\nu^1\big)} + \frac{1}{2}\log\frac{1}{\delta_\Omega\big(w_\nu^2\big)} - l(\nu) + C - \log C_0 \\ &\leq K_\Omega\big(w_\nu^1, w_\nu^2\big) + (K + C - \log C_0) - l(\nu) \\ &\leq K_D\big(z_\nu^1, z_\nu^2\big) + (K + C - \log C_0) - l(\nu) \quad \forall \nu \geq N. \end{aligned}$$

The above estimate implies that $\{l(\nu) : \nu \in \mathbb{Z}_+\}$ is bounded, which contradicts (7.3).

Thus, the map $\widehat{F} : D \cup \{p\} \to \overline{\Omega}$ as defined by

$$\widehat{F}(z) := \begin{cases} F(z), & \text{if } z \in D, \\ q, & \text{if } z = p, \end{cases} \tag{7.5}$$

is continuous and $\widehat{F}\big|_D = F$.

Step 2. *Proving that F extends continuously to a ∂D-neighbourhood of p*

Let W be the neighbourhood of q with $W \Subset \mathscr{O}$ and let $C,\ \nu > 0$ be the constants as given by Lemma 5.4. Then,

$$k_\Omega(w; v) \geq c\|v\|\left(\log\frac{1}{\delta_\Omega(w)}\right)^{1+\nu} \quad \forall w \in W \cap \Omega \text{ and } \forall v \in \mathbb{C}^n.$$

Note that the function $r \mapsto \big(\log(1/r)\big)^{-(1+\nu)},\ 0 < r < 1$, is integrable at zero, and its value approaches 0 as $r \to 0^+$. Thus, it satisfies the conditions on M featured in Theorem 1.5.

Now, $\partial\Omega \cap \mathscr{O}$ being $\mathcal{C}^2$ smooth, Ω must satisfy a uniform interior cone condition in W. Also, since the map $\widehat{F}$ as defined by (7.5) is continuous, we can find an open ball U^* with centre p with $U^* \Subset U$ such that

$$F(U^* \cap D) \subset \widehat{F}(U^* \cap (D \cup \{p\})) \Subset W.$$

Hence, by Theorem 1.5, F extends continuously to a ∂D-neighbourhood of p. $\square$

Now that we have seen the argument for Theorem 1.7, we can make the following

Remark 7.1. We see that the two key properties that the pair (Ω, q) must possess, in addition to the issue of the regularity $\partial\Omega$ near q that was discussed in Section 1, that make the above proof work are (i) a localization result for K_Ω akin to Result 5.6, and (ii) whatever that leads to the estimate (5.3). Readers familiar with the results in [5] might notice that a weaker condition than $\mathscr{O} \cap \Omega$ being log-type convex provides just these two ingredients. That said:

- Replacing log-type convexity of $\mathscr{O} \cap \Omega$ by the property alluded to only ensures the continuous extension of F to $D \cup \{p\}$. It is unclear if the continuous extension of F to $D \cup \mathscr{V}$ can be achieved without imposing *some* additional constraint on k_Ω.
- Our intention in Theorem 1.7 is to present — in terms of the assumptions made — a result in the same spirit as Result 1.1. In the latter case, a lower bound akin to (1.3) is automatic.

It is for these reasons that we highlight Theorem 1.7 in Section 1. Also, while some conditions are now known that will ensure that $\mathscr{O} \cap \Omega$ is convex and has the visibility property (i.e., the property introduced in [5]), the latter property is slightly non-explicit. Moreover, the latter is not the last word on properties that will produce a localization result akin to Result 5.6; this will be discussed in forthcoming work. We also refer the reader to [20, Proposition 12] by Nikolov–Andreev for a result with assumptions similar to those in Theorem 1.7, but applied globally.

The discussion in Remark 7.1 provides enough clues for us to conclude with one last result. We refer the reader to [5, Section 2] for definitions.

Theorem 7.2. *Let D and Ω be domains in $\mathbb{C}^n$, $n \geq 2$, Ω bounded, and let $F : D \to \Omega$ be a proper holomorphic map. Let $p \in \partial D$ and $q \in C(F, p)$. Assume that there is a continuous, negative plurisubharmonic function ρ on D and a neighbourhood U of p such that $\partial D \cap U$ is a $\mathcal{C}^{1,\,\mathrm{Dini}}$ submanifold of U, and $\rho(z) \geq -\delta_D(z)$ for all $z \in U \cap D$. Suppose there exists a neighbourhood $\mathscr{O}$ of q such that*

- *$\partial\Omega \cap \mathscr{O}$ is a $\mathcal{C}^2$ submanifold of $\mathscr{O}$, and*
- *$\mathscr{O} \cap \Omega$ is convex and has the visibility property.*

Then, F extends to a continuous map on $D \cup \{p\}$.

Since the proof of the above is very similar to the argument in Step 1 of the proof in Section 7, we only mention the differences in the argument. Firstly, the role of Result 5.6 is played by [5, Theorem 1.4]. Secondly, the visibility property of $\mathscr{O} \cap \Omega$ ensures that (5.3) is true.

Acknowledgments

I am grateful to my thesis advisor, Prof. Gautam Bharali, for suggesting various ideas and for the invaluable discussions over the course of this project. I am also grateful to him for his help with the writing of this paper. This work is supported by a scholarship from the National Board for Higher Mathematics (NBHM) (Ref. No. 0203/16(19)/2018-R&D-II/10706) and by the DST-FIST programme (grant no. DST FIST-2021 [TPN-700661]).

References

[1] François Berteloot, A remark on local continuous extension of proper holomorphic mappings, in: The Madison Symposium on Complex Analysis, Madison, WI, 1991, in: Contemp. Math., vol. 137, American Mathematical Society, Providence, RI, 1992, pp. 79–83.

[2] F. Berteloot, A. Sukhov, On the continuous extension of holomorphic correspondences, Ann. Sc. Norm. Super. Pisa, Cl. Sci. 24 (4) (1997) 747–766.

[3] Gautam Bharali, Complex geodesics, their boundary regularity, and a Hardy–Littlewood-type lemma, Ann. Acad. Sci. Fenn., Math. 41 (1) (2016) 253–263.

[4] Gautam Bharali, Andrew Zimmer, Goldilocks domains, a weak notion of visibility, and applications, Adv. Math. 310 (2017) 377–425.

[5] Filippo Bracci, Nikolai Nikolov, Pascal J. Thomas, Visibility of Kobayashi geodesics in convex domains and related properties, Math. Z. 301 (2) (2022) 2011–2035.

[6] E.F. Collingwood, A.J. Lohwater, The Theory of Cluster Sets, Cambridge Tracts in Mathematics and Mathematical Physics, vol. 56, 1966.
[7] Klas Diederich, John E. Fornæss, Proper holomorphic maps onto pseudoconvex domains with real-analytic boundary, Ann. Math (2) 110 (3) (1979) 575–592.
[8] John Erik Fornæss, Berit Stensønes, Lectures on Counterexamples in Several Complex Variables, Math. Notes, vol. 33, Princeton Univ. Press, NJ, 1987.
[9] Franc Forstnerič, Proper holomorphic mappings: a survey, in: Several Complex Variables, Stockholm, 1987/1988, in: Math. Notes, vol. 38, Princeton Univ. Press, Princeton, NJ, 1993, pp. 297–363.
[10] Franc Forstnerič, Jean-Pierre Rosay, Localization of the Kobayashi metric and the boundary continuity of proper holomorphic mappings, Math. Ann. 279 (2) (1987) 239–252.
[11] Ian Graham, Distortion theorems for holomorphic maps between convex domains in $\mathbb{C}^n$, Complex Var. Theory Appl. 15 (1) (1990) 37–42.
[12] Ian Graham, Sharp constants for the Koebe theorem and for estimates of intrinsic metrics on convex domains, in: Several Complex Variables and Complex Geometry, Part 2, Santa Cruz, CA, 1989, in: Proc. Sympos. Pure Math., vol. 52, Amer. Math. Soc., Providence, RI, 1991, pp. 233–238.
[13] G.M. Henkin, An analytic polyhedron is not holomorphically equivalent to a strictly pseudoconvex domain, Dokl. Akad. Nauk SSSR 210 (1973) 1026–1029.
[14] Steven G. Krantz, Function Theory of Several Complex Variables, second edition, AMS Chelsea Publishing, 2001.
[15] Igor Kukavica, Kaj Nyström, Unique continuation on the boundary for Dini domains, Proc. Am. Math. Soc. 126 (2) (1998) 441–446.
[16] Jinsong Liu, Hongyu Wang, Localization of the Kobayashi metric and applications, Math. Z. 297 (1–2) (2021) 867–883.
[17] Prachi Mahajan, On isometries of the Kobayashi and Carathéodory metrics, Ann. Pol. Math. 104 (2) (2012) 121–151.
[18] Anwoy Maitra, On the continuous extension of Kobayashi isometries, Proc. Am. Math. Soc. 148 (8) (2020) 3437–3451.
[19] Peter R. Mercer, A general Hopf lemma and proper holomorphic mappings between convex domains in $\mathbb{C}^n$, Proc. Am. Math. Soc. 119 (2) (1993) 573–578.
[20] Nikolai Nikolov, Lyubomir Andreev, Estimates of the Kobayashi and quasi-hyperbolic distances, Ann. Mat. Pura Appl. 196 (1) (2017) 43–50.
[21] S.I. Pinčuk, Proper holomorphic maps of strictly pseudoconvex domains, Sib. Mat. Zh. 15 (1974) 909–917, 959.
[22] Nessim Sibony, A class of hyperbolic manifolds, in: Recent Developments in Several Complex Variables, Proc. Conf., Princeton Univ., Princeton, NJ, 1979, in: Ann. of Math. Stud., vol. 100, Princeton Univ. Press, Princeton, NJ, 1981, pp. 357–372.
[23] A.B. Sukhov, On the boundary regularity of holomorphic mappings, Mat. Sb. 185 (12) (1994) 131–142 (Russian).
[24] Norbert Vormoor, Topologische Fortsetzung biholomorpher Funktionen auf dem Rand bei beschrankten streng-pseudokonvexen Gebieten in $\mathbb{C}^n$, Math. Ann. 204 (1973) 239–261.
[25] S.E. Warschawski, On differentiability at the boundary in conformal mapping, Proc. Am. Math. Soc. 12 (1961) 614–620.