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Compositional Approximately Bisimilar Abstractions of Interconnected Systems ⋆

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Abstract: This paper formulates and studies the concepts of approximate (alternating) bisimulation relations characterizing equivalence relations between interconnected systems and their abstractions. These equivalence relations guarantee that the abstraction preserves the dynamics of the original model. We develop a compositional approach to abstraction-based controller synthesis based on the notions of approximate composition and incremental input-to-state stability. In particular, given a large system consisting of interconnected components, we provide conditions under which the notion of approximate (alternating) simulation relation is preserved when moving from the subsystems to the large interconnected system. The effectiveness of the proposed results has been evaluated through traffic congestion control.

Keywords: Symbolic control, Compositional abstraction, Interconnected Systems

1. INTRODUCTION

Model verification and control is an important concept in formal methods, where models are used to represent part of the system that is formalized by a set of properties. It stands for the ability to prove that some properties hold true for a model based on the assumptions of the system and the properties themselves. Model verification and control also serve as powerful tools for validating the correctness and performance of the system. It is a way to ensure that the system meets its specifications when designing a specific control software, see Girard and Pappas (2007); Pola et al. (2008); Tabuada (2009); Julius et al. (2009); Zamani et al. (2011); Hashimoto et al. (2019). These contributions provide a systematic foundation for model verification and control of various classes of dynamical systems, including linear, stochastic, and nonlinear systems. Model verification is performed by establishing exact or approximate (bi)simulation relations between an original system and its corresponding abstraction, see Girard and Pappas (2007); Pola et al. (2008); Tabuada (2009). Abstraction-based controller synthesis responds to the problem of synthesizing controllers that satisfy spatiotemporal logic specifications, see Meyer et al. (2017); Saoud et al. (2018, 2021). These specifications are usually expressed using temporal logic formula.

The abstraction construction procedure generally suffers from scalability issues, making the construction of abstractions for large interconnected systems challenging; see Saoud (2019) and references therein. To solve this problem, many compositional approaches have been proposed in the literature. In such approaches, one starts from a large system consisting of interconnected subsystems. Then, an abstraction for the large system is constructed from the abstraction of its subsystems. In this context, (Rungger and Zamani (2016)) relied on the notion of a simulation function and a small-gain type condition to provide a compositional framework that constrains the behavior of the bottom-up system and its abstraction. Zamani and Arcak (2017) developed compositional frameworks that quantify the joint dissipativity properties of control subsystems and their abstractions. Swikir and Zamani (2019) has studied the problem of designing controllers of interconnected systems with alternating simulation functions and a small-gain type condition. Finally, Saoud et al. (2021) proposed a compositional abstraction framework using the concept of approximate composition, which does not rely on the small-gain condition and results in a more general framework.

However, all the aforementioned approaches make it possible to compositionally construct an abstraction that is related to the original large system by an approximate (alternating) simulation relation, and cannot be directly generalized to the compositional construction of abstractions that are related to the original system by an approximate (alternating) bisimulation relation. The question of compositional construction of approximately bisimilar abstractions has been only explored in (Tazaki and Imura (2008)). Indeed, given a large system consisting of interconnected components, the authors in (Tazaki and Imura (2008)) show that if each subsystem is related to its

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abstraction by an interconnection compatible approximate bisimulation relation, then the interconnected system is related to the global abstraction by an approximate bisimulation relation. Moreover, their framework is limited to finite abstractions of interconnected linear subsystems.

This paper proposes, for the first time in the literature, an approach to compositionally construct approximately bisimilar abstractions for nonlinear systems. Indeed, given a large system consisting of interconnected components, we provide conditions under which the concept of approximate (alternating) simulation relation is preserved when going from the subsystems to the large interconnected system. We rely on the notion of approximate composition introduced in Saoud et al. (2021). This notion allows the distance between inputs and outputs of neighboring components to be bounded by a given parameter called the approximate composition parameter. Indeed, we observe that the behavior of interconnected systems tolerating some composition parameter error becomes more conservative (and less deterministic) as the approximate composition parameters become large. In this paper, we first show how to measure the conservatism (in terms of approximate (alternating) simulation relations) of interconnected systems when enlarging their approximate composition parameters for the case where the subsystems are Incrementally Input-to-State Stable (δ-ISS). Indeed, we show that if a collection of δ-ISS systems tolerates certain approximate composition errors between adjacent components, then it is approximately (alternatingly) bisimilar to any admissible composition. This preliminary result provides a systematic basis for developing a new framework capable of constructing an abstract system with certain compatibility errors related to the original large exactly compatible system by an approximate (alternating) bisimulation relation. Due to space constraints, the proofs are omitted and provided in (Belamfedel et al., 2022).

2. PRELIMINARIES AND PROBLEM STATEMENT

Notations: The symbols $\mathbb{N}, \mathbb{N}_0$, $\mathbb{R}$, and $\mathbb{R}_0^+$ denote the set of positive integers, non-negative integers, real, and non-negative real numbers, respectively. For any $x_1, x_2, x_3 \in X$, the map $\mathbf{d}_X : X \times X \rightarrow \mathbb{R}_0^+$ is a *pseudometric* if the following conditions hold: (i) $x_1 = x_2$ implies $\mathbf{d}_X(x_1, x_2) = 0$; (ii) $\mathbf{d}_X(x_1, x_2) = \mathbf{d}_X(x_2, x_1)$; (iii) $\mathbf{d}_X(x_1, x_3) \leq \mathbf{d}_X(x_1, x_2) + \mathbf{d}_X(x_2, x_3)$. We identify a relation $\mathcal{R} \subseteq A \times B$ defined by $b \in \mathcal{R}(a)$ if and only if $(a, b) \in \mathcal{R}$. Given a relation $\mathcal{R} \subseteq A \times B$, $\mathcal{R}^{-1}$ denotes the inverse relation of $\mathcal{R}$, i.e. $\mathcal{R}^{-1} = \{(b, a) \in B \times A \mid \quad (a, b) \in R\}$. For $x \in \mathbb{R}^n$, $\|x\|$ denotes its infinity norm. The null vector of dimension $N \in \mathbb{N}_0$ is denoted by $\mathbf{0}_N := (0, \ldots, 0)$. The identity map is denoted by $\mathrm{id}(s) = s$. For a discrete-time signal, $\mathbf{x} : \mathbb{N}_0 \rightarrow X \subseteq \mathbb{R}^n$, $\|\mathbf{x}\|_k = \sup_{j=0,1,\ldots,k} \|\mathbf{x}(j)\|$. The function $\alpha : \mathbb{R}_{\geq 0} \rightarrow \mathbb{R}_{\geq 0}$ is of class $\mathcal{K}$ if it is continuous, $\alpha(0) = 0$, and strictly increasing. If $\alpha \in \mathcal{K}$ is unbounded, it is of class $\mathcal{K}_\infty$. A function $\sigma : \mathbb{R}_{\geq 0} \rightarrow \mathbb{R}_{\geq 0}$ is of class $\mathcal{L}$ if it is continuous, strictly decreasing, and $\lim_{t \rightarrow \infty} \sigma(t) = 0$. A function $\beta : \mathbb{R}_{\geq 0} \times \mathbb{R}_{\geq 0} \rightarrow \mathbb{R}_{\geq 0}$ is of class $\mathcal{KL}$ if it is class $\mathcal{K}$ in its first argument and class $\mathcal{L}$ in its second argument. For a $n \times m$ matrix A, $|A|$ stands for its infinity norm.

2.1 Transition system

First, we define the *transition systems* adopted from Tabuada (2009), which allows us to represent concrete dynamical systems and their abstractions in a unified way.

Definition 1. A transition system is a tuple $S = \big(X, X^0, U^{ext}, U^{int}, \Delta, Y, H\big)$, where X is the set of states (possibly infinite), $X^0 \subseteq X$ is the set of initial states, U^{ext} and U^{int} is the set of external and internal inputs (possibly infinite), respectively, $\Delta \subseteq X \times U^{ext} \times U^{int} \times X$ is the transition relation, Y is the set of outputs, and $H : X \rightarrow Y$ is the output map.

The set of admissible inputs for $x \in X$ is denoted by $U_S(x) := \left\{\left(u^{ext}, u^{int}\right) \in U^{ext} \times U^{int} \mid \Delta\left(x, u^{ext}, u^{int}\right) \neq \emptyset\right\}$. Denote by $x' \in \Delta\left(x, u^{ext}, u^{int}\right)$ as an alternative representation for a transition $\left(x, u^{ext}, u^{int}, x'\right) \in \Delta$, where state x' is called a $\left(u^{ext}, u^{int}\right)$-successor (or simply successor) of state x, for some input $\left(u^{ext}, u^{int}\right) \in U^{ext} \times U^{int}$. A transition system S is said to be:

- ***pseudometric***, if the state set X, input sets $U^i, i \in \{ext, int\}$ and the output set Y are equipped with pseudometrics $\mathbf{d}_X : X \times X \rightarrow \mathbb{R}_0^+$, $\mathbf{d}_{U^i} : U^i \times U^i \rightarrow \mathbb{R}_0^+$ and $\mathbf{d}_Y : Y \times Y \rightarrow \mathbb{R}_0^+$, respectively;
- ***finite***, if X, U^{int}, and U^{ext} are finite sets;
- ***deterministic***, if there exists at most one $\left(u^{ext}, u^{int}\right)$-successor of x, for any $x \in X$ and $\left(u^{ext}, u^{int}\right) \in U^{ext} \times U^{int}$.

For a deterministic transition system $S = \big(X, X^0, U^{ext}, U^{int}, \Delta, Y, H\big)$, the notation $\mathbf{x}\left(k, x, \mathbf{u}^{int}, \mathbf{u}^{ext}\right)$ represents the state reached at k^{th} transition from an initial state $x \in X^0$ under input signals $\mathbf{u}^{int} : \mathbb{N}_0 \rightarrow U^{int}$ and $\mathbf{u}^{ext} : \mathbb{N}_0 \rightarrow U^{ext}$.

2.2 Approximate (alternating) bisimulation relations

In the following, we introduce a novel notion of approximate (alternating) bisimulation relations, allowing us to relate two transition systems.

For two transition systems $S_1 = (X_1, X_1^0, U_1^{ext}, U_1^{int}, \Delta_1, Y_1, H_1)$ and $S_2 = (X_2, X_2^0, U_2^{ext}, U_2^{int}, \Delta_2, Y_2, H_2)$ such that Y_1 and Y_2 are subsets of the same pseudometric space Y equipped with a pseudometric $\mathbf{d}$ and U_j^{ext} (respectively U_j^{int}), $j \in \{1, 2\}$, are subsets of the same pseudometric space U^{ext} (respectively U^{int}) equipped with a pseudometric $\mathbf{d}_{u^{ext}}$ (respectively $\mathbf{d}_{u^{int}}$), we introduce the following relations.

Definition 2. For $\varepsilon, \mu \geq 0$, S_2 is said to be (ε, μ)-approximately simulated by S_1, if there exists a relation $\mathcal{R} \subseteq X_1 \times X_2$ satisfying,

(i) $\forall x_1^0 \in X_1^0, \exists x_2^0 \in X_2^0$ such that $\left(x_1^0, x_2^0\right) \in \mathcal{R}$;
(ii) $\forall \left(x_1, x_2\right) \in \mathcal{R}, \mathbf{d}\left(H_1\left(x_1\right), H_2\left(x_2\right)\right) \leq \varepsilon$;
(iii) $\forall \left(x_1, x_2\right) \in \mathcal{R}, \forall \left(u_1^{ext}, u_1^{\mathrm{int}}\right) \in U_{S_1}(x_1)$, $\forall x_1' \in \Delta_1\left(x_1, u_1^{ext}, u_1^{\mathrm{int}}\right), \exists \left(u_2^{ext}, u_2^{\mathrm{int}}\right) \in U_{S_2}\left(x_2\right)$ with
$$\max\left(\mathbf{d}_{u^{ext}}\left(u_1^{ext}, u_2^{ext}\right), \mathbf{d}_{u^{int}}\left(u_1^{int}, u_2^{int}\right)\right) \leq \mu$$
and $\exists x_2' \in \Delta_2\left(x_2, u_2^{ext}, u_2^{int}\right)$ satisfying $(x_1', x_2') \in \mathcal{R}$.

Moreover, S_2 is said to be (ε, μ)-approximately bisimilar to S_1, if S_1 is (ε, μ)-approximately simulated by S_2, and

S_2 is (ε,μ)-approximately simulated by S_1. Simulation and bisimulation relations are denoted respectively by, $S_2 \preccurlyeq^{\varepsilon,\mu} S_1$ and $S_2 \approx^{\varepsilon,\mu} S_1$.

For verification problems, approximate (bi)simulation relations are mainly used. The concept of approximate alternating (bi)simulation relations introduced in Tabuada (2009) are used if the goal is to synthesize controllers.

Definition 3. For $\varepsilon, \mu \geq 0$, S_2 is said to be (ε,μ)-approximately alternatingly simulated by S_1, if there exists a relation $\mathcal{R} \subseteq X_1 \times X_2$ satisfying,

(i) $\forall x_2^0 \in X_2^0, \exists x_1^0 \in X_1^0$ such that $(x_1^0, x_2^0) \in \mathcal{R}$;
(ii) $\forall (x_1,x_2) \in \mathcal{R}, \mathbf{d}(H_1(x_1), H_2(x_2)) \leq \varepsilon$;
(iii) $\forall (x_1,x_2) \in \mathcal{R}, \forall (u_2^{ext}, u_2^{int}) \in U_{S_2}(x_2)$, $\exists (u_1^{ext}, u_1^{int}) \in U_{S_1}(x_1)$ with $\max(\mathbf{d}_{u^{ext}}(u_1^{ext}, u_2^{ext}), \mathbf{d}_{u^{int}}(u_1^{int}, u_2^{int})) \leq \mu$ such that $\forall x_1' \in \Delta_1(x_1, u_1^{ext}, u_1^{int})$, $\exists x_2' \in \Delta_2(x_2, u_2^{ext}, u_2^{int})$ satisfying $(x_1', x_2') \in \mathcal{R}$.

Moreover, S_2 is said to be (ε,μ)-approximately alternatingly bisimilar to S_1, if S_1 is (ε,μ)-approximately alternatingly simulated by S_2, and S_2 is (ε,μ)-approximately alternatingly simulated by S_1. The alternating simulation and alternating bisimulation relations are denoted respectively by $S_2 \preccurlyeq_{\mathcal{A}}^{\varepsilon,\mu} S_1$ and $S_2 \approx_{\mathcal{A}}^{\varepsilon,\mu} S_1$.

Remark 1. Definitions 2 and 3 define the notions of simulation and bisimulation for subsystems. These definitions extend naturally from component to interconnected systems by considering systems without internal inputs.

Contrarily to the concepts of approximate (bi)-simulation relation introduced in Tabuada (2009) and Girard and Pappas (2007), the concept of approximate (bi)-simulation relation introduced in *Definition 2* is more relaxed since it allows a mismatch on the choice of inputs for the transition systems. In particular, when $\mu = 0$ and $\mathbf{d}_{\mathbf{u}^{int}}$ is metric, the relation proposed in *Definition 2* reduces to the notion of approximate bisimulation introduced in Girard and Pappas (2007), and when $\mu = \infty$, it covers the approximate bisimulation relation given in Tabuada (2009). Furthermore, the concept of approximate alternating bisimulation of *Definition 3* includes the one in Pola and Tabuada (2009) by taking $\mu = \infty$.

To gather all the ingredients to conduct our main results, the following two propositions are needed. These properties mainly show the ordering and the transitivity properties of the introduced relationships.

Proposition 2. Given three pseudometric transition systems S_1, S_2 and S_3. For any $\mu, \mu' \geq 0$ and $\varepsilon, \varepsilon' \geq 0$. The following statements hold:

- if $S_1 \preccurlyeq^{\varepsilon,\mu} S_2$ and $S_2 \preccurlyeq^{\varepsilon',\mu'} S_3$, then $S_1 \preccurlyeq^{\varepsilon+\varepsilon',\mu+\mu'} S_3$
- if $S_1 \preccurlyeq_{\mathcal{A}}^{\varepsilon,\mu} S_2$ and $S_2 \preccurlyeq_{\mathcal{A}}^{\varepsilon',\mu'} S_3$, then $S_1 \preccurlyeq_{\mathcal{A}}^{\varepsilon+\varepsilon',\mu+\mu'} S_3$.

Proposition 3. Given two pseudometric transition systems S_1 and S_2. For any $\mu' \geq \mu \geq 0$ and $\varepsilon' \geq \varepsilon \geq 0$. The following statement holds:

- if $S_1 \preccurlyeq^{\varepsilon,\mu} S_2$ then $S_1 \preccurlyeq^{\varepsilon',\mu'} S_2$
- if $S_1 \preccurlyeq_{\mathcal{A}}^{\varepsilon,\mu} S_2$ then $S_1 \preccurlyeq_{\mathcal{A}}^{\varepsilon',\mu'} S_2$.

3. INCREMENTAL INPUT-TO-STATE STABILITY FOR TRANSITION SYSTEMS

Next, we introduce the concept of global incremental input-to-state stability ($\delta - ISS$) for transition systems.

Definition 4. Consider a deterministic and pseudometric transition system $S = (X, X^0, U^{ext}, U^{int}, \Delta, Y, H)$. The transition system S is said to be globally incrementally Input-to-State Stable (δ-ISS) if there exists a function β of class $\mathcal{KL}$ and a function γ of class $\mathcal{K}$ such that, for any initial states $x_1, x_2 \in X$, for any input signals $\mathbf{u}_1^{int}, \mathbf{u}_2^{int} : \mathbb{N}_0 \to U^{int}$, $\mathbf{u}_1^{ext}, \mathbf{u}_2^{ext} : \mathbb{N}_0 \to U^{ext}$, the following inequality holds:

$$\mathbf{d}_X(\mathbf{x}(k, x_1, \mathbf{u}_1^{int}, \mathbf{u}_1^{ext}), \mathbf{x}(k, x_2, \mathbf{u}_2^{int}, \mathbf{u}_2^{ext})) \leq \beta(\mathbf{d}_X(x_1,x_2), k) + \gamma^{int}\left(\left\|\mathbf{u}_1^{int} - \mathbf{u}_2^{int}\right\|_{k-1}\right) + \gamma^{ext}\left(\left\|\mathbf{u}_1^{ext} - \mathbf{u}_2^{ext}\right\|_{k-1}\right) \quad (1)$$

for all $k \in \mathbb{N}_0$, such that $\mathbf{x}(k, x_1, \mathbf{u}_1^{int}, \mathbf{u}_1^{ext}) \in X$ and $\mathbf{x}(k, x_2, \mathbf{u}_2^{int}, \mathbf{u}_2^{ext}) \in X$, and where the second and third term of the sum in the right-hand side of (1) is taken equal to 0 for $k = 0$.

In the rest of the section, we show how to construct the maps β and γ characterizing the δ-ISS properties in (1) for discrete-time control systems (Σ_{nl}) defined as below:

$$(\Sigma_{nl}) : x(k+1) = f(x(k), \mathbf{u}^{ext}(k), \mathbf{u}^{int}(k)), \; k \in \mathbb{N}_0, \quad (2)$$

where $x(k) \in \mathcal{X}$, $u^{ext}(k) \in \mathcal{U}^{ext}$, and $u^{int}(k) \in \mathcal{U}^{int}$ are state, external and internal inputs, respectively. The discrete-time control system Σ_{nl} can be represented as a transition system $S = (X, X^0, U^{ext}, U^{int}, \Delta, Y, H)$ with $X^0 = X = \mathcal{X}$, $U^{ext} = \mathcal{U}^{ext}$, $U^{int} = \mathcal{U}^{int}$ the transition $(x, u^{ext}, u^{int}, x') \in \Delta$ iff $x' = f(x, u^{ext}, u^{int})$, for $x, x' \in X$, $u^{ext} \in U^{ext}$ and $u^{int} \in U^{int}$, $Y = X$, and $H(x) = x$. In the rest of the paper, the discrete-time dynamical system Σ_{nl} and its transition system's representation S can be used interchangeably.

(i) $\delta - ISS$ for discrete-time linear systems: Consider a linear discrete-time system:

$$(\Sigma_l) : \; x(k+1) = Ax(k) + B\mathbf{u}^{ext}(k) + D\mathbf{u}^{int}(k), \quad (3)$$

where $A \in \mathbb{R}^{n\times n}, B \in \mathbb{R}^{n\times m}$, $C \in \mathbb{R}^{q\times n}$ and $D \in \mathbb{R}^{n\times p}$, $\mathbf{x}, \mathbf{u}^{ext}$, and $\mathbf{u}^{int}$ denote the state signal, the external input signal and the internal input signal, respectively.

The following result provides conditions for the system Σ_l in (3) to be δ-ISS.

Theorem 4. Consider a system Σ_l as in (3). If all the eigen values of the matrix A are inside the unit disk, then Σ_l is δ-ISS with functions β, γ^{ext}, and γ^{int} defined, for $(r,k) \in \mathbb{R}_0^+ \times \mathbb{N}_0$, by:

$$\beta(r,k) = |A^k| r, \quad \gamma^{ext}(r) = \frac{|B| r}{1 - |A|}, \quad \gamma^{int}(r) = \frac{|D| r}{1 - |A|}.$$

(ii) $\delta - ISS$ for Lipschitz nonlinear systems:
Consider the discrete-time nonlinear system (2) satisfies the following Lipschitz continuity assumption:

Assumption 5. There exist constants $L^x, L^u, L^w \in \mathbb{R}^+$ such that:

$$\|f(x^a, u^{ext,a}, u^{int,a}) - f(x^b, u^{ext,b}, u^{int,b})\| \leq L^x \|x^a - x^b\| + L^{u^{ext}} \|u^{ext,a} - u^{ext,b}\| + L^{u^{int}} \|u^{int,a} - u^{int,b}\| \quad (4)$$

$\forall x^a, x^b \in X, \forall u^{int,a}, u^{int,b} \in U^{int}$ and $\forall u^{ext,a}, u^{ext,b} \in U^{ext}$.

The following result is adapted from *Theorem 1* in Bayer et al. (2013).

Theorem 6. Given a system Σ_{nl} (2) satisfying Assumption 5. If the constant $L^x < 1$, then Σ_{nl} is δ-ISS with functions β, γ^{ext}, and γ^{int} defined, for $(r,k) \in \mathbb{R}_0^+ \times \mathbb{N}_0$, by:

$$\beta(r,k) = (L^x)^k r, \gamma^{ext}(r) = \frac{(L^{u^{ext}})r}{1-L^x}, \gamma^{int}(r) = \frac{(L^{u^{int}})r}{1-L^x}.$$

4. COMPOSITIONAL BISIMILAR ABSTRACTIONS FOR INTERCONNECTED SYSTEMS

In this section, we consider networks of interconnected transition systems and provide conditions to preserve approximate (alternating) bisimulation relations from the subsystems to the global interconnected system.

4.1 Interconnected system

An interconnected system is composed of a collection of $N \in \mathbb{N}$ transition systems $\{S_i\}_{i\in I}$, a set of vertices $I = \{1,\dots,N\}$ and a binary connectivity relation $\mathcal{I} \subseteq I \times I$ where each vertex $i \in I$ is labelled with the system S_i. For $i \in I$, we define $\mathcal{N}(i) = \{j \in I | (j,i) \in \mathcal{I}\}$ as the set of neighbouring components from where the incoming edges come. The i^{th} subsystem is described by $S_i = \left(X_i, X_i^0, U_i^{ext}, U_i^{\mathrm{int}}, \Delta_i, Y_i, H_i\right)$, where H_i is an identity map $H_i(x) = x$.

Definition 5. Given a collection of transition systems $\{S_i\}_{i\in I}$, where $S_i = \left(X_i, X_i^0, U_i^{ext}, U_i^{int}, \Delta_i, Y_i, H_i\right)$ such that for all $i \in I, \prod_{j\in\mathcal{N}(i)} Y_j$ and U_i^{int} are subsets of the same pseudometric space equipped with the following pseudometric:

for $u_i^{l,int} = (y_{j_1}^l, \dots, y_{j_k}^l), l \in \{1,2\}$, with $\mathcal{N}(i) = \{j_1,\dots,j_k\}$,

$$\mathbf{d}_{U_i^{\mathrm{int}}}\left(u_i^{1,\mathrm{int}}, u_i^{2,int}\right) = \max_{j\in\mathcal{N}(i)} \left\{\mathbf{d}_{Y_j}\left(y_j^1, y_j^2\right)\right\}.$$

Let $M := (\mu_1,\dots,\mu_N) \in \left(\mathbb{R}_0^+\right)^N$. We say that $\{S_i\}_{i\in I}$ is compatible for M-approximate composition with respect to $\mathcal{I}$, if for each $i \in I$ and for each $\prod_{j\in\mathcal{N}(i)}\{y_j\} \in \prod_{j\in\mathcal{N}(i)} Y^j$, where the term $\prod_{j\in\mathcal{N}(i)}\{y_j\}$ can be formally defined as $\prod_{j\in\mathcal{N}(i)}\{y_j\} = \left(y_{j_1}, y_{j_2}, \dots, y_{j_p}\right)$ with $\mathcal{N}(i) = \{j_1, j_2, \dots, j_p\}$, there exists $u_i^{int} \in U_i^{int}$ such that $\mathbf{d}_{U_i^{\mathrm{int}}}\left(u_i^{\mathrm{int}}, \prod_{j\in\mathcal{N}(i)}\{y_j\}\right) \leq \mu_i$. We denote M-approximate composed system by $\langle S_i\rangle_{i\in I}^{M,\mathcal{I}}$ and is given by the tuple $\langle S_i\rangle_{i\in I}^{M,\mathcal{I}} = \left(X, X^0, U^{ext}, \Delta_M, Y, H\right)$, where:

- $X = \prod_{i\in I} X_i$; $X^0 = \prod_{i\in I} X_i^0$; $U^{ext} = \prod_{i\in I} U_i^{ext}$; $Y = \prod_{i\in I} Y_i$;
- $H(x) = H(x_1,\dots,x_N) = (H_1(x_1),\dots,H_N(x_N)) = (x_1,\dots,x_N)$
- for $x = (x_1,\dots,x_N), x' = (x'_1,\dots,x'_N)$ and $u^{ext} = (u_1^{ext},\dots,u_N^{ext}), x' \in \Delta_M(x, u^{ext})$ if and only if for all $i \in I$, and for all $\prod_{j\in\mathcal{N}(i)}\{y_j\} = \prod_{j\in\mathcal{N}(i)}\{H_j(x_j)\} \in \prod_{j\in\mathcal{N}(i)} Y_j$, there exists $u_i^{int} \in U_i^{int}$ with $\mathbf{d}_{U_i^{int}}\left(u_i^{int}, \prod_{j\in\mathcal{N}(i)}\{y_j\}\right) \leq \mu_i, (u_i^{ext}, u_i^{int}) \in U_{S_i}(x_i)$ and $x'_i \in \Delta_i\left(x_i, u_i^{ext}, u_i^{int}\right)$.

For the illustration of the notion of approximate composition, we kindly refer interested readers to Example 1 in the preprint by Belamfedel et al. (2022).

We equip the composed output space with the metric:

$$for\ y^j \in Y\ with\ y^j = (y_1^j,\dots,y_N^j), j\in\{1,2\},$$
$$\mathbf{d}\left(y^1,y^2\right) = \max_{i\in I}\left\{\mathbf{d}_{Y_i}\left(y_i^1,y_i^2\right)\right\} \quad (5)$$

Similarly, we equip the composed input and the state spaces with the pseudometric:

$$for\ u^{j,ext} \in U^{ext}\ with\ u^{j,ext} = (u_1^{j,ext},\dots,u_N^{j,ext}), j\in\{1,2\},$$
$$\mathbf{d}_{U^{ext}}(u^{1,ext},u^{2,ext}) = \max_{i\in I}\left\{\mathbf{d}_{U_i^{ext}}\left(u_i^{1,ext},u_i^{2,,ext}\right)\right\}, \quad (6)$$

$$for\ u^{j,int} \in U^{int}\ with\ u^{j,int} = (u_1^{j,int},\dots,u_N^{j,int}), j\in\{1,2\},$$
$$\mathbf{d}_{U^{int}}(u^{1,int},u^{2,int}) = \max_{i\in I}\left\{\mathbf{d}_{U_i^{int}}\left(u_i^{1,int},u_i^{2,,int}\right)\right\}. \quad (7)$$

The behavior of interconnected systems tolerating some composition parameter error becomes more conservative (and less deterministic) when the approximate composition parameters become large. The following result shows that under the δ-ISS property, we can measure the conservatism of the approximate composition when increasing the approximate composition parameter.

Theorem 7. Consider a collection of transition systems $\{S_i\}_{i\in I}$ and $\bar{M} = (\bar{\mu}_1,\dots,\bar{\mu}_N \in (\mathbb{R}_0^+)^N$. If each subsystem of $\{S_i\}_{i\in I}$ is δ-ISS and $\{S_i\}_{i\in I}$ is compatible for $\bar{M}$-approximate composition with respect to $\mathcal{I}$, then it is also compatible for M-approximate composition with respect to $\mathcal{I}$, for any $M = (\mu_1,\dots,\mu_N) \in (\mathbb{R}_0^+)^N$ such that $\bar{M} \geq M$ (i.e., $\bar{\mu}_i \geq \mu_i, i\in I$). Moreover, for any $\varepsilon \geq 0$ such that

$$\beta_i(\varepsilon,1) + \gamma_i^{int}(\varepsilon + \bar{\mu}_i - \mu_i) \leq \varepsilon, \quad \forall i \in I \quad (8)$$

the relation $\mathcal{R} = \{(x,x') \in X\times X \,|\, \mathbf{d}(H(x),H(x')) \leq \varepsilon\}$ is a $(\varepsilon,0)$-approximate bisimulation relation between $S_{\bar{M}} = \langle S_i\rangle_{i\in I}^{\bar{M},\mathcal{I}}$ and $S_M = \langle S^i\rangle_{i\in I}^{M,\mathcal{I}}$.

Remark 8. Note that the results in Saoud et al. (2021) shows a simulation relation $S_M \preccurlyeq^{\varepsilon,0} S_{\bar{M}}$. The previous result shows that under the δ-ISS property of each subsystem the symmetrical relation $S_{\bar{M}} \preccurlyeq^{\varepsilon,0} S_M$ holds and thus $S_{\bar{M}} \approx^{\varepsilon,0} S_M$. Indeed, while any trajectory of the system S_M is a trajectory of the system $S_{\bar{M}}$, the proposed result shows that under the δ-ISS property, one can measure the conservatism between S_M and $S_{\bar{M}}$, thereby measuring the conservatism of the approximate composition.

Theorem 9. Consider a collection of transition systems $\{S_i\}_{i\in I}$ and $\bar{M} = (\bar{\mu}_1,\dots,\bar{\mu}_N) \in \left(\mathbb{R}_0^+\right)^N$. If each subsystem of $\{S_i\}_{i\in I}$ is δ-ISS and $\{S_i\}_{i\in I}$ is compatible for $\bar{M}$-approximate composition with respect to $\mathcal{I}$, then it is also compatible for M-approximate composition with respect to $\mathcal{I}$, for any $M = (\mu_1,\dots,\mu_N) \in \left(\mathbb{R}_0^+\right)^N$ such that $\bar{M} \geq M$ (i.e., $\bar{\mu}_i \geq \mu_i, i\in I$). Moreover, for any $\varepsilon \geq 0$ such that

$$\beta_i(\varepsilon,1) + \gamma_i^{int}(\varepsilon + \bar{\mu}_i - \mu_i) \leq \varepsilon, \quad \forall i \in I \quad (9)$$

the relation $\mathcal{R} = \{(x,x') \in X\times X \,|\, \mathbf{d}(H(x),H(x')) \leq \varepsilon\}$ is a $(\varepsilon,0)$-approximate alternating bisimulation relation between $S_{\bar{M}} = \langle S_i\rangle_{i\in I}^{\bar{M},\mathcal{I}}$ and $S_M = \langle S^i\rangle_{i\in I}^{M,\mathcal{I}}$.

Remark 10. Note that we have the alternating simulation relation from $S_M \preccurlyeq_{\mathcal{A}}^{\varepsilon,0} S_{\bar{M}}$ without any stability requirement. The proposed result shows the symmetrical version $S_{\bar{M}} \preccurlyeq_{\mathcal{A}}^{\varepsilon,0} S_M$ under the δ-ISS property of each subsystem.

4.2 Approximate (Alternating) Bisimilar Composition

The compositionality results for approximate bisimulation relation is stated as follows.

Theorem 11. Let $\{S_i\}_{i\in I}$ and $\{\hat{S}_i\}_{i\in I}$ be two collections of transition systems with $S_i = (X_i, X_i^0, U_i^{ext}, U_i^{\text{int}}, \Delta_i, Y_i, H_i)$ and $\hat{S}_i = (\hat{X}_i, \hat{X}_i^0, \hat{U}_i^{ext}, \hat{U}_i^{int}, \hat{\Delta}_i, \hat{Y}_i, \hat{H}_i)$. Consider positive constants ε_i, μ_i, for $i \in I$, with $\varepsilon = \max_{i\in I} \varepsilon_i$, $\mu = \max_{i\in I} \mu_i$ and consider $M = (\delta_1, \ldots, \delta_N)$ and $\hat{M} = (\mu_1 + \delta_1 + \varepsilon, \ldots, \mu_N + \delta_N + \varepsilon)$. Let the following hold:

(i) For all $i \in I$, S_i is δ-ISS and satisfies the following inequality $\max_{i\in I} \left(\beta_i(\varepsilon, 1) + \gamma_i^{int}(2\varepsilon + \mu_i)\right) \leq \varepsilon$;
(ii) For all $i \in I$, S_i is (ε_i, μ_i)- approximately bisimilar to $\hat{S}_i$, and we denote $S_i \approx^{\varepsilon_i,\mu_i} \hat{S}_i$;
(iii) $\{S_i\}_{i\in I}$ are compatible for M-approximate composition with respect to $\mathcal{I}$;
(iv) $\{\hat{S}_i\}_{i\in I}$ are compatible for $\hat{M}$-approximate composition with respect to $\mathcal{I}$;

then, $S_M = \langle S_i \rangle_{i\in I}^{M,\mathcal{I}}$ is $(2\varepsilon, \mu)$-approximately bisimilar to $\hat{S}_{\hat{M}} = \langle \hat{S}_i \rangle_{i\in I}^{\hat{M},\mathcal{I}}$.

We now present the analogous result for approximate alternating bisimulation relations.

Theorem 12. Let $\{S_i\}_{i\in I}$ and $\{\hat{S}_i\}_{i\in I}$ be two collections of transition systems with $S_i = (X_i, X_i^0, U_i^{ext}, U_i^{\text{int}}, \Delta_i, Y_i, H_i)$ and $\hat{S}_i = (\hat{X}_i, \hat{X}_i^0, \hat{U}_i^{ext}, \hat{U}_i^{int}, \hat{\Delta}_i, \hat{Y}_i, \hat{H}_i)$. Consider positive constants ε_i, μ_i, for $i \in I$, with $\varepsilon = \max_{i\in I} \varepsilon_i$, $\mu = \max_{i\in I} \mu_i$ and consider $M = (\delta_1, \ldots, \delta_N)$ and $\hat{M} = (\mu_1 + \delta_1 + \varepsilon, \ldots, \mu_N + \delta_N + \varepsilon)$. Let the following hold:

(i) For all $i \in I$, S_i is δ-ISS and satisfies the following inequality $\max_{i\in I} \left(\beta_i(\varepsilon, 1) + \gamma_i^{int}(2\varepsilon + \mu_i)\right) \leq \varepsilon$;
(ii) For all $i \in I$, S_i is (ε_i, μ_i)- approximately alternatingly bisimilar to $\hat{S}_i$, and we denote $S_i \approx^{\varepsilon_i,\mu_i} \hat{S}_i$;
(iii) $\{S_i\}_{i\in I}$ are compatible for M-approximate composition with respect to $\mathcal{I}$;
(iv) $\{\hat{S}_i\}_{i\in I}$ are compatible for $\hat{M}$-approximate composition with respect to $\mathcal{I}$;

then, $S_M = \langle S_i \rangle_{i\in I}^{M,\mathcal{I}}$ is $(2\varepsilon, \mu)$-approximately alternatingly bisimilar to $\hat{S}_{\hat{M}} = \langle \hat{S}_i \rangle_{i\in I}^{\hat{M},\mathcal{I}}$.

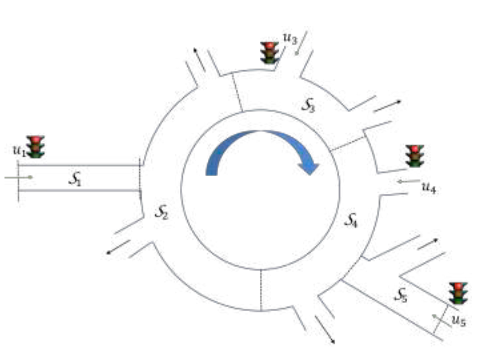

Fig. 1. Traffic flow network where the clockwise flow of traffic is allowed and $\mathcal{S}_i$ represents road sections.

5. CASE STUDY: TRAFFIC FLOW MODEL

This section demonstrates the engineering relevance of the proposed compositional results using a traffic flow example. The implementation was performed in MATLAB on an Apple M1 Max processor with 64 GB of memory.

5.1 Model description and control objective

Consider the traffic flow model (Saoud et al. (2021)), described as:

$$x_1(k+1) = \left(1 - \frac{Tv}{1.6l}\right) x_1(k) + 5u_1(k),$$
$$x_2(k+1) = \frac{Tv}{l} x_1(k) + \left(1 - \frac{Tv}{l} - q\right) x_2(k) + \frac{Tv}{l} x_4(k),$$
$$x_3(k+1) = \frac{Tv}{l} x_2(k) + \left(1 - \frac{Tv}{l} - q\right) x_3(k) + 8u_3(k),$$
$$x_4(k+1) = \frac{Tv}{l} x_3(k) + \left(1 - \frac{Tv}{l} - q\right) x_4(k) + 8u_4(k),$$
$$x_5(k+1) = \frac{Tv}{l} x_4(k) + \left(1 - \frac{Tv}{l} - q\right) x_5(k) + 8u_5(k),$$

where the state $x_i(k), i \in I = \{1, 2, 3, 4, 5\}$, represents the traffic density in the i^{th} road section, expressed in vehicles per section, $l = 0.25$ km is the length of the road, $v = 70$ km/hr is the flow speed, $T = \frac{10}{3600}$ hours is the discrete-time interval, and $q = 0.25$ is the ratio representing the percentage of vehicles leaving the section of road. For each i^{th} difference equation, the states x_j with $j \neq i$, $i = \{1, 2, 3, 4, 5\}$ represent the internal inputs. The external inputs $\mathbf{u}_1(k), \mathbf{u}_3(k), \mathbf{u}_4(k), \mathbf{u}_5(k) \in U = \{0, 1\}$, where 0 represents the red signal, and 1 represents the green signal in the traffic model. We consider the compact state-space $X = [0, 40]^5$. The control objective is to synthesize the controller to stay inside a safe region $\mathfrak{S} = [2, 25] \times [5, 25]^4$.

The proposed model can be seen as an exact composition of 5 subsystems $S = \langle S_i \rangle_{i\in I}^{\mathbf{0}_5,\mathcal{I}}$, with,
$\mathcal{I} = \{(1,1), (1,2), (2,2), (2,4), (3,3), (2,3), (4,4), (3,4), (5,5), (4,5)\}$.

5.2 Abstraction and controller synthesis

First one can check that each subsystem S_i, $i \in I$ is δ-ISS with $\beta_1(r, k) = (0.513)^k r$, $\beta_2(r, k) = \beta_3(r, k) = \beta_4(r, k) = \beta_5(r, k) = (0.0287)^k s$, $\gamma_1^{int}(r) = 0.01r$ and $\gamma_2^{int}(r) = \gamma_3^{int}(r) = \gamma_4^{int}(r) = \gamma_5^{int}(r) = 0.195r$. We compute local abstraction $\hat{S}_i$ for each subsystem S_i, $i \in I$, using the symbolic approach presented in Girard et al. (2009). Each abstraction $\hat{S}_i$ is related to the original system S_i, $i \in I$, by an (ε_i, μ_i)-approximate bisimulation relation, with $\varepsilon_i = 1$ and $\mu_i = 1$. We then compose the local abstractions in order to compute the global abstraction using an $\hat{M}$-approximate composition, with $\hat{M} = (1, 1, 1, 1, 1)$. One can also check that for the chosen values of ε_i and μ_i, $i \in I$, condition (i) of Theorem 12 is satisfied. Hence, in view of Theorem 12, we have that $\hat{S} \approx_{\mathcal{A}}^{(\varepsilon,\mu')} S$ [1], where $S = \langle S^i \rangle_{i\in I}^{\mathbf{0}_4,\mathcal{I}}$ and $\hat{S} = \langle \hat{S}_i \rangle_{i\in I}^{\hat{M},\mathcal{I}}$.

The computation time of the abstractions of the five components $\{1, 2, 3, 4, 5\}$ are given by 0.22 seconds, 0.25

[1] Given the safety specification for the original system $\mathfrak{S}$ and since the original system is related to the compositional abstraction by an $\varepsilon-$approximate bisimulation relation, the abstract specification is a deflated version of the original one.

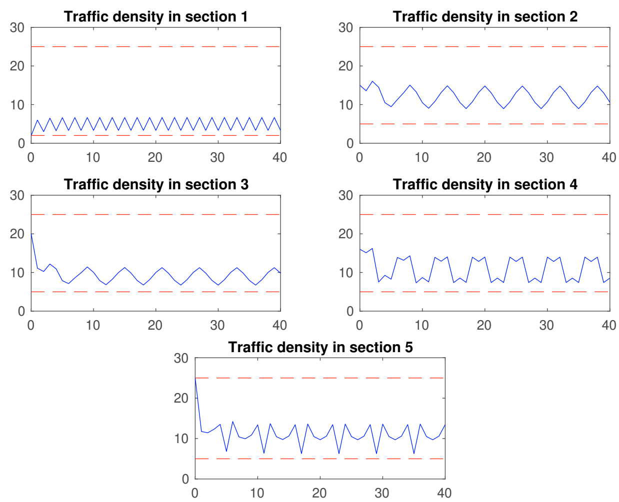

Fig. 2. The evolution of traffic densities in each section of the road.

seconds, 0.16 seconds, 0.14 seconds and 0.15 seconds, respectively, and the composition of the global abstraction from local ones using an approximate composition takes less than 138 seconds. This resulted in 139 seconds to compute an abstraction compositionally. Constructing an abstraction monolithically, using the same discretization parameters, took 241 seconds. Hence, the proposed compositional approach is two times faster in this scenario.

Figure 2 shows the evolution of traffic densities in each section starting from the initial condition $x = [2, 15, 20, 16, 25]$ using a safety controller synthesized for the constructed compositional abstraction. The dashed red lines represent the boundary of the safe set for each section. One can readily see that all the trajectories evolve within the safe region.

6. CONCLUSION

This paper studied the problem of abstraction of interconnected nonlinear systems. A compositional framework for constructing abstractions is proposed based on the notion of approximate composition and the δ-ISS property. In particular, given a large system consisting of interconnected components, we provided conditions under which the concept of approximate (alternating) simulation relation is preserved when going from the subsystems to the large interconnected system. A numerical result is proposed showing the merits of the theoretical results.

REFERENCES

Bayer, F., Bürger, M., and Allgöwer, F. (2013). Discrete-time incremental ISS: A framework for robust NMPC. In *2013 European Control Conference (ECC)*, 2068–2073. IEEE.

Belamfedel, A.S., Saharsh, Jagtap, P., and Saoud, A. (2022). Compositional approximately bisimilar abstractions of interconnected systems. *arXiv preprint arXiv:2211.08655*.

Girard, A. and Pappas, G.J. (2007). Approximation metrics for discrete and continuous systems. *IEEE Transactions on Automatic Control*, 52(5), 782–798.

Girard, A., Pola, G., and Tabuada, P. (2009). Approximately bisimilar symbolic models for incrementally stable switched systems. *IEEE Transactions on Automatic Control*, 55(1), 116–126.

Hashimoto, K., Saoud, A., Kishida, M., Ushio, T., and Dimarogonas, D.V. (2019). A symbolic approach to the self-triggered design for networked control systems. *IEEE Control Systems Letters*, 3(4), 1050–1055.

Julius, A.A., D'Innocenzo, A., Di Benedetto, M.D., and Pappas, G.J. (2009). Approximate equivalence and synchronization of metric transition systems. *Systems & Control Letters*, 58(2), 94–101.

Meyer, P.J., Girard, A., and Witrant, E. (2017). Compositional abstraction and safety synthesis using overlapping symbolic models. *IEEE Transactions on Automatic Control*, 63(6), 1835–1841.

Pola, G., Girard, A., and Tabuada, P. (2008). Approximately bisimilar symbolic models for nonlinear control systems. *Automatica*, 44(10), 2508–2516.

Pola, G. and Tabuada, P. (2009). Symbolic models for nonlinear control systems: Alternating approximate bisimulations. *SIAM Journal on Control and Optimization*, 48(2), 719–733.

Rungger, M. and Zamani, M. (2016). Compositional construction of approximate abstractions of interconnected control systems. *IEEE Transactions on Control of Network Systems*, 5(1), 116–127.

Saoud, A. (2019). *Compositional and efficient controller synthesis for cyber-physical systems*. Ph.D. thesis, Université Paris-Saclay (ComUE).

Saoud, A., Girard, A., and Fribourg, L. (2018). On the composition of discrete and continuous-time assume-guarantee contracts for invariance. In *2018 European Control Conference (ECC)*, 435–440. IEEE.

Saoud, A., Jagtap, P., Zamani, M., and Girard, A. (2021). Compositional abstraction-based synthesis for interconnected systems: An approximate composition approach. *IEEE Transactions on Control of Network Systems*, 8(2), 702–712.

Swikir, A. and Zamani, M. (2019). Compositional synthesis of finite abstractions for networks of systems: A small-gain approach. *Automatica*, 107, 551–561.

Tabuada, P. (2009). *Verification and control of hybrid systems: a symbolic approach*. Springer Science & Business Media.

Tazaki, Y. and Imura, J.i. (2008). Bisimilar finite abstractions of interconnected systems. In *International Workshop on Hybrid Systems: Computation and Control*, 514–527. Springer.

Zamani, M. and Arcak, M. (2017). Compositional abstraction for networks of control systems: A dissipativity approach. *IEEE Transactions on Control of Network Systems*, 5(3), 1003–1015.

Zamani, M., Pola, G., Mazo, M., and Tabuada, P. (2011). Symbolic models for nonlinear control systems without stability assumptions. *IEEE Transactions on Automatic Control*, 57(7), 1804–1809.