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Thermal one-point functions: CFT's with fermions, large d and large spin

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Abstract: We apply the OPE inversion formula on thermal two-point functions of fermions to obtain thermal one-point function of fermion bi-linears appearing in the corresponding OPE. We primarily focus on the OPE channel which contains the stress tensor of the theory. We apply our formalism to the mean field theory of fermions and verify that the inversion formula reproduces the spectrum as well as their corresponding thermal one-point functions. We then examine the large N critical Gross-Neveu model in $d = 2k + 1$ dimensions with k even and at finite temperature. We show that stress tensor evaluated from the inversion formula agrees with that evaluated from the partition function at the critical point. We demonstrate the expectation values of 3 different classes of higher spin currents are all related to each other by numerical constants, spin and the thermal mass. We evaluate the ratio of the thermal expectation values of higher spin currents at the critical point to the Gaussian fixed point or the Stefan-Boltzmann result, both for the large N critical $O(N)$ model and the Gross-Neveu model in odd dimensions. This ratio is always less than one and it approaches unity on increasing the spin with the dimension d held fixed. The ratio however approaches zero when the dimension d is increased with the spin held fixed.

Keywords: Thermal Field Theory, $1/N$ Expansion, Scale and Conformal Symmetries

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1 Introduction

Given a quantum field theory, it is usually important to understand its behaviour at finite temperature. That is when one of the directions of the Euclidean theory is taken to be a circle of radius β, the inverse temperature. This question is particularly relevant when the quantum field theory is a conformal field theory. Usually critical points of quantum field theories occur at finite temperature. Furthermore, studying conformal field theories which arise in AdS/CFT context implies that one is studying properties of AdS black holes.

It is possible to use the symmetries of the conformal field theory to constraint conformal field theories on $S^1 \times R^{d-1}$ where the circle S^1 is of length β. Such a program was initiated in [1] and pursued in [2–7]. Under some reasonable assumptions of analyticity of the finite temperature 2-point functions of primary scalar operators, a thermal inversion formula was derived [1]. This inversion formula allowed one to obtain the thermal one-point functions for all operators which appear in the OPE of the given 2 point function. The inversion formula was applied to fermionic 2-point functions in [2], however the OPE channel studied was the scalar channel, that is the spinor indices of the fermionic operators in the 2-point function were contracted. This channel for instance does not contain the stress tensor of the theory.

In this paper we apply the OPE inversion formula on fermionic 2-point functions and focus on the OPE channel which contains the stress tensor. We will see that there are 2 classes of operators that exist in this channel, these operators are schematically of the form

$$\begin{aligned}\mathcal{O}_+[n,l] &= \bar{\psi}\gamma_{\mu_1}\partial_{\mu_2}\cdots\partial_{\mu_l}\partial^{2n}\psi, \\ \mathcal{O}_-[n,l] &= \bar{\psi}\gamma^{\mu}\partial_{\mu}\partial_{\mu_1}\partial_{\mu_2}\cdots\partial_{\mu_l}\partial^{2n}\psi,\end{aligned} \tag{1.1}$$

ψ is a Dirac spinor and these operators are rank l symmetric traceless tensors, γ^{μ} are the Dirac matrices. To isolate the one-point functions of operators belonging to each of these classes we need to apply the OPE inversion formula to 2 related thermal 2-point functions. There is a third class of symmetric traceless tensors schematically of the form

$$\mathcal{O}_0[n,l] = \bar{\psi}\partial_{\mu_1}\partial_{\mu_2}\cdots\partial_{\mu_l}\partial^{2n}\psi. \tag{1.2}$$

These occur in the scalar channel of the OPE expansion of 2 fermionic operators, to isolate the one-point function of these operators we apply the OPE inversion formula to thermal 2-point functions in which the spinor indices are contracted [2].

After presenting the general formalism for evaluating thermal one-point functions belonging to the classes in (1.1) we apply it to the mean field theory of fermions (MFT). We show that the expectation values of operators obtained by expanding the MFT correlator in the short distance limit agrees precisely with that using the OPE inversion formula. For $d = 2$ this check is done for all the operators in the class (1.1), for $d > 2$ we perform this check for $n = 0, 1$. In MFT, operators in the class (1.2) do not appear in the OPE.

We then examine the large N critical Gross-Neveu model in $d = 2k + 1$ dimensions. We show that the gap equation of the model can be obtained by either demanding operators $\bar{\psi}\psi$ or $\bar{\psi}\gamma^{\mu}\partial_{\mu}\psi$ do not occur in the spectrum. The gap equation has a real solution for the thermal mass for k even. We show that the one-point function of the operator $\mathcal{O}_+[0, 2]$ precisely agrees with the stress tensor obtained from the partition function of the theory. The form of expectation value of $\mathcal{O}_+[0, 2]$ appears manifestly different from that of stress tensor from the partition function. But on substituting the value of thermal mass from the gap equation they precisely agree. Finally we show that for the large N critical Gross-Neveu model, the expectation values of the three classes of operators in (1.1), (1.2) all are related by numerical factors, spin and the thermal mass. One such relation we prove using the OPE inversion formula is

$$a_{\mathcal{O}_0}[0,l] = m_{\rm th}\, a_{\mathcal{O}_+}[0,l]. \tag{1.3}$$

Here $m_{\rm th}$ is the thermal mass and $a_{\mathcal{O}_0}$ and $a_{\mathcal{O}_+}$ refer to the thermal expectation values. Our analysis shows it is sufficient to work with the operators $\mathcal{O}_+[0, l]$ in this model. Finally, we quote here the result for the thermal expectation values of the operators in the class $\mathcal{O}_+[0, l]$ for the critical large N Gross-Neveu model.

$$a_{\mathcal{O}_+}[n=0,l] = \frac{l}{2\pi^k \left(k-\frac{1}{2}\right)_l}\left(\frac{m_{\rm th}}{2}\right)^{l+k-1}\sum_{n=0}^{l+k-1}\frac{(l+k-n)_{2n}}{(2m_{\rm th})^n n!}\mathrm{Li}_{k+n}(-e^{-m_{\rm th}}). \tag{1.4}$$

As we have seen the inversion formula applied to the critical Gross-Neveu as well as the $O(N)$ model [1, 2] yields reasonably compact expressions for the one-point function of higher spin currents. Since the method can be easily applied to these classes of CFT in any dimensions we can study the behaviour of these one-point functions for arbitrary dimensions $d = 2k+1$, and arbitrary spin. Motivations to study this include the results from earlier works related to large d conformal field theories in [8–13], and the recent conjectures in [14, 15], that conformal field theories with a stress tensor in higher dimensions are trivial, or non-unitary. The large spin exploration is natural due to the observation in [1], that thermal one-point functions with large spin are universal. Here we see that these one-point functions asymptote to their Stefan-Boltzmann values at large spin.

To study the dependence of the one-point functions on spin l and dimension d, we chose the ratio of a given one-point function of spin l operator at the non-trivial fixed point of say the $O(N)$ model or the Gross-Neveu model to Gaussian fixed point in $d = 2k + 1$ dimensions. We denote this ratio by

$$r(l,d) = \frac{a_{\mathcal{O}}[l]_{\mathrm{m_{th}}\neq 0}}{a_{\mathcal{O}}[l]_{m_{\mathrm{th}}=0}}, \qquad l = 2, 4, \cdots . \tag{1.5}$$

Note that setting $m_{\mathrm{th}} = 0$ takes the one-point function to the free theory or the Stefan-Boltzmann result, while we define the non-trivial fixed point by choosing the value of m_{th} which satisfies the gap equation. This ratio is analogous to the famous ratio between the stress tensor of $\mathcal{N} = 4$ super-Yang-Mills at strong coupling and the Stefan-Boltzmann result which is $3/4$ or the ratio between the stress tensor of the critical $O(N)$ model at strong coupling to the Stefan-Boltzmann result with is $4/5$ for $d = 3$. Here we examine the ratio at arbitrary spins not just $l = 2$.

For the critical $O(N)$ model at large N, a real solution to the gap equation exists in $2k + 1$ dimensions with k odd. The ratio (1.5) is always less than unity, and as k is increased with the spin l held fixed, the ratio vanishes. The same behaviour is seen for the critical Gross-Neveu model at large N which has a real solution to the gap equation in $2k + 1$ dimensions with k even. The fact that on increasing the dimensions the ratio vanishes seems to indicate that the number of degrees of freedom at the non-trivial fixed point decreases. It will be interesting to see if such behaviour is true in general not just for the models studied in this paper. When the dimension is fixed and the spin l is increased, we see that for both models the ratio (1.5) tends to unity. This is consistent with the perturbative analysis of [1], for one-point functions at large spin. Their analysis isolated a universal contribution to the one-point functions at large spin.

The organization of the paper is as follows. In the section 2 we discuss the OPE expansion of the two-point function of spinor operators and briefly review the OPE inversion formula. In section 3 we apply our formalism to the MFT of fermions and then in section 4 to the Gross-Neveu model to obtain thermal one-point functions using the OPE inversion formula. In section 5 we study the behaviour of the one-point functions of both the critical $O(N)$ model and the Gross-Neveu model both at large d and at large l. Section 6 contains the conclusions. The appendix A provides the derivation of the gap equation for the Gross-Neveu model and its stress tensor from the partition function.

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2 Inversion formula for fermionic operators

In this section we wish to obtain the Euclidean inversion formulae for CFT's at finite temperature with only fermionic operators generalising the discussion of [1]. Consider the following fermion bi-linears

$$\begin{aligned}
\mathcal{O}_0 &= \bar{\psi}\partial^{\mu_1}\cdots\partial^{\mu_J}\partial^{2n}\psi - \text{Traces}, && l = J, \\
\mathcal{O}_+ &= \frac{1}{J+1}\left(\bar{\psi}\gamma_\mu\partial^{\mu_1}\cdots\partial^{\mu_J}\partial^{2n}\psi + \text{cyclic}\right) - \text{Traces}, && l = J+1, \\
\mathcal{O}_- &= \bar{\psi}\gamma^\mu\partial_\mu\partial^{\mu_1}\cdots\partial^{\mu_{J-1}}\partial^{2n}\psi - \text{Trace}, && l = J-1,
\end{aligned} \tag{2.1}$$

where $n = 0, 1, \ldots$, l is the spin and J the number of derivatives and $\bar{\psi} = \psi^\dagger$. These are the possible symmetric traceless tensors formed out of bi-linears of fermions which can have non-trivial expectation value in the thermal vacuum. In this section we obtain the Euclidean inversion formula which relates one-point functions of the above fermion bi-linears to the two-point function of the fermions. We test the inversion formula by considering the mean field theory of fermions. We then apply it the Gross-Neveu model at large N to derive the one-point functions at finite temperature for the operators listed in (2.1).

2.1 OPE expansion of fermionic correlators

Consider the following two-point functions in a CFT at finite temperature

$$\begin{aligned}
g_1(x) &= \langle\bar{\psi}(x)\psi(0)\rangle_{S^1_\beta\times R^{d-1}}, \\
g_2(x) &= \langle\bar{\psi}(x)\frac{\gamma^\mu x_\mu}{|x|}\psi(0)\rangle_{S^1_\beta\times R^{d-1}}, \\
g_3(x) &= \langle\partial_\mu\bar{\psi}(x)\gamma^\mu\psi(0)\rangle_{S^1_\beta\times R^{d-1}},
\end{aligned} \tag{2.2}$$

where $x = (\tau, x^1\cdots x^{d-1})$ and $|x|^2 = \tau^2 + (x^1)^2 + \cdots (x^{(d-1)})^2$. We will derive inversion formulae relating these two-point functions to the one-point functions in (2.1). To this we would need the OPE expansions of these correlators. Let the OPE of the fermion bi-linear be given by

$$\psi^\dagger_\alpha(x)\psi_\beta(0) = \sum_{\mathcal{O}\in\psi^\dagger\times\psi}\frac{f_{\psi^\dagger\psi\mathcal{O}}}{c_{\mathcal{O}}}|x|^{\Delta_{\mathcal{O}}-2\Delta_\psi-J}x_{\mu_1}\cdots x_{\mu_J}\mathcal{O}^{\mu_1\cdots\mu_J}_{\beta\alpha}(0) + \cdots. \tag{2.3}$$

Here $\mathcal{O}^{\beta\alpha\mu_1\cdots\mu_J}(0)$ are all the operators that occur in the OPE of the fermions. The tensor indices are symmetric and traceless. The $\cdots$ refers to other representations which are anti-symmetric in any pair of the tensor indices. These vanish in the thermal vacuum and therefore not relevant for our purpose. The representations of SO(d) which acquire non-trivial expectation values in the vacuum $S^1_\beta \times R^{d-1}$ are those which contain the trivial representation under $O(d-1)$. The fermion bi-linear indices in (2.3) together with the tensor indices can be combined into irreducible representations of SO(d), we will do this subsequently for each of the correlators. The coefficients $f_{\psi^\dagger\psi\mathcal{O}}$, are the structure constants and $c_{\mathcal{O}}$ is the normalization of the two-point function of the operator $\mathcal{O}$.

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Let us substitute the OPE (2.3) into the correlators given in (2.2). For the first correlator we obtain

$$g_1(x) = \sum_{\mathcal{O}\in\psi^\dagger\times\psi} \frac{f_{\psi^\dagger\psi\mathcal{O}}}{c_{\mathcal{O}}}|x|^{\Delta_{\mathcal{O}}-2\Delta_\psi-J}x_{\mu_1}\cdots x_{\mu_J}\langle\mathcal{O}_\alpha^{\alpha;\ \mu_1\cdots\mu_J}\rangle. \tag{2.4}$$

The thermal expectation values on the right hand side of (2.4) are the one-point functions which we are interested in computing. In a theory of only fermions, this class of operators can be written as the fermion bi-linears[1]

$$\mathcal{O}_0^{\mu_1\cdots\mu_J} \equiv \mathcal{O}_\alpha^{\alpha;\mu_1\cdots\mu_J} = \psi^\dagger\partial^{\mu_1}\cdots\partial^{\mu_J}\psi - \text{traces}. \tag{2.5}$$

Using translational invariance and spatial rotational invariance of the thermal vacuum, we have the following result for thermal one-point functions of symmetric traceless tensors.

$$\langle\mathcal{O}^{\mu_1\cdots\mu_J}(x)\rangle = b_{\mathcal{O}}T^{\Delta_{\mathcal{O}}}(e^{\mu_1}e^{\mu_2}\cdots e^{\mu_J} - \text{Traces}). \tag{2.6}$$

Here e^μ is the unit vector in the thermal direction τ. Now we also have the identity

$$|x|^{-J}(x_{\mu_1}\cdots x_{\mu_J})(e^{\mu_1}e^{\mu_2}\cdots e^{\mu_J} - \text{Traces}) = \frac{J!}{2^J(\nu)_J}C_J^{(\nu)}(\eta), \tag{2.7}$$

where

$$\nu = \frac{d-2}{2}, \qquad (a)_n = \frac{\Gamma(a+n)}{\Gamma(a)}, \qquad \eta = \frac{\tau}{|x|}, \tag{2.8}$$

and $C_J^{(\nu)}(\eta)$ is the Gegenbauer polynomial of degree J. Using the property (2.6) and (2.7) in the expression (2.4), we obtain

$$g_1(x) = \sum_{\mathcal{O}\in\psi^\dagger\times\psi}|x|^{\Delta_{\mathcal{O}}-2\Delta_\psi}a_{\mathcal{O}_0}C_J^{(\nu)}(\eta), \tag{2.9}$$

$$a_{\mathcal{O}_0} = b_{\mathcal{O}_0}T^{\Delta_{\mathcal{O}}}\frac{J!}{2^J(\nu)_J}\frac{f_{\psi^\dagger\psi\mathcal{O}}}{c_{\mathcal{O}}}.$$

The above equation is the OPE expansion of the finite temperature two-point function $g_1(x)$, in terms of thermal one-point functions of fermion bi-linears $a_{\mathcal{O}_0}$.

Let us now repeat the analysis for the two-point function $g_2(x)$. Substituting the OPE (2.3), we obtain

$$g_2(x) = \sum_{\mathcal{O}\in\psi^\dagger\times\psi}|x|^{\Delta_{\mathcal{O}}-2\Delta_\psi-J-1}x_{\mu_1}\cdots x_{\mu_J}x_\nu\langle\gamma^{\nu\ \alpha\beta}\mathcal{O}_{\beta\alpha}^{\mu_1\cdots\mu_J}\rangle. \tag{2.10}$$

Now the operator on the r.h.s. can be decomposed into various irreducible representations,

$$\begin{aligned}\gamma^{\nu\ \alpha\beta}\mathcal{O}_{\beta\alpha}^{\mu_1\cdots\mu_J} = &\left[\frac{1}{J+1}\left(\gamma^{\nu\ \alpha\beta}\mathcal{O}_{\beta\alpha}^{\mu_1\cdots\mu_J} + \gamma^{\nu_1\ \alpha\beta}\mathcal{O}_{\beta\alpha}^{\mu_2\cdots\mu_J\mu} + \text{cyclic}\right) - \text{Traces}\right] + \text{Traces}\\ &+ \frac{1}{J+1}\left(\gamma^{\nu\ \alpha\beta}\mathcal{O}_{\beta\alpha}^{\mu_1\cdots\mu_J} - \gamma^{\nu_1\ \alpha\beta}\mathcal{O}_{\beta\alpha}^{\mu\mu_2\cdots\mu_J}\right) + \cdots(J-1)\text{terms}),\end{aligned} \tag{2.11}$$

[1]In general these operators are just bosonic traceless symmetric tensors. For instance if there are Yukawa couplings in the theory they could be also be made of bosonic bi-linears. In this work we will restrict our attention to theories without such couplings.

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where 'Traces' are the terms subtracted to ensure that the term in the square bracket on the first line is a rank $J+1$ traceless symmetric tensor. The 'Traces' are given by

$$\text{Traces} = \frac{2}{(J+1)(d+J-1)}\left(\delta^{\mu\mu_1}\gamma^{\rho\,\alpha\beta}\mathcal{O}_{\beta\alpha}^{\rho\mu_2\cdots\mu_J} + \delta^{\mu_1\mu_2}\gamma^{\rho\,\alpha\beta}\mathcal{O}_{\beta\alpha}^{\rho\mu_3\cdots\mu_J\mu_1} + \text{cyclic}\right). \tag{2.12}$$

The equation (2.11) essentially writes the tensor product of a vector with a symmetric traceless tensor of rank J as a sum of symmetric traceless tensors of rank $J+1$ and rank $J-1$ together with tensors which are anti-symmetric in two of the indices. Now since thermal expectation values are non-zero only for symmetric tensors, the tensors which are anti-symmetric in any two of the indices can be ignored. Using the property (2.6) and the identity (2.7) in the expression the correlator $g_2(x)$ given in (2.10) we obtain

$$g_2(x) = \sum_{\mathcal{O}\in\psi^\dagger\times\psi} |x|^{\Delta_\mathcal{O}-2\Delta_\psi}\left(a_{\mathcal{O}_+}C_{J+1}^{(\nu)}(\eta) + \frac{2J}{(J+1)(d+J-1)}a_{\mathcal{O}_-}C_{J-1}^{(\nu)}(\eta)\right), \tag{2.13}$$

where

$$\begin{aligned} a_{\mathcal{O}_+} &= b_{\mathcal{O}_+}T^{\Delta_{\mathcal{O}_+}}\frac{(J+1)!}{2^{J+1}(\nu)_{J+1}}\frac{f_{\psi^\dagger\psi\mathcal{O}}}{c_\mathcal{O}}, \\ a_{\mathcal{O}_-} &= b_{\mathcal{O}_-}T^{\Delta_{\mathcal{O}_-}}\frac{(J-1)!}{2^{J-1}(\nu)_{J-1}}\frac{f_{\psi^\dagger\psi\mathcal{O}}}{c_\mathcal{O}}. \end{aligned} \tag{2.14}$$

Here again we restrict our attention to the case where the operators on the R.H.S of (2.13) are fermion bi-linears given by

$$\begin{aligned} \mathcal{O}_+ &= \frac{1}{J+1}\left(\psi^\dagger\gamma^\mu\partial^{\mu_1}\cdots\partial^{\mu_J}\psi + \text{cyclic}\right) - (\text{Traces}), \\ \mathcal{O}_- &= \psi^\dagger\gamma^\mu\partial_\mu\partial^{\mu_1}\cdots\partial^{\mu_{J-1}}\psi. \end{aligned} \tag{2.15}$$

Finally let us examine the correlator $g_3(x)$. From (2.3), we obtain the OPE

$$\begin{aligned} \partial_\mu\psi^\dagger(x)\gamma^\mu\psi = \sum_{\mathcal{O}\in\psi^\dagger\times\psi}\frac{f_{\psi^\dagger\psi\mathcal{O}}}{c_\mathcal{O}}\Big(&|x|^{\Delta_\mathcal{O}-2\Delta_\psi-J-2}(\Delta_\mathcal{O}-2\Delta_\psi-J)x_\mu x_{\mu_1}\cdots x_{\mu_J}\mathcal{O}_+^{\mu\mu_1\cdots x_{\mu_J}} \\ &+\left(J+\frac{2J(\Delta_\mathcal{O}-2\Delta_\psi-J)}{(J+1)(J+d-1)}\right)|x|^{\Delta_\mathcal{O}-2\Delta_\psi-J}x_{\mu_1}\cdots x_{\mu_{J-1}}\mathcal{O}_-^{\mu_1\cdots x_{\mu_{J-1}}}+\cdots\Big). \end{aligned} \tag{2.16}$$

We can now take thermal expectation values and obtain

$$\begin{aligned} g_3(x) = \sum_{\mathcal{O}\in\psi^\dagger\times\psi}|x|^{\Delta_\mathcal{O}-2\Delta_\psi-1}\Big[&(\Delta_\mathcal{O}-2\Delta_\psi-J)a_{\mathcal{O}_+}C_{J+1}^{(\nu)}(\eta) \\ &+\left(J+\frac{2J(\Delta_\mathcal{O}-2\Delta_\psi-J)}{(J+1)(J+d-1)}\right)a_{\mathcal{O}_-}C_{J-1}^{(\nu)}(\eta)\Big]. \end{aligned} \tag{2.17}$$

It is important to realise that due to the presence of operators belonging to the class $\mathcal{O}_+$ as well as $\mathcal{O}_-$ in the OPE expansions of $g_2(x)$ and $g_3(x)$ given in (2.13) and (2.17), the OPE inversion formulas for the one-point functions will involve both these correlators.

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2.2 Euclidean inversion formulas

In this section we briefly review the Euclidean inversion formula introduced in [1] and obtain the expressions relating the one-point functions listed in (2.1) to the thermal 2-point functions. One difference we need to keep track is the fact that the 2 point functions $g_2(x)$ and $g_3(x)$ involve a linear combination of one-point functions of operators belonging to class $\mathcal{O}_+$ and $\mathcal{O}_-$ and therefore the inversion formula for these operators will involve linear combinations of $g_2(x)$ and $g_3(x)$.

Consider the OPE expansion of a correlator given in the form

$$g(x) = \sum_{\mathcal{O}} |x|^{\Delta_{\mathcal{O}}-2\Delta_\psi} a_{\mathcal{O}} C_l^{(\nu)}(\eta). \tag{2.18}$$

The expansions in (2.9), (2.13) and (2.17) are of this form. By introducing the spectral function $\hat{a}(\Delta, l)$ we can write the OPE expansion in terms of an integral

$$g(x) = \sum_{l=0}^{\infty} \oint_{-\epsilon-i\infty}^{-\epsilon+i\infty} \frac{d\Delta}{2\pi i} \hat{a}(\Delta, l) C_l^{(\nu)}(\eta)|x|^{\Delta-2\Delta_\psi}. \tag{2.19}$$

Here the spectral function should have poles of the form

$$\hat{a}(\Delta, l) \sim -\frac{a_{\mathcal{O}}}{\Delta - \Delta_{\mathcal{O}}}. \tag{2.20}$$

The contour in (2.19) is chosen to encircle the right-half of the Δ plane when $|x| < 1$ and demanding the spectral function does not grow exponentially in this region. Deforming the contour to encircle the poles results in the sum given in (2.18). The contour in (2.19) has been chosen so that all the physical poles along with the unit operator are included.

We can invert the equation (2.19) using the orthogonality of the Gegenbauer polynomials.

$$\hat{a}(\Delta, l) = \frac{1}{N_l} \int_{|x|<1} d^d x C_l^{(\nu)}(\eta)|x|^{2\Delta_\psi-\Delta-d} g(x). \tag{2.21}$$

Here we first use the property

$$\int_{S^{d-1}} d\Omega C_l^{(\nu)}(\eta) C_{l'}^{(\nu)}(\eta) = N_l \delta_{ll'}, \tag{2.22}$$

$$N_l = \frac{4^{1-\nu}\pi^{\nu+\frac{3}{2}}\Gamma(l+2\nu)}{l!(l+\nu)\Gamma(\nu)^2\Gamma\left(\nu+\frac{1}{2}\right)},$$

to fix on to a particular l. Then the integral over x functions as the Laplace transform which picks out the relevant pole. It can be seen that (2.21) is consistent, by substituting for $g(x)$ from (2.19).

Now the Euclidean inversion formula (2.21) is cast as an integral over the 2 dimensional plane using rotational invariance. Let us first discuss the case $d > 2$. Using the spatial $\mathrm{SO}(d-1)$ rotational invariance we can choose to write the d component vector x as $x = (\tau, x_E, 0, \cdots)$. So the relevant kinematics can be parametrized by introducing the following complex variables, as well as polar coordinates.

$$\begin{aligned} z &= \tau + i x_E, & \bar{z} &= \tau - i x_E, \\ z &= r w, & \bar{z} &= r w^{-1}. \end{aligned} \tag{2.23}$$

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Note that in these variables

$$\eta = \frac{\tau}{|x|} = \cos\theta = \frac{1}{2}(w + w^{-1}). \tag{2.24}$$

Therefore the Gegenbauer polynomials are functions of the polar angle θ. It can be written in terms of the hypergeometric function

$$C_l^{(\nu)}\left(\frac{1}{2}(w+w^{-1})\right) = \frac{\Gamma(l+2\nu)}{\Gamma(\nu)\Gamma(l+\nu+1)}\left(F_l(w^{-1})e^{i\pi\nu}+F_l(w)e^{-i\pi\nu}\right), \qquad \mathrm{Im}\, w>0 \tag{2.25}$$

where

$$F_l(w) = w^{l+2\nu}{}_2F_1(l+2\nu, \nu, l+\nu+1, w^2). \tag{2.26}$$

The representation in terms of hypergeometric function allows to continue w to the entire complex plane. For $\mathrm{Im}\, w < 0$, the phases of the two terms in (2.25) are exchanged. We see using (2.23), that $g(x)$ is a function of $g(z,\bar{z}) = g(r,\theta)$. These observations allow us to perform all the remaining $d-2$ angular integrals in (2.21) leaving the integral over the complex plane $(z,\bar{z})$.

Consider $g(z,\bar{z}) = g(rw, rw^{-1})$ as a function in the complex w plane. We assume the following analytic properties in the w plane [1]: the 2 point function is analytic in the w plane except at the branch cuts $(-\infty,-1/r)$, $(-r,0)$, $(r,0)$, $(1/r,\infty)$. The second assumption is that at large w the growth of $g(rw,rw^{-1})$ is bounded by the polynomial w^{l_0} for a fixed l_0. At small w the growth of $g(rw,rw^{-1})$ is bounded by w^{-l_0}. These 2 properties allow one to deform the integral contour over w along the branches $(-\infty,-\frac{1}{r})\cup(\frac{1}{r},\infty)$, together with the circle at ∞ from that of the unit circle.[2] Using these methods and a change of variables the inversion formula can be written as

$$\hat{a}(\Delta,l) = \hat{a}_{\text{disc}}(\Delta,l) + \theta(l_0-l)\hat{a}_{\text{arcs}}(\Delta,l), \tag{2.27}$$

$$\hat{a}_{\text{disc}}(\Delta,l) = (1+(-1)^l)K_l\int_0^1\frac{d\bar{z}}{\bar{z}}\int_1^{\frac{1}{\bar{z}}}\frac{dz}{z}(z\bar{z})^{\Delta_\psi-\frac{\Delta}{2}-\nu}(z-\bar{z})^{2\nu}F_l\left(\sqrt{\frac{\bar{z}}{z}}\right)\mathrm{Disc}[g(z,\bar{z})],$$

$$K_l = \frac{\Gamma(l+1)\Gamma(\nu)}{4\pi\Gamma(l+\nu)}.$$

Here the discontinuity across the branch cuts is given by

$$\mathrm{Disc}[g(z,\bar{z})] = \frac{1}{i}(g(z+i\epsilon,\bar{z}) - g(z-i\epsilon,\bar{z})). \tag{2.28}$$

For $l<l_0$ the contribution from the arcs, which essentially becomes an integral over the circle at infinity is given by the term $\hat{a}_{\text{arcs}}(\Delta,l)$ which needs to be evaluated by performing the following integral in (2.27) over the circle at infinity in the w-plane explicitly.

$$\hat{a}_{\text{arcs}}(\Delta,l) = 2K_l\int_0^1\frac{dr}{r^{\Delta+1-2\Delta_\psi}}\times \oint\frac{dw}{iw}\lim_{|w|\to\infty}\left[\left(\frac{w-w^{-1}}{i}\right)^{2\nu}F_l(w^{-1})e^{i\pi\nu}g(r,w)\right]. \tag{2.29}$$

[2]The integration contour can also be deformed towards the origin and this again can be related to the integral along the contour deformation mentioned above as it is illustrated in detail in [1].

The expression in (2.27) is the form of the inversion formula we will use for $d>3$. Note that it can be applied to the three correlators in (2.2) as their OPE expansions (2.9), (2.13) and (2.17) are of the form (2.18).

For $d=2$, we need to treat the normalization of the Gegenbauer polynomials carefully. but in the end the inversion formula is very similar. It is best to first re-examine the OPE representation of the thermal two-point function which can be written explicitly as

$$g(x)=\sum_{\mathcal{O}}|x|^{\Delta_{\mathcal{O}}-2\Delta_\psi}b_{\mathcal{O}}T^{\Delta_{\mathcal{O}}}\frac{l!}{2^l(\nu)_l}\frac{f_{\psi^\dagger\psi\mathcal{O}}}{c_{\mathcal{O}}}C_l^{(\nu)}(\eta). \tag{2.30}$$

In the limit $d\to 2$ or $\nu\to 0$, the Gegenbauer polynomials take the limiting form

$$\begin{aligned}&\lim_{\nu\to 0}C_l^{(\nu)}(\eta)=\frac{2\nu}{l}\left(\frac{w^l+w^{-l}}{2}\right)=\frac{2\nu}{l}\cos(l\theta), \qquad l>0,\\ &\lim_{\nu\to 0}C_0^{(\nu)}(\eta)=1.\end{aligned} \tag{2.31}$$

From (2.26) we also obtain the following limit

$$\lim_{\nu\to 0}F_l(w)=w^l. \tag{2.32}$$

Taking the limit $\nu\to 0$ in (2.30), we obtain

$$\begin{aligned} g(x)&=\sum_{\mathcal{O}}|x|^{\Delta_{\mathcal{O}}-2\Delta_\psi}a_{\mathcal{O}}|_{d=2}, & a_{\mathcal{O}}|_{d=2}&=\frac{b_{\mathcal{O}}}{2^{l-1}}T^{\Delta_{\mathcal{O}}}\frac{f_{\psi^\dagger\psi\mathcal{O}}}{c_{\mathcal{O}}},\\ a_{\mathcal{O}}|_{d=2}&=\lim_{\nu\to 0}\frac{2\nu}{l}a_{\mathcal{O}}|_\nu, \quad \text{for } l>0, & a_{\mathcal{O}}|_{d=2}&=\lim_{\nu\to 0}a_{\mathcal{O}}|_\nu, \quad \text{for } l=0.\end{aligned} \tag{2.33}$$

Going through a similar analysis we obtain

$$\hat{a}(\Delta,l)_{\text{disc}}|_{d=2}=(1+(-1)^l)\frac{1}{2\pi}\int_0^1\frac{d\bar{z}}{\bar{z}}\int_1^{\frac{1}{\bar{z}}}\frac{dz}{z}z^{\Delta_\psi-\bar{h}}\bar{z}^{\Delta_\psi-h}\text{Disc}[g(z,\bar{z})], \tag{2.34}$$

with

$$h=\frac{\Delta-l}{2}, \qquad \text{and} \quad \bar{h}=\frac{\Delta+l}{2}. \tag{2.35}$$

The contribution from the arcs is given by

$$\hat{a}_{\text{arcs}}(\Delta,l)|_{d=2}=\frac{1}{2\pi}\int_0^1\frac{1}{r^{\Delta+1-2\Delta_\psi}}\oint\frac{dw}{iw}\lim_{|w|\to\infty}w^{-l}. \tag{2.36}$$

3 Mean field theory of fermions

In mean field theory, a $2n$ point function is given by pairwise contraction of the n two-point functions. Therefore, the 2 point function at finite temperature can be obtained by using the method of images. Consider a fermionic operator ψ of dimension Δ_ψ in MFT, then the thermal two-point function is given by

$$\langle\psi_\alpha(x)\psi_\beta^\dagger(0)\rangle_\beta=\frac{1}{2^{[\frac{d}{2}]}}\sum_{m\in\mathbb{Z}}(-1)^m\frac{\gamma^\mu_{\alpha\beta}x_\mu^{(m)}}{|x^{(m)}|^{2\Delta_\psi+1}}, \qquad \text{where} \qquad x_\mu^{(m)}\equiv\{(\tau+m),\vec{x}\}. \tag{3.1}$$

We have set the inverse temperature $\beta = 1$. This correlator obeys anti-periodic boundary conditions along the thermal circle. We have normalized the correlator, by $2^{[\frac{d}{2}]}$, the dimension of the Dirac spinor. This is for convenience so that this factor cancels on taking the trace over the γ-matrices. We will account for this while comparing with our results from the partition function. Note that scalar correlator $g_1(x)$ vanishes, this implies in MFT the class of operators $\mathcal{O}_0$ has zero expectation value in the thermal vacuum.

$\boldsymbol{g_2(\tau, \vec{x})}$**.** Let us evaluate the correlator $g_2(x)$ for MFT[3]

$$g_2(\tau, \vec{x}) = \frac{1}{|x|^{2\Delta_\psi}} + \sum_{\substack{m=-\infty \\ m\neq 0}}^{\infty} (-1)^m \frac{m\tau + |x|^2}{[(\tau+m)^2 + \vec{x}^2]^{\Delta_\psi + \frac{1}{2}} |x|}. \tag{3.2}$$

We can systematically expand this correlator in small x so as to compare with the OPE expansion in (2.13). This expansion is facilitated by the identity

$$\frac{1}{(1-2xy+y^2)^\alpha} = \sum_{j=0}^{\infty} C_j^{(\alpha)} y^j, \tag{3.3}$$

where $C_j^{(\alpha)}$ are Gegenbauer polynomials of order j with index α. Once the OPE expansion and the thermal expectation value of the operators in the class $\mathcal{O}_+$, $\mathcal{O}_-$ are obtained, they can be compared against the same obtained from the inversion formula. This will provide an important check on the inversion formula. Indeed, when the formula is applied to $g_2(\tau, x)$ given in (3.2), this case would be a distinct check from that done in [1]. Proceeding with the expansion we obtain

$$\begin{aligned} g_2(\tau, \vec{x}) &= \frac{1}{|x|^{2\Delta_\psi}} + \sum_{\substack{m=-\infty \\ m\neq 0}}^{\infty} (-1)^m \, (m\eta + |x|) \times \sum_{j=0}^{\infty} (-1)^j C_j^{(\Delta_\psi + \frac{1}{2})}(\eta) \, \frac{\mathrm{sgn}(m)^j |x|^j}{|m|^{2\Delta_\psi + 1 + j}}, \\ &= \frac{1}{|x|^{2\Delta_\psi}} - \sum_{j=1,3,\ldots} 2\eta \, C_j^{(\Delta_\psi + \frac{1}{2})}(\eta) |x|^j (2^{1-2\Delta\psi - j} - 1)\zeta(2\Delta_\psi + j) \\ &\quad + \sum_{j=0,2,4,\ldots} 2\, C_j^{(\Delta_\psi + \frac{1}{2})}(\eta) |x|^{j+1} (2^{-2\Delta\psi - j} - 1)\zeta(2\Delta_\psi + 1 + j). \end{aligned} \tag{3.4}$$

In the second line we have cancelled off terms which occur with equal and opposite signs and then performed the sum over m. The recurrence relation

$$2(n+\lambda)\eta C^{(\lambda)}(\eta) = (n+1) C_{n+1}^{(\lambda)}(\eta) + (n-1+2\eta) C_{n-1}^{(\lambda)}(\eta), \tag{3.5}$$

can be used to remove the explicit factor of η in the second line of (3.4). This results in

$$\begin{aligned} g_2(\tau, \vec{x}) = &\frac{1}{|x|^{2\Delta_\psi}} + \sum_{j=0,2,4,\ldots} 2\,|x|^{j+1} (2^{-2\Delta_\psi - j} - 1)\zeta(2\Delta_\psi + 1 + j) C_j^{(\Delta_\psi + \frac{1}{2})}(\eta) \\ &- \sum_{j=1,3,\ldots} 2|x|^j \frac{(2^{1-2\Delta\psi - j} - 1)}{2\,(j + \Delta_\psi + \frac{1}{2})} \zeta(2\Delta_\psi + j) \Big((j+1) C_{j+1}^{(\Delta_\psi + \frac{1}{2})}(\eta) + (j + 2\Delta_\psi) C_{j-1}^{(\Delta_\psi + \frac{1}{2})}(\eta) \Big). \end{aligned} \tag{3.6}$$

[3]We have kept track of the ordering of the ψ and $\bar{\psi}$, which is different in the definition of $g_2(x)$ in (2.2).

To compare with the OPE expansion of the correlator, we would need the index of the Gegenbauer polynomials to be ν instead of $\Delta_\psi + \frac{1}{2}$. For this we can use the following identity

$$C_j^{(\Delta)}(\eta) = \sum_{l=j,j-2,\dots,j \bmod 2} \frac{(l+\nu)(\Delta)_{\frac{j+l}{2}}(\Delta-\nu)_{\frac{j-l}{2}}}{\left(\frac{j-l}{2}\right)!(\nu)_{\frac{j+l+2}{2}}} C_l^{(\nu)}(\eta). \tag{3.7}$$

Then expressing $j = 2n + l$ and grouping terms with same summation ranges we obtain

$$\begin{aligned} g_2(\tau,\vec{x}) = &\frac{1}{|x|^{2\Delta_\psi}} + \\ &\sum_{\substack{n=0 \\ l=0,2,\dots}}^{\infty} 2|x|^{2n+l+1}(2^{-2\Delta_\psi-2n-l}-1)\zeta(2\Delta_\psi+1+2n+l)\left(1-\frac{2n+l+1+2\Delta_\psi}{2\left(2n+l+\Delta_\psi+\frac{3}{2}\right)}\right) \\ &\qquad \times \frac{(l+\nu)\left(\Delta_\psi+\frac{1}{2}-\nu\right)_n\left(\Delta_\psi+\frac{1}{2}\right)_{n+l}}{n!(\nu)_{n+l+1}} C_l^{(\nu)}(\eta) \\ &-\sum_{\substack{n=0 \\ l=0,2,\dots \\ n+l\neq 0}}^{\infty} 2|x|^{2n+l-1}\frac{(2^{2-2\Delta_\psi-2n-l}-1)}{2\left(2n+l+\Delta_\psi-\frac{1}{2}\right)}\zeta(2\Delta_\psi+2n+l-1)(2n+l) \\ &\qquad \times \frac{(l+\nu)\left(\Delta_\psi+\frac{1}{2}\right)_{n+l}\left(\Delta_\psi+\frac{1}{2}-\nu\right)_n}{n!(\nu)_{n+l+1}} C_l^{(\nu)}(\eta). \end{aligned} \tag{3.8}$$

To compare with the OPE expansion, it is useful to separate the $n = 0, l = 2, 4, \cdots$ terms from the last line of the above equation. Then combine the rest with the terms in the first summation. We can group these terms by shifting $n \to n-1$ in the first summation. These manipulations lead to

$$\begin{aligned} g_2(\tau,\vec{x}) = &\frac{1}{|x|^{2\Delta_\psi}} + \\ &\sum_{\substack{n=1 \\ l=0,2,\dots}}^{\infty} \frac{2(l+\nu)(l+2n)\left(1-2^{-2\Delta_\psi-l-2n+2}\right)\zeta(l+2n+2\Delta_\psi-1)|x|^{l+2n-1}C_l^{(\nu)}(\eta)}{\Gamma(n+1)(2\Delta+2l+4n-1)(\nu)_{l+n+1}} \\ &\qquad \times \left(\Delta_\psi+\frac{1}{2}\right)_{l+n}\left(\Delta_\psi-\nu+\frac{1}{2}\right)_n\left(1-\frac{4n(l+\nu+n)}{(2\Delta_\psi+2l+2n-1)(2\Delta_\psi-2\nu+2n-1)}\right) \\ &+\sum_{l=2,4,\dots} \frac{2l\left(2^{-2\Delta_\psi-l+2}-1\right)(l+\nu)\left(\Delta_\psi+\frac{1}{2}\right)_l\zeta(l+2\Delta_\psi-1)|x|^{l-1}C_l^{(\nu)}(\eta)}{(-2\Delta_\psi-2l+1)(\nu)_{l+1}}. \end{aligned} \tag{3.9}$$

We re-write the OPE expansion in (2.13) as

$$\begin{aligned} g_2(\tau,\vec{x}) = &\sum_{n=0}^{\infty}\frac{|x|^{2n+1}}{d}a_{\mathcal{O}_-}[n,l=0]C_0^{\nu}(\eta) + \sum_{l=1}^{\infty}|x|^{l-1}a_{\mathcal{O}_+}[n=0,l]C_l^{\nu}(\eta) \\ &+\sum_{\substack{n=1 \\ l=1}}^{\infty}|x|^{2n+l-1}\left(a_{\mathcal{O}_+}[n,l]+\frac{2(l+1)}{(l+2)(d+l)}a_{\mathcal{O}_-}[n-1,l]\right)C_l^{\nu}(\eta). \end{aligned} \tag{3.10}$$

Here we have re-labelled the sum over J as l and separated out the $l = 0$ term as well as the $n = 0$ term. We have also used

$$\Delta_{\mathcal{O}_+[n,l]} = 2\Delta_\psi + l - 1 + 2n, \qquad \Delta_{\mathcal{O}_-[n,l]} = 2\Delta_\psi + l + 1 + 2n, \tag{3.11}$$

where l refers to the spin of the operators, J refers to the number of derivatives in the operators. Now comparing (3.9) and (3.10) we see that only operators with even spins have non-trivial expectation values in MFT. We also obtain

$$\begin{aligned} a_{\mathcal{O}_-}[n, l = 0] = &\frac{2\nu(\nu+1)(2\nu+1-2\Delta_\psi)\left(2^{-2(\Delta_\psi+n)}-1\right)\left(\Delta_\psi+\frac{1}{2}\right)_n\left(\Delta_\psi-\nu+\frac{1}{2}\right)_n}{n!(\nu+n+1)(\nu)_{n+1}} \\ &\times\zeta(2n+2\Delta_\psi+1), \end{aligned} \tag{3.12}$$

and

$$a_{\mathcal{O}_+}[n = 0, l] = \frac{2l\left(2^{-2\Delta_\psi-l+2}-1\right)(l+\nu)\left(\Delta_\psi+\frac{1}{2}\right)_l\zeta(l+2\Delta_\psi-1)}{(-2\Delta_\psi-2l+1)(\nu)_{l+1}}, \qquad l = 2, 4, 6\cdots. \tag{3.13}$$

Finally for operators with $n \geq 1$ and $l = 2, 4, 6, \cdots$ we get a single linear equation

$$\begin{aligned} &a_{\mathcal{O}_+}[n,l]+\frac{2(l+1)}{(l+2)(d+l)}a_{\mathcal{O}_-}[n-1,l] \\ &=\frac{(l+\nu)(l+2n)\Gamma(\nu)2^{-l-2(\Delta_\psi+n)}\left(2^{l+2(\Delta_\psi+n)}-4\right)\Gamma\left(l+n+\Delta_\psi-\frac{1}{2}\right)\Gamma\left(n+\Delta_\psi-\nu-\frac{1}{2}\right)}{\Gamma\left(\Delta_\psi+\frac{1}{2}\right)\Gamma(n+1)\Gamma\left(\Delta_\psi-\nu-\frac{1}{2}\right)\Gamma(l+n+\nu+1)} \\ &\times\zeta(l+2n+2\Delta_\psi-1). \end{aligned} \tag{3.14}$$

At this point there are some observations we can make: note that for $\Delta_\psi = \nu + \frac{1}{2}$, the MFT reduces to the theory of free fermions in d dimensions. Therefore by equations of motion we must have

$$\begin{aligned} a_{\mathcal{O}_-[n,l]} &= 0, \quad \text{for } \Delta_\psi = \nu + \frac{1}{2} \text{ and } n = 0, 1, \cdots, \\ a_{\mathcal{O}_+[n,l]} &= 0, \quad \text{for } \Delta_\psi = \nu + \frac{1}{2} \text{ and } n = 1, 2, \cdots. \end{aligned} \tag{3.15}$$

It can be easily seen that (3.12) satisfies this requirement and (3.14) is consistent with (3.15). Next, notice that the class of operators $a_{\mathcal{O}_+[n=0,l]}$ has the stress tensor, let us examine the one-point function for the free field case

$$a_{\mathcal{O}_+[n=0,l]} = -\frac{2l(2^{3-(d+l)}-1)}{(d-2)}\zeta(l-2+d) \qquad \text{for } \Delta_\psi = \nu + \frac{1}{2}. \tag{3.16}$$

We see that the result is proportional to $\zeta(d)$ for $l = 2$, which is the result expected for the stress tensor of free fermions. The reason the above expression has a divergence at $d = 2$ is due to the behaviour of the Gegenbauer polynomials in the limit $\nu \to 0$ as shown in (2.31).

Using the relation given in (2.33) which relates the one-point function in $d = 2$ to that in arbitrary d, we obtain

$$a_{\mathcal{O}_+[n=0,l]}|_{d=2} = -2(2^{1-l} - 1)\zeta(l) \qquad \text{for } \Delta_\psi = \frac{1}{2}. \tag{3.17}$$

To solve for $a_{\mathcal{O}_-[n,l]}$ and $a_{\mathcal{O}_+[n,l]}$ for $n \geq 1$ we need the correlator $g_3(\tau, \vec{x})$.

$\boldsymbol{g_3(\tau, x)}$**.** From the MFT correlator, we see that

$$\begin{aligned} g_3(\tau, \vec{x}) &= \langle \partial_\mu \psi_\alpha(x)\, \psi^\dagger_\beta(0) \rangle \gamma^\mu_{\beta\alpha}, \\ &= \sum_{m\in\mathbb{Z}} (-1)^m \frac{d - 2\Delta_\psi - 1}{[(\tau+m)^2 + \vec{x}^2]^{\Delta_\psi + \frac{1}{2}}}. \end{aligned} \tag{3.18}$$

We can again perform the expansion in small x in terms of Gegenbauer polynomials as done for the correlator $g_2(\tau, \vec{x})$. This results in

$$\begin{aligned} g_3(\tau,\vec{x}) = (d-2\Delta_\psi-1)\Bigg(&\frac{1}{|x|^{2\Delta_\psi+1}} \\ &+\sum_{n=0}^{\infty}\sum_{l=0,2,\cdots} \frac{2\zeta(2\Delta_\psi+1+2n+l)(l+\nu)\left(\Delta_\psi+\frac{1}{2}\right)_{l+n}\left(\Delta_\psi+\frac{1}{2}-\nu\right)_n}{n!(\nu)_{l+n+1}} \\ &\times(2^{-2\Delta_\psi-2n-l}-1)C_l^\nu(\eta)|x|^{2n+l}\Bigg). \end{aligned} \tag{3.19}$$

We can rewrite the OPE expansion of the correlator $g_3(\tau, \vec{x})$ given in (2.17) as

$$\begin{aligned} g_3(\tau,\vec{x}) = &\sum_{n=0}^{\infty} |x|^{2n}\left(1+\frac{2n}{d}\right) a_{\mathcal{O}_-}[n,l=0] C_0^{(\nu)}(\eta) \\ &+ \sum_{\substack{n=0\\l=1}}^{\infty} |x|^{2n+l}\left[2(n+1)a_{\mathcal{O}_+}[n+1,l] + \left(l+1+\frac{4n(l+1)}{(l+2)(l+d)}\right)a_{\mathcal{O}_-}[n,l]\right] C_l^{(\nu)}(\eta). \end{aligned} \tag{3.20}$$

To obtain this expansion from (2.17) we replace the number of derivatives J by the appropriate spin l of the operators $\mathcal{O}_+$, $\mathcal{O}_-$ and write their conformal dimensions as in (3.11). We have also separated out the $l = 0$ contribution. Now comparing (3.20) and (3.19), we see that only even spin operators have non-trivial expectation values. For $l = 0$ we obtain

$$\begin{aligned} a_{\mathcal{O}_-}[n, l=0] = &\frac{2\nu(\nu+1)\left(2^{-2(\Delta+n)}-1\right)\left(\Delta+\frac{1}{2}\right)_n\left(\Delta-\nu+\frac{1}{2}\right)_n(d-2\Delta_\psi-1)}{n!(\nu+n+1)(\nu)_{n+1}} \\ &\times\zeta(2n+2\Delta+1). \end{aligned} \tag{3.21}$$

From $g_2(\tau, \vec{x})$, we had already obtained the expectation value of $a_{\mathcal{O}_-}[n, l = 0]$ in (3.12). Note that the above equation is identical to that in (3.12) which serves as an important

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consistency check of our methods. Now for $l = 2, 4, \ldots$ and $n = 0, 1, \cdots$ we have

$$\begin{aligned} &2(n+1)a_{\mathcal{O}_+}[n+1,l] + \left(l+1+\frac{4n(l+1)}{(l+2)(l+d)}\right)a_{\mathcal{O}_-}[n,l] \\ &= \frac{2(-2\Delta_\psi+2\nu+1)(l+\nu)\left(2^{-l-2(\Delta_\psi+n)}-1\right)\left(\Delta_\psi+\frac{1}{2}\right)_{l+n}\left(\Delta_\psi-\nu+\frac{1}{2}\right)_n}{n!(\nu)_{l+n+1}} \\ &\qquad\qquad\qquad\qquad \times\zeta(l+2n+2\Delta+1). \end{aligned} \tag{3.22}$$

From the OPE expansion of $g_2(\tau,\vec{x})$ we obtained the linear relation between the expectation values given in (3.14). By replacing $n \to n+1$ we arrive at the equation

$$\begin{aligned} &a_{\mathcal{O}_+}[n+1,l] + \frac{2(l+1)}{(l+2)(d+l)}a_{\mathcal{O}_-}[n,l] \\ &= \frac{(l+\nu)(l+2n+2)\Gamma(\nu)(1-2^{-l-2(\Delta+n)})\Gamma\left(l+n+\Delta+\frac{1}{2}\right)\Gamma\left(n+\Delta-\nu+\frac{1}{2}\right)}{\Gamma\left(\Delta+\frac{1}{2}\right)\Gamma(n+2)\Gamma\left(\Delta-\nu-\frac{1}{2}\right)\Gamma(l+n+\nu+2)} \\ &\times\zeta(l+2n+2\Delta+1), \end{aligned} \tag{3.23}$$

for $l = 2, 4, \ldots$ and $n = 0, 1, \cdots$.

The equations (3.22) and (3.23) are linear independent equations from which we can solve for the expectation values $a_{\mathcal{O}_+}[n+1,l], a_{\mathcal{O}_-}[n,l]$ for $n = 0, 1, \cdots$ and $l = 2, 4, \cdots$. We can write the explicit values of these expectation values but they are not very illustrative. For $a_{\mathcal{O}_+}[n=0,l]$, $l = 2, 4, \cdots$, the expression is given in (3.13), while for $a_{\mathcal{O}_-}[n,0]$, $n = 0, 1, 2\cdots$ the expectation value can be read out form (3.12). This completes the analysis of obtaining the one-point functions by directly expanding the two-point functions in small $|x|$.

3.1 OPE inversion in $d = 2$

Consider the correlator $g_2(x)$ given in (3.2), its OPE expansion is of the form (2.18). Therefore we can apply the inversion formula to obtain the coefficient $\hat{a}(\Delta, l)$ which contains the information of the one-point functions as residues of the poles in the complex Δ-plane. Using the co-ordinates $z, \bar{z}$ defined in (2.23), the MFT thermal two-point function can be written as

$$g_2(z,\bar{z}) = \frac{1}{|x|^{2\Delta_\psi}} + \sum_{\substack{m=-\infty \\ m\neq 0}}^{\infty} \frac{(-1)^m}{[(m-z)(m-\bar{z})]^{\Delta_\psi+\frac{1}{2}}}\left(-\frac{m}{2}\sqrt{\frac{z}{\bar{z}}} - \frac{m}{2}\sqrt{\frac{\bar{z}}{z}} + \sqrt{z\bar{z}}\right). \tag{3.24}$$

Now on substituting $z = rw$ and $\bar{z} = rw^{-1}$, we see that the correlator vanishes in w plane at large $|w|$ as well as small $|w|$ whenever $\Delta_\psi \geq 1/2$. In this domain, there is no contribution from the circle or arcs at infinity in the w-plane and the entire contribution to $\hat{a}(\Delta, l)$ arises from the discontinuity across the branch cuts.[4] Therefore we have

$$\hat{a}(\Delta_{\mathcal{O}}, l) = \frac{(1+(-1)^l)}{2\pi}\int_0^1 \frac{d\bar{z}}{\bar{z}}\int_1^{1/\bar{z}} \frac{dz}{z} z^{\Delta_\psi-\bar{h}}\bar{z}^{\Delta_\psi-h}\text{Disc}[g_2(z,\bar{z})]. \tag{3.25}$$

[4] In the domain $0 < \Delta_\psi < 1/2$ we need to evaluate the contributions from the arcs at infinity for only $l = 0$.

In this expression, the branch cut of (3.24) in the w-plane from $-m$ to $-\infty$ has already been taken care by symmetry with the branch cut from $+m$ to ∞ by the inclusion of $(-1)^l$. Therefore we need to restrict ourselves to the branch cut on the positive real axis of the w-plane. It is easy to see that this arises solely due to the following discontinuity

$$\text{Disc}\left[\frac{1}{((m-z)(m-\bar{z}))^{\Delta_\psi+\frac{1}{2}}}\right] = \frac{2\sin\left(\pi\left(\Delta_\psi+\frac{1}{2}\right)\right)}{[(z-m)(m-\bar{z})]^{\Delta_\psi+\frac{1}{2}}}\theta(z-m). \tag{3.26}$$

We are therefore led to perform the integral

$$\begin{aligned}\hat{a}(\Delta_{\mathcal{O}},l) = &\frac{(1+(-1)^l)}{2\pi}\times \\ &\sum_{m=1}^{\infty}\int_0^1\frac{d\bar{z}}{\bar{z}}\int_m^{1/\bar{z}}\frac{dz}{z}z^{\Delta_\psi-\bar{h}}\bar{z}^{\Delta_\psi-h}(-1)^m\left(-\frac{m}{2}\sqrt{\frac{\bar{z}}{z}}-\frac{m}{2}\sqrt{\frac{z}{\bar{z}}}+\sqrt{z\bar{z}}\right)\frac{2\sin\left(\pi\left(\Delta_\psi+\frac{1}{2}\right)\right)}{[(z-m)(m-\bar{z})]^{\Delta_\psi+\frac{1}{2}}}, \\ \equiv &I_1+I_2+I_3.\end{aligned} \tag{3.27}$$

Each of the integrals that occur can be carried out term by term. Let us examine the first integral which is given by

$$\begin{aligned}I_1 = &-\frac{(1+(-1)^l)}{2\pi}\sin\left(\pi\left(\Delta_\psi+\frac{1}{2}\right)\right)\times \\ &\sum_{m=1}^{\infty}\int_0^1\frac{d\bar{z}}{\bar{z}}\int_m^{1/\bar{z}}\frac{dz}{z}z^{\Delta_\psi-\bar{h}-\frac{1}{2}}\bar{z}^{\Delta_\psi-h+\frac{1}{2}}\frac{(-1)^m m}{[(z-m)(m-\bar{z})]^{\Delta_\psi+\frac{1}{2}}}.\end{aligned} \tag{3.28}$$

From the integrand it is easy to see that the poles in the $\Delta_{\mathcal{O}}$ plane arise due to the small $\bar{z}$ regime. Since we are only interested in the residues at these poles, we can take the upper limit of the integral over z to infinity. Considering the 2nd term in (3.28), we have

$$\int_m^{\infty}\frac{z^{\Delta_\psi-\bar{h}-\frac{3}{2}}}{(z-m)^{\Delta_\psi+\frac{1}{2}}}dz = \frac{\Gamma\left(\frac{1}{2}-\Delta_\psi\right)\Gamma(\bar{h}+1)}{\Gamma\left(\bar{h}-\Delta_\psi+\frac{3}{2}\right)}m^{-\bar{h}-1}. \tag{3.29}$$

To perform the integral over $\bar{z}$, we first expand in small $\bar{z}$ and integrate term by term, this leads to

$$(-1)^{m+1}m\int_0^1 d\bar{z}\frac{\bar{z}^{\Delta_\psi-h-\frac{1}{2}}}{(m-\bar{z})^{\Delta_\psi+\frac{1}{2}}} = \sum_{n=0}^{\infty}\frac{\left(\Delta_\psi+\frac{1}{2}\right)_n}{\Gamma(n+1)\left(\Delta_\psi-h+\frac{1}{2}+n\right)}(-1)^{(m+1)}m^{-\Delta_\psi-n+\frac{1}{2}}. \tag{3.30}$$

We can use (3.29), (3.30) to obtain the integral

$$\begin{aligned}I_1 = &\sum_{n=0}^{\infty}\frac{\left(\Delta_\psi+\frac{1}{2}\right)_n\Gamma(\bar{h}+1)\left(1-2^{-\Delta_\psi-\bar{h}+\frac{1}{2}-n}\right)\zeta\left(\Delta_\psi+\frac{1}{2}+\bar{h}+n\right)}{\Gamma\left(\Delta_\psi+\frac{1}{2}\right)\Gamma\left(\bar{h}-\Delta_\psi+\frac{3}{2}\right)\Gamma(n+1)\left(\Delta_\psi-h+\frac{1}{2}+n\right)}, \\ &\text{for} \qquad l=0,2,4,\cdots,\end{aligned} \tag{3.31}$$

where we have summed over m. We can use the same methods to obtain the remaining two integrals in (3.27). The second integral is given by

$$I_2 = \sum_{n=0}^{\infty} \frac{\left(\Delta_\psi + \frac{1}{2}\right)_n \Gamma(\bar{h}) \left(1 - 2^{-\Delta_\psi - \bar{h} + \frac{3}{2} - n}\right) \zeta\left(\Delta_\psi - \frac{1}{2} + \bar{h} + n\right)}{\Gamma\left(\Delta_\psi + \frac{1}{2}\right) \Gamma\left(\bar{h} - \Delta_\psi + \frac{1}{2}\right) \Gamma(n+1) \left(\Delta_\psi - h - \frac{1}{2} + n\right)}, \tag{3.32}$$
$$\text{for} \qquad l = 0, 2, 4, \cdots,$$

while the 3rd integral is given by

$$I_3 = \sum_{n=0}^{\infty} \frac{-2\left(\Delta_\psi + \frac{1}{2}\right)_n \Gamma(\bar{h}) \left(1 - 2^{-\Delta_\psi - \bar{h} + \frac{1}{2} - n}\right) \zeta\left(\Delta_\psi + \frac{1}{2} + \bar{h} + n\right)}{\Gamma\left(\Delta_\psi + \frac{1}{2}\right) \Gamma\left(\bar{h} - \Delta_\psi + \frac{1}{2}\right) \Gamma(n+1) \left(\Delta_\psi - h + \frac{1}{2} + n\right)}, \tag{3.33}$$
$$\text{for} \qquad l = 0, 2, 4, \cdots.$$

We examine the residues at the poles in $\Delta_{\mathcal{O}}$ plane to obtain the one-point functions. Consider the residue for the operator with dimension $\Delta_{\mathcal{O}} = 2\Delta_\psi + l - 1$, these poles occur only in I_2 with $n = 0$. We obtain the residue

$$a_{\mathcal{O}_+}[n=0,l] = \frac{2\left(\Delta_\psi + \frac{1}{2}\right)_l (1 - 2^{-2\Delta_\psi - l + 2})\zeta(2\Delta_\psi + l - 1)}{\Gamma(l)\left(\Delta_\psi + l - \frac{1}{2}\right)}, \qquad l = 2, 4, \cdots. \tag{3.34}$$

Comparing this equation with (3.13), we see that it agrees precisely with the one-point function obtained by the brute force expansion of the thermal two-point function. To do this, we relate the residue to the one-point function using (2.20) and the one-point function in arbitrary d to $d = 2$ using (2.33). Let us now look at the residue at $\Delta_{\mathcal{O}} = 2\Delta_\psi + 2n + 1$, these poles arise in all the terms I_1, I_2 and I_3. Adding up the contribution from these terms and using the OPE expansion in (3.10) and (2.20) to identify the residues we get

$$a_{\mathcal{O}_-}[n, l=0] = -\frac{2\left[\left(\Delta_\psi + \frac{1}{2}\right)_n\right]^2 (1 - 2\Delta_\psi)(1 - 2^{-2(\Delta_\psi + n)})\zeta(2\Delta_\psi + 2n + 1)}{n!(n+1)!}. \tag{3.35}$$

Again this agrees with the one-point function obtained by the small x expansion in (3.12) after using the relation to $d = 2$ in (2.33). Finally let us examine the residues at $\Delta_{\mathcal{O}} = 2\Delta_\psi + 2n + l$ with $l = 2, 4, \cdots$ and $n = 0, 1, 2, \cdots$. Again these poles arise from all terms I_1, I_2, I_3, the contribution from I_2 can be isolated easily once one makes a shift $n \to n+1$ in I_2. Summing these residues and again using (3.10) and (2.20), we obtain the relation,

$$\begin{aligned} &a_{\mathcal{O}_+}[n+1, l] + \frac{2(l+1)}{(l+2)^2} a_{\mathcal{O}_-}[n, l] \\ &= \frac{(2\Delta_\psi - 1)(l + 2n + 2)(1 - 2^{-l - 2(\Delta_\psi + n)})\left(\Delta_\psi + \frac{1}{2}\right)_n \left(\Delta_\psi + \frac{1}{2}\right)_{l+n} \zeta(l + 2n + 2\Delta_\psi + 1)}{\Gamma(n+2)\Gamma(l+n+2)}. \end{aligned} \tag{3.36}$$

We see that this equation coincides precisely with (3.23) once the $d = 2$ limit is taken and we use the relation (2.33).

Let us now consider the correlator $g_3(x)$, we can write its OPE expansion given in (3.20) as,

$$g_3(x) = \sum_{n,l=0}^{\infty} c[n,l]\,|x|^{2n+l} C_l^{(\nu)}(\eta), \tag{3.37}$$

with

$$\begin{aligned} c[n,l=0] &= \left(1+\frac{2n}{d}\right) a_{\mathcal{O}_-}[n,l=0], \\ c[n,l\geq 1] &= \left(2(n+1)a_{\mathcal{O}_+}[n+1,l] + \left(l+1+\frac{4n(l+1)}{(l+2)(l+d)}\right)a_{\mathcal{O}_-}[n,l]\right). \end{aligned} \tag{3.38}$$

Now let us examine the MFT thermal correlator $g_3(x)$ which is given by,

$$g_3(\tau,\vec{x}) = \sum_{m\in\mathbb{Z}} (-1)^m \frac{1-2\Delta_\psi}{[(\tau+m)^2+\vec{x}^2]^{\Delta_\psi+\frac{1}{2}}}. \tag{3.39}$$

This correlator is very similar to the one studied in [1] in MFT for bosonic operators of dimensions Δ_ψ To obtain the two-point function (3.39) from a bosonic MFT correlator, we need to shift $\Delta_\psi \to \Delta_\psi + \frac{1}{2}$, there is also an insertion of $(-1)^m$ with an overall factor $1-2\Delta_\psi$. Taking these changes into account and applying the inversion formula we obtain

$$\begin{aligned} c[n,l] &= 4(d-2\Delta_\psi-1)\zeta(2\Delta_\psi+1+l+2n)\frac{\left(\Delta_\psi+\frac{1}{2}\right)_{n+l}\left(\Delta_\psi+\frac{1}{2}\right)_n}{n!\Gamma(n+l+1)}(2^{-2\Delta_\psi-l-2n}-1), \\ l &= 0,2,4,\cdots \end{aligned} \tag{3.40}$$

We can now use (3.38), to identify the one point functions of interest, for $l=0$, we obtain

$$a_{\mathcal{O}_-}[n,l=0] = \frac{4(1-2\Delta_\psi)\left(2^{-2(\Delta_\psi+n)}-1\right)\left(\left(\Delta_\psi+\frac{1}{2}\right)_n\right)^2\zeta(2n+2\Delta_\psi+1)}{(n+1)\Gamma(n+1)^2}. \tag{3.41}$$

We see that it agrees with the small $|x|$ expansion of the correlator in (3.12) after using the relation to $d=2$ in (2.33). For $l\geq 2$ we obtain the relation

$$\begin{aligned} &2(n+1)a_{\mathcal{O}_+}[n+1,l]+\left(l+1+\frac{4n(l+1)}{(l+2)^2}\right)a_{\mathcal{O}_-}[n,l] \\ &=4(1-2\Delta_\psi)\zeta(2\Delta_\psi+1+l+2n)\frac{\left(\Delta_\psi+\frac{1}{2}\right)_{n+l}\left(\Delta_\psi+\frac{1}{2}\right)_n}{n!\Gamma(n+l+1)}(2^{-2\Delta_\psi-l-2n}-1). \end{aligned} \tag{3.42}$$

Again this equation precisely agrees with the second linear equation (3.23) between one point functions of $\mathcal{O}_-$, $\mathcal{O}_+$ obtained using the brute force expansion in small $|x|$, when $d=2$ and after using the relation (2.33).

3.2 OPE inversion in $d > 2$

The inversion formula for $d > 2$ dimensions is given in (2.27).

$$\hat{a}(\Delta, l) = \hat{a}_{\rm disc}(\Delta, l) + \theta(l_0 - l)\hat{a}_{\rm arcs}(\Delta - l). \tag{3.43}$$

The correlator of interest is the mean field theory correlator $g_2(x)$ given in (3.24), the form of the MFT correlator is invariant across dimensions once we choose the kinematics as discussed around (2.23). The contribution from the $\hat{a}_{\rm arcs}$ at infinity is given by (2.29)

$$\hat{a}_{\rm arcs}(\Delta, l) = 2K_l \int_0^1 \frac{dr}{r^{\Delta+1-2\Delta_\psi}} \times \oint \frac{dw}{iw} \lim_{|w|\to\infty} \left[(w - w^{-1})^{2\nu} F_l(w^{-1}) e^{i\pi\nu} g(r, w)\right]. \tag{3.44}$$

Using the definition of $F_l(w)$ given in (2.26), we see that the contribution reduces to

$$\hat{a}_{\rm arcs}(\Delta, l) = 2K_l \int_0^1 \frac{dr}{r^{\Delta+1-2\Delta_\psi}} \times \oint \frac{dw}{iw} \lim_{|w|\to\infty} w^{-l} g_2(r, w). \tag{3.45}$$

From the expression of the 2-point function for g_2 given in (3.24), it vanishes as $|w|^{-\Delta+\frac{1}{2}}$ for large $|w|$. Therefore again as in the case of $d = 2$, if $\Delta_\psi \geq \frac{1}{2}$, we see that there is no contribution from the arcs at infinity and for $0 < \Delta_\psi < \frac{1}{2}$, we just need to include the arc contribution for the $l = 0$ case. As in $d = 2$ dimensions, we will take $\Delta_\psi \geq \frac{1}{2}$. We are thus led to evaluating only the contribution from the discontinuity across the branch cuts, which is given by

$$\hat{a}_{\rm disc}(\Delta, l) = (1 + (-1)^l) K_l \int_0^1 \frac{d\bar{z}}{\bar{z}} \int_1^{\frac{1}{\bar{z}}} \frac{dz}{z} (z\bar{z})^{\Delta_\psi - \frac{\Delta}{2} - \nu} F_l\left(\sqrt{\frac{\bar{z}}{z}}\right) \mathrm{Disc}[g(z, \bar{z})], \tag{3.46}$$
$$K_l = \frac{\Gamma(l+1)\Gamma(\nu)}{4\pi\Gamma(l+\nu)}.$$

The discontinuity across the branch cuts is given in (3.26). Substituting this, we obtain the following expression for the one point function

$$\begin{aligned}\hat{a}(\Delta, l) = (1 + (-1)^l) K_l \sum_{m=1}^{\infty} \int_0^1 \frac{d\bar{z}}{\bar{z}} \int_1^{\max(m, 1/\bar{z})} \frac{dz}{z} (z\bar{z})^{\Delta_\psi - \frac{\Delta}{2} - \nu} (z - \bar{z})^{2\nu} F_J\left(\sqrt{\frac{\bar{z}}{z}}\right) \\ \times \frac{(-1)^m 2 \sin\left(\pi\left(\Delta_\psi + \frac{1}{2}\right)\right)}{[(z - m)(m - \bar{z})]^{\Delta_\psi + \frac{1}{2}}} \left(-\frac{m}{2}\sqrt{\frac{z}{\bar{z}}} - \frac{m}{2}\sqrt{\frac{\bar{z}}{z}} + \sqrt{z\bar{z}}\right), \\ \equiv I_1 + I_2 + I_3.\end{aligned} \tag{3.47}$$

The last line defines the three integrals which must be done to obtain $\hat{a}(\Delta, l)$. Let us examine the first integral

$$\begin{aligned}I_1 = 2K_l \sin\left(\pi\left(\Delta_\psi + \frac{1}{2}\right)\right) \\ \times \int_0^1 \frac{d\bar{z}}{\bar{z}} \int_m^{\max\left(m, \frac{1}{\bar{z}}\right)} \frac{dz}{z} \frac{(-1)^m (-m) F_J\left(\sqrt{\frac{\bar{z}}{z}}\right) (z - \bar{z})^{2\nu} (z\bar{z})^{-\frac{\Delta}{2} + \Delta_\psi - \nu}}{(z - m)^{\Delta_\psi + \frac{1}{2}} (m - \bar{z})^{\Delta_\psi + \frac{1}{2}}} \sqrt{\frac{z}{\bar{z}}}, \\ l = 0, 2, \cdots.\end{aligned} \tag{3.48}$$

It is convenient to define new variables

$$\bar{z} = mz'\bar{z}', \qquad z = mz'. \tag{3.49}$$

Using the $(z', \bar{z}')$ variables we obtain the integral

$$\begin{aligned} I_1(m) = 2m^{-\Delta}(-1)^{m+1}K_l \sin\left(\pi\left(\Delta_\psi + \frac{1}{2}\right)\right) \int_0^1 \frac{d\bar{z}}{\bar{z}} \int_1^{\max\left(\frac{1}{m\sqrt{\bar{z}}},1\right)} \frac{dz}{z} \Bigg[\\ (z-1)^{-\Delta_\psi-\frac{1}{2}}(1-z\bar{z})^{-\Delta_\psi-\frac{1}{2}}(z-z\bar{z})^{2\nu}\left(z^2\bar{z}\right)^{-\frac{\Delta}{2}+\Delta_\psi-\nu}\bar{z}^{-\frac{1}{2}}F_J\left(\sqrt{\bar{z}}\right)\Bigg]. \end{aligned} \tag{3.50}$$

Here we have re-named the primed variables and isolated the m-th term in the sum. As in the case of $d = 2$ and in [1], we expect the poles to arise at $\bar{z} \to 0$ in the integrand. Therefore, we can take the upper limit of the z integration to ∞ and the integrals over z and $\bar{z}$ factorize. Performing the integral over z from 1 to ∞ we obtain

$$\begin{aligned} I_1(m) = & \frac{2\pi K_l m^{-\Delta}(-1)^{m+1}\Gamma(\Delta+1)}{\Gamma\left(\frac{1}{2}+\Delta_\psi\right)\Gamma\left(\Delta-\Delta_\psi+\frac{3}{2}\right)} \int_0^1 d\bar{z}\sqrt{\frac{1}{\bar{z}}}(-\bar{z})^{-\Delta_\psi-\frac{1}{2}}(1-\bar{z})^{2\nu}\bar{z}^{-\frac{\Delta}{2}+\Delta_\psi-\nu-1} \\ & \times {}_2F_1\left(\Delta+1, \Delta_\psi+\frac{1}{2}; \Delta-\Delta_\psi+\frac{3}{2}; \frac{1}{\bar{z}}\right) F_J\left(\sqrt{\bar{z}}\right). \end{aligned} \tag{3.51}$$

We can now expand in small $\bar{z}$ and perform the integrals term by term. The leading and the sub-leading terms are given by

$$\begin{aligned} I_1(m) = & \frac{2\pi K_l m^{-\Delta}(-1)^{m+1}\Gamma\left(\Delta-\Delta_\psi+\frac{1}{2}\right)}{\Gamma\left(\frac{1}{2}+\Delta_\psi\right)\Gamma(1+\Delta-2\Delta_\psi)} \int_0^1 d\bar{z}\bar{z}^{\left(-\frac{\Delta}{2}+\frac{l}{2}+\Delta_\psi-\frac{3}{2}\right)} \\ & \times \left[1+\bar{z}\left(\frac{\nu(l+2\nu)}{l+\nu+1}+\frac{(\Delta-2\Delta_\psi)(1+2\Delta_\psi)}{2\Delta-2\Delta_\psi-1}\right)+O(\bar{z}^2)\right]. \end{aligned} \tag{3.52}$$

Performing the sum over m and the integral over $\bar{z}$, we obtain

$$\begin{aligned} I_1 = & \frac{\Gamma(\nu)\Gamma(l+1)\Gamma\left(\Delta-\Delta_\psi+\frac{1}{2}\right)(1-2^{1-\Delta})\zeta(\Delta)}{\Gamma(l+\nu)\Gamma\left(\frac{1}{2}+\Delta_\psi\right)\Gamma(1+\Delta-2\Delta_\psi)} \\ & \times\left[\frac{1}{(-\Delta+l+2\Delta_\psi-1)}+\left(\frac{\nu(l+2\nu)}{l+\nu+1}+\frac{(\Delta-2\Delta_\psi)(1+2\Delta_\psi)}{2\Delta-2\Delta_\psi-1}-2\nu\right)\frac{1}{(-\Delta+l+2\Delta_\psi+1)}\right] \\ & +\cdots. \end{aligned} \tag{3.53}$$

Let us evaluate the residue at the pole $\Delta = 2\Delta_\psi + l - 1$, from the general OPE expansion in (3.10) we see residue results in following one point function

$$a_{\mathcal{O}_+}[n=0,l] = -\frac{l\left(\Delta_\psi+\frac{1}{2}\right)_l(1-2^{-\Delta_\psi-l+2})\zeta(2\Delta_\psi+l-1)}{(\nu)_l\left(\Delta_\psi+l-\frac{1}{2}\right)}. \tag{3.54}$$

This result precisely coincides with the one point function obtained in (3.13) by the directly expanding the correlator $g_2(x)$ in the small x expansion.

We can proceed on similar lines and evaluate the leading contribution of I_2 in the small $\bar{z}$ expansion of the integrand. This allows us to obtain the residues at $\Delta = 2\Delta_\psi + l + 1$. These are given by

$$I_2 = \frac{\Gamma(\nu)\Gamma(l+1)\Gamma\left(\Delta - \Delta_\psi + \frac{1}{2}\right)(1-2^{1-\Delta})\zeta(\Delta)}{\Gamma(l+\nu)\Gamma\left(\frac{1}{2}+\Delta_\psi\right)\Gamma(1+\Delta-2\Delta_\psi)(-\Delta+l+2\Delta_\psi+1)} + \cdots, \tag{3.55}$$

and

$$I_3 = -2\frac{\Gamma(\nu)\Gamma(l+1)\Gamma\left(\Delta - \Delta_\psi - \frac{1}{2}\right)(1-2^{1-\Delta})\zeta(\Delta)}{\Gamma(l+\nu)\Gamma\left(\frac{1}{2}+\Delta_\psi\right)\Gamma(\Delta-2\Delta_\psi)(-\Delta+l+2\Delta_\psi+1)} + \cdots. \tag{3.56}$$

From the expression of I_1 in (3.53) we see that too contains a pole at $\Delta = 2\Delta_\psi + l + 1$. Adding up the residue at $\Delta = 2\Delta_\psi + 1, l = 0$ allows us to obtain the one point function

$$a_{\mathcal{O}_-}[n=0, l=0] = 2(1-2^{-2\Delta_\psi})(2\Delta_\psi - 2\nu - 1)\zeta(2\Delta_\psi + 1). \tag{3.57}$$

Here we have related the residue to the one point function using (3.10). We see that the result is in precise agreement with (3.12) which is obtained by the brute force expansion of the correlator $g_2(x)$. Finally for $l > 0$ we can combine the all the residues at $\Delta = 2\Delta_\psi + l + 1$ from I_1, I_2, I_3 from (3.53), (3.55), (3.56) to obtain the equation

$$\begin{aligned} a_{O_+}[1, l] + \frac{2(l+1)}{(l+2)(d+l)} a_{\mathcal{O}_-}[0, l] = \\ \frac{(l+2)\left(\Delta_\psi + \frac{1}{2}\right)_l \left(\Delta_\psi - \nu - \frac{1}{2}\right)(1-2^{-\Delta_\psi - l})\zeta(2\Delta_\psi + l + 1)}{(\nu)_l (l+\nu+1)}. \end{aligned} \tag{3.58}$$

Again we have used the OPE expansion (3.10) to identify the linear combination of the one-point functions with the residue. The result precisely coincides with the equation (3.23) obtained by the brute force expansion of $g(x)$ in small $|x|$.

Let us finally examine the correlator $g_3(x)$ given in (3.18). As we have discussed for the case of $d = 2$, this two-point function is similar to the two-point function of scalars in MFT studied in [1] but with a $(-1)^m$ inserted in the sum over images. We also need to replace $\Delta_\psi \to \Delta_\psi + \frac{1}{2}$ and multiply by the overall factor $2\nu + 1 - 2\Delta_\psi$. Consider the expansion of $g_3(x)$ given in (3.37), using the inversion formula given in [1] together with the modifications mentioned above, we obtain the residue at $\Delta = 2\Delta_\psi + 1 + l$,[5]

$$c(0, l) = 2\frac{\left(\Delta_\psi + \frac{1}{2}\right)_l}{(\nu)_l}(2\Delta_\psi - 2\nu - 1)(1-2^{-(2\Delta_\psi + l)})\zeta(2\Delta_\psi + 1 + l). \tag{3.59}$$

We can now use the equations (3.38) to identify the one point functions. We obtain for $l = 0$,

$$c(0,0) = a_{\mathcal{O}_-}[n=0, l=0] = 2(1-2^{-2\Delta_\psi})(2\Delta_\psi - 2\nu - 1)\zeta(2\Delta_\psi + 1). \tag{3.60}$$

[5]This is equation (4.19) of [1].

For $l = 2, 4, \cdots$. we get

$$\begin{aligned} c(0, l \geq 2) &= 2a_{\mathcal{O}_+}[1, l] + (l+1)a_{\mathcal{O}_-}[0, l] \\ &= 2\frac{\left(\Delta_\psi + \frac{1}{2}\right)_l}{(\nu)_l}(2\Delta_\psi - 2\nu - 1)(1 - 2^{-(2\Delta_\psi + l)})\zeta(2\Delta_\psi + 1 + l). \end{aligned} \tag{3.61}$$

Note that this equation precisely coincides with the linear equation relating these one point functions given in (3.22) obtained by the small x expansion of the correlator $g_3(x)$.

This concludes the discussion of using the inversion formula for the MFT of fermions. It is important to mention that the application of the inversion formula on the correlator $g_2(x)$ is not related to the one studied for the scalars in [1]. The MFT correlator $g_2(x)$ in (3.24) has factors which involve $\sqrt{z}, \sqrt{\bar{z}}$, inspite of this, the location of branch cuts in the complex w-plane falls into the general discussion of [1]. Therefore we could apply the inversion formula leading to results which agree with the brute force small x expansion of the correlator $g_2(x)$.

4 The critical Gross-Neveu model at large N

In this section we study thermal one-point functions in the critical U(N) Gross-Neveu model at large N in arbitrary odd $d = 2k + 1$ dimensions. The theory is defined using the action

$$S = \int d^d x \left[i\bar{\psi}_a \gamma^\mu \partial_\mu \psi_a + \frac{\lambda}{N}(\bar{\psi}_a \psi_a)^2 \right], \tag{4.1}$$

where N is the number of fermions and $a = 1, 2, \cdots, N$. For $d > 3$, the interaction is non-renormalizable, nevertheless in [2, 16, 17], it has been argued that on choosing a definite prescription to evaluate the partition function one is led to a gap equation which is independent of the cutoff at large N.[6] The derivation of the gap equation is given in the appendix A, the equation is given by

$$2(2m_{\rm th})^k \sum_{n=0}^{k-1} \frac{(k-n)_{2n}}{(2m_{\rm th})^n n!} \mathrm{Li}_{k+n}(-e^{-m_{\rm th}}) + \frac{(m_{\rm th})^{2k}\sqrt{\pi}}{\Gamma\left(\frac{1}{2}+k\right)\cos \pi k} = 0. \tag{4.2}$$

The gap equation has a real solution for the thermal mass $m_{\rm th}$ in dimensions $d = 2k + 1$, with $k = 2, 4, 6, \cdots$. While for $k = 1, 3, 5, \cdots$, the gap equation has complex solutions for $m_{\rm th}$. The table 1 lists the thermal masses of the Gross-Neveu model in various dimensions.

In [1], it was observed that for the bosonic $O(N)$ model in $d = 3$ the corresponding gap equation can be obtained by demanding the scalar bilinear $\phi^a \phi_a$ with dimension $\Delta = 1$ does not exist in the spectrum at the critical point. This observation was also seen to hold true for the $O(N)$ model in arbitrary odd dimensions in [2]. This paper also studied the application of the inversion formula for the correlator $g_1(x)$ in Gross-Neveu model in arbitrary odd dimensions. Similar to the bosonic case, it was observed that the gap equation of the Gross-Neveu model in (4.2) can be obtained by demanding that the operator $\mathcal{O}_0[0, 0]$ or schematically the bilinear $\bar{\psi}\psi$ with dimensions $\Delta = 2k$ does not exist in the spectrum at the critical point.

[6] A similar procedure has been followed to obtain the gap equation and the thermal mass for the bosonic $O(N)$ model in arbitrary d dimensions [2, 18].

In this section, we study the correlator $g_2(x)$ in detail and also examine the correlator $g_1(x)$ and $g_3(x)$. We see that the absence of the operator $\mathcal{O}_-[0,0]$ which is schematically of the form $\bar{\psi}\gamma^\mu\partial_\mu\psi$ in the spectrum also leads to the identical gap equation in (4.2). We then evaluate the one-point functions of operators $\mathcal{O}_+[0,l]$ or operators of the form

$$\mathcal{O}_+[0,l]: \quad \bar{\psi}\gamma_{\mu_1}\partial_{\mu_2}\cdots\partial_{\mu_l}\psi, \qquad \Delta = 2k+l-1. \tag{4.3}$$

We show that the one-point functions of these operators are given by

$$a_{\mathcal{O}_+}[0,l] = \frac{l\,(m_{\rm th})^{l+k-1}}{\pi^k 2^{l+k}\left(k-\frac{1}{2}\right)_l}\sum_{n=0}^{l+k-1}\frac{(k+l-n)_{2n}}{2^n n!(m_{\rm th})^n}\mathrm{Li}_{k+n}(-e^{-m_{\rm th}}), \tag{4.4}$$
$$l = 2,4,\cdots.$$

The one-point function $a_{\mathcal{O}_+}[0,l=2]$ corresponds to the stress tensor. In the appendix A, we have evaluated the stress tensor directly from the partition function. This is given by

$$T_{00} = \frac{(m_{\rm th})^{2k+1}}{2^{k+2}\pi^{k-\frac{1}{2}}\Gamma\left(k+\frac{3}{2}\right)\cos\pi k} \tag{4.5}$$
$$+\frac{(m_{\rm th})^{k+1}}{\pi^k}\sum_{n=0}^{k+1}\frac{[(k+n)^2+(k-n)](k-n+2)_{2n-2}}{2^n n!(m_{\rm th})^n}\mathrm{Li}_{k+n}(-e^{-m_{\rm th}}).$$

Manifestly the one-point function $a_{\mathcal{O}_+}[0,l=2]$ does not seem to agree with the stress tensor in (4.5), however on substituting the value of $m_{\rm th}$ from the gap equation (4.2) and scaling by the overall dimension dependent constant they precisely coincide. The overall scaling is because the one-point function $a_{\mathcal{O}_+}[0,l=2]$ also contains the structure constant $f_{\psi^\dagger\psi T}$ and the normalisation of the two point function c_T. The table 1 also compares the values of $a_{\mathcal{O}_+}[0,l=2]$ with the stress tensor.

Our analysis also shows that the one-point functions in (4.4) are related to that of one-point functions of $\mathcal{O}_0[0,l]$ or operators of the form

$$\mathcal{O}_0[0,l]: \quad \bar{\psi}\partial_{\mu_1}\partial_{\mu_2}\cdots\partial_{\mu_l}\psi \qquad \Delta_{\mathcal{O}} = 2k+l. \tag{4.6}$$

The relation is given by

$$a_{\mathcal{O}_0}[0,l] = m_{th}a_{\mathcal{O}_+}[0,l]. \tag{4.7}$$

Similarly evaluating the one-point functions $a_{\mathcal{O}_+}[1,l]$ and $a_{\mathcal{O}_-}[0,l]$, we see that they are related to the one-point functions $a_{\mathcal{O}_0}[0,l]$ by factors which depend on l and $m_{\rm th}$. Thus by explicit calculation, we see that the one-point functions of all fermion bi-linears are related to $a_{\mathcal{O}_0}[0,l]$ which occurs in the OPE expansion of $g_1(x)$. It is interesting to contrast this with MFT, in which $g_1(x)$ trivially vanished and did not contain any one-point functions.

4.1 OPE inversion on $g_2(x)$

The two-point function of fermions with a thermal mass $m_{\rm th}$ and at finite temperature is given by

$$\langle\psi_\alpha(x)\psi^\dagger_\beta(0)\rangle = \frac{i}{2^{\frac{d-1}{2}}}\sum_{n,\,k_0=2\pi\left(n+\frac{1}{2}\right)}\int\frac{d^{d-1}k}{(2\pi)^{d-1}}\frac{\gamma^\mu_{\alpha\beta}k_\mu - im_{th}\delta_{\alpha\beta}}{k^2+m^2_{th}}e^{ikx}. \tag{4.8}$$

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d	$m_{th}(\in \mathbb{R})$	$a_{\mathcal{O}_+}[0,l]$	T_{00}	$a_T = 2^k k(2k-1) a_{\mathcal{O}_+}[0,l]$
5	1.48051	-0.04048	-0.971519	-0.971519
9	2.86003	-0.0125136	-5.6061	-5.6061
13	4.24178	-0.0159513	-67.3783	-67.3783
17	5.6273	-0.0460522	-1414.72	-1414.72
21	7.01451	-0.23691	-46093.1	-46093.1
25	8.40257	-1.90627	-2.15503×10^6	-2.15503×10^6
29	9.7911	-22.0996	-1.36866×10^8	-1.36866×10^8
33	11.1799	-348.833	-1.13391×10^{10}	-1.13391×10^{10}
37	12.569	-7193.26	-1.18797×10^{12}	-1.18797×10^{12}
41	13.9581	-187758.14	-1.53565×10^{14}	-1.53565×10^{14}
45	15.3474	-6.05104×10^6	-2.40094×10^{16}	-2.40094×10^{16}
49	16.7368	-2.35981×10^8	-4.46587×10^{18}	-4.46587×10^{18}
53	18.1262	-1.09528×10^{10}	-9.74651×10^{20}	-9.74651×10^{20}
57	19.5156	-5.96646×10^{11}	-2.46648×10^{23}	-2.46648×10^{23}
61	20.9051	-3.76957×10^{13}	-7.16416×10^{25}	-7.16416×10^{25}
65	22.2946	-2.73404×10^{15}	-2.36731×10^{28}	-2.36731×10^{28}
69	23.6842	-2.25624×10^{17}	-8.82995×10^{30}	-8.82995×10^{30}
73	25.0737	-2.10199×10^{19}	-3.69208×10^{33}	-3.69208×10^{33}
77	26.4633	-2.19548×10^{21}	-1.71994×10^{36}	-1.71994×10^{36}
81	27.8529	-2.55503×10^{23}	-8.87735×10^{38}	-8.87735×10^{38}

Table 1. The table shows the agreement in the value of stress tensor evaluated from the partition function given in (4.5) with the one-point function from OPE inversion formula given in (4.4) at $l = 2$. We need the 2^k, the dimension of the Dirac spinor in the last column, since we had factored this out in the two point function (4.8), the rest of the factors in the last column are because of the presence of the structure constant $f_{\psi^\dagger\psi\mathcal{O}}$ and the normalization of the 2-pt functions $c_{\mathcal{O}}$ in $a_{\mathcal{O}_+}[0,2]$.

Note that this correlator is anti-periodic under the shift $\tau \to \tau + 1$, again we have divided by the dimension of the Dirac spinor so that traces over γ matrices gives unity. It is useful for us to write down the correlator with $\bar{\psi}$ and ψ inter-changed for the constructions of $g_2(x)$ and $g_3(x)$. From (4.8) we obtain

$$\langle \psi^\dagger_\beta(x)\psi_\alpha(0)\rangle = \frac{i}{2^{\frac{d-1}{2}}} \sum_{n,\, k_0 = 2\pi(n+\frac{1}{2})} \int \frac{d^{d-1}k}{(2\pi)^{d-1}} \frac{\gamma^\mu_{\alpha\beta} k_\mu + i m_{th}\delta_{\alpha\beta}}{k^2 + m_{th}^2} e^{ikx}. \tag{4.9}$$

To derive (4.9) from (4.8), we inter-change the fermions, change the dummy variables of integrations and summations and also use translation invariance. Using this we can write down the correlator $g_2(x)$ using the definition in (2.2)

$$\begin{aligned} g_2(\tau, \vec{x}) &= \frac{i}{|x|} \sum_{n,\, k_0 = 2\pi(n+\frac{1}{2})} \int \frac{d^{d-1}k}{(2\pi)^{d-1}} \frac{k_\mu x^\mu}{k^2 + m_{th}^2} e^{ikx}, \\ &= \frac{x^\mu \partial_\mu}{|x|} \sum_{n,\, k_0 = 2\pi(n+\frac{1}{2})} \int \frac{d^{d-1}k}{(2\pi)^{d-1}} \frac{e^{ikx}}{k^2 + m_{th}^2}. \end{aligned} \tag{4.10}$$

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To perform the integral, we first use the Poisson re-summation formula to convert the sum over Matsubara frequencies to sum over images in τ.

$$\sum_{n\in\mathbb{Z}} f\left[(2n+1)\pi\right] = \sum_{n\in\mathbb{Z}} (-1)^n \int_{-\infty}^{\infty} \frac{d\omega}{2\pi} f(\omega) e^{in\omega}. \tag{4.11}$$

Applying the re-summation on the integral in (4.10) and performing the resultant integral, we obtain

$$\begin{aligned}\sum_{n,\, k_0=2\pi\left(n+\frac{1}{2}\right)} \int \frac{d^{d-1}k}{(2\pi)^{d-1}} \frac{e^{i\vec{k}.\vec{x}} e^{ik_0\tau}}{k_0^2+\vec{k}^2+m_{\rm th}^2} &= \sum_{n\in\mathbb{Z}} (-1)^n \int \frac{d^{d-1}k d\omega}{(2\pi)^d} \frac{e^{i\vec{k}\cdot\vec{x}} e^{i\omega(\tau+n)}}{\omega^2+\vec{k}^2+m_{\rm th}^2}, \\ &= \sum_{n\in\mathbb{Z}} (-1)^n (2\pi)^{-\frac{d}{2}} \left(\frac{|x^{(n)}|}{m_{\rm th}}\right)^{1-\frac{d}{2}} K_{\frac{d}{2}-1}(m_{\rm th}|x^{(n)}|).\end{aligned} \tag{4.12}$$

Here $x^{(n)} = (\tau+n, \vec{x})$, choose the configuration given in (2.23) we can express the correlator as a function of $(z, \bar{z})$

$$g_2(z,\bar{z}) = \sum_{m\in\mathbb{Z}} (-1)^{m+1} \left(\frac{m_{\rm th}}{2\pi}\right)^{\frac{d}{2}} \left[-\frac{m}{2}\sqrt{\frac{z}{\bar{z}}} - \frac{m}{2}\sqrt{\frac{\bar{z}}{z}} + \sqrt{z\bar{z}}\right] \frac{K_{\frac{d}{2}}\left(m_{\rm th}\sqrt{(m-z)(m-\bar{z})}\right)}{\left(\sqrt{(m-z)(m-\bar{z})}\right)^{d/2}}. \tag{4.13}$$

One consistency check of this correlator is the following, on taking the $m_{\rm th} \to 0$ limit it is proportional to the MFT correlator in (3.2) or (3.24) with $\Delta_\psi = k$

$$g_2(z,\bar{z})|_{\text{Gross−Neveu},\, m_{\rm th}\to 0} = -\frac{\Gamma\left(k+\frac{1}{2}\right)}{2\pi^{k+\frac{1}{2}}} g_2(z,\bar{z})|_{\text{MFT},\, \Delta_\psi=k}. \tag{4.14}$$

On comparing the two-point function in (4.13), with the corresponding one for the bosonic $O(N)$ model studied in [1], we have an insertion of $(-1)^m$ since we are dealing with fermions. We also have the factor in the square brackets in addition to the Bessel function. Examining this factor in the w-plane where w is defined as (2.23), we see that this factor does not affect the branch cut structure present in the Bessel function together with the factor $\left(\sqrt{(m-z)(m-\bar{z})}\right)^{-d/2}$. Therefore, the branch cuts in the w-plane are as assumed in section 2.2 and we can proceed to apply the inversion formula. Let us first write down the contribution from the discontinuities

$$\hat{a}_{\rm disc}(\Delta, l) = 2K_l \int_0^1 \frac{d\bar{z}}{\bar{z}} \int_1^{\frac{1}{\bar{z}}} \frac{dz}{z} (z-\bar{z})^{2\nu} (z\bar{z})^{\frac{1}{2}-\frac{\Delta}{2}} F_l\left(\sqrt{\frac{\bar{z}}{z}}\right) \text{Disc}[g_2(z,\bar{z})]. \tag{4.15}$$

Here we have substituted $\Delta_\psi = \frac{d-1}{2}$ and $l \in 2\mathbb{Z}$. At this point we note that we have taken some input from the perturbative results of [8, 19], that the fundamental field does not acquire anomalous dimensions at large N for the Gross-Neveu model. It should also be noted that the branch cut for each term in (4.13) depends on m and the integration range

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in z depends on m for each term. As we have mentioned earlier, the branch cut in the w-plane can be obtained from the relation

$$\text{Disc}\left[\frac{K_{k+\frac{1}{2}}\left(\sqrt{-x}\right)}{(-x)^{\frac{k}{2}+\frac{1}{4}}}\right]=\pi\frac{J_{-\frac{1}{2}-k}(\sqrt{x})}{x^{\frac{k}{2}+\frac{1}{4}}}\theta(x). \tag{4.16}$$

Substituting this relation in (4.15) we obtain

$$\begin{aligned}\hat{a}_{\text{disc}}(\Delta,l)=&2\pi K_l\left(\frac{m_{\text{th}}}{2\pi}\right)^{\frac{d}{2}}\sum_{m=1}^{\infty}\int_0^1\frac{d\bar{z}}{\bar{z}}\int_1^{\max\left(m,\frac{1}{\bar{z}}\right)}\frac{dz}{z}(z-\bar{z})^{2\nu}(z\bar{z})^{\frac{1}{2}-\frac{\Delta}{2}}F_l\left(\sqrt{\frac{\bar{z}}{z}}\right)\\&\times\frac{(-1)^{m+1}J_{-\frac{1}{2}-k}\left(m_{\text{th}}\sqrt{(z-m)(m-\bar{z})}\right)}{\left[(z-m)(m-\bar{z})\right]^{\frac{k}{2}+\frac{1}{4}}}\left[-\frac{m}{2}\sqrt{\frac{z}{\bar{z}}}-\frac{m}{2}\sqrt{\frac{\bar{z}}{z}}+\sqrt{z\bar{z}}\right].\\\equiv&I_1+I_2+I_3.\end{aligned}\tag{4.17}$$

Just as before the last line defines the 3 integrals required to obtain $\hat{a}(\Delta,l)$. Consider the m-th term of the first integral,

$$\begin{aligned}I_1(m)=\pi m(-1)^m K_l\left(\frac{m_{\text{th}}}{2\pi}\right)^{\frac{d}{2}}\int_0^1\frac{d\bar{z}}{\bar{z}}\int_1^{\max\left(m,\frac{1}{\bar{z}}\right)}\frac{dz}{z}\Bigg[&(z-\bar{z})^{2\nu}(z\bar{z})^{\frac{1}{2}-\frac{\Delta}{2}}F_l\left(\sqrt{\frac{\bar{z}}{z}}\right)\sqrt{\frac{z}{\bar{z}}}\\&\times\frac{J_{-\frac{1}{2}-k}\left(m_{\text{th}}\sqrt{(z-m)(m-\bar{z})}\right)}{\left[(z-m)(m-\bar{z})\right]^{\frac{k}{2}+\frac{1}{4}}}\Bigg].\end{aligned}\tag{4.18}$$

We follow the same procedure as in the application of the inversion formula for the MFT to find the leading poles. We first change variables as in (3.49), this leads to

$$\begin{aligned}I_1(m)=\pi m^{k+\frac{1}{2}-\Delta}(-1)^m\left(\frac{m_{\text{th}}}{2\pi}\right)^{\frac{d}{2}}K_l\int_0^1 d\bar{z}\int_1^{\max\left(\frac{1}{m\sqrt{\bar{z}}},1\right)}dz\,\Big[&\bar{z}^{-1-\frac{\Delta}{2}}z^{2k-1-\Delta}(z-1)^{-\frac{k}{2}-\frac{1}{4}}\\\times(1-\bar{z})^{2k-1}(1-z\bar{z})^{-\frac{k}{2}-\frac{1}{4}}F_l\left(\sqrt{\bar{z}}\right)J_{-\frac{1}{2}-k}&\left(m_{\text{th}}m\sqrt{(z-1)(1-z\bar{z})}\right)\Big].\end{aligned}\tag{4.19}$$

Now we can expand in small $\bar{z}$, this decouples the integrals and then we perform the integral term by term in $\bar{z}$. The leading pole is given by

$$\begin{aligned}I_1(m)^{(0)}=&4\pi m^{k+\frac{1}{2}-\Delta}(-1)^m\left(\frac{m_{\text{th}}}{2\pi}\right)^{k+\frac{1}{2}}K_l\frac{1}{-\Delta+l+2k-1}\\&\times\int_0^{\infty}y^{-k+\frac{1}{2}}(1+y^2)^{2k-1-\Delta}J_{-\frac{1}{2}-k}(m_{\text{th}}m\,y).\end{aligned}\tag{4.20}$$

To obtain the above equation, we have also made a change of variables to $y=\sqrt{z-1}$. The superscript denotes the fact that we are focussing on the leading term in the small $\bar{z}$ expansion. The integral over y is known and we obtain

$$I_1^{(0)}=\sum_{m=1}^{\infty}\frac{2(-1)^m\pi^{\frac{1}{2}-k}2^{-\Delta+k+\frac{1}{2}}m^{\frac{1}{2}-k}m_{\text{th}}^{\Delta-k+\frac{1}{2}}K_l}{\Gamma(1+\Delta-2k)(-\Delta+l+2k-1)}K_{\Delta-k+\frac{1}{2}}(m_{\text{th}}m). \tag{4.21}$$

We also need the first sub-leading term in the small $\bar{z}$ expansion. After performing the $\bar{z}$ integral, the first sub-leading contribution is given by

$$I_1^{(1)}(m) = 4\pi m^{k+\frac{1}{2}-\Delta}(-1)^m\left(\frac{m_{\rm th}}{2\pi}\right)^{k+\frac{1}{2}} K_l \frac{1}{-\Delta+l+2k+1}\times\mathcal{I}, \tag{4.22}$$
$$\mathcal{I} = \int_0^\infty dy y^{-k+\frac{1}{2}}(1+y^2)^{2k-1-\Delta}\left[\left(-\frac{(2k-1)(2+l)}{1+2k+2l}+\left(k+\frac{1}{2}\right)(1+y^2)\right)J_{-\frac{1}{2}-k}(y) + \frac{m m_{\rm th}}{2}y(1+y^2)J_{\frac{1}{2}-k}(y)\right].$$

Integrating over z we obtain

$$I_1^{(1)}(m) = \frac{2(-1)^m\pi^{\frac{1}{2}-k}2^{-\Delta+k+\frac{1}{2}}m^{\frac{1}{2}-k}m_{\rm th}^{\Delta-k+\frac{1}{2}}K_l}{(-\Delta+l+2k+1)}\left[-\frac{(2k-1)(2+l)}{1+2k+2l}\frac{K_{\frac{1}{2}+\Delta-k}(m m_{\rm th})}{\Gamma(1+\Delta-2k)} + (2k+1)(m m_{\rm th})^{-1}\frac{K_{-\frac{1}{2}+\Delta-k}(m m_{\rm th})}{\Gamma(\Delta-2k)} + \frac{K_{-\frac{3}{2}+\Delta-k}(m m_{\rm th})}{\Gamma(\Delta-2k)}\right]. \tag{4.23}$$

Let us evaluate the residue at the pole $\Delta = 2k+l-1$, using the OPE expansion in (3.10) we can identify the one-point functions for the following operators

$$a_{\mathcal{O}_+}[n=0,l]|_{\rm disc} = \sum_{m=1}^{\infty}\frac{2(-1)^m m^{\frac{1}{2}-k}\pi^{\frac{1}{2}-k}K_l m_{\rm th}^{l+k-\frac{1}{2}}}{2^{l+k-\frac{3}{2}}\Gamma(l)}K_{l+k-\frac{1}{2}}(m m_{\rm th}). \tag{4.24}$$

To perform the sum over m, we use the following property of the Bessel function with half integer orders.

$$K_{l+\frac{1}{2}}(x) = e^{-x}\sum_{n=0}^{l}\frac{\sqrt{\pi}(l+1-n)_{2n}}{(2x)^{n+\frac{1}{2}}n!}, \qquad l\in\mathbb{Z}. \tag{4.25}$$

Substituting this identity in (4.24) and performing the sum over m, we obtain

$$a_{\mathcal{O}_+}[n=0,l] = \frac{l}{2\pi^k\left(k-\frac{1}{2}\right)_l}\left(\frac{m_{\rm th}}{2}\right)^{l+k-1}\sum_{n=0}^{l+k-1}\frac{(l+k-n)_{2n}}{(2m_{\rm th})^n n!}{\rm Li}_{k+n}(-e^{-m_{\rm th}}), \tag{4.26}$$

where $l = 2,4,,\cdots$. We will show subsequently that the contribution from the arcs at infinity vanishes for $l>0$, Therefore we have identified these residues to be the complete contribution to the one-point function of the operators $\mathcal{O}_+[0,l]$. A simple check is to observe that the one-point functions in (4.26) coincides with the MFT expression in (3.54) on taking $m_{\rm th}\to 0$ and taking $\Delta_\psi = k$ together with using the relation (4.14).

Similarly we can evaluate the contribution of the leading expansion in $\bar{z}$ in the integrands of I_2 and I_3. This results in poles at $\Delta = 2k+l+1$, the contributions are

$$I_2(m) = \frac{2(-1)^m\pi^{\frac{1}{2}-k}2^{-\Delta+k+\frac{1}{2}}m^{\frac{1}{2}-k}m_{\rm th}^{\Delta-k+\frac{1}{2}}K_l}{(-\Delta+l+2k+1)\Gamma(1+\Delta-2k)}K_{\frac{1}{2}+\Delta-k}(m m_{\rm th}), \tag{4.27}$$

and

$$I_3(m) = \frac{8(-1)^{m+1}\pi^{\frac{1}{2}-k}2^{-\Delta+k+\frac{1}{2}}m^{-\frac{1}{2}-k}m_{\rm th}^{\Delta-k-\frac{1}{2}}K_l}{(-\Delta+l+2k+1)\Gamma(\Delta-2k)}K_{-\frac{1}{2}+\Delta-k}(m m_{\rm th}). \tag{4.28}$$

The sum of the residues of the poles from $I_1^{(1)}(m), I_2(m)$ and $I_3(m)$ is given by

$$
\begin{aligned}
&-I(m)|_{\text{ Res at }\Delta=2k+l+1} \\
&=\frac{(-1)^m K_l \pi^{\frac{1}{2}-k} 2^{-k-l+\frac{1}{2}} m^{-\frac{1}{2}-k} (m_{\text{th}})^{l+k+\frac{1}{2}}}{\Gamma(l+1)} \\
&\quad\times\left[\frac{3-2k}{1+2k+2l}(mm_{\text{th}})K_{l+k+\frac{3}{2}}(mm_{\text{th}})+(2k-3)K_{l+k+\frac{1}{2}}(mm_{\text{th}})+(mm_{\text{th}})K_{l+k-\frac{1}{2}}(mm_{\text{th}})\right].
\end{aligned}
\tag{4.29}
$$

Then using the recurrence relation of the Bessel functions,

$$
\frac{2\nu}{x}K_\nu(x) = -K_{\nu-1} + K_{\nu+1}, \tag{4.30}
$$

we can simply this to

$$
\begin{aligned}
-I(m)|_{\text{ Res at }\Delta=2k+l+1} = &\frac{(-1)^m K_l \pi^{\frac{1}{2}-k} 2^{-k-l+\frac{3}{2}} m^{\frac{1}{2}-k} {m_{\text{th}}}^{l+k+\frac{3}{2}} (l+2)}{\Gamma(l+1)(1+2k+2l)} \\
&\times K_{l+k-\frac{1}{2}}(mm_{\text{th}}).
\end{aligned}
\tag{4.31}
$$

We can sum over m using the identity (4.25) which results in the complete contribution from the discontinuity across the cuts to the residues at $\Delta = 2k + l + 1$.

$$
\begin{aligned}
-\hat{a}(\Delta, l)_{\text{disc}}|_{\text{ Res at }\Delta=2k+l+1} = &\frac{(l+2)}{\pi^k(1+2k+2l)\left(k-\frac{1}{2}\right)_l}\left(\frac{m_{\text{th}}}{2}\right)^{l+k+1} \\
&\times\sum_{n=0}^{l+k-1}\frac{(l+k-n)_{2n}}{(2m_{\text{th}})^n n!}\text{Li}_{k+n}(-e^{-m_{\text{th}}}).
\end{aligned}
\tag{4.32}
$$

Finally for $l > 0$, from the residues at $\Delta = 2k+l+1$ given in (4.32) and the OPE expansion in (3.10), we obtain

$$
\begin{aligned}
&a_{\mathcal{O}_+}[1,l] + \frac{2(l+1)}{(l+2)(2k+1+l)}a_{\mathcal{O}_-}[0,l] = \\
&\frac{(l+2)}{\pi^k(1+2k+2l)\left(k-\frac{1}{2}\right)_l}\left(\frac{m_{\text{th}}}{2}\right)^{l+k+1}\sum_{n=0}^{l+k-1}\frac{(l+k-n)_{2n}}{(2m_{\text{th}})^n n!}\text{Li}_{k+n}(-e^{-m_{\text{th}}}),
\end{aligned}
\tag{4.33}
$$

where again $l = 0, 2, \cdots$.

Contribution from the arcs. To complete the evaluation of the one-point function we need to evaluate the contribution from the arc at infinity. This is given by the expression in (3.45). Taking the limit $w \to \infty$ in $g_2(x)$, it can be seen only the mode $m = 0$ contributes, this is due to fact the argument of the Bessel function has a square root, which ensures that it vanishes on the arc exponentially for $m > 0$. Furthermore due to the integration over the full circle in the w plane only $l = 0$ contributes. Due to these reasons the contribution from the arcs is given by

$$
\hat{a}_{\text{arc}}(\Delta, 0) = -4\pi K_0\left(\frac{m_{\text{th}}}{2\pi}\right)^{\frac{d}{2}}\int_0^1 dr\frac{K_{k+\frac{1}{2}}(m_{\text{th}}r)}{r^{\Delta-k+\frac{1}{2}}}. \tag{4.34}
$$

To obtain the location of the pole in the Δ plane we can push the upper limit of the integral to ∞. This does not change either the location of the poles nor their residues. This results in the following

$$\hat{a}_{\rm arc}(\Delta,0) = -\frac{1}{4\pi^{k+\frac{1}{2}}}\left(\frac{m_{\rm th}}{2}\right)^{\Delta}\Gamma\left(-\frac{\Delta}{2}\right)\Gamma\left(\frac{1}{2}-\frac{\Delta}{2}+k\right). \tag{4.35}$$

We therefore obtain the residue at $\Delta = 2k+1$

$$\hat{a}(\Delta,0)_{\rm arc}|\ \text{Res at }_{\Delta=2k+1} = -\frac{1}{2\pi^{k+\frac{1}{2}}}\left(\frac{m_{\rm th}}{2}\right)^{2k+1}\Gamma\left(-k-\frac{1}{2}\right). \tag{4.36}$$

Gap equation from OPE inversion. Similar to the analysis [1] for the bosonic critical $O(N)$ model, let us demand that the low lying operator $\mathcal{O}_-[l=0,n=0]$ with $\Delta = 2k+1$ does not occur in the spectrum. These are the operators which are schematically of the form $\bar{\psi}\gamma^\mu\partial_\mu\psi$. This implies that contribution to the residues from the discontinuity of the branch cuts in (4.32) together with the arcs at infinity in (4.36) at the pole $\Delta = 2k+1$ must vanish.

$$[-\hat{a}(\Delta,0)_{\rm disc} - \hat{a}(\Delta,0)_{\rm arc}]\ |\ \text{Res at }_{\Delta=2k+1} = 0. \tag{4.37}$$

Note that this combination is the one-point function $a_{\mathcal{O}_-}[0,0]$ as can be seen from the OPE expansion (3.10). Using (4.32) and (4.36) in (4.2) we obtain the equation

$$2(2m_{\rm th})^k\sum_{n=0}^{k-1}\frac{(k-n)_{2n}}{(2m_{\rm th})^n n!}{\rm Li}_{k+n}(-e^{-m_{\rm th}}) + \frac{(m_{\rm th})^{2k}\sqrt{\pi}}{\Gamma\left(\frac{1}{2}+k\right)\cos\pi k} = 0, \tag{4.38}$$

which precisely agrees with that obtained from the partition function in (4.2).

It is important to mention that this gap equation was also obtained [2] for the Gross-Neveu model by demanding operators $\mathcal{O}_0[0,0]$ which are schematically of the form $\bar{\psi}\psi$.[7] We will review this subsequently. Heuristically this fact could have been anticipated, at the large N saddle point $\bar{\psi}\gamma^\mu\partial_\mu\psi \sim \zeta\bar{\psi}\psi = m_{\rm th}\bar{\psi}\psi$ where ζ is the field introduced by the Hubbard-Stratonovich transformation to linearise the 4-fermi interaction. Therefore the vanishing of the one-point function $\mathcal{O}_0[0,0]$ implies that the one-point function $\mathcal{O}_-[0,0]$ also vanishes. The fact that the explicit computation does indeed bear out this expectation is an important consistency check of the OPE inversion formula developed for the correlator $g_2(x)$.

4.2 OPE inversion on $g_3(x)$

It can be easily seen that using the definition of $g_3(x)$ in (2.2) and (4.8), this correlator is given by

$$g_3(x) = \sum_{n,k_0=2\pi\left(n+\frac{1}{2}\right)}\partial^2\left[\int\frac{d^{d-1}x}{(2\pi)^{d-1}}\left(\frac{1}{k^2+m_{th}^2}\right)e^{ikx}\right]. \tag{4.39}$$

[7] See equation (27) of [2].

We can use the equation (4.12) which relates the term in the square brackets to the Bessel function to take the derivatives. This leads to

$$g_3(x) = \sum_{m\in\mathbb{Z}} \frac{(-1)^m}{(2\pi)^{\frac{d}{2}}} m_{th}^{\frac{d}{2}+1} |x^{(m)}|^{\frac{2-d}{2}} K_{\frac{d-2}{2}}\left(m_{th}|x^{(m)}|\right). \tag{4.40}$$

Thus $g_3(x)$ is a simpler correlator and similar to that encountered in the bosonic $O(N)$ model. We can apply the inversion formula as before

$$\hat{a}_{\rm disc}(\Delta, l) = 2K_l \int_0^1 \frac{d\bar{z}}{\bar{z}} \int_1^{\frac{1}{\bar{z}}} \frac{dz}{z} (z-\bar{z})^{2\nu} (z\bar{z})^{\frac{1}{2}-\frac{\Delta}{2}} F_l\left(\sqrt{\frac{\bar{z}}{z}}\right) {\rm Disc}[g_3(z,\bar{z})], \tag{4.41}$$

where l is even. Since $g_3(x)$ is still a sum of BesselK functions, we can use (4.16) to obtain the discontinuity across the branch cuts. Proceeding along the similar lines, the contribution to the residue of $\hat{a}_{\rm disc}(\Delta, l)$ at the poles $\Delta = 2k+l$ is given by

$$-\hat{a}_{\rm disc}(\Delta, l)\Big|_{\text{Res at }\Delta=2k+l} = \frac{2}{\pi^k \left(k+\frac{1}{2}\right)_l} \left(\frac{m_{\rm th}}{2}\right)^{l+k+1} \sum_{n=0}^{k+l-1} \frac{(l+k-n)_{2n}}{(2m_{\rm th})^n n!} {\rm Li}_{k+n}(-e^{-m_{\rm th}}). \tag{4.42}$$

Similar to the discussion for $g_2(x)$, the residue at $\Delta = 2k, l = 0$ receives contribution from the arcs at infinity which is given by

$$-\hat{a}_{\rm arc}(\Delta, 0)\Big|_{\text{Res at }\Delta=2k} = \left(\frac{m_{\rm th}}{2}\right)^{2k+1} \frac{\Gamma\left(-k+\frac{1}{2}\right)}{\pi^{k+\frac{1}{2}}}. \tag{4.43}$$

Now the expectation value $a_{\mathcal{O}_-}[0,0]$ which refers to the operator schematically of the form $\bar{\psi}\gamma^\mu\partial_\mu\psi$ is given by the combination

$$\begin{aligned} a_{\mathcal{O}_-}[0,0] &= -\left[\hat{a}_{\rm disc}(\Delta,0) + a_{\rm arc}(\Delta,0)\right]\Big|_{\text{Res at }\Delta=2k}, \qquad (4.44)\\ &= \frac{2}{\pi^k}\left(\frac{m_{\rm th}}{2}\right)^{k+1} \sum_{n=0}^{k-1} \frac{(k-n)_{2n}}{(2m_{\rm th})^n n!} {\rm Li}_{k+n}(-e^{-m_{\rm th}}) + \left(\frac{m_{\rm th}}{2}\right)^{2k+1} \frac{\Gamma\left(-k+\frac{1}{2}\right)}{\pi^{k+\frac{1}{2}}},\\ &= 0. \end{aligned}$$

Again, demanding that this operator does not exist in the spectrum results in the same gap equation (4.38) obtained by considering the correlator $g_2(x)$. This agreement is necessary for the consistency of obtaining the expectation value $a_{\mathcal{O}_-}[0,0]$ both from $g_2(x)$ and $g_3(x)$. Finally using the OPE expansion of $g_3(x)$ in (3.20), (3.37), we see that poles at $\Delta = 2k$ correspond to the coefficient $c[0,0]$. Then from the relations in (3.38), we obtain the linear relation between the expectation values $a_{\mathcal{O}_-}[0,0]$ and $a_{\mathcal{O}_+}[0,0]$

$$2a_{\mathcal{O}_+}[1,l] + (l+1)a_{\mathcal{O}_-}[0,l] = \frac{2}{\pi^k \left(k+\frac{1}{2}\right)_l} \left(\frac{m_{\rm th}}{2}\right)^{l+k+1} \sum_{n=0}^{k+l-1} \frac{(l+k-n)_{2n}}{(2m_{\rm th})^n n!} {\rm Li}_{k+n}(-e^{-m_{\rm th}}). \tag{4.45}$$

We can use the equations (4.33) and (4.45), to obtain values $a_{\mathcal{O}_+}[1,l]+$ and $a_{\mathcal{O}_-}[0,l]$, but it is not illustrative. What is important to note is that on comparing the l.h.s. of (4.33) and (4.45) and the expectation value (4.26), we see that for a given l, $a_{\mathcal{O}_+}[1,l]+$ and $a_{\mathcal{O}_-}[0,l]$ are proportional to $m_{\rm th}^2 a_{\mathcal{O}_+}[0,l]$.

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4.3 OPE inversion on $g_1(x)$

Finally for completeness, let us give the results for $g_1(x)$, the correlator studied in [2]. From (2.2) and (4.9), we see that

$$g_1(x) = \sum_{n,k_0=2\pi\left(n+\frac{1}{2}\right)} \int \frac{d^{d-1}x}{(2\pi)^{d-1}} \left(\frac{m_{th}}{k^2+m_{th}^2}\right) e^{ikx}. \tag{4.46}$$

Performing the Poisson re-summation and then evaluating the Fourier transform we obtain

$$g_1(x) = \sum_{n\in\mathbb{Z}} \frac{(-1)^n}{(2\pi)^{\frac{d}{2}}} m_{th}^{\frac{d}{2}} |x^{(n)}|^{\frac{2-d}{2}} K_{\frac{d-2}{2}} \left(m_{th}|x^{(n)}|\right). \tag{4.47}$$

From the OPE expansion in (2.9), we see that this correlator contains the information of the one-point functions of the operators $\mathcal{O}_0[0,l]$. These operators are schematically represented by the traceless symmetric bi-linears given in (2.1). Again using the OPE inversion formula on $g_1(x)$, the residues at poles $\Delta = 2k + l$ lead to the following one-point function

$$a_{\mathcal{O}_0}[0,l] = \frac{1}{\pi^k \left(k+\frac{1}{2}\right)_l} \left(\frac{m_{\rm th}}{2}\right)^{l+k} \sum_{n=0}^{l+k-1} \frac{(l+k-n)_{2n} \mathrm{Li}_{k+n}\left(-e^{-m_{th}}\right)}{(2m_{\rm th})^n}, \tag{4.48}$$
$$l = 2, 4, \cdots.$$

For $l = 0$, as before there is a contribution to the residue both from the arc as well as the disc. The residue at the $\Delta = 2k$ pole is given by

$$-\hat{a}_{\rm arc}(\Delta,0)\Big|_{\text{Res at }\Delta=2k} = \frac{m^{2k}}{2^{2k+1}\pi^{k+\frac{1}{2}}} \Gamma\left(\frac{1}{2}-k\right). \tag{4.49}$$

Combining the contribution of the arc and the disc results in the expectation value of the operator $\bar{\psi}\psi$. Demanding that this operator does not exist in the spectrum results in the gap equation

$$a_{\mathcal{O}_0}[0,0] = -\left[\hat{a}_{\rm disc}(\Delta,0) + \hat{a}_{\rm arc}(\Delta,0)\right]\Big|_{\text{Res at }\Delta=2k} = 0, \tag{4.50}$$
$$\text{thus,} \quad \frac{1}{\pi^k}\left(\frac{m_{\rm th}}{2}\right)^k \sum_{n=0}^{k-1} \frac{(k-n)_{2n}\mathrm{Li}_{k+n}\left(-e^{-m_{th}}\right)}{(2m_{\rm th})^n} + \frac{m^{2k}}{2^{2k+1}\pi^{k+\frac{1}{2}}}\Gamma\left(\frac{1}{2}-k\right) = 0.$$

Properties of the one-point functions in the GN model. Comparing the equation obtained by demanding the operators $\mathcal{O}_-[0,0]$ and $\mathcal{O}_0[0,0]$ do not exist, equations (4.44) and (4.50) respectively, we see that the equations are related by just an overall multiplicative factor of $m_{\rm th}$. Therefore, the gap equation is same and one obtains no new conditions which is important for the consistency of starting with the thermal propagator with one parameter $m_{\rm th}$. As explained earlier, this might have been expected by the large N saddle point equations of motion $\bar{\psi}\gamma^\mu\partial_\mu\psi \sim m_{\rm th}\bar{\psi}\psi$. What is perhaps more non-trivial is the following observation. Comparing the one-point functions of operators $\mathcal{O}_+[0,l]$ in (4.26) and operators $\mathcal{O}_0[0,l]$ in (4.48) we see that

$$a_{\mathcal{O}_0}[0,l] = m_{\rm th} a_{\mathcal{O}_+}[0,l], \qquad l = 2, 4, \cdots. \tag{4.51}$$

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Here there is no obvious equation of motion relating these expectation values. Such a relation must be specific to the critical Gross-Neveu model at large N. Observe that for the MFT of fermions the correlator $g_0(x)$ vanishes, so all expectation values $a_{\mathcal{O}_0}[0,l]$ vanish. However the expectation values $a_{\mathcal{O}_+}[0,l]$ are non-trivial and are given by (3.54). Of course for the MFT of free fermions, we have $m_{\rm th}=0$, and therefore the fact that $a_{\mathcal{O}_0}[0,l]$ vanishes and $a_{\mathcal{O}_+}[0,l]$ does not, is consistent with (4.51).

For the critical Gross-Neveu model it is likely there are more relations of the kind (4.51). From the observations made after equation (4.45), we know that the expectation values $a_{\mathcal{O}_+}[1,l]$ and $a_{\mathcal{O}_-}[0,l]$ are also proportional to $m_{\rm th}^2 a_{\mathcal{O}_+}[0,l]$. Here the proportionality constants involve spin and numerical factors. This and the relation (4.51) hint that the expectation values of bilinears of the form $\mathcal{O}_+[n,l], n>0$, $\mathcal{O}_-[n,l], n\geq 0$ and $\mathcal{O}_0[n,l], n\geq 0$ are all related to the expectation value $a_{\mathcal{O}_+}[0,l]$. It will be interesting to prove this.

5 Large d and large spin behaviour of one-point functions

As we have mentioned in the introduction, since the OPE inversion formula provides compact expressions for one-point functions, we study their behaviour at large dimensions d and spin l. As we have discussed in section 2, the one-point function $a_{\mathcal{O}}$ is proportional to the thermal expectation value of the corresponding operator. The proportionality constants involve the structure constant, $f_{\bar\psi\psi\mathcal{O}}$ and the normalization of the two-point function of $\mathcal{O}$, $c_{\mathcal{O}}$. To eliminate this dependence, we use the following, for the $O(N)$ model or the Gross-Neveu models OPE coefficients, anomalous dimensions are the same both in the Gaussian fixed point as well as the critical fixed point at large N [8, 19–23]. Therefore, we study the ratio of the one-point functions $a_{\mathcal{O}}$ at these fixed points.

$$r(l,k)=\frac{a_{\mathcal{O}}[l]_{m_{\rm th}\neq 0,k}}{a_{\mathcal{O}}[l]_{m_{\rm th}=0,k}}. \tag{5.1}$$

Here $m_{\rm th}\neq 0,k$ denotes evaluating the one-point function for the real positive solution of the gap equation at dimension $d=2k+1$. We discuss two cases, first we examine the behaviour of $r(l,k)$ by increasing k keeping l fixed and in the second case we study $r(l,k)$ by increasing l at fixed k.

In the rest of the section we study the one-point function of the stress tensor in detail for the two theories. The energy density for conformal field theories on $S_1\times R^{d-1}$ can be written as

$$T_{00}=-E=\frac{(d-1)\Gamma\left(\frac{d}{2}\right)}{\pi^{\frac{d}{2}}\beta^d}\times c(m_{\rm th},d). \tag{5.2}$$

For free bosons $c(m_{\rm th}=0,d)=-\zeta(d)$, which is the Stefan-Boltzmann value.[8] Therefore, we can consider $c(m_{\rm th},d)$ as a rough measure of the degrees of freedom which is seen on heating the system. We plot the $c(m_{\rm th},d)$ as a function of d, we see that $c(m_{\rm th},d)$ vanishes as d increases. For fermions we can further write $c(m_{\rm th},d)=2^{\frac{d-1}{2}}\tilde c(m_{\rm th},d)$, where the factor $2^{\frac{d-1}{2}}$ is due to the dimension of the spinor in odd d dimensions. For free fermions, the Stefan-Boltzmann value is $\tilde c(m_{\rm th},d)=-(1-2^{-(d-1)})\zeta(d)$. Again we see that the $\tilde c(m_{\rm th},d)$ vanishes as d increases, that is the degrees of freedom seen by heating the system again decreases.

[8]We are examining the energy density divided by N.

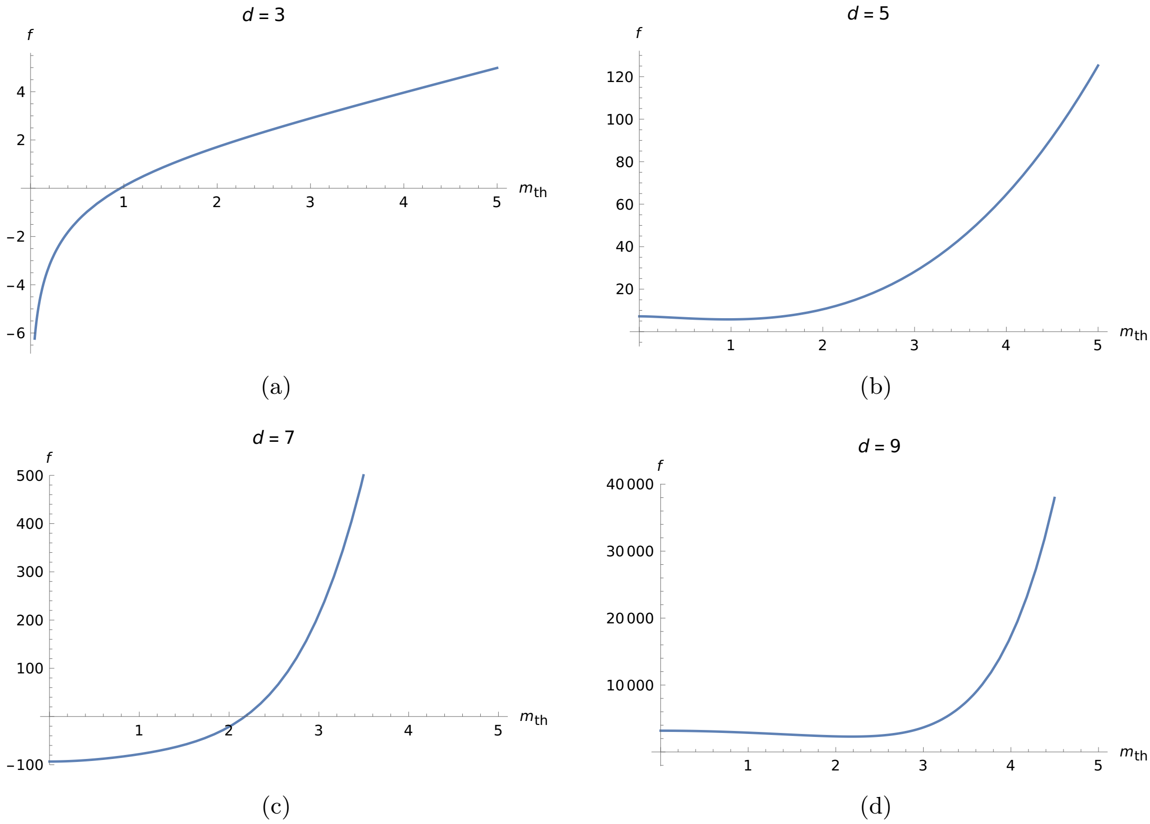

Figure 1. The l.h.s. of the gap equation (5.3) for the $O(N)$ model (denoted by f in the figure) has been plotted as a function of $m_{\rm th}$, for $d=3$ and $d=7$ the graph cuts the x-axis only once while for $d=5,9$, it is always positive. Similar trend follows for higher values of d, which is possible to check numerically.

5.1 $O(N)$ model

In this section we will numerically study the behaviour of one-point functions for $O(N)$ model with increasing d at a fixed l and vice versa. For this analysis, one should first look for the existence of real solutions of the gap equation at various odd dimensions. The gap equation for the $O(N)$ model at strong coupling in odd dimension can be derived using the standard field theoretic technique elaborated in appendix A. The same gap equation is obtained in [2] by demanding the absence of the operator ϕ^2 in the ope of the two-point function for critical $O(N)$ model using the formalism of inversion formula.

$$(m_{\rm th})^{2k}+2\sqrt{\pi}\sum_{m=0}^{k-1}\frac{(2m_{\rm th})^{k-m}(k-m)_{2m}{\rm Li}_{k+m}\left(e^{-m_{\rm th}}\right)}{\Gamma\left(\frac{1}{2}-k\right)m!}=0. \tag{5.3}$$

For $d=3,7,11,15,\ldots$, the gap equation is observed to have only one positive real solution for $m_{\rm th}$ and for $d=5,9,13,\ldots$ no real solution for $m_{\rm th}$ exists. We have shown this in figure 1. This phenomenon was noticed in [2].

The one-point function for double twist operator of kind $\phi\partial_{\mu_1}\cdots\partial_{\mu_l}\phi$ is given by [2],

$$a_l=\frac{(1+(-1)^l)}{2^{2l+k}l!\left(k-\frac{1}{2}\right)_l}\sum_{n=0}^{k+l-1}\frac{2^{n+1}m_{th}^n(2(k+l-1)-n)!{\rm Li}_{2k-n+l-1}\left(e^{-m_{th}}\right)}{n!(k-n+l-1)!}. \tag{5.4}$$

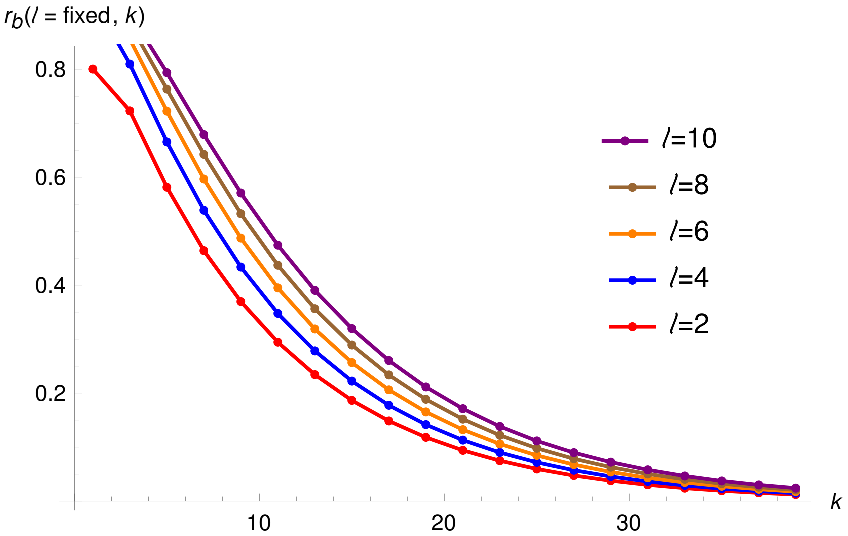

Figure 2. The ratio $r_b(l,k)$ for the $O(N)$ model is plotted against $k(\in$ odd integers) for various fixed values of spin l. The plot shows that the one-point functions at fixed l becomes smaller in comparison with the same for the free theory with increasing d.

The ratio of one-point functions at the non-trivial solution of the gap equation to the same at $m_{\rm th}=0$, the Stefan-Boltzmann value is given by

$$r_b(l,k) = \frac{a_l|_{m_{\rm th}\neq 0,k}}{a_l|_{m_{\rm th}=0,k}}. \tag{5.5}$$

$m_{\rm th}\neq 0, k$ denotes the real positive solution of the gap equation at dimension $d=2k+1$ can be obtained by solving the gap equation numerically.

We examine the behaviour of $r_b(l,k)$ with increasing k at fixed l; we restrict k to be odd, as the real solution for $m_{\rm th}$ exists only for odd k. The result of this analysis is shown in figure 2. The observation is that the ratio of the one-point function of double twist operator at fixed value of spin l evaluated at the Gaussian fixed point to that at the free theory limit keeps on decreasing with increasing k, but this ratio falls slower for higher values of l

$$\lim_{k\to\infty} r_b(l=\text{fixed},k)\to 0. \tag{5.6}$$

Now, we keep the k fixed and increase l and observe how $r_b(l,k)$ behaves. The result of our numerical study is described in figure 3. At a fixed k, $r_b(l,k)$ saturates at 1 for large values of l, which indicates that the one-point function of large spin operator evaluated at the critical point of $O(N)$ model is equal to that in free theory.

$$\lim_{l\to\infty} r_b(l,k=\text{fixed}) = 1. \tag{5.7}$$

Let us study the behaviour of the coeffcient $c(m_{\rm th},k)$ which determines energy density as defined in (5.2). This coefficient can be extracted from the following expression for the energy density which can be derived from the partition function.

$$E=\sum_{m=0}^{k+1}\frac{m_{\rm th}^{k-m+1}\left((k+m)^2+k-m\right)\Gamma(k+m)\mathrm{Li}_{k+m}\left(e^{-m_{\rm th}}\right)}{\pi^k 2^{k+m-1}m!\Gamma(k-m+2)}-\frac{m_{\rm th}^{2k+1}\Gamma\left(-k-\frac{1}{2}\right)}{\pi^{k+\frac{1}{2}}2^{2k+1}}. \tag{5.8}$$

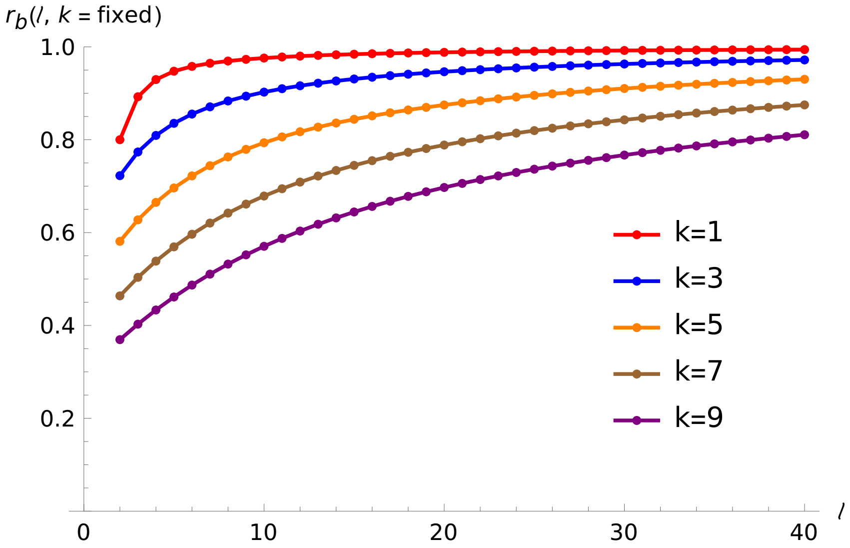

Figure 3. Plot showing the nature of the ratio $r_b(l,k)$ for the $O(N)$ model versus l, keeping k fixed.

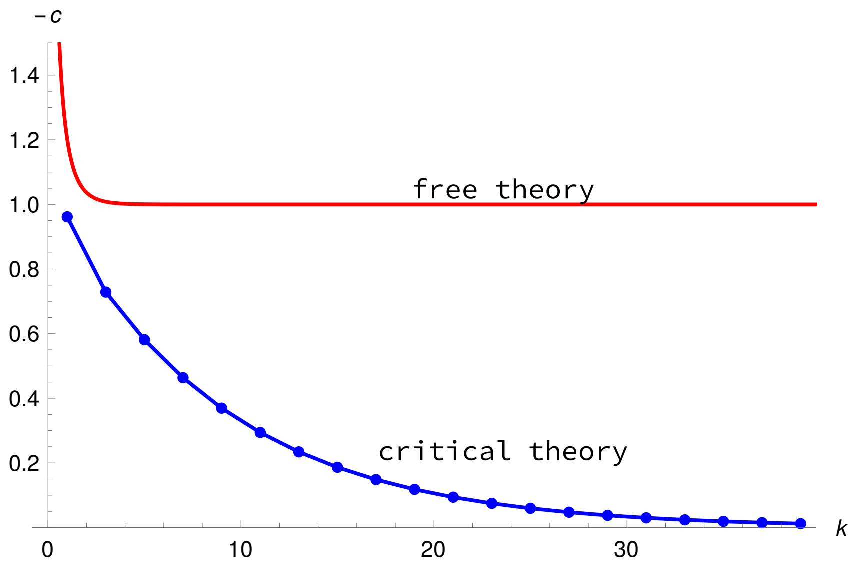

Figure 4. Plot of $c(m_{\text{th}}, k)$ for the free theory and the critical theory in the $O(N)$ model.

The energy density at the Gaussian fixed point is obtained by taking $m_{\text{th}} \to 0$ in the above expression, this results in

$$-c(m_{\text{th}} = 0, k) = \zeta(2k+1). \tag{5.9}$$

At the non-trivial fixed point this coefficient can be evaluated numerically. In figure 4, we plot this measure of degrees of freedom for the critical fixed point and also the Stefan-Boltzmann value for reference. Note that the $-c(m_{\text{th}} = 0, k)$ tends to zero for the critical point while the Stefan-Boltzmann value tends to one for large dimension d.

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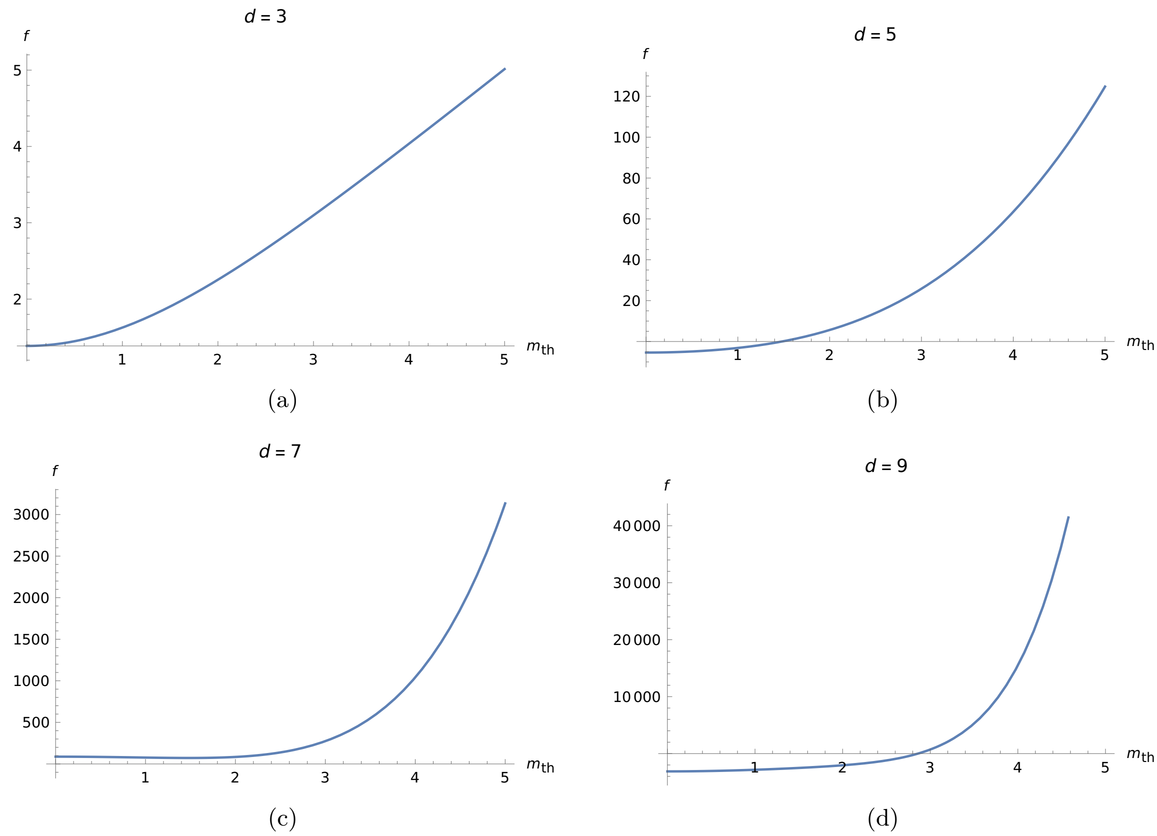

Figure 5. The l.h.s. of the gap equation ([5.10](#)) for the Gross-Neveu model (denoted by f in the figure) has been plotted as a function of m_{th}, for $d = 5$ and $d = 9$ the graph cuts the x-axis only once while for $d = 3, 7$, it is always positive.

5.2 Gross-Neveu model

We repeat the same analysis for the Gross-Neveu model. The gap equation for this model at strong coupling in odd d dimensions is given by,

$$(m_{\rm th})^{2k} + 2\sqrt{\pi}\sum_{m=0}^{k-1} \frac{(k-m)_{2m}(2m_{\rm th})^{k-m}\mathrm{Li}_{k+m}\left(-e^{-m_{\rm th}}\right)}{m!\Gamma\left(\frac{1}{2}-k\right)} = 0. \tag{5.10}$$

In contrast to the case of $O(N)$ model, the above gap equation has a positive real solution for $m_{\rm th}$ in $d = 5, 9, 13, \ldots$ and no real solution is found for $d = 3, 7, 11, \ldots$. This is seen in figure 5.

For the Gross-Neveu model we have the general expression for the thermal expectation value of the operators of the kind $\mathcal{O}_+$ for arbitrary l in equation (4.26). The ratio of one-point functions of the fermionic operators $\mathcal{O}_+$ at the non-trivial critical point to the same at the Gaussian fixed point is given by,

$$r_f(l,k) = \frac{a_{\mathcal{O}+}[0,l]_{m_{\rm th},k}}{a_{\mathcal{O}+}[0,l]_{m_{\rm th}=0}}. \tag{5.11}$$

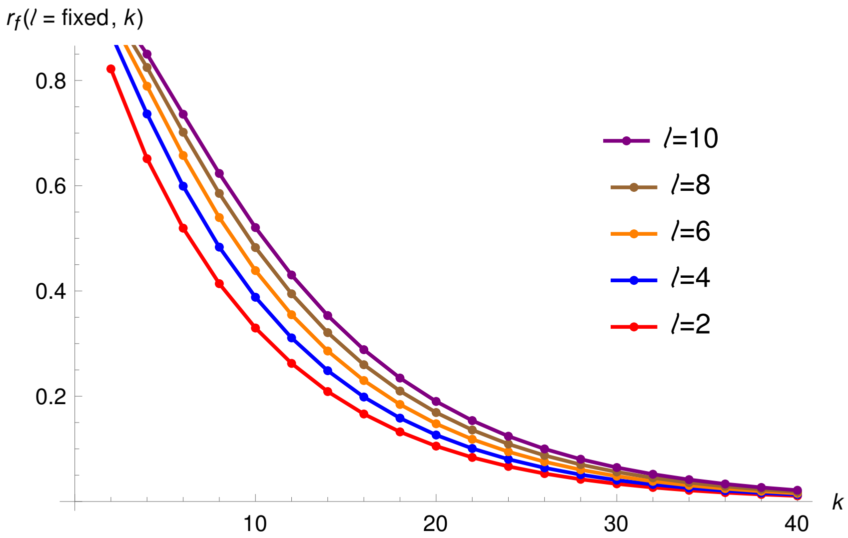

Figure 6. The ratio $r_f(l,k)$ for the Gross-Neveu model is plotted against k for various fixed values of spin l. The plot shows that the one-point functions at fixed l become smaller in comparison with the same for the free theory with increasing d.

Just as in the $O(N)$ model, first we examine the behaviour of $r_f(l,k)$ with increasing k keeping l fixed. But here we restrict k to be even, as the real solution for $m_{\rm th}$ exists only for even k. The result of this analysis is shown in figure 6. The results are identical to that of $O(N)$ model. The ratio of the one-point function of double twist operators at fixed value of spin l evaluated at the critical point to that at the Gaussian fixed point decreases with increasing k, but this ratio falls slower for higher values of l.

$$\lim_{k\to\infty} r_f(l=\text{fixed},k) \to 0. \tag{5.12}$$

Next, we keep the k fixed and increase l and observe how $r(l,k)$ behaves. The result of our numerical study is described in figure 7. At a fixed k, $r(l,k)$ tends to 1 for large values of l, which indicates that the one-point function of large spin operators evaluated at the critical point of Gross-Neveu model is equal to that in free theory.

$$\lim_{l\to\infty} r_f(l,k=\text{fixed}) = 1. \tag{5.13}$$

For a conformal field theory of fermions we can define the energy density

$$T_{00} = -E = \frac{(d-1)\Gamma\left(\frac{d}{2}\right)}{\pi^{\frac{d}{2}}\beta^d} \times 2^k\, \tilde{c}(m_{\rm th},k). \tag{5.14}$$

Here we factor out 2^k which is the number of components of the Dirac spinor in $2k+1$ dimensions. We can study the behaviour of the degrees of freedom by studying $\tilde{c}(m_{\rm th},k)$. The free energy density per fermions for the Gross-Neveu model is given by,

$$E = \sum_{m=0}^{k+1} \frac{\left((k+m)^2+(k-m)\right)\Gamma(k+m)\mathrm{Li}_{k+m}\left(-e^{-m_{\rm th}}\right)}{m_{\rm th}^{m-k-1}\pi^k 2^m m!\Gamma(k-m+2)} - \frac{m_{th}^{2k+1}\Gamma\left(-k-\frac{1}{2}\right)}{2^{k+2}\pi^{k+\frac{1}{2}}}. \tag{5.15}$$

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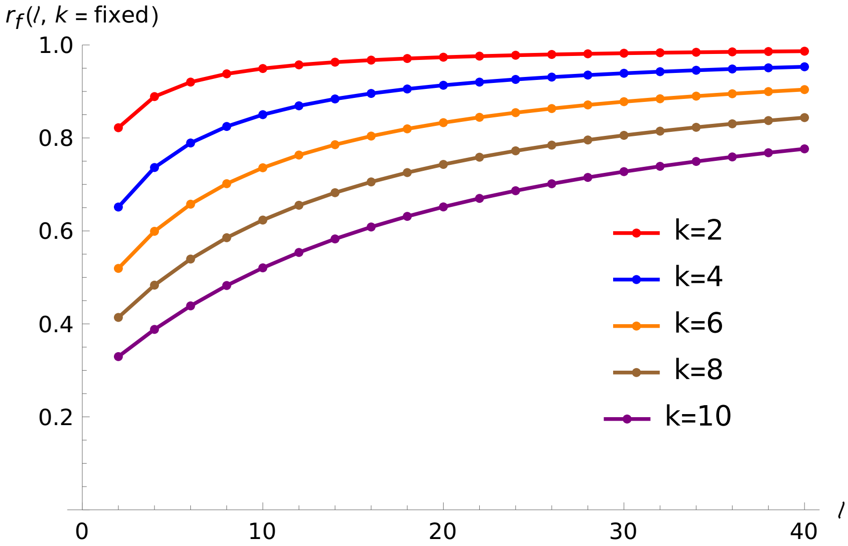

Figure 7. Plot of the ratio $r_f(l,k)$ for the Gross-Neveu model versus l, keeping k fixed.

The Stefan-Boltzmann value at the Gaussian fixed point can be obtained by taking $m_{\rm th} = 0$ in the above expression, this yields

$$\tilde{c}(m_{\rm th} = 0, k) = -(1 - 2^{-2k})\zeta(2k+1). \tag{5.16}$$

At the non-trivial fixed point, we solve the gap equation for $m_{\rm th}$ numerically and the substitute in (5.15) to obtain $\tilde{c}(m_{\rm th}, k)$. The result of this analysis is shown in figure 8. We have also plotted the corresponding behaviour of the Stefan-Boltzmann value for fermions. Again the effective degrees of freedom measured by the energy density decreases for the critical point as the dimension d is increased. It will be interesting to understand this behaviour of one-point functions more deeply and see if this behaviour is seen for all CFT's which are not free.

6 Conclusions

In this paper we have used the OPE inversion formula on thermal two-point functions of fermions in the channel which contains the stress tensor. The OPE inversion formula was applied to the MFT of fermions and the large N critical Gross-Neveu model. We studied the properties of the resulting thermal one-point functions. The inversion formula made it easy to study the behaviour of the one-point functions at large spin as well as arbitrary dimensions.

There are other theories where the OPE inversion formula could be applied. One such class of theories are the large N Chern-Simons matter theories [24, 25]. These models are connected to the Gross-Neveu or the $O(N)$ vector model. It would be interesting to obtain thermal one-point functions in these models and study their spin dependence and also see how the boson-fermion duality exhibited by these models are reflected in the one-point functions. Thermal correlators in these models are known in momentum space [24, 26–30], so perhaps one way of proceeding in these models is to derive a OPE inversion formula directly in momentum space.

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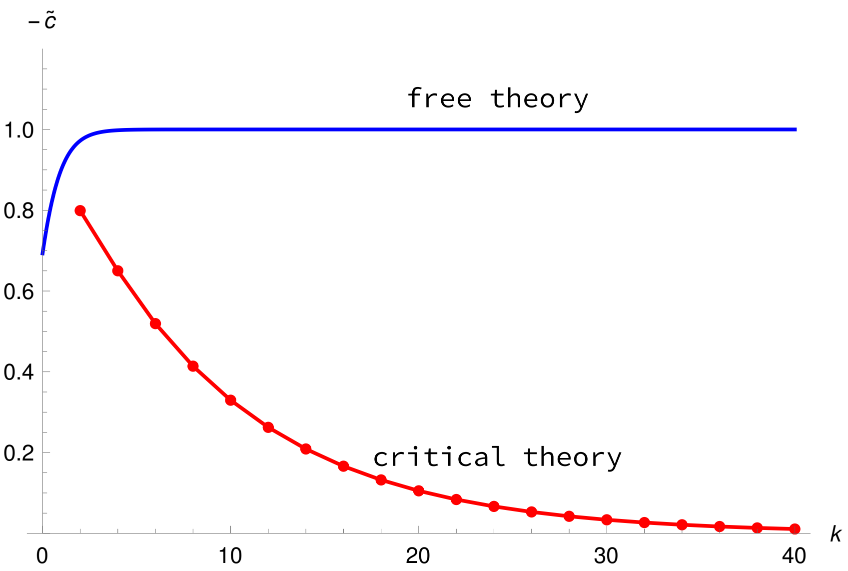

Figure 8. Plot of $\tilde{c}(m_{\rm th}, k)$ vs k for both the free theory and critical Gross-Neveu model.

One of our motivations to study thermal one-point functions in detail was that one-point functions of conformal primaries evaluated in AdS black holes can be used to probe the interior geometry of black holes [31–36]. The time to the singularity in the interior could be obtained as a phase factor by suitable analytic continuation of the conformal dimension. However the one-point functions, even that of the fermionic MFT studied here or the bosonic one in [1], do not exhibit this feature. It will be interesting to see if the application of the OPE inversion formula to the holographic two-point functions evaluated in [37–39] can be used to show that holographic one-point functions contain information about the interior geometry of the black hole.

In this paper we studied one-point functions in the geometry $S^1 \times R^{d-1}$, other geometries which are relevant to holography and evaluation of entanglement entropy are hyperbolic cylinders or the $S^1 \times AdS_{d-1}$ geometry. Black holes with hyperbolic horizons are dual to conformal field theories on this background. It should be possible to obtain an OPE inversion formula for field theories on such backgrounds. Recently a proposal to write two-point functions on such curved backgrounds have been given in [40]. It would be interesting to use the OPE inversion formula to these two-point functions and study the properties of the resulting one-points functions in these geometries.

A Gross-Neveu model: partition function, gap equation

In this appendix we evaluate the partition function of the critical Gross-Neveu model at large N. We obtain the gap equation as the saddle point equation at large N and then evaluate its stress tensor. We begin with the Lagrangian of N massless Dirac fermions $\psi^a, a = 1, 2, \cdots N$ transforming in the fundamental of $\mathrm{U}(N)$ along with the 4-fermi interaction.

$$S = \int d^{2k+1}x \left[\bar{\psi}(i\gamma^\mu \partial_\mu)\psi + \frac{\lambda}{N}(\bar{\psi}\psi)^2 \right] . \tag{A.1}$$

Since we are in Euclidean space $\bar{\psi} = \psi^\dagger$ and the γ matrices obey

$$\{\gamma^\mu, \gamma^\nu\} = 2\delta^{\mu\nu}. \tag{A.2}$$

The partition function of the theory $S^1 \times R^{2k}$ is given by

$$\tilde{Z} = \int \mathcal{D}\bar{\psi}\mathcal{D}\psi e^{-S(\psi,\bar{\psi})}. \tag{A.3}$$

We first linearise the theory using the Hubbard-Stratonovich transformation

$$\begin{aligned}\tilde{Z} &= \int \mathcal{D}\bar{\psi}\mathcal{D}\psi \exp\left[-\int_0^\beta d\tau d^{2k}x \left(i\bar{\psi}\gamma^\mu\partial_\mu\psi + \frac{\lambda}{N}(\bar{\psi}\psi)^2\right)\right], \\ &= \int \mathcal{D}\bar{\psi}\mathcal{D}\psi\mathcal{D}\zeta \left[-\int_0^\beta d\tau d^{2k}x \left(i\bar{\psi}\gamma^\mu\partial_\mu\psi + \frac{\zeta^2 N}{\lambda} + i\zeta(\bar{\psi}\psi)\right)\right].\end{aligned} \tag{A.4}$$

In the second line we have absorbed the normalization of the Gaussian integral over ζ into the measure. We can separate the zero mode of ζ and the non-zero modes and write the partition function

$$\zeta = \tilde{\zeta} + \zeta_0. \tag{A.5}$$

Here ζ_0 is the zero mode, substituting for ζ we can write the partition function as

$$\begin{aligned}\tilde{Z} &= \int d\zeta_0 \mathcal{D}\bar{\psi}\mathcal{D}\psi \left[\exp\left(-\frac{\zeta_0^2 N\beta V_{2k}}{4\lambda}\right)\exp(-S_0 - S_I)\right], \\ S_0 &= \int d\tau d^{2k}x \left[i\bar{\psi}\gamma^\mu\partial_\mu\psi + i\zeta_0\bar{\psi}\psi + \frac{\tilde{\zeta}^2 N}{4\lambda}\right], \qquad S_I = \int d\tau d^{2k}x \, i\tilde{\zeta}\bar{\psi}\psi.\end{aligned} \tag{A.6}$$

We can neglect S_I in the leading large N limit, observe that on canonically normalising the quadratic term in $\tilde{\zeta}$, the interaction S_I acquires a factor of $\frac{1}{\sqrt{N}}$. After performing the Gaussian integration in $\tilde{\zeta}$, we are left with

$$\tilde{Z} = \int d\zeta_0 \exp\left[-\beta V_{2k} N \left(\frac{\zeta_0^2}{4\lambda} - \frac{1}{\beta}\log Z(\zeta_0)\right)\right], \tag{A.7}$$

where

$$\log Z(\zeta_0) = 2^{k-1} \sum_{n=-\infty}^{\infty} \int \frac{d^{2k}p}{(2\pi)^{2k}} \log\left[\frac{4\pi^2\left(n+\frac{1}{2}\right)^2}{\beta^2} + \vec{p}^2 + \zeta_0^2\right]. \tag{A.8}$$

The 2^{k-1} factor arises from the fact that the Dirac operator is a $2^k \times 2^k$ dimensional matrix. After evaluating the Matsubara sum, we obtain

$$\log Z(\zeta_0) = \frac{2^{k-1}}{\beta^{2k}} \int \frac{d^{2k}p}{(2\pi)^{2k}} \left[\sqrt{\vec{p}^2 + \zeta_0^2\beta} + 2\log\left(1 + e^{-\sqrt{\vec{p}^2+\zeta_0^2\beta^2}}\right)\right]. \tag{A.9}$$

To integrate the first term in the square bracket we resort to the analytical continuation of the integral

$$\int_0^\infty dx \, \frac{x^{2k-1}}{(x^2+1)^a} = \frac{\Gamma(a-k)\Gamma(k)}{2\Gamma(a)}. \tag{A.10}$$

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The integral involving the second term in the square bracket of (A.9) is convergent and after some straightforward manipulations can be written in terms of Polylogarithms. This leads us to

$$\log Z(\zeta_0) = -\frac{m^{2k+1}\beta}{\pi^k 2^{k+2}}\left[\sum_{n=0}^{k}\frac{(k-n+1)_{2n}(\zeta_0\beta)^{-k-n-1}\mathrm{Li}_{k+n+1}\left(-e^{-\zeta_0\beta}\right)}{2^{n-k-2}n!}+\frac{\Gamma\left(-k-\frac{1}{2}\right)}{\sqrt{\pi}}\right]. \tag{A.11}$$

We can obtain the partition function $\tilde{Z}$ in (A.7) by using the saddle point approximation to perform the integral over ζ_0. The saddle point $\zeta_0^* = m_{\rm th}$ is determined by the equation

$$m_{\rm th} = 2\lambda\frac{1}{\beta}\frac{\partial}{\partial m_{\rm th}}Z(m_{\rm th}). \tag{A.12}$$

For the critical Gross-Neveu model we take the large λ limit. To the leading order in the large λ expansion, the saddle point equation reduces to

$$\frac{\partial}{\partial m_{\rm th}}Z(m_{\rm th}) = 0. \tag{A.13}$$

This results in the following gap equation for the critical value $m_{\rm th}$.

$$2(2m_{\rm th}\beta)^k\sum_{n=0}^{k-1}\frac{(k-n)_{2n}}{(2m_{\rm th}\beta)^n n!}\mathrm{Li}_{k+n}(-e^{-m_{\rm th}\beta}) + \frac{(m_{\rm th}\beta)^{2k}}{\sqrt{\pi}}\Gamma\left(\frac{1}{2}-k\right) = 0. \tag{A.14}$$

We can evaluate the stress tensor at the critical point from the partition function by

$$T_{00} = -\frac{\partial}{\partial\beta}\log Z(m_{\rm th}). \tag{A.15}$$

The above expression yields the energy density divided by the number of fermions N. Performing this differentiation on the partition function given in (A.11), we obtain

$$\begin{aligned} T_{00} = &-\frac{m_{\rm th}^{2k+1}\Gamma\left(-k-\frac{1}{2}\right)}{2^{k+2}\pi^{k+\frac{1}{2}}} \\ &+\frac{m_{\rm th}^{k+1}}{\pi^k\beta^k}\sum_{n=0}^{k+1}\frac{\left[(k+n)^2+(k-n)\right](k-n+2)_{2n-2}}{(2m_{\rm th}\beta)^n n!}\mathrm{Li}_{k+n}\big(-e^{-m_{\rm th}\beta}\big). \end{aligned} \tag{A.16}$$

Note that as expected for the critical theory, this energy density can be written as

$$T_{00} = T^d\, H(m_{\rm th}\beta), \tag{A.17}$$

where H is a function of $m_{\rm th}\beta$, the dimensionless number which is the root of the gap equation (A.14).

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