Linear Algebra and its Applications 683 (2024) 31–45

Contents lists available at ScienceDirect

Linear Algebra and its Applications

journal homepage: www.elsevier.com/locate/laa

Improved bounds for the numerical radius via polar decomposition of operators

Pintu Bhunia [1]

Department of Mathematics, Indian Institute of Science, Bengaluru 560012, Karnataka, India

A R T I C L E   I N F O

Article history:
Received 10 April 2023
Received in revised form 30 August 2023
Accepted 23 November 2023
Available online 27 November 2023
Submitted by C.-K. Li

MSC:
47A12
47A30
15A60

Keywords:
Numerical radius
Operator norm
Polar decomposition
Inequality

A B S T R A C T

Using the polar decomposition of a bounded linear operator A defined on a complex Hilbert space, we obtain several numerical radius inequalities of the operator A, which generalize and improve the earlier related ones. Among other bounds, we show that if $w(A)$ is the numerical radius of A, then

$$w(A) \le \frac{1}{2}\|A\|^{1/2}\left\||A|^t + |A^*|^{1-t}\right\|,$$

for all $t \in [0,1]$. Also, we obtain some upper bounds for the numerical radius involving the spectral radius and the Aluthge transform of operators. It is shown that

$$w(A) \le \|A\|^{1/2}\left(\frac{1}{2}\left\|\frac{|A|+|A^*|}{2}\right\| + \frac{1}{2}\left\|\widetilde{A}\right\|\right)^{1/2},$$

where $\widetilde{A} = |A|^{1/2}U|A|^{1/2}$ is the Aluthge transform of A and $A = U|A|$ is the polar decomposition of A. Other related results are also provided.

E-mail addresses: pintubhunia5206@gmail.com, pintubhunia@iisc.ac.in.

[1] Dr. Pintu Bhunia would like to sincerely acknowledge Prof. Kallol Paul for his valuable comments on this article. Dr. Pintu Bhunia also would like to thank SERB, Govt. of India for the financial support in the form of National Post Doctoral Fellowship (N-PDF, File No. PDF/2022/000325) under the mentorship of Prof. Apoorva Khare.

https://doi.org/10.1016/j.laa.2023.11.021
0024-3795/

1. Introduction

The purpose of the article is to develop numerical radius inequalities of bounded linear operators defined on a complex Hilbert space. The study of numerical range and the associated inequalities is useful in investigating many properties of linear operators and has various applications in numerous fields of sciences. In recent times the numerical radius has found its application in quantum information theory, in particular, quantum error correction [14], additive uncertainty relations [25], multi-observable quantum uncertainty relations [18]. Applying the numerical radius inequalities one can also estimate the roots of polynomials using the notion of the Frobenius companion matrix, see [7]. Before proceeding further we introduce the necessary notations and terminologies.

Let $\mathcal{B}(\mathcal{H})$ denote the C^*-algebra of all bounded linear operators on a complex Hilbert space $\mathcal{H}$, with inner product $\langle \cdot, \cdot \rangle$ and the corresponding norm $\| \cdot \|$. For $A \in \mathcal{B}(\mathcal{H})$, let $|A| = (A^*A)^{1/2}$ and $|A^*| = (AA^*)^{1/2}$, where A^* is the adjoint of A. For $t \in [0, 1]$, the t-Aluthge transform (generalized Aluthge transform) of $A \in \mathcal{B}(\mathcal{H})$ is $\widetilde{A_t} = |A|^t U |A|^{1-t}$, where $A = U|A|$ is the polar decomposition of A and U is the partial isometry. In particular, for $t = \frac{1}{2}$, $\widetilde{A} = \widetilde{A_{\frac{1}{2}}} = |A|^{1/2} U |A|^{1/2}$ is the Aluthge transform of A.
Let $\|A\|$, $w(A)$ and $r(A)$ denote the operator norm, the numerical radius and the spectral radius of A, respectively. The numerical radius of A is defined as

$$w(A) = \sup\{|\langle Ax, x \rangle| : x \in \mathcal{H}, \|x\| = 1\}$$

and it is the radius of the smallest disc with center at origin that contains the numerical range. Note that the numerical range $W(A)$ is defined as $W(A) = \{\langle Ax, x \rangle : x \in \mathcal{H}, \|x\| = 1\}$. It is well known that the numerical radius $w(\cdot) : \mathcal{B}(\mathcal{H}) \to \mathbb{R}$ defines a norm on $\mathcal{B}(\mathcal{H})$ and is equivalent to the operator norm. For every $A \in \mathcal{B}(\mathcal{H})$,

$$\frac{1}{2}\|A\| \leq w(A) \leq \|A\|, \tag{1.1}$$

holds. The spectral radius of A is defined as $r(A) = \sup\{|\lambda| : \lambda \in \sigma(A)\}$, where $\sigma(A)$ is the spectrum of A. Since, the spectrum $\sigma(A)$ is contained in the closure of the numerical range (i.e., $\sigma(A) \subset \overline{W(A)}$), we have

$$r(A) \leq w(A).$$

Therefore, for every $A \in \mathcal{B}(\mathcal{H})$, $r(A) \leq w(A) \leq \|A\|$ holds, and $r(A) = w(A) = \|A\|$, when $A \in \mathcal{B}(\mathcal{H})$ is a normal operator. Also, note that $r(A) = r(\widetilde{A})$, $w(\widetilde{A}) \leq w(A)$ and $\|\widetilde{A}\| \leq \|A^2\|^{1/2} \leq \|A\|$. For more details about the numerical range, the numerical radius and related inequalities, the readers can follow the books [7,26]. Various refinements of the numerical radius bounds in (1.1) have been studied over the years. Kittaneh in [21, 2003] and [20, 2005], respectively, developed the following bounds

$$w(A) \leq \frac{1}{2}\|A\| + \|A^2\|^{1/2} \tag{1.2}$$

and

$$w^2(A) \leq \frac{1}{2}\left\||A|^2 + |A^*|^2\right\|. \tag{1.3}$$

The bounds in (1.2) and (1.3) improve the same in (1.1). Dragomir in [16, 2008] proved that

$$w^2(A) \leq \frac{1}{2}\|A\|^2 + \frac{1}{2}w(A^2). \tag{1.4}$$

Clearly, the bound in (1.4) improves the same in (1.1). After that, Abu-Omar and Kittaneh in [1, 2015] developed

$$w^2(A) \leq \frac{1}{4}\left\||A|^2 + |A^*|^2\right\| + \frac{1}{2}w(A^2), \tag{1.5}$$

which improves both the bounds in (1.2), (1.3) and (1.4). Further, Bhunia and Paul in [12, 2021] proved that

$$w^2(A) \leq \frac{1}{4}\left\||A|^2 + |A^*|^2\right\| + \frac{1}{2}w(|A||A^*|) \tag{1.6}$$

and

$$w(A) \leq \frac{1}{\sqrt{2}}w(|A| + i|A^*|). \tag{1.7}$$

The bound in (1.6), is incomparable with the bound in (1.5), refines both the bounds in (1.2) and (1.3). The bound in (1.7) refines the bound (1.3). The same authors in [10, 2021] obtained an improvement of (1.2) by using the spectral radius, namely,

$$w(A) \leq \frac{1}{2}\|A\| + \frac{1}{2}r^{1/2}(|A||A^*|). \tag{1.8}$$

Further, Bhunia [5, 2023] obtained an improvement of the second inequality in (1.1), namely,

$$w(A) \leq \|A\|^{1/2}\left\|\alpha|A| + (1-\alpha)|A^*|\right\|^{1/2}, \tag{1.9}$$

for all $\alpha \in [0, 1]$. Recently, Kittaneh, Moradi and Sababheh [19, 2023] also developed the following nice improvement of the second inequality in (1.1):

$$w(A) \leq \frac{1}{2}\|A\|^{1/2}\left\||A|^{1/2} + |A^*|^{1/2}\right\|. \tag{1.10}$$

Some generalizations of the inequalities in (1.3), (1.5) (1.6) and (1.7) and the other improvements also studied, see [2–4,9,11–13,17,27].

In this paper, we obtain various numerical radius inequalities of bounded linear operators, which generalize and improve on the bounds in (1.1), (1.2), (1.3), (1.6), (1.7), (1.8), (1.9) and (1.10). Other bounds are also developed which refine the existing ones.

2. Main results

We begin our study with the following known lemmas. First lemma is known as McCarthy inequality.

Lemma 2.1. *[24] Let $A \in \mathcal{B}(\mathcal{H})$ be positive, and let $x \in \mathcal{H}$ with $\|x\| = 1$. Then*

$$\langle Ax, x\rangle^p \leq \langle A^p x, x\rangle,$$

for all $p \geq 1$.

Second lemma involves 2×2 positive operator matrix.

Lemma 2.2. *[23, Lemma 1] Let $A, B, C \in \mathcal{B}(\mathcal{H})$, where A and B are positive. Then the operator matrix $\begin{bmatrix} A & C^* \\ C & B \end{bmatrix} \in \mathcal{B}(\mathcal{H} \oplus \mathcal{H})$ is positive if and only if*

$$|\langle Cx, y\rangle|^2 \leq \langle Ax, x\rangle\langle By, y\rangle.$$

Third lemma is named as Buzano's inequality.

Lemma 2.3. *[15] Let $x, y, z \in \mathcal{H}$, where $\|z\| = 1$. Then*

$$|\langle x, z\rangle\langle z, y\rangle| \leq \frac{\|x\|\|y\| + |\langle x, y\rangle|}{2}.$$

By using the above lemmas we first prove the following proposition.

Proposition 2.4. *Let $A, B, C \in \mathcal{B}(\mathcal{H})$, where A and B are positive. If $\begin{bmatrix} A & C^* \\ C & B \end{bmatrix} \in \mathcal{B}(\mathcal{H} \oplus \mathcal{H})$ is positive, then the following bounds hold:*
(i) $w^2(C) \leq \frac{1}{2}\left\|A^2 + B^2\right\|$.
(ii) $w^2(C) \leq \frac{1}{2}\|A\|\,\|B\| + \frac{1}{2}w(AB)$.
(iii) $w^2(C) \leq \frac{1}{4}\left\|A^2 + B^2\right\| + \frac{1}{2}w(AB)$.
(iv) $w^2(C) \leq \|\alpha A + (1-\alpha)B\|\,\|A\|^{1-\alpha}\|B\|^{\alpha}$, for all $\alpha \in [0, 1]$.

Proof. Take $x \in \mathcal{H}$ with $\|x\| = 1$.
(i) From Lemma 2.2, we have

$$\begin{aligned}|\langle Cx, x\rangle|^2 &\leq \langle Ax, x\rangle\langle Bx, x\rangle \\ &\leq \frac{1}{2}\left(\langle Ax, x\rangle^2 + \langle Bx, x\rangle^2\right) \\ &\leq \frac{1}{2}\left(\langle A^2x, x\rangle + \langle B^2x, x\rangle\right) \text{ (by Lemma 2.1)} \\ &\leq \frac{1}{2}\left\|A^2 + B^2\right\|.\end{aligned}$$

This implies, $w^2(C) \leq \frac{1}{2}\left\|A^2 + B^2\right\|$.
(ii) From Lemma 2.2, we have

$$\begin{aligned}|\langle Cx, x\rangle|^2 &\leq \langle Ax, x\rangle\langle x, Bx\rangle \\ &\leq \frac{1}{2}\left(\|Ax\|\|Bx\| + |\langle Ax, Bx\rangle|\right) \text{ (by Lemma 2.3)} \\ &\leq \frac{1}{2}\left(\|A\|\|B\| + w(AB)\right).\end{aligned}$$

This gives, $w^2(C) \leq \frac{1}{2}\|A\|\|B\| + \frac{1}{2}w(AB)$.
(iii) From Lemma 2.2, we have

$$\begin{aligned}|\langle Cx, x\rangle|^2 &\leq \langle Ax, x\rangle\langle x, Bx\rangle \\ &\leq \frac{1}{2}\left(\|Ax\|\|Bx\| + |\langle Ax, Bx\rangle|\right) \text{ (by Lemma 2.3)} \\ &\leq \frac{1}{2}\left(\frac{\|Ax\|^2 + \|Bx\|^2}{2} + |\langle Ax, Bx\rangle|\right) \\ &\leq \frac{1}{4}\left\|A^2 + B^2\right\| + \frac{1}{2}w(AB).\end{aligned}$$

This implies, $w^2(C) \leq \frac{1}{4}\left\|A^2 + B^2\right\| + \frac{1}{2}w(AB)$.
(iv) From Lemma 2.2, we have

$$\begin{aligned}|\langle Cx, x\rangle|^2 &\leq \langle Ax, x\rangle\langle Bx, x\rangle \\ &= \langle Ax, x\rangle^\alpha\langle Bx, x\rangle^{1-\alpha}\langle Ax, x\rangle^{1-\alpha}\langle Bx, x\rangle^\alpha \\ &\leq \left(\alpha\langle Ax, x\rangle + (1-\alpha)\langle Bx, x\rangle\right)\langle Ax, x\rangle^{1-\alpha}\langle Bx, x\rangle^\alpha \\ &\leq \|\alpha A + (1-\alpha)B\|\,\|A\|^{1-\alpha}\|B\|^\alpha.\end{aligned}$$

This implies, $w^2(C) \leq \|\alpha A + (1-\alpha)B\|\,\|A\|^{1-\alpha}\|B\|^\alpha$, as desired. □

By applying Proposition 2.4, we prove the following lemma.

Lemma 2.5. *Let $B, C \in \mathcal{B}(\mathcal{H})$. Then $w(BC)$ satisfies the following bounds:*
(i) $w^2(BC) \leq \frac{1}{2}\left\| |B^*|^4 + |C|^4 \right\|$.
(ii) $w^2(BC) \leq \frac{1}{2}\|B\|^2\|C\|^2 + \frac{1}{2}w\left(B(CB)^*C\right)$.
(iii) $w^2(BC) \leq \frac{1}{4}\left\| |B^*|^4 + |C|^4 \right\| + \frac{1}{2}w\left(B(CB)^*C\right)$.
(iv) $w^2(BC) \leq \left\|\alpha|B^*|^2 + (1-\alpha)|C|^2\right\| \|B\|^{2(1-\alpha)}\|C\|^{2\alpha}$, *for all* $\alpha \in [0,1]$.

Proof. Following Lemma 2.2, it is easy to observe that the operator matrix $\begin{bmatrix} BB^* & BC \\ C^*B^* & C^*C \end{bmatrix} \in \mathcal{B}(\mathcal{H} \oplus \mathcal{H})$ is positive. Using this positive operator matrix in Proposition 2.4, we obtain the desired upper bounds of $w(BC)$. □

Now, we are in a position to obtain our first aim result.

Theorem 2.6. *If $A \in \mathcal{B}(\mathcal{H})$, then*

$$w^2(A) \leq \frac{1}{2}\|A\|^2 + \frac{1}{2}w\left(|A|^{2t}|A^*|^{2(1-t)}\right),$$

for all $t \in [0,1]$. In particular, for $t = \frac{1}{2}$

$$w^2(A) \leq \frac{1}{2}\|A\|^2 + \frac{1}{2}w\left(|A||A^*|\right). \tag{2.1}$$

Proof. Let $A = U|A|$ be the polar decomposition of A. By taking $B = U|A|^{1-t}$ and $C = |A|^t$ in (ii) of Lemma 2.5, we obtain

$$w^2(A) \leq \frac{1}{2}\|A\|^2 + \frac{1}{2}w\left(U|A|^{1-t}\widetilde{A_t}^*|A|^t\right), \tag{2.2}$$

where $\widetilde{A_t} = |A|^tU|A|^{1-t}$ is the t-Aluthge transform of A. Now, it is easy to see that $U|A|^{1-t}\widetilde{A_t}^*|A|^t = |A^*|^{2(1-t)}|A|^{2t}$. This completes the proof. □

Remark 2.7. (i) Since $w(|A||A^*|) \leq \|A^2\|$,

$$\begin{aligned} w^2(A) &\leq \frac{1}{2}\|A\|^2 + \frac{1}{2}w\left(|A||A^*|\right) \\ &\leq \frac{1}{2}\|A\|^2 + \frac{1}{2}\|A^2\| \\ &\leq \|A\|^2. \end{aligned}$$

Therefore, the bound in (2.1) refines the second bound in (1.1).
(ii) It follows from (2.2) that

$$w^2(A) \leq \frac{1}{2}\|A\|^2 + \frac{1}{2}\left\||A|^t\widetilde{A_t}|A|^{1-t}\right\|, \tag{2.3}$$

for all $t \in [0,1]$. In particular, for $t = \frac{1}{2}$

$$w^2(A) \leq \frac{1}{2}\|A\|^2 + \frac{1}{2}\left\||A|^{1/2}\widetilde{A}|A|^{1/2}\right\|, \tag{2.4}$$

where $\widetilde{A} = \widetilde{A_{\frac{1}{2}}} = |A|^{1/2}U|A|^{1/2}$ is the Aluthge transform of A.
(iii) From (i) and (ii) we deduce that if $w(A) = \|A\|$, then

$$\|A^2\| = \|A\|^2 = \left\||A|^{1/2}\widetilde{A}|A|^{1/2}\right\| = \|A\|\left\|\widetilde{A}\right\|.$$

Next theorem reads as:

Theorem 2.8. *If $A \in \mathcal{B}(\mathcal{H})$, then*

$$w^2(A) \leq \frac{1}{4}\left\||A|^{4t} + |A^*|^{4(1-t)}\right\| + \frac{1}{2}w\left(|A|^{2t}|A^*|^{2(1-t)}\right),$$

for all $t \in [0,1]$. In particular, for $t = \frac{1}{2}$

$$w^2(A) \leq \frac{1}{4}\left\||A|^2 + |A^*|^2\right\| + \frac{1}{2}w\left(|A||A^*|\right). \tag{2.5}$$

Proof. Let $A = U|A|$ be the polar decomposition of A. By taking $B = U|A|^{1-t}$ and $C = |A|^t$ in (iii) of Lemma 2.5 and using similar arguments as in the proof of Theorem 2.6, we obtain the desired results. $\square$

Remark 2.9. (i) The bound in (2.5) was also developed in [12, Theorem 2.5] using different technique.
(ii) Using similar arguments as (2.3) and (2.4), we can obtain

$$w^2(A) \leq \frac{1}{4}\left\||A|^{4t} + |A^*|^{4(1-t)}\right\| + \frac{1}{2}\left\||A|^t\widetilde{A_t}|A|^{1-t}\right\|, \tag{2.6}$$

for all $t \in [0,1]$. In particular, for $t = \frac{1}{2}$

$$w^2(A) \leq \frac{1}{4}\left\||A|^2 + |A^*|^2\right\| + \frac{1}{2}\left\||A|^{1/2}\widetilde{A}|A|^{1/2}\right\|. \tag{2.7}$$

Next result reads as follows:

Theorem 2.10. *If $A \in \mathcal{B}(\mathcal{H})$, then*

$$w(A) \leq \left\|\alpha|A|^{2(1-t)} + (1-\alpha)|A^*|^{2t}\right\|^{1/2}\|A\|^{(1-\alpha)(1-t)+\alpha t},$$

for all $\alpha, t \in [0,1]$. In particular, for $t = \frac{1}{2}$

$$w(A) \leq \|\alpha|A| + (1-\alpha)|A^*|\|^{1/2} \|A\|^{1/2}, \tag{2.8}$$

for all $\alpha \in [0,1]$. *Also, in particular, for* $\alpha = \frac{1}{2}$

$$w(A) \leq \left\| \frac{|A|^{2t} + |A^*|^{2(1-t)}}{2} \right\|^{1/2} \|A\|^{1/2}, \tag{2.9}$$

for all $t \in [0,1]$.

Proof. Let $A = U|A|$ be the polar decomposition of A. The proof follows from (iv) of Lemma 2.5 by taking $B = U|A|^{1-t}$ and $C = |A|^t$. $\square$

Note that, the bound in (2.8) was also obtained in [5, Theorem 2.8] using different technique. For our next result we need the following lemma.

Lemma 2.11. *[8, Theorem 2.5] Let* $B, C \in \mathcal{B}(\mathcal{H})$ *be such that* $|B|C = C^*|B|$. *If* $f, g : [0,\infty) \to [0,\infty)$ *are continuous functions with* $f(\lambda)g(\lambda) = \lambda$, *for all* $\lambda \geq 0$, *then*

$$w^p(BC) \leq r^p(C) w\left(\begin{bmatrix} 0 & f^{2p}(|B|) \\ g^{2p}(|B^*|) & 0 \end{bmatrix}\right) = \frac{1}{2} r^p(C) \left\| f^{2p}(|B|) + g^{2p}(|B^*|) \right\|,$$

for all $p \geq 1$.

Using the above lemma we obtain the following bound.

Theorem 2.12. *Let* $f, g : [0,\infty) \to [0,\infty)$ *be continuous functions with* $f(\lambda)g(\lambda) = \lambda$, *for all* $\lambda \geq 0$. *If* $A \in \mathcal{B}(\mathcal{H})$, *then*

$$w^p(A) \leq \frac{1}{2} \|A\|^{pt} \left\| f^{2p}\left(|A|^{1-t}\right) + g^{2p}\left(|A^*|^{1-t}\right) \right\|,$$

for all $p \geq 1$, *and for all* $t \in [0,1]$. *In particular, for* $p = 1$

$$w(A) \leq \frac{1}{2} \|A\|^{t} \left\| f^{2}\left(|A|^{1-t}\right) + g^{2}\left(|A^*|^{1-t}\right) \right\|, \tag{2.10}$$

for all $t \in [0,1]$.

Proof. Let $A = U|A|$ be the polar decomposition of A. By taking $B = U|A|^{1-t}$ and $C = |A|^t$ in Lemma 2.11, we get

$$w^p(A) \leq \frac{1}{2} r^p(|A|^t) \left\| f^{2p}\left(|A|^{1-t}\right) + g^{2p}\left(|A^*|^{1-t}\right) \right\|.$$

Since $r(|A|^t) = \||A|^t\| = \|A\|^t$, we obtain the desired results. $\square$

Considering $f(\lambda) = \lambda^{\alpha}$ and $g(\lambda) = \lambda^{1-\alpha}$, $0 \leq \alpha \leq 1$, in (2.10) we obtain the following corollary.

Corollary 2.13. *If $A \in \mathcal{B}(\mathcal{H})$, then*

$$w(A) \leq \frac{1}{2}\|A\|^t \left\| |A|^{2\alpha(1-t)} + |A^*|^{2(1-\alpha)(1-t)} \right\|,$$

for all $\alpha, t \in [0,1]$. In particular, for $t = \frac{1}{2}$

$$w(A) \leq \frac{1}{2}\|A\|^{1/2} \left\| |A|^{\alpha} + |A^*|^{1-\alpha} \right\|, \tag{2.11}$$

for all $\alpha \in [0,1]$.

Remark 2.14. (i) Let $A \in \mathcal{B}(\mathcal{H})$. In particular, considering $\alpha = \frac{1}{2}$ in (2.11) we obtain the following bound

$$w(A) \leq \frac{1}{2}\|A\|^{1/2} \left\| |A|^{1/2} + |A^*|^{1/2} \right\|, \tag{2.12}$$

which was recently proved by Kittaneh et al. [19].
(ii) It follows from the bound in (2.11) that

$$w(A) \leq \frac{1}{2}\|A\|^{1/2} \min_{\alpha \in [0,1]} \left\| |A|^{\alpha} + |A^*|^{1-\alpha} \right\|, \tag{2.13}$$

for every $A \in \mathcal{B}(\mathcal{H})$. Clearly, the bound in (2.13) is sharper than that of the bound in (2.12). Considering $A = \begin{bmatrix} 0 & 2 & 0 \\ 0 & 0 & 3 \\ 0 & 0 & 0 \end{bmatrix} \oplus [1]$ (defined on $\mathbb{C}^3 \oplus \mathbb{C}$), we have, $\left\| |A|^{\alpha} + |A^*|^{1-\alpha} \right\| = \max\left\{2^{1-\alpha}, 2^{\alpha} + 3^{1-\alpha}, 3^{\alpha}, 2\right\}$. Clearly,

$$\left\| |A|^{1/2} + |A^*|^{1/2} \right\| = \sqrt{2} + \sqrt{3} \approxeq 3.14626436994$$

and

$$\left\| |A|^{\alpha_0} + |A^*|^{1-\alpha_0} \right\| = 2^{\alpha_0} + 3^{1-\alpha_0} \approxeq 2.98118458519, \text{ where } \alpha_0 = \frac{87}{100}.$$

Hence,

$$\min_{\alpha \in [0,1]} \left\| |A|^{\alpha} + |A^*|^{1-\alpha} \right\| < \left\| |A|^{1/2} + |A^*|^{1/2} \right\|.$$

This implies that the bound in (2.13) is a non-trivial refinement of the bound in (2.12).
(iii) From the bound in (2.13) we conclude that if $w(A) = \|A\|$, then

$$\min_{\alpha \in [0,1]} \left\| |A|^{\alpha} + |A^*|^{1-\alpha} \right\| = \left\| |A|^{1/2} + |A^*|^{1/2} \right\| = 2\|A\|^{1/2}.$$

Next, we need the following lemma.

Lemma 2.15. *[8, Corollary 2.7] Let $B, C \in \mathcal{B}(\mathcal{H})$ be such that $|B|C = C^*|B|$. Then*

$$w(BC) \le \frac{1}{4}\left(\left\||B|^2 + |B^*|^2\right\| + 2\|B^2\|\right)^{1/2}\left(\left\||C|^2 + |C^*|^2\right\| + 2\|C^2\|\right)^{1/2}.$$

Now, we obtain an upper bound of $w(A)$ using the Aluthge transform of A.

Theorem 2.16. *If $A \in \mathcal{B}(\mathcal{H})$, then*

$$w(A) \le \|A\|^{1/2}\left(\frac{1}{2}\left\|\frac{|A| + |A^*|}{2}\right\| + \frac{1}{2}\left\|\widetilde{A}\right\|\right)^{1/2},$$

where $\widetilde{A} = |A|^{1/2}U|A|^{1/2}$ is the Aluthge transform of A and $A = U|A|$ is the polar decomposition of A.

Proof. By taking $B = U|A|^{1/2}$ and $C = |A|^{1/2}$ in Lemma 2.15, we obtain the desired inequality. □

Remark 2.17. (i) Clearly,

$$\|A\|^{1/2}\left(\frac{1}{2}\left\|\frac{|A| + |A^*|}{2}\right\| + \frac{1}{2}\left\|\widetilde{A}\right\|\right)^{1/2} \le \|A\|^{1/2}\left(\frac{1}{2}\|A\| + \frac{1}{2}\left\|\widetilde{A}\right\|\right)^{1/2}.$$

Since $\|\widetilde{A}\| \le \|A^2\|^{1/2}$,

$$\|A\|^{1/2}\left(\frac{1}{2}\|A\| + \frac{1}{2}\|\widetilde{A}\|\right)^{1/2} \le \|A\|^{1/2}\left(\frac{1}{2}\|A\| + \frac{1}{2}\|A^2\|^{1/2}\right)^{1/2} \le \|A\|.$$

Therefore, the bound obtained in Theorem 2.16 refines the second bound in (1.1).
(ii) Following [22, Corollary 2], we have $\||A| + |A^*|\| \le \|A\| + \left\||A|^{1/2}|A^*|^{1/2}\right\|$. Also, it is easy to observe that $\left\||A|^{1/2}|A^*|^{1/2}\right\| = r^{1/2}\left(|A||A^*|\right)$. Therefore, from Theorem 2.16, we derive that

$$\begin{aligned} w(A) &\le \|A\|^{1/2}\left(\frac{1}{2}\left\|\frac{|A| + |A^*|}{2}\right\| + \frac{1}{2}\left\|\widetilde{A}\right\|\right)^{1/2} \\ &\le \|A\|^{1/2}\left(\frac{1}{2}\left(\frac{1}{2}\|A\| + \frac{1}{2}r^{1/2}\left(|A||A^*|\right)\right) + \frac{1}{2}\|\widetilde{A}\|\right)^{1/2} \\ &\le \|A\|^{1/2}\left(\frac{1}{2}\left(\frac{1}{2}\|A\| + \frac{1}{2}w^{1/2}\left(|A||A^*|\right)\right) + \frac{1}{2}\|\widetilde{A}\|\right)^{1/2} \\ &\le \|A\|^{1/2}\left(\frac{1}{2}\left(\frac{1}{2}\|A\| + \frac{1}{2}\left\|A^2\right\|^{1/2}\right) + \frac{1}{2}\|\widetilde{A}\|\right)^{1/2}. \end{aligned}$$

Based on the inequalities in (ii) of Remark 2.17 and the first inequality in (1.1), we obtain the following proposition.

Proposition 2.18. *Let $A \in \mathcal{B}(\mathcal{H})$. If $A^2 = 0$, then*

$$w(A) = \frac{1}{2}\|A\| \text{ and } \||A| + |A^*|\| = \|A\|.$$

The converse of the above proposition may not hold. For example, considering $A = \begin{bmatrix} 0 & 2 \\ 0 & 0 \end{bmatrix} \oplus [1]$ (defined on $\mathbb{C}^2 \oplus \mathbb{C}$), we see that $w(A) = 1 = \frac{1}{2}\|A\|$ and $\||A| + |A^*|\| = 2 = \|A\|$, but $A^2 \neq 0$. To develop our next result we need the following lemma.

Lemma 2.19. *[10, Corollary 2.13] Let $B, C \in \mathcal{B}(\mathcal{H})$ be such that $|B|C = C^*|B|$. Then*

$$w(BC) \leq \frac{1}{2} r(C) \left(\|B\| + r^{1/2}(|B||B^*|) \right).$$

Now, we prove the following theorem.

Theorem 2.20. *If $A \in \mathcal{B}(\mathcal{H})$, then*

$$w(A) \leq \frac{1}{2}\|A\| + \frac{1}{2}\|A\|^t r^{1/2} \left(|A|^{1-t}|A^*|^{1-t} \right),$$

for all $t \in [0, 1]$. In particular, for $t = \frac{1}{2}$

$$w(A) \leq \frac{1}{2}\|A\| + \frac{1}{2}\|A\|^{1/2} r^{1/2} \left(|A|^{1/2}|A^*|^{1/2} \right). \tag{2.14}$$

Proof. Let $A = U|A|$ be the polar decomposition of A. By putting $B = U|A|^{1-t}$ and $C = |A|^t$ in Lemma 2.19, we get the desired results. $\square$

Remark 2.21. Let $A \in \mathcal{B}(\mathcal{H})$. For $0 \leq t \leq 1$, we see that

$$\begin{aligned} r^{1/2} \left(|A|^{1-t}|A^*|^{1-t} \right) &\leq w^{1/2} \left(|A|^{1-t}|A^*|^{1-t} \right) \\ &\leq \left\| |A|^{1-t}|A^*|^{1-t} \right\|^{1/2} \\ &\leq \||A||A^*|\|^{(1-t)/2} = \left\| A^2 \right\|^{(1-t)/2}. \end{aligned}$$

Therefore, it follows from Theorem 2.20 that, for all $t \in [0, 1]$,

$$\begin{aligned} w(A) &\leq \frac{1}{2}\|A\| + \frac{1}{2}\|A\|^t r^{1/2} \left(|A|^{1-t}|A^*|^{1-t} \right) \\ &\leq \frac{1}{2}\|A\| + \frac{1}{2}\|A\|^t w^{1/2} \left(|A|^{1-t}|A^*|^{1-t} \right) \end{aligned}$$

$$\leq \frac{1}{2}\|A\| + \frac{1}{2}\|A\|^t \left\| |A|^{1-t}|A^*|^{1-t} \right\|^{1/2}$$
$$\leq \frac{1}{2}\|A\| + \frac{1}{2}\|A\|^t \left\| A^2 \right\|^{(1-t)/2}.$$

In particular, considering $t = 0$, we get

$$w(A) \leq \frac{1}{2}\|A\| + \frac{1}{2}r^{1/2}\left(|A||A^*|\right)$$
$$\leq \frac{1}{2}\|A\| + \frac{1}{2}\|A^2\|^{1/2},$$

which was also proved in [10, Theorem 2.1 and Remark 2.2] by using different approach.

The next lemma that is needed for our purpose is as follows.

Lemma 2.22. *[12, Corollary 2.17] Let $B, C \in \mathcal{B}(\mathcal{H})$. Then*

$$w^r(BC) \leq \frac{1}{2}w^2\left(|C|^r + i|B^*|^r\right),$$

for all $r \geq 2$.

Using the above lemma we prove the following theorem.

Theorem 2.23. *If $A \in \mathcal{B}(\mathcal{H})$, then*

$$w^r(A) \leq \frac{1}{2}w^2\left(|A|^{rt} + i|A^*|^{r(1-t)}\right)$$
$$\leq \frac{1}{2}\left\| |A|^{2rt} + |A^*|^{2r(1-t)} \right\|,$$

for all $t \in [0, 1]$ and for all $r \geq 2$. In particular, for $r = 2$

$$w(A) \leq \frac{1}{\sqrt{2}}w\left(|A|^{2t} + i|A^*|^{2(1-t)}\right) \tag{2.15}$$
$$\leq \frac{1}{\sqrt{2}}\left\| |A|^{4t} + |A^*|^{4(1-t)} \right\|^{1/2}.$$

for all $t \in [0, 1]$.

Proof. Let $A = U|A|$ be the polar decomposition of A. By considering $B = U|A|^{1-t}$ and $C = |A|^t$ in Lemma 2.22, we obtain the desired first inequality. The next inequalities follow easily. □

In particular, considering $t = \frac{1}{2}$ in the inequality (2.15), we get

$$\begin{aligned} w^2(A) &\le \frac{1}{2} w^2 \left(|A| + i|A^*|\right) \\ &\le \frac{1}{2} \left\| |A|^2 + |A^*|^2 \right\|, \end{aligned}$$

which was also proved in [12, Corollary 2.15 and Remark 2.16] using different technique. To prove our final result we need the following lemma.

Lemma 2.24. *[6, Corollary 2.11] Let $B, C \in \mathcal{B}(\mathcal{H})$ be such that $|B|C = C^*|B|$. Then*

$$w(BC) \le \frac{1}{\sqrt{2}} r(C) w \left(|B| + i|B^*|\right).$$

Theorem 2.25. *If $A \in \mathcal{B}(\mathcal{H})$, then*

$$\begin{aligned} w(A) &\le \frac{1}{\sqrt{2}} \|A\|^t w \left(|A|^{1-t} + i|A^*|^{1-t}\right) \\ &\le \|A\|^t \left\| \frac{|A|^{2(1-t)} + |A^*|^{2(1-t)}}{2} \right\|^{1/2}, \end{aligned}$$

for all $t \in [0,1]$. In particular, for $t = \frac{1}{2}$

$$\begin{aligned} w(A) &\le \frac{1}{\sqrt{2}} \|A\|^{1/2} w \left(|A|^{1/2} + i|A^*|^{1/2}\right) \qquad (2.16) \\ &\le \|A\|^{1/2} \left\| \frac{|A| + |A^*|}{2} \right\|^{1/2}. \end{aligned}$$

Proof. Let $A = U|A|$ be the polar decomposition of A. By considering $B = U|A|^{1-t}$ and $C = |A|^t$ in Lemma 2.24, we obtain the desired first inequality. The next inequalities follow easily. $\square$

Remark 2.26. We would like to remark that the bound (2.16) is sharper than the bound

$$w(A) \le \|A\|^{1/2} \left\| \frac{|A| + |A^*|}{2} \right\|^{1/2}. \qquad (2.17)$$

The bound (2.17) follows from the bounds $w(A) \le \|A\|$ (see in (1.1)) and $w(A) \le \left\| \frac{|A|+|A^*|}{2} \right\|$ (see in [21]).

Declaration of competing interest

The author declares that there is no competing interest.

Data availability

No data was used for the research described in the article.

References

[1] A. Abu-Omar, F. Kittaneh, Upper and lower bounds for the numerical radius with an application to involution operators, Rocky Mt. J. Math. 45 (4) (2015) 1055–1065.

[2] A. Abu-Omar, F. Kittaneh, A numerical radius inequality involving the generalized Aluthge transform, Stud. Math. 216 (1) (2013) 69–75.

[3] S. Bag, P. Bhunia, K. Paul, Bounds of numerical radius of bounded linear operator using t-Aluthge transform, Math. Inequal. Appl. 23 (3) (2020) 991–1004.

[4] P. Bhunia, K. Paul, Refinement of numerical radius inequalities of complex Hilbert space operators, Acta Sci. Math. (Szeged) (2023), https://doi.org/10.1007/s44146-023-00070-1.

[5] P. Bhunia, Numerical radius inequalities of bounded linear operators and (α, β)-normal operators, https://doi.org/10.48550/arXiv.2301.03877, 2023.

[6] P. Bhunia, S. Jana, K. Paul, Numerical radius inequalities and estimation of zeros of polynomials, Georgian Math. J. (2023), https://doi.org/10.1515/gmj-2023-2037.

[7] P. Bhunia, S.S. Dragomir, M.S. Moslehian, K. Paul, Lectures on Numerical Radius Inequalities, Infosys Science Foundation Series in Mathematical Sciences, vol. XII+ 209, Springer, Cham, ISBN 978-3-031-13670-2, 2022.

[8] P. Bhunia, K. Paul, Some improvements of numerical radius inequalities of operators and operator matrices, Linear Multilinear Algebra 70 (10) (2022) 1995–2013.

[9] P. Bhunia, K. Paul, Development of inequalities and characterization of equality conditions for the numerical radius, Linear Algebra Appl. 630 (2021) 306–315.

[10] P. Bhunia, K. Paul, Furtherance of numerical radius inequalities of Hilbert space operators, Arch. Math. (Basel) 117 (5) (2021) 537–546.

[11] P. Bhunia, K. Paul, Proper improvement of well-known numerical radius inequalities and their applications, Results Math. 76 (4) (2021) 177.

[12] P. Bhunia, K. Paul, New upper bounds for the numerical radius of Hilbert space operators, Bull. Sci. Math. 167 (2021) 102959.

[13] P. Bhunia, S. Bag, K. Paul, Numerical radius inequalities and its applications in estimation of zeros of polynomials, Linear Algebra Appl. 573 (2019) 166–177.

[14] S. Botelho-Andrade, C.-K. Li, Generalized numerical ranges, dilation, and quantum error correction, in: Recent Trends in Operator Theory and Applications, in: Contemp. Math., vol. 737, Amer. Math. Soc., Providence, RI, 2019, pp. 25–42.

[15] M.L. Buzano, Generalizzatione della disuguaglianza di Cauchy-Schwarz, Rend. Semin. Mat. (Torino) 31 (1971/73) (1974) 405–409.

[16] S.S. Dragomir, Some inequalities for the norm and the numerical radius of linear operators in Hilbert spaces, Tamkang J. Math. 39 (1) (2008) 1–7.

[17] M. El-Haddad, F. Kittaneh, Numerical radius inequalities for Hilbert space operators. II, Stud. Math. 182 (2) (2007) 133–140.

[18] K. He, J.C. Hou, Applying the theory of numerical radius of operators to obtain multi-observable quantum uncertainty relations, Acta Math. Sin. Engl. Ser. 38 (2022) 1241–1254.

[19] F. Kittaneh, H.R. Moradi, M. Sababheh, Sharper bounds for the numerical radius, Linear Multilinear Algebra (2023), https://doi.org/10.1080/03081087.2023.2177248.

[20] F. Kittaneh, Numerical radius inequalities for Hilbert space operators, Stud. Math. 168 (1) (2005) 73–80.

[21] F. Kittaneh, Numerical radius inequality and an estimate for the numerical radius of the Frobenius companion matrix, Stud. Math. 158 (1) (2003) 11–17.

[22] F. Kittaneh, Norm inequalities for certain operator sums, J. Funct. Anal. 143 (1997) 337–348.

[23] F. Kittaneh, Notes on some inequalities for Hilbert space operators, Publ. Res. Inst. Math. Sci. 24 (2) (1988) 283–293.

[24] C.A. McCarthy, C_p, Isr. J. Math. 5 (1967) 249–271.

[25] K. Szymański, K. Życzkowski, Geometric and algebraic origins of additive uncertainty relations, J. Phys. A 53 (1) (2020) 015302.

[26] P.Y. Wu, H-L. Gau, Numerical Ranges of Hilbert Space Operators, Encyclopedia of Mathematics and Its Applications, vol. 179, Cambridge University Press, Cambridge, 2021, xviii+483 pp. ISBN: 978-1-108-47906-6 47-02.

[27] T. Yamazaki, On upper and lower bounds for the numerical radius and an equality condition, Stud. Math. 178 (1) (2007) 83–89.