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Existence of higher extremal Kähler metrics on a minimal ruled surface



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ABSTRACT

In this paper we prove that on a special type of minimal ruled surface, which is an example of a 'pseudo-Hirzebruch surface', every Kähler class admits a certain kind of 'higher extremal Kähler metric', which is a Kähler metric whose corresponding top Chern form and volume form satisfy a nice equation motivated by analogy with the equation characterizing an extremal Kähler metric. From an already proven result, it will follow that this specific higher extremal Kähler metric cannot be a 'higher constant scalar curvature Kähler (hcscK) metric', which is defined, again by analogy with the definition of a constant scalar curvature Kähler (cscK) metric, to be a Kähler metric whose top Chern form is harmonic. By doing a certain set of computations involving the top Bando-Futaki invariant we will conclude that hcscK metrics do not exist in any Kähler class on this surface.

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1. Introduction

1.1. Background of the study

Extremal Kähler metrics were introduced by Calabi [6,7] as candidates for ‘canonical metrics’ (Tian [20]) in a given Kähler class on a compact Kähler manifold. These are generalizations of constant scalar curvature Kähler (cscK) metrics which themselves include Kähler-Einstein metrics as a special case (see Aubin [2], Barth, Hulek et al. [4], Székelyhidi [19] and Tian [20] and the references therein). Extremal Kähler metrics can be equivalently defined as those Kähler metrics for which the gradient of the scalar curvature (which is expressed in terms of the first Chern form) is a holomorphic vector field (viz. $\nabla^{1,0}S(\omega) = \nabla^{(1,0)}\left(\frac{2n\pi c_1(\omega)\wedge\omega^{n-1}}{\omega^n}\right)$ is a holomorphic vector field) (see [6,7], Pingali [18], [19] and [20]). Special cases of these are cscK metrics which can be characterized as those metrics for which the corresponding first Chern form is harmonic (viz. $\Delta c_1(\omega) = -\bar{\partial}^*\bar{\partial}c_1(\omega) = 0$) (see Bando [3], [6,7], Futaki [9], Pingali [18], [19] and [20]). These definitions (and even the definition of a Kähler-Einstein metric ([9] and [19])) involve the first Chern form which is the same as the Ricci form for Kähler manifolds [19]; this fact and the fact, that the first Chern class of a Kähler manifold does not depend on the choice of the Kähler metric on the manifold [19], give some interesting results in the theory of extremal Kähler metrics (see [6,7], [9], [19] and Yau [22]).

Taking the analogy of these definitions to the top Chern form, Pingali [18] defined *higher extremal Kähler metrics* and *higher constant scalar curvature Kähler (hcscK) metrics* by considering the following equation (where $c_n(\omega)$ is the top Chern form of a Kähler metric ω on a compact Kähler n -manifold M and λ is a smooth real-valued function on M):

$$c_n(\omega) = \lambda\omega^n \quad (1.1.1)$$

where ω is said to be *higher extremal Kähler* if $\nabla^{1,0}\lambda = (\bar{\partial}\lambda)^\sharp$ is a holomorphic vector field on M and ω is said to be *hcscK* if λ is a constant, which is equivalent to saying $c_n(\omega)$ is a harmonic top form on M (see Bando [3]). The primary motivations for studying these types of Kähler metrics were that firstly the top Chern class of a Kähler manifold is the same as its Euler class (see Barth, Hulek et al. [4]) and secondly Yau [23] had stated that the behaviour of the higher Chern forms is mysterious in general and was as then unexplored, so some interesting results were expected from the study of these objects.

Earlier, hcscK metrics were studied by Bando [3], who defined obstructions for the existence of the same in a Kähler class on a compact Kähler manifold (which are now called as Bando-Futaki invariants), and also by Futaki [10,11], whereas a version of higher extremal Kähler metrics called ‘perturbed extremal Kähler metrics’ (and analogously a version of hcscK metrics called ‘perturbed constant scalar curvature Kähler (perturbed cscK) metrics’) were studied by Futaki [10,11]. But Futaki’s results [10,11] do not seem to apply in our case as mentioned by Pingali [18].

This paper answers the problem of finding higher extremal Kähler metrics with some nice enough symmetries on a certain example of minimal ruled surface, which comes from a special family of minimal ruled surfaces called ‘pseudo-Hirzebruch surfaces’, which like Hirzebruch surfaces are some of the most important examples of compact Kähler surfaces (see Barth, Hulek et al. [4] and Tønnesen-Friedman [21]). The problem of constructing higher extremal Kähler metrics on this minimal ruled surface was started by Pingali [18] motivated by the analogy with the problem of constructing extremal Kähler metrics on this surface which was dealt with first by Tønnesen-Friedman [21] (and later by Apostolov, Calderbank et al. [1]). Pingali [18] had proven the existence of a higher extremal Kähler metric with some required properties in one specific Kähler class on this Kähler surface, while we will prove in this paper that in every Kähler class on this Kähler surface there exists a higher extremal Kähler representative (which is constructed by a certain method which imposes those required nice symmetries on the metric), and it was already proven in [18] that this constructed metric, if it does exist, cannot be hcscK. Then after proving some facts about the top Bando-Futaki invariant on a general compact Kähler manifold we will be able to conclude that hcscK metrics do not exist in any Kähler class on our Kähler surface. We will then generalize our results for all pseudo-Hirzebruch surfaces. Finally we will do a brief comparison of our results in the higher extremal Kähler case with those gotten by Tønnesen-Friedman [21] (and Apostolov, Calderbank et al. [1]) in the usual extremal Kähler analogue of this problem.

1.2. Overview of the paper

In this paper we will consider the minimal ruled surface $X := \mathbb{P}(L \oplus \mathcal{O})$ where L is a degree -1 holomorphic line bundle on a genus 2 Riemann surface Σ , Σ is equipped with a Kähler metric ω_Σ of constant scalar curvature -2 and L is equipped with a Hermitian metric h whose curvature form is $-\omega_\Sigma$. The problem of finding extremal Kähler metrics on X was studied by Tønnesen-Friedman [21], more general results regarding the same were already proven in [21], then further results in more generalized settings were proven by Apostolov, Calderbank et al. [1] and a complete exposition about the specific case of X is contained in Székelyhidi [19] where the momentum construction method outlined in Hwang-Singer [12] is used to obtain the required symmetries on the metric. In this paper we will deal with the higher extremal Kähler analogue of this problem and prove some important results on the same.

It has been proven using the Leray-Hirsch Theorem and the Nakai-Moishezon Criterion (proven by Fujiki [8] and Tønnesen-Friedman [21] and briefly explained in Subsection 2.1) that the Kähler cone (i.e. the set of all Kähler classes) of X is precisely the following set:

$$H^{(1,1)}(X, \mathbb{R})^+ = \{aC + bS_\infty \mid a, b > 0\} \subseteq H^{(1,1)}(X, \mathbb{R}) \subseteq H^2(X, \mathbb{R}) = \mathbb{R}C \oplus \mathbb{R}S_\infty \quad (1.2.1)$$

where C is the Poincaré dual of a typical fibre of X and S_∞ is the image of Σ in X sitting as the ‘infinity divisor’ of X . Following the usual extremal Kähler case as in Székelyhidi [19], Pingali [18] took $a = 2\pi$, $b = 2m\pi$ where $m > 0$ and posed the problem of finding a higher extremal Kähler metric ω on X , analogously using the momentum construction method of Hwang-Singer [12] (described in Subsection 2.2), satisfying the following:

$$[\omega] = 2\pi (C + mS_\infty), \quad c_2(\omega) = \frac{\lambda}{2(2\pi)^2} \omega^2, \quad \nabla^{1,0} \lambda \in \mathfrak{h}(X) \quad (1.2.2)$$

where $[\omega]$ denotes the Kähler class of ω and $\mathfrak{h}(X)$ denotes the set of all holomorphic vector fields on X .

Pingali [18] solved the problem (1.2.2) for $m = 1$ (i.e. constructed the required higher extremal Kähler metric ω in the Kähler class $2\pi(C + S_\infty)$) and conjectured that there might exist a maximum value of m beyond which there may not exist a solution to the problem (1.2.2), meaning there perhaps may not exist higher extremal Kähler metrics in the Kähler classes $2\pi(C + mS_\infty)$ for values of m larger than this maximum value. This was expected by analogy with the usual extremal Kähler analogue of the problem (1.2.2) (refer to Apostolov, Calderbank et al. [1], Székelyhidi [19] and Tønnesen-Friedman [21]). But on the contrary we will show (in Subsections 2.3, 3.1 and 3.2) that higher extremal Kähler metrics with the required properties exist in the Kähler classes $2\pi(C + mS_\infty)$ for all positive values of m . Since being a higher extremal Kähler metric is a scale-invariant property (as will be seen in Subsection 2.3) an appropriate rescaling procedure applied on the Kähler metric and its top Chern form will enable us to construct the required higher extremal Kähler metrics in the Kähler classes $aC + bS_\infty$ with $a, b > 0$, which exhaust the Kähler cone of X .

As will be seen in Subsections 2.2 and 2.3, solving the problem (1.2.2) eventually boils down to solving an ODE BVP for the ‘momentum profile’ ϕ of the Kähler metric ω (a smooth real-valued function of a real variable γ which appears in the momentum construction method), depending on $m > 0$ and one more real parameter C , and the ODE in our case is not readily integrable and also not autonomous (it is a version of Chini’s Equation) unlike the usual extremal Kähler case given in [19] and [21], and hence requires a very delicate analysis for the existence of a solution satisfying all the boundary conditions (see Pingali [18]). Pingali [18] used some explicit numerical estimates specific to the case $m = 1$ and managed to solve it for $m = 1$, but our method of proving the existence of a solution to the required ODE BVP (explained in Subsections 3.1 and 3.2) is completely different from that of Pingali [18] as we are doing it for an arbitrary positive value of m . Roughly speaking, we will fix $m > 0$ and drop the final boundary condition and look at the resultant ODE IVP, and then find the smooth solutions to the ODE IVP depending on the parameter C and study the variation of the final boundary value of these solutions w.r.t. C , which will eventually lead us to a value of C for which the required final boundary condition holds.

Then after constructing the higher extremal Kähler metric with the required nice symmetries in each Kähler class of X we will tackle the problem of (non-)existence of hscK metrics on X in Section 4. We will first prove that the top Bando-Futaki invariant $\mathcal{F}_n : \mathfrak{h}(M) \times H^{(1,1)}(M, \mathbb{R})^+ \rightarrow \mathbb{R}$ for a compact Kähler n -manifold M can be re-expressed in the following way, if the Kähler metric ω on M is higher extremal Kähler:

$$\mathcal{F}_n(Y, [\omega]) = - \int_M (\lambda - \lambda_0)^2 \omega^n =: -\|\lambda - \lambda_0\|_{\mathcal{L}^2(M, \omega)}^2 \tag{1.2.3}$$

where $\lambda \in \mathcal{C}^\infty(M, \mathbb{R})$ satisfies equation (1.1.1) and $\lambda_0 \in \mathbb{R}$ is a constant. Then it will follow that given ω is higher extremal Kähler, $\mathcal{F}_n(\cdot, [\omega]) \equiv 0$ if and only if ω is hscK, and further in the Kähler class of an hscK metric on M every higher extremal Kähler representative is hscK. This statement along with the fact, that our constructed higher extremal Kähler metrics on our surface X are not hscK (proven by Pingali [18] given their existence), will help us in concluding the complete non-existence of hscK metrics (even without the nice symmetries imposed by the momentum construction method) on X .

In Section 5 we will take the genus of the Riemann surface Σ to be $g \geq 2$, the constant scalar curvature of the Kähler metric ω_Σ on Σ to be $-2(g - 1)$, the degree of the holomorphic line bundle L to be $d \neq 0$ and the curvature form of the Hermitian metric h on L to be $d\omega_\Sigma$. We will see that for the minimal ruled surface $X := \mathbb{P}(L \oplus \mathcal{O})$ in this general setting as well, the same analysis and all the arguments as in the special case of $g = 2$ and $d = -1$ (shown in Sections 2, 3 and 4) go through well and we can obtain all the results about the existence of higher extremal Kähler metrics and the non-existence of hscK metrics on X in the general case as well.

We will finally summarize our results on higher extremal Kähler (and hscK) metrics and the top Bando-Futaki invariant and compare them with the analogous results of Apostolov, Calderbank et al. [1], Calabi [7], LeBrun-Simanca [14] and Tønnesen-Friedman [21] on extremal Kähler (and cscK) metrics and the Futaki invariant (exposition contained in Székelyhidi [19]) in Section 6.

2. Constructing higher extremal Kähler metrics on a minimal ruled surface

2.1. The minimal ruled surface and its Kähler cone

Let (Σ, ω_Σ) be a genus 2 (compact) Riemann surface equipped with a Kähler metric of constant scalar curvature -2 (and hence area 2π). Let L be a degree -1 holomorphic line bundle on Σ equipped with a Hermitian metric h whose curvature form is $-\omega_\Sigma$. Let X be the minimal ruled (complex) surface $\mathbb{P}(L \oplus \mathcal{O})$ where \mathcal{O} is the trivial line bundle on Σ and \mathbb{P} denotes vector bundle projectivization. This surface X is an example of a ‘pseudo-Hirzebruch surface’ (see definition in Tønnesen-Friedman [21] and in Section 5).

Let C be the Poincaré dual of a typical fibre of X (i.e. C is a copy of the Riemann sphere S^2 sitting in X), S_∞ (called the *infinity divisor* of X) be the image of the subbundle $L \oplus \{0\} \subseteq L \oplus \mathcal{O}$ under the bundle projectivization to $X = \mathbb{P}(L \oplus \mathcal{O})$ and similarly S_0 (called the *zero divisor* of X) be the image of the subbundle $\{0\} \oplus \mathcal{O} \subseteq L \oplus \mathcal{O}$ under the bundle projectivization to $X = \mathbb{P}(L \oplus \mathcal{O})$, so that S_∞ and S_0 are actually copies of Σ sitting in X as its infinity and zero divisors respectively (and Σ is identified with S_0 as a (complex) curve in X). We have the following intersection formulae (refer to Barth, Hulek et al. [4], Székelyhidi [19] and Tønnesen-Friedman [21]):

$$C^2 = 0, \quad S_\infty^2 = 1, \quad S_0^2 = -1, \quad C \cdot S_\infty = 1, \quad C \cdot S_0 = 1, \quad S_\infty \cdot S_0 = 0 \quad (2.1.1)$$

and also the following intersection formulae (again refer to [4], [19] and [21]), where $c_1(L) \in H^{(1,1)}(\Sigma, \mathbb{R}) = H^2(\Sigma, \mathbb{R})$ is the first Chern class of L , $[\omega_\Sigma] \in H^{(1,1)}(\Sigma, \mathbb{R}) = H^2(\Sigma, \mathbb{R})$ is the Kähler class of ω_Σ and $[\Sigma] \in H^2(\Sigma, \mathbb{R})$ is the fundamental class of Σ (and $[\Sigma]$ is identified with S_0 in $H^2(X, \mathbb{R})$):

$$c_1(L) \cdot [\Sigma] = -1, \quad [\omega_\Sigma] \cdot [\Sigma] = 2\pi, \quad C \cdot [\Sigma] = 1, \quad S_\infty \cdot [\Sigma] = 0, \quad S_0 \cdot [\Sigma] = -1 \quad (2.1.2)$$

By the Leray-Hirsch Theorem we have $H^2(X, \mathbb{R}) = \mathbb{R}C \oplus \mathbb{R}S_\infty$. So starting with a general cohomology class $\alpha := aC + bS_\infty \in H^{(1,1)}(X, \mathbb{R}) \subseteq H^2(X, \mathbb{R})$ we have the following criterion for determining when α will be a Kähler class on X (integral cohomology version attributed to Nakai-Moishezon and explained in Barth, Hulek et al. [4] with all the relevant references, while real cohomology version proven in Fujiki [8], Lamari [13], LeBrun-Singer [16] and Tønnesen-Friedman [21]):

Theorem 2.1.1 (*Nakai-Moishezon Criterion*). α is a Kähler class on X if and only if the following conditions are satisfied:

- (1) $\alpha^2 > 0$.
- (2) $\alpha \cdot \Gamma > 0$ for every irreducible (complex) curve $\Gamma \subseteq X$.

It was further shown by Fujiki [8] that it suffices to check the intersection number of α against $\Gamma = C, S_\infty, S_0$ and $[\Sigma]$ only in Theorem 2.1.1 (see the explanation given in Tønnesen-Friedman [21]):

Corollary 2.1.1 (*Fujiki, Tønnesen-Friedman*). α is a Kähler class on X if and only if the following conditions are satisfied:

- (1) $\alpha^2 > 0$.
- (2) $\alpha \cdot C > 0, \alpha \cdot S_\infty > 0, \alpha \cdot S_0 > 0$ and $\alpha \cdot [\Sigma] > 0$.

From Corollary 2.1.1 and the intersection formulae (2.1.1) and (2.1.2) we derive the following inequalities characterizing all Kähler classes $\alpha = aC + bS_\infty$ on X :

$$2ab + b^2 > 0, \quad b > 0, \quad a + b > 0, \quad a > 0, \quad a > 0 \tag{2.1.3}$$

which simply boil down to $a > 0, b > 0$. Therefore the Kähler cone of X is precisely ([8] and [21]):

$$H^{(1,1)}(X, \mathbb{R})^+ = \{aC + bS_\infty \mid a, b > 0\} \tag{2.1.4}$$

Following Székelyhidi [19] we will first consider Kähler classes only of the form $\beta := 2\pi(C + mS_\infty)$ with $m > 0$, and after obtaining results about the existence of higher extremal Kähler metrics and the non-existence of hcscK metrics in these Kähler classes, we will generalize these results to the general Kähler classes which are of the form $\alpha = aC + bS_\infty$ with $a, b > 0$ by using a simple rescaling argument.

2.2. Description of the momentum construction method

We give here a brief description of the momentum construction method attributed to Hwang-Singer [12] which was applied by Pingali [18] to define the problem of construction of higher extremal Kähler metrics possessing some nice symmetries along the fibres and the zero and infinity divisors of the minimal ruled surface X described in Subsection 2.1. In this description we are following [18].

We write an ansatz for a Kähler metric on X , which is supposed to lie in a required Kähler class and be higher extremal Kähler, in a way similar to Hwang-Singer [12] and Székelyhidi [19]. The idea is to first consider an ansatz for a metric on the total space of L minus its zero section and then extend this metric across the zero and infinity divisors of $X = \mathbb{P}(L \oplus \mathcal{O})$, and this is done by taking the pullback of L to its total space minus the zero section and then adding the curvature of the resultant bundle to the pullback of ω_Σ to X .

Let $p : X \rightarrow \Sigma$ be the fibre bundle projection, z be a coordinate on Σ , w be a coordinate on the fibres of L , $s := \ln |(z, w)|_h^2 = \ln |w|^2 + \ln h(z)$ be the coordinate on the total space of L minus the zero section, f be a strictly convex smooth function of s such that $s + f(s)$ is strictly increasing, and ω be a Kähler metric on X given by the following ansatz (as in [18] and [19]):

$$\omega = p^*\omega_\Sigma + \sqrt{-1}\partial\bar{\partial}f(s) \tag{2.2.1}$$

From the computations done in [18] and [19] we get the following expression for ω :

$$\omega = (1 + f'(s))p^*\omega_\Sigma + f''(s)\sqrt{-1}\frac{dw \wedge d\bar{w}}{|w|^2} \tag{2.2.2}$$

We want ω to be in the Kähler class $2\pi(C + mS_\infty)$ where $m \in \mathbb{R}_{>0}$, and for that to happen we must have $0 \leq f'(s) \leq m$ (which we get by integrating equation (2.2.2) over

X and computing the areas of C and S_∞ using the intersection formulae (2.1.1) and (2.1.2), as done in [18] and [19]).

We compute the curvature form matrix of ω given by $\Theta(\omega) := \bar{\partial}(H^{-1}\partial H)(\omega)$, where $H(\omega)$ is the Hermitian matrix of ω , as follows:

$$\omega^2 = 2(1 + f'(s))f''(s)\mathbf{p}^*\omega_\Sigma\sqrt{-1}\frac{dw \wedge d\bar{w}}{|w|^2} \tag{2.2.3}$$

$$\Theta(\omega) = \begin{bmatrix} -\partial\bar{\partial}\ln(1 + f'(s)) + 2\sqrt{-1}\mathbf{p}^*\omega_\Sigma & 0 \\ 0 & -\partial\bar{\partial}\ln(f''(s)) \end{bmatrix} \tag{2.2.4}$$

This is where the conditions $1 + f'(s) > 0$ and $f''(s) > 0$ are needed. We then use the Legendre Transform $F(\tau)$ in the variable $\tau := f'(s)$ as follows:

$$f(s) + F(\tau) = s\tau \tag{2.2.5}$$

We then define the *momentum profile* of ω as $\phi(\tau) := \frac{1}{F''(\tau)} = f''(s)$ (again as in [18] and [19]). We then write down the curvature form matrix $\sqrt{-1}\Theta(\omega)$ in terms of $\phi(\gamma)$, where $\gamma := \tau + 1 \in [1, m + 1]$, as follows:

$$\sqrt{-1}\Theta(\omega) = \begin{bmatrix} \frac{\phi}{\gamma} \left(\frac{\phi}{\gamma} - \phi' \right) \sqrt{-1}\frac{dw \wedge d\bar{w}}{|w|^2} - \left(\frac{\phi}{\gamma} + 2 \right) \mathbf{p}^*\omega_\Sigma & 0 \\ 0 & -\phi''\phi\sqrt{-1}\frac{dw \wedge d\bar{w}}{|w|^2} - \phi'\mathbf{p}^*\omega_\Sigma \end{bmatrix} \tag{2.2.6}$$

The top Chern form of ω is given by $c_2(\omega) := \frac{1}{(2\pi)^2} \det(\sqrt{-1}\Theta(\omega))$ and in terms of $\phi(\gamma)$ is given by:

$$c_2(\omega) = \frac{1}{(2\pi)^2}\mathbf{p}^*\omega_\Sigma\sqrt{-1}\frac{dw \wedge d\bar{w}}{|w|^2}\frac{\phi}{\gamma^2}(\gamma(\phi + 2\gamma)\phi'' + \phi'(\phi'\gamma - \phi)) \tag{2.2.7}$$

In order for ω to be a higher extremal Kähler metric we need to have:

$$c_2(\omega) = \frac{\lambda}{2(2\pi)^2}\omega^2 \tag{2.2.8}$$

where $\nabla^{1,0}\lambda$ is a holomorphic vector field. Comparing equations (2.2.3), (2.2.7) and (2.2.8) we do the following calculation:

$$\nabla^{1,0}\lambda = \lambda'\nabla^{1,0}\gamma = \lambda'\nabla^{1,0}\tau = \lambda'\nabla^{(1,0)}(f'(s)) = \lambda'w\frac{\partial}{\partial w} \tag{2.2.9}$$

So $\nabla^{1,0}\lambda$ is a holomorphic vector field if and only if λ' is a constant i.e. $\lambda = A\gamma + B$ for some $A, B \in \mathbb{R}$.

So finally ω is a higher extremal Kähler metric on X if and only if its momentum profile $\phi(\gamma)$, $\gamma \in [1, m + 1]$ satisfies the following ODE, for some $C \in \mathbb{R}$ (which is obtained by

substituting the value of λ in equations (2.2.7) and (2.2.8) and integrating w.r.t. γ , as done in [18]:

$$(2\gamma + \phi)\phi' = A\frac{\gamma^4}{3} + B\frac{\gamma^3}{2} + C\gamma \tag{2.2.10}$$

with the following boundary conditions (which are required for ω to extend smoothly to the zero and infinity divisors of X as shown in [18] and [19]):

$$\begin{aligned} \phi(1) &= \phi(m+1) = 0 \\ \phi'(1) &= -\phi'(m+1) = 1 \end{aligned} \tag{2.2.11}$$

and with $\phi > 0$ on $(1, m+1)$, as $\phi = f'' > 0$.

Now finally the problem boils down to solving the ODE (2.2.10) for $\phi(\gamma)$ on $[1, m+1]$ with the boundary conditions (2.2.11) and with $\phi(\gamma) > 0$ on $(1, m+1)$, for some constants $A, B, C \in \mathbb{R}$.

2.3. Analysis of the ODE BVP for the momentum profile

Following Pingali [18] we define the polynomial $p(\gamma) := A\frac{\gamma^3}{3} + B\frac{\gamma^2}{2} + C$ and the transformation $v := \frac{(2\gamma+\phi)^2}{2}$, $\gamma \in [1, m+1]$ after which the ODE BVP (2.2.10) and (2.2.11) with $\phi > 0$ on $(1, m+1)$, obtained in Subsection 2.2, reduces to the following:

$$\begin{aligned} v' &= 2\sqrt{2}\sqrt{v} + p(\gamma)\gamma \text{ on } [1, m+1] \\ v(1) &= 2, \quad v(m+1) = 2(m+1)^2 \\ v'(1) &= 6, \quad v'(m+1) = 2(m+1) \\ v(\gamma) &> 2\gamma^2 \text{ on } (1, m+1) \end{aligned} \tag{2.3.1}$$

As shown in [18], imposing the boundary conditions on the ODE in (2.3.1) gives us A, B as linear functions of C as follows:

$$\begin{aligned} A(C) &= \frac{3C}{m} \left[1 - \frac{1}{(m+1)^2} \right] - \frac{6}{m} \left[1 + \frac{1}{(m+1)^2} \right] \\ B(C) &= -2C \left[1 + \frac{1}{m} - \frac{1}{m(m+1)^2} \right] + 4 \left[1 + \frac{1}{m} + \frac{1}{m(m+1)^2} \right] \end{aligned} \tag{2.3.2}$$

Now we observe that if we solve the ODE in (2.3.1) with the boundary condition $v(1) = 2$ then equations (2.3.2) will imply that $v'(1) = 6$, and similarly solving the ODE in (2.3.1) with the boundary condition $v(m+1) = 2(m+1)^2$ will give us $v'(m+1) = 2(m+1)$ after applying equations (2.3.2). So after this the ODE BVP (2.3.1) reduces to the following:

$$\begin{aligned}
 v' &= 2\sqrt{2}\sqrt{v} + p(\gamma)\gamma \text{ on } [1, m+1] \\
 v(1) &= 2, \quad v(m+1) = 2(m+1)^2 \\
 v(\gamma) &> 2\gamma^2 \text{ on } (1, m+1)
 \end{aligned} \tag{2.3.3}$$

where $p(\gamma) = A(C)\frac{\gamma^3}{3} + B(C)\frac{\gamma^2}{2} + C$ after considering the equations (2.3.2).

We then analyze the polynomials $p(\gamma)$ and $p(\gamma)\gamma$ on $[1, m+1]$ and get the following result (proven by Pingali [18]):

Lemma 2.3.1 (Pingali). *The polynomial $p(\gamma)$ satisfying $p(1) = 2$ and $p(m+1) = -2$ has exactly one root γ_0 in $[1, m+1]$ and has at most one critical point $\gamma' := -\frac{B}{A}$ in $[1, m+1]$. Further the polynomial $p(\gamma)\gamma$ also has the same γ_0 as its unique root in $[1, m+1]$ but has at most 3 critical points in $[1, m+1]$. In particular both $p(\gamma)$ and $p(\gamma)\gamma$ are strictly positive on $[1, \gamma_0)$ and strictly negative on $(\gamma_0, m+1]$.*

By using Lemma 2.3.1 we can observe that if v is a smooth solution of the ODE in (2.3.3) on $[1, m+1]$ satisfying both the boundary conditions viz. $v(1) = 2$ and $v(m+1) = 2(m+1)^2$, then integrating the expression for v' in (2.3.3) on $[1, \gamma_0]$ and $[\gamma_0, m+1]$ separately and noting the sign of $p(\gamma)\gamma$ on both the subintervals will help us conclude $v(\gamma) \geq 2\gamma^2$ on $[1, m+1]$, and then rewriting the equation of v' in (2.3.3) as $2\sqrt{v}(\sqrt{v} - \sqrt{2}\gamma)' = p(\gamma)\gamma$ and using the uniqueness of γ_0 will help us conclude $v(\gamma) > 2\gamma^2$ on $(1, m+1)$ (see [18]). Thus the ODE BVP (2.3.3) further reduces to the following:

$$\begin{aligned}
 v' &= 2\sqrt{2}\sqrt{v} + p(\gamma)\gamma \text{ on } [1, m+1] \\
 v(1) &= 2, \quad v(m+1) = 2(m+1)^2
 \end{aligned} \tag{2.3.4}$$

Now finally our problem has been reduced to solving the ODE BVP (2.3.4), depending on the parameter $C \in \mathbb{R}$, for a fixed $m \in \mathbb{R}_{>0}$ (indicating the Kähler class under consideration). By following the strategy of Pingali [18] we drop the final boundary condition $v(m+1) = 2(m+1)^2$ in the ODE BVP (2.3.4) and consider the following ODE IVP:

$$\begin{aligned}
 v' &= 2\sqrt{2}\sqrt{v} + p(\gamma)\gamma \text{ on } [1, m+1] \\
 v(1) &= 2
 \end{aligned} \tag{2.3.5}$$

We first get a smooth solution $v(\cdot; C)$ on $[1, m+1]$ for the ODE IVP (2.3.5) for each C in an interval of the real line. Then we find a C_1 in that interval such that $v(m+1; C_1) > 2(m+1)^2$ and another C_2 in the same interval such that $v(m+1; C_2) < 2(m+1)^2$, thereby proving that there exists a C in the interval such that $v(m+1; C) = 2(m+1)^2$. The first part was done by Pingali [18] (see the following result where m is fixed) and this paper's main goal is to do the second part which requires some deeper analysis of

the variation of the ODE IVP (2.3.5) w.r.t. the parameter C . Note that this will be done for an arbitrary $m > 0$.

Theorem 2.3.1 (Pingali). *There exists an $M > 2$ (depending only on m) such that for any $C < M$ there exists a unique smooth solution $v(\cdot; C)$ on $[1, m + 1]$ for the ODE IVP (2.3.5). Further there exists a $C_1 < M$ such that $v(m + 1; C_1) > 2(m + 1)^2$.*

We will prove the following main result in this paper (where m is fixed):

Theorem 2.3.2. *There exists a unique $M = M(m) > 2$ such that for every $C \in (-\infty, M)$ there exists a unique smooth solution $v(\cdot; C) : [1, m + 1] \rightarrow \mathbb{R}$ for the ODE IVP (2.3.5) such that $\lim_{C \rightarrow -\infty} v(m + 1; C) = \infty$ and $\lim_{C \rightarrow M^-} v(m + 1; C) = 0$, and for every $C \geq M$ there exists a unique smooth solution $v(\cdot; C) : [1, \gamma_{*,C}] \rightarrow \mathbb{R}$ for (2.3.5), for a unique $\gamma_{*,C} \in (1, m + 1]$, such that $v(\cdot; C)$ cannot be continued smoothly beyond $\gamma_{*,C}$. Further there exists a unique $C = C(m) \in (-\infty, M)$ such that $v(m + 1; C) = 2(m + 1)^2$, and further the C with this property has to be strictly greater than 2.*

We will then have answered the question of the existence of higher extremal Kähler metrics in the Kähler classes of the form $2\pi(C + mS_\infty)$ on X affirmatively:

Corollary 2.3.1. *For each $m > 0$ there exists a higher extremal Kähler metric ω on X satisfying the following:*

$$[\omega] = 2\pi(C + mS_\infty), \quad c_2(\omega) = \frac{\lambda}{2(2\pi)^2} \omega^2, \quad \nabla^{1,0} \lambda \in \mathfrak{h}(X) \tag{2.3.6}$$

Remark 2.3.1. Note that Pingali [18] had proven for $m = 1$ that there exists a $C_2 < M$ such that $v(m + 1; C_2) < 2(m + 1)^2$, thus proving the existence of a $C < M$ satisfying $v(m + 1; C) = 2(m + 1)^2$, and hence proving the existence of the required higher extremal Kähler metric in the Kähler class $2\pi(C + S_\infty)$. But this was done by using some explicit numerical bounds on $p(\gamma)\gamma$ and v' specifically applicable for $m = 1$, and the author was not able to generalize those arguments and estimates for a general $m > 0$. So we will adopt a completely different strategy of proof for Theorem 2.3.2.

Now we observe that on a compact Kähler n -manifold M if we rescale a Kähler metric ω by a factor of $k > 0$ then the Kähler class and the Hermitian matrix get multiplied by k whereas the connection and hence the curvature remain unchanged. So the curvature form matrix and hence the top Chern form (as defined in Subsection 2.2) of $k\omega$ are exactly the same as that of ω . Thus if ω satisfying equation (1.1.1) is a higher extremal Kähler metric on M then so is $k\omega$ as seen by the following:

$$c_n(k\omega) = c_n(\omega) = \lambda \omega^n = \frac{\lambda}{k^n} (k\omega)^n \tag{2.3.7}$$

So on our surface X if $\omega \in 2\pi(C + mS_\infty)$ is a higher extremal Kähler metric satisfying the equations (2.3.6) then for any $k > 0$ the rescaled Kähler metric $\eta := \frac{k}{2\pi}\omega \in k(C + mS_\infty)$ is also higher extremal Kähler and satisfies the equation $c_2(\eta) = \frac{\lambda}{2k^2}\eta^2$.

This allows us to generalize Corollary 2.3.1 to the general Kähler classes on X which are of the form $aC + bS_\infty$ where $a, b > 0$:

Corollary 2.3.2. *For all $a, b > 0$ there exists a higher extremal Kähler metric η on X satisfying the following:*

$$[\eta] = aC + bS_\infty, \quad c_2(\eta) = \frac{\lambda}{2a^2}\eta^2, \quad \nabla^{1,0}\lambda \in \mathfrak{h}(X) \tag{2.3.8}$$

3. Proof of the main result

3.1. First part of the proof

Throughout Subsections 3.1 and 3.2, $m > 0$ is fixed. The final goal over here is to prove Theorem 2.3.2.

Motivation 3.1.1. We will first prove that for every $C \in \mathbb{R}$ there exists a unique \mathcal{C}^1 solution v to the ODE IVP (2.3.5) on a non-degenerate interval containing 1, and in fact this v exists and is strictly increasing on $[1, \gamma_0]$ where γ_0 is the unique root of $p(\gamma)\gamma$ in $[1, m + 1]$. We will then prove that the \mathcal{C}^1 solution v defined on any interval is always strictly positive on the interval, and as a consequence is smooth (i.e. \mathcal{C}^∞) on the interval. We will finally prove a necessary and sufficient condition for the continuation of the solution v defined a priori on $[1, \tilde{r})$ for a given $\tilde{r} \in (1, m + 1]$.

As was noted by Pingali [18] if v is a \mathcal{C}^1 solution to (2.3.5) on any interval then substituting $\sqrt{v} < v + 1$ and $|p(\gamma)\gamma| \leq l$ (for some $l > 0$) in the expression for $v' = (v + 1)'$ in (2.3.5) and applying Grönwall’s inequality will give us a $K > 0$ such that $v(\gamma) \leq K$ on the interval.

Thus solutions to (2.3.5) are always bounded above (and always bounded below by 0) on any interval on which they exist. So by standard ODE Theory the existence of a strictly positive lower bound on a solution of (2.3.5) is a sufficient condition for the continuation of the solution beyond its prior interval of definition (Pingali [18]).

Lemma 3.1.1 (Pingali). *For a given $C \in \mathbb{R}$ let v be a \mathcal{C}^1 solution to the ODE IVP (2.3.5) existing on $[1, \tilde{r}) \subseteq [1, m + 1]$. If there exists an $\epsilon > 0$ such that $v(\gamma) \geq \epsilon$ on $[1, \tilde{r})$ then v can be continued beyond \tilde{r} .*

We will prove the converse of Lemma 3.1.1 (viz. Theorem 3.1.1), but before that we prove some more basic results:

Lemma 3.1.2. *For every $C \in \mathbb{R}$ there exists a unique C^1 solution v to the ODE IVP (2.3.5) on $[1, r]$ for some $r \in (1, m + 1]$ such that $v' > 0$ on $[1, r)$. If $[1, r'] \subseteq [1, m + 1]$ is the maximal interval of existence of v then $[1, \gamma_0] \subseteq [1, r')$ and $v' > 0$ on $[1, \gamma_0]$. Similarly if $[1, m + 1]$ is the maximal interval of existence of v then $[1, \gamma_0] \subseteq [1, m + 1]$ and $v' > 0$ on $[1, \gamma_0]$.*

Proof. As noted in Subsection 2.3, $v(1) = 2$ will imply $v'(1) = 6$ in (2.3.5). Since $\sqrt{v(1)} = \sqrt{2} > 0$ so the RHS of (2.3.5) is continuous in γ and locally Lipschitz in v in a neighbourhood of 1 and by standard ODE Theory there exists a unique C^1 solution v to (2.3.5) on $[1, r)$ for some $r \in (1, m + 1]$. Since $v'(1) = 6 > 0$ so this $r \in (1, m + 1]$ can be chosen so that $v' > 0$ on $[1, r)$.

Let $[1, r'] \subseteq [1, m + 1]$ be the maximal interval of existence of v . If $\gamma_0 \geq r'$ then by Lemma 2.3.1, $p(\gamma) \geq 0$ on $[1, r')$ and hence $v' \geq 0$ on $[1, r')$. So $v(\gamma) \geq v(1) = 2 > 0$ on $[1, r')$ and by Lemma 3.1.1, v can be continued beyond r' contradicting the maximality of r' . So $[1, \gamma_0] \subseteq [1, r')$ and $v' \geq 0$ on $[1, \gamma_0]$, but as $\sqrt{v(\gamma)} > 0$ on $[1, \gamma_0]$ so $v' > 0$ on $[1, \gamma_0]$.

If $[1, m + 1]$ is the maximal interval of existence of v then by Lemma 2.3.1, $[1, \gamma_0] \subseteq [1, m + 1]$ and by the same arguments as in the above case, $v' > 0$ on $[1, \gamma_0]$. \square

Remark 3.1.1. Note that $v' > 0$ on $[1, \gamma_0]$ actually implies $v' > 0$ on $[1, \gamma'_0] \subseteq [1, r')$ (or $[1, \gamma'_0] \subseteq [1, m + 1]$) for some $\gamma'_0 > \gamma_0$.

Observe that if there exists a C^1 solution v to (2.3.5) on any interval then $v \geq 0$ on the interval, but Lemma 3.1.3 is saying that $v > 0$ on the interval.

Lemma 3.1.3 (Positivity of Solutions). *For a given $C \in \mathbb{R}$ let v be the unique C^1 solution to the ODE IVP (2.3.5) existing on some interval containing 1.*

- (1) *If $[1, r'] \subseteq [1, m + 1]$ is the maximal interval of existence of v then $v(\gamma) > 0$ for all $\gamma \in [1, r')$ and $\lim_{\gamma \rightarrow r'} v(\gamma) = 0$ and $\lim_{\gamma \rightarrow r'} v'(\gamma) < 0$.*
- (2) *If $[1, m + 1]$ is the maximal interval of existence of v then $v(\gamma) > 0$ for all $\gamma \in [1, m + 1]$.*

Proof.

Case. (1) If $[1, r')$ is the maximal interval of existence of v then by Lemma 3.1.2, $\gamma_0 \in [1, r')$ and $v'(\gamma_0) > 0$ and also $v(\gamma_0) \geq v(1) = 2 > 0$. Since v cannot be continued beyond r' so by Lemma 3.1.1, $\inf_{\gamma \in [1, r')} v(\gamma) = 0$. If $t_0 \in (1, r')$ is such that $v(t_0) = 0$ then t_0 must be a point of local minimum of v and so $v'(t_0) = 0$ which will imply $p(t_0)t_0 = 0$ and then $t_0 = \gamma_0$ (by the uniqueness of γ_0) which contradicts the first assertion above. So $v(\gamma) > 0$ for all $\gamma \in [1, r')$.

Now since v and $p(\gamma)\gamma$ are bounded on $[1, r']$, from the expression of v' in (2.3.5) we get v' is bounded, thereby implying v is Lipschitz on $[1, r']$. So $\lim_{\gamma \rightarrow r'} v(\gamma)$ exists and as $v > 0$ on $[1, r')$ so $\lim_{\gamma \rightarrow r'} v(\gamma) = \inf_{\gamma \in [1, r')} v(\gamma) = 0$.

Now as $\sqrt{\cdot}$ on $[0, \infty)$, v on $[1, r')$ and $p(\gamma)\gamma$ on $[1, m + 1]$ are uniformly continuous so from (2.3.5), v' is uniformly continuous on $[1, r')$ and so $\lim_{\gamma \rightarrow r'} v'(\gamma)$ exists. Since $\lim_{\gamma \rightarrow r'} v(\gamma) = 0$ and $v > 0$ on $[1, r')$ so $\lim_{\gamma \rightarrow r'} v'(\gamma) \leq 0$. If $\lim_{\gamma \rightarrow r'} v'(\gamma) = 0$ then from (2.3.5) we will get r' is a root of $p(\gamma)\gamma$ which is not possible by Lemma 3.1.2. So $\lim_{\gamma \rightarrow r'} v'(\gamma) < 0$.

Case. (2) If $[1, m + 1]$ is the maximal interval of existence of v then $v \geq 0$ on $[1, m + 1]$ and $v(1) = 2 > 0$ and from Lemma 3.1.2, $v'(\gamma_0) > 0$ and $v(\gamma_0) > 0$. So by the same argument as in Case (1), there cannot exist a $t_0 \in (1, m + 1)$ such that $v(t_0) = 0$. So $v(\gamma) > 0$ for all $\gamma \in [1, m + 1)$.

Let if possible $v(m + 1) = 0$. Since $v \geq 0$ and v is C^1 on $[1, m + 1]$ so $v'(m + 1) \leq 0$. If $v'(m + 1) = 0$ then from (2.3.5) we will get $m + 1$ is a root of $p(\gamma)\gamma$ which is not possible by Lemma 2.3.1. So $v'(m + 1) < 0$ i.e. v is strictly decreasing in a neighbourhood of $m + 1$. Now v is the C^1 solution of (2.3.5) and $v > 0$ on $[1, m + 1)$, and for v to be extendable as the C^1 solution to (2.3.5) on an interval strictly containing $[1, m + 1)$ we must have $v \geq 0$ on the larger interval which will not be possible with $v'(m + 1) < 0$. So in that case, v will exist as the C^1 solution of (2.3.5) maximally on $[1, m + 1)$, a contradiction to the hypothesis. So $v(m + 1) > 0$ and hence $v(\gamma) > 0$ for all $\gamma \in [1, m + 1]$. \square

Before proceeding further let us observe the following:

Corollary 3.1.1 (Smoothness of Solutions). *Let v be the C^1 solution of the ODE IVP (2.3.5) on a non-degenerate interval J . Then v' is bounded and uniformly continuous on J , and $v^{(i)}$ exists on J for all $i \in \mathbb{N}_{\geq 2}$ i.e. v is C^∞ on J . For all $i \in \mathbb{N}_{\geq 2}$ if $\inf_J v > 0$ then $v^{(i)}$ is bounded on J and if $\inf_J v = 0$ then $v^{(i)}$ is unbounded on J .*

Proof. By Lemma 3.1.3, $v > 0$ on J and so $\frac{1}{\sqrt{v}}$ makes sense. Considering the ODE in (2.3.5) and its $(i - 1)^{\text{th}}$ derivative, substituting the implied property of $\sqrt{\cdot}$, v and $p(\gamma)\gamma$ in both the cases into these ODEs and using induction will give us the required results for all $i \in \mathbb{N}_{\geq 2}$. \square

Observe one more thing that the solution v cannot be constant on any non-degenerate interval J , as that would imply (again from (2.3.5)) that the polynomial $p(\gamma)\gamma$ is a constant polynomial which is not possible by Lemma 2.3.1.

We now have the following necessary and sufficient condition for the continuation of the solution to (2.3.5) defined a priori on some interval:

Theorem 3.1.1 (Criterion for Continuation of Solutions). *For any $C \in \mathbb{R}$ if v is the smooth solution to (2.3.5) defined on an interval $[1, \tilde{r}) \subseteq [1, m + 1]$ then:*

- (1) v can be continued beyond \tilde{r} if and only if $\inf_{\gamma \in [1, \tilde{r})} v(\gamma) = \epsilon_1 > 0$ if and only if $\lim_{\gamma \rightarrow \tilde{r}} v(\gamma) = \epsilon_2 > 0$.
- (2) $[1, \tilde{r})$ is the maximal interval of existence of v if and only if $\inf_{\gamma \in [1, \tilde{r})} v(\gamma) = 0$ if and only if $\lim_{\gamma \rightarrow \tilde{r}} v(\gamma) = 0$.

Proof. From Lemma 3.1.3 and Corollary 3.1.1, $v > 0$ and v is Lipschitz on $[1, \tilde{r})$ so $\inf_{\gamma \in [1, \tilde{r})} v(\gamma)$ and $\lim_{\gamma \rightarrow \tilde{r}} v(\gamma)$ both exist and are non-negative. So again by using Lemma 3.1.3 and Corollary 3.1.1 it can be easily checked that either both $\inf_{\gamma \in [1, \tilde{r})} v(\gamma)$ and $\lim_{\gamma \rightarrow \tilde{r}} v(\gamma)$ are simultaneously strictly positive or both are simultaneously zero. From Lemma 3.1.1 we already have that if both are simultaneously positive then v can be continued beyond \tilde{r} . For proving the converse let $\inf_{\gamma \in [1, \tilde{r})} v(\gamma) = \lim_{\gamma \rightarrow \tilde{r}} v(\gamma) = 0$. Let if possible v be extendable as the smooth solution of (2.3.5) to an interval $[1, r')$ with $[1, \tilde{r}) \subseteq [1, r') \subseteq [1, m + 1]$. Then by Lemma 3.1.3, $v > 0$ on $[1, r')$ and as $\tilde{r} \in [1, r')$ and v is continuous on $[1, r')$ so $0 = \lim_{\gamma \rightarrow \tilde{r}} v(\gamma) = v(\tilde{r}) > 0$, a contradiction. So $[1, \tilde{r})$ is the maximal interval of existence of v . \square

So if the smooth solution v to (2.3.5) cannot be defined on $[1, m + 1]$ then there exists a unique $\gamma_* \in (1, m + 1]$ such that $[1, \gamma_*)$ is the maximal interval of existence of v .

Remark 3.1.2. Lemma 3.1.3 and Theorem 3.1.1 are together saying that the solution of (2.3.5) continues to exist as long as it is strictly positive, but the moment it attains zero, it ‘breaks down’ i.e. it cannot be continued further.

So finally for every $C \in \mathbb{R}$ considering the ODE IVP (2.3.5) depending on C we have exactly one of the following two scenarios (as a consequence of Lemma 3.1.2 and Theorem 3.1.1):

- (1) There exists a unique smooth solution $v_C := v(\cdot; C)$ on $[1, m + 1]$.
- (2) There exists a unique smooth solution $v_C := v(\cdot; C)$ with maximal interval of existence $[1, \gamma_{*,C})$ for a unique $\gamma_{*,C} := \gamma_*(C) \in (1, m + 1]$.

Motivation 3.1.2. In order to prove Theorem 2.3.2 we will first show that the set of all $C \in \mathbb{R}$, for which the condition (1) above holds true, is precisely the interval $(-\infty, M)$ for a unique $M > 2$, and then we will check the limits of $v_C(m + 1)$ as $C \rightarrow -\infty$ and as $C \rightarrow M$ respectively to conclude that the range of the function $(-\infty, M) \rightarrow \mathbb{R}$, $C \mapsto v_C(m + 1)$ is precisely the interval $(0, \infty)$. For this we will prove some preparatory results in the remainder of Subsection 3.1 and in Subsection 3.2.

Let $(\mathcal{C}[1, m + 1], \|\cdot\|_\infty)$ be the Banach space of all continuous functions on $[1, m + 1]$. For each $C \in \mathbb{R}$ define $u(\cdot; C) : [1, m + 1] \rightarrow \mathbb{R}$ as follows:

- (1) If the smooth solution v_C to (2.3.5) exists on $[1, m + 1]$ then $u(\gamma; C) := v_C(\gamma)$ for all $\gamma \in [1, m + 1]$.
- (2) If the smooth solution v_C to (2.3.5) has maximal interval of existence $[1, \gamma_{*,C})$ then $u(\gamma; C) := v_C(\gamma)$ for all $\gamma \in [1, \gamma_{*,C})$ and $u(\gamma; C) := 0$ for all $\gamma \in [\gamma_{*,C}, m + 1]$.

By Lemma 3.1.3, $u(\cdot; C)$ is continuous on $[1, m + 1]$ in the Case (2) above as well, and hence $u(\cdot; C) \in \mathcal{C}[1, m + 1]$ in both the Cases (1) and (2) above. Thus we get a function $\Phi : \mathbb{R} \rightarrow \mathcal{C}[1, m + 1]$ defined as $\Phi(C) := u(\cdot; C)$ for all $C \in \mathbb{R}$. It can be readily checked from (2.3.5) that Φ is well-defined and injective.

Motivation 3.1.3. We want to prove that Φ is continuous and considering the pointwise partial order \leq on $\mathcal{C}[1, m + 1]$, Φ is monotone decreasing.

For a given $C \in \mathbb{R}$ let $\gamma_{0,C} := \gamma_0(C)$ be the unique root of the polynomial $p_C(\gamma)\gamma := p(\gamma; C)\gamma$ in $[1, m + 1]$ and similarly let $u_C := u(\cdot; C)$ on $[1, m + 1]$.

Theorem 3.1.2. *Let $(C_n) \rightarrow C_0$ and $u_n := u_{C_n}$ and $u_0 := u_{C_0}$. Then there exists a subsequence (u_{n_k}) of (u_n) such that $(u_{n_k}) \rightarrow u_0$ uniformly on $[1, m + 1]$.*

Proof. Let $v_n := v_{C_n}$ and $v_0 := v_{C_0}$, and $p_n(\gamma)\gamma := p_{C_n}(\gamma)\gamma$ and $p_0(\gamma)\gamma := p_{C_0}(\gamma)\gamma$. Then from the expressions (2.3.2), $(A(C_n)) \rightarrow A(C_0)$ and $(B(C_n)) \rightarrow B(C_0)$, and hence $(p_n(\gamma)\gamma) \rightarrow p_0(\gamma)\gamma$ uniformly on $[1, m + 1]$. Let $\gamma_{0,n} := \gamma_{0,C_n}$ and $\gamma_{0,0} := \gamma_{0,C_0}$ be the roots of $p_n(\gamma)\gamma$ and $p_0(\gamma)\gamma$ in $[1, m + 1]$ respectively. Then we can verify that $\gamma_{0,n} \rightarrow \gamma_{0,0}$. Also note that $u_n(1) = v_n(1) = 2 = v_0(1) = u_0(1)$ and $u'_n(1) = v'_n(1) = 6 = v'_0(1) = u'_0(1)$. After this the proof of Theorem 3.1.2 will be divided into the following 3 Cases:

Case. (1) The solution v_n exists on $[1, m + 1]$ for all $n \in \mathbb{N}$ and $\inf_{\gamma \in [1, m+1], n \in \mathbb{N}} v_n(\gamma) = \epsilon > 0$.

Here $u_n = v_n$ for all $n \in \mathbb{N}$. Since $(p_n(\gamma)\gamma)$ is uniformly norm bounded, substituting $|p_n(\gamma)\gamma| \leq l$ (for some $l > 0$) and $\sqrt{v_n} < v_n + 1$ for all $n \in \mathbb{N}$ in the expression for $v'_n = (v_n + 1)'$ in (2.3.5) and using Grönwall's inequality will yield a $K > 0$ such that $\epsilon \leq v_n(\gamma) \leq K$ for all $\gamma \in [1, m + 1]$ and for all $n \in \mathbb{N}$, thereby proving that (v_n) is uniformly norm bounded on $[1, m + 1]$. Now substituting $v_n \leq K$ and $|p_n(\gamma)\gamma| \leq l$ in the expression for v'_n in (2.3.5) will give us that (v'_n) is also uniformly norm bounded, implying that (v_n) is uniformly equicontinuous on $[1, m + 1]$. So by Arzelà-Ascoli Theorem there exists a subsequence (v_{n_k}) of (v_n) converging uniformly on $[1, m + 1]$ to some $w \in \mathcal{C}[1, m + 1]$. As $(p_n(\gamma)\gamma)$ and $(\sqrt{v_{n_k}})$ are uniformly convergent so (v'_{n_k}) is uniformly convergent on $[1, m + 1]$ (again from (2.3.5)). Then by standard Uniform Convergence Theory w is differentiable and satisfies the ODE IVP (2.3.5) for $C = C_0$ on $[1, m + 1]$. As $\inf_{\gamma \in [1, m+1], n \in \mathbb{N}} v_n(\gamma) = \epsilon > 0$ so $\inf_{\gamma \in [1, m+1]} w(\gamma) = \tilde{\epsilon} > 0$ and so by Theorem 3.1.1, w

is differentiable and satisfies (2.3.5) on $[1, m + 1]$, and hence the solution v_0 exists on $[1, m + 1]$ and by uniqueness, $w = v_0$ and by definition, $u_0 = v_0$ as well. Thus $(u_{n_k}) \rightarrow u_0$ uniformly on $[1, m + 1]$.

Case. (2) The solution v_n exists on $[1, m + 1]$ for all $n \in \mathbb{N}$ and $\inf_{\gamma \in [1, m+1], n \in \mathbb{N}} v_n(\gamma) = 0$.

Here also $u_n = v_n$ for all $n \in \mathbb{N}$. The Grönwall’s inequality argument as in Case (1) above will prove $0 \leq v_n(\gamma) \leq K$ i.e. (v_n) is uniformly norm bounded on $[1, m + 1]$. Once again by substituting this in (2.3.5) with $C = C_n$, (v'_n) will be uniformly norm bounded on $[1, m + 1]$ and so there will exist a subsequence $(v_{n_k}) \rightarrow w \in \mathcal{C}[1, m + 1]$ uniformly on $[1, m + 1]$. By the same arguments as in Case (1), (v'_{n_k}) is uniformly convergent on $[1, m + 1]$ and w is differentiable and satisfies the ODE IVP (2.3.5) for $C = C_0$ on $[1, m + 1]$.

Claim. *There exists an $\epsilon_0 > 0$ such that for any sequence (t_n) where $t_n \in (1, m + 1)$ is a local minimum of v_n we have $v_n(t_n) \geq \epsilon_0$.*

If the above Claim is false then there exists a sequence (t_n) of respective local minima of v_n such that $\lim_{n \rightarrow \infty} v_n(t_n) = 0$. Passing to a subsequence if necessary, let $(t_n) \rightarrow t_0 \in [1, m + 1]$. Note also that $v'_n(t_n) = 0$. By the uniform norm boundedness of (v'_n) let $R > 0$ be the uniform Lipschitz constant for (v_n) . For any $n, j \in \mathbb{N}$ considering the following estimates:

$$\begin{aligned} |v_n(t_0)| &\leq |v_n(t_0) - v_n(t_j)| + |v_n(t_j) - v_n(t_n)| + |v_n(t_n)| \\ &\leq |v_n(t_0) - v_n(t_j)| + R|t_j - t_n| + |v_n(t_n)| \end{aligned} \tag{3.1.1}$$

we get $\lim_{n \rightarrow \infty} v_n(t_0) = 0$. Now as both $(\sqrt{v_n})$ and $(p_n(\gamma)\gamma)$ are uniformly equicontinuous so (v'_n) is also uniformly equicontinuous on $[1, m + 1]$ (from (2.3.5)). On similar lines as the estimates (3.1.1), considering the following estimates:

$$\begin{aligned} |v'_n(t_0)| &\leq |v'_n(t_0) - v'_n(t_j)| + |v'_n(t_j) - v'_n(t_n)| + |v'_n(t_n)| \\ &= |v'_n(t_0) - v'_n(t_j)| + |v'_n(t_j) - v'_n(t_n)| \end{aligned} \tag{3.1.2}$$

we get $\lim_{n \rightarrow \infty} v'_n(t_0) = 0$. With both these limits, substituting $\gamma = t_0$ in the ODE in (2.3.5) with $C = C_n$ and taking limits as $n \rightarrow \infty$ will imply that t_0 is a root of $p_0(\gamma)\gamma$ and so $t_0 = \gamma_{0,0} \in (1, m + 1)$. Now as w is a subsequential uniform limit of (v_n) and w satisfies the ODE IVP (2.3.5) with $C = C_0$ on $[1, m + 1]$, we will get $w(\gamma_{0,0}) = w(t_0) = 0$ and $w'(\gamma_{0,0}) = w'(t_0) = 0$ which contradicts Lemmas 3.1.2 and 3.1.3. Hence the Claim.

As $\inf_{\gamma \in [1, m+1], n \in \mathbb{N}} v_n(\gamma) = 0$, we must have $\inf_{n \in \mathbb{N}} v_n(m + 1) = 0$. As $(v_{n_k}) \rightarrow w$ uniformly on $[1, m + 1]$ so $w(m + 1) = 0$ and hence by Theorem 3.1.1, $[1, m + 1]$ is the maximal interval of existence of w as the smooth solution to (2.3.5) with $C = C_0$, and so $w =$

$v_0 = u_0$ on $[1, m + 1)$ and $w = u_0$ on $[1, m + 1]$ by continuity. With this, $(u_{n_k}) \rightarrow u_0$ uniformly on $[1, m + 1]$.

Case. (3) The solution v_n has maximal interval of existence $[1, \gamma_{\star, n})$ with $\gamma_{\star, n} := \gamma_{\star, C_n} \in (1, m + 1]$ for all $n \in \mathbb{N}$. W.l.o.g. $(\gamma_{\star, n})$ is a monotone sequence converging to some $\sigma \in [1, m + 1]$.

Here $u_n = v_n$ on $[1, \gamma_{\star, n})$ and $u_n = 0$ on $[\gamma_{\star, n}, m + 1]$ for all $n \in \mathbb{N}$. As was noted in the beginning, the polynomials $(p_n(\gamma) \gamma) \rightarrow p_0(\gamma) \gamma$ uniformly on $[1, m + 1]$ and their respective roots $\gamma_{0, n} \rightarrow \gamma_{0, 0}$. By Lemmas 2.3.1 and 3.1.2, $1 < \gamma_{0, n} < \gamma_{\star, n} \leq m + 1$ for all $n \in \mathbb{N}$ and taking limits as $n \rightarrow \infty$ we get $1 < \gamma_{0, 0} \leq \sigma \leq m + 1$ ($\gamma_{0, 0} > 1$ by Lemma 2.3.1) and specifically $\sigma \in (1, m + 1]$.

Claim. *There exists a $\tilde{K} > 0$ such that for any $n \in \mathbb{N}$ and any local maximum $t \in (1, \gamma_{\star, n})$ of v_n we have $v_n(t) \leq \tilde{K}$.*

For any $n \in \mathbb{N}$ if $t \in (1, \gamma_{\star, n})$ is a local maximum of v_n then $v'_n(t) = 0$ implies $v_n(t) = \frac{p_n(t)^2 t^2}{8}$ (from (2.3.5) with $C = C_n$), and as $(p_n(\gamma) \gamma)$ is uniformly norm bounded on $[1, m + 1]$ in all cases so there exists a $\tilde{K} > 0$ such that $v_n(t) = \frac{p_n(t)^2 t^2}{8} \leq \tilde{K}$ for all $n \in \mathbb{N}$.

Looking at the definition of u_n in this case we will observe for each $n \in \mathbb{N}$ that $\max_{\gamma \in [1, m+1]} u_n(\gamma) = \sup_{\gamma \in [1, \gamma_{\star, n})} v_n(\gamma) = v_n(t) \leq \tilde{K}$ for some local maximum $t \in (1, \gamma_{\star, n})$ of v_n . Note that the supremum of v_n on $[1, \gamma_{\star, n})$ will be attained at an interior point only (which will then be a local maximum of v_n), because $v_n(1) = 2$ and $v'_n(1) = 6$ and from Lemma 3.1.3, $\lim_{\gamma \rightarrow \gamma_{\star, n}} v_n(\gamma) = 0$ and $\lim_{\gamma \rightarrow \gamma_{\star, n}} v'_n(\gamma) < 0$. Thus we see $0 \leq u_n(\gamma) \leq \tilde{K}$ for all $\gamma \in [1, m + 1]$ and for all $n \in \mathbb{N}$ i.e. (u_n) is uniformly norm bounded on $[1, m + 1]$. By substituting the uniform norm bounds on $(\sqrt{v_n})$ and $(p_n(\gamma) \gamma)$ in the expression for v'_n in (2.3.5) on $[1, \gamma_{\star, n})$ we get an $\tilde{R} > 0$ such that for each $n \in \mathbb{N}$, $|v'_n(\gamma)| \leq \tilde{R}$ for all $\gamma \in [1, \gamma_{\star, n})$, and so $\left| \lim_{\gamma \rightarrow \gamma_{\star, n}} v'_n(\gamma) \right| \leq \tilde{R}$. So by its definition, (u_n) is uniformly Lipschitz on $[1, m + 1]$. So we extract a subsequence $(u_{n_k}) \rightarrow w \in \mathcal{C}[1, m + 1]$ uniformly on $[1, m + 1]$. As $u_{n_k} \geq 0$ so $w \geq 0$.

Claim. $w(\sigma) = 0$.

Note that $u_n(\gamma_{\star, n}) = 0$ for all $n \in \mathbb{N}$. Then using the following estimates for $j, k \in \mathbb{N}$:

$$\begin{aligned} |w(\sigma)| &\leq |w(\sigma) - u_{n_k}(\sigma)| + |u_{n_k}(\sigma) - u_{n_k}(\gamma_{\star, n_j})| \\ &\quad + |u_{n_k}(\gamma_{\star, n_j}) - u_{n_k}(\gamma_{\star, n_k})| + |u_{n_k}(\gamma_{\star, n_k})| \\ &\leq |w(\sigma) - u_{n_k}(\sigma)| + |u_{n_k}(\sigma) - u_{n_k}(\gamma_{\star, n_j})| + \tilde{R}|\gamma_{\star, n_j} - \gamma_{\star, n_k}| \end{aligned} \tag{3.1.3}$$

it can be easily seen that $w(\sigma) = 0$.

After this the proof of $w = u_0$ in Case (3) will depend upon whether $(\gamma_{*,n})$ is increasing or decreasing.

Case. (3.1) $(\gamma_{*,n})$ decreases to σ .

So $[1, \sigma] = \bigcap_{k \in \mathbb{N}} [1, \gamma_{*,n_k})$ and so each v_{n_k} will satisfy the ODE IVP (2.3.5) with $C = C_{n_k}$ on $[1, \sigma]$. But $v_{n_k} = u_{n_k}$ on $[1, \gamma_{*,n_k})$ and $(u_{n_k}) \rightarrow w$ uniformly on $[1, m + 1]$. So $(\sqrt{v_{n_k}}) \rightarrow \sqrt{w}$ uniformly on $[1, \sigma]$ and this will imply (again from (2.3.5) with $C = C_{n_k}$) that (v'_{n_k}) is uniformly convergent on $[1, \sigma]$. Hence w is differentiable and satisfies (2.3.5) for $C = C_0$ on $[1, \sigma]$. As $w(\sigma) = 0$, from Lemma 3.1.3 and Theorem 3.1.1, $[1, \sigma)$ is the maximal interval of existence of the solution v_0 and $w = v_0$ on $[1, \sigma)$ and $\sigma = \gamma_{*,0}$. Now note that $u_{n_k} \equiv 0$ on $[\gamma_{*,n_k}, m + 1]$ for each $k \in \mathbb{N}$ and so (the uniform limit) $w \equiv 0$ on $(\sigma, m + 1] = \bigcup_{k \in \mathbb{N}} [\gamma_{*,n_k}, m + 1]$, as $(\gamma_{*,n})$ is decreasing to σ . So by its definition, $w = u_0$ on $[1, m + 1]$.

Case. (3.2) $(\gamma_{*,n})$ increases to σ .

So $[1, \sigma) = \bigcup_{k \in \mathbb{N}} [1, \gamma_{*,n_k})$ and $[1, \gamma_{*,n_k}) = \bigcap_{j \geq k} [1, \gamma_{*,n_j})$ for each $k \in \mathbb{N}$. So for a fixed $k \in \mathbb{N}$, v_{n_j} satisfies the ODE IVP (2.3.5) with $C = C_{n_j}$ on $[1, \gamma_{*,n_k})$ for all $j \geq k$. By using the same set of arguments as in Case (3.1) for the sequence $(v_{n_j})_{j \geq k}$ converging uniformly to w on $[1, \gamma_{*,n_k})$ we will see that $(v'_{n_j})_{j \geq k}$ is uniformly convergent on $[1, \gamma_{*,n_k})$. So w is differentiable and satisfies (2.3.5) for $C = C_0$ on $[1, \gamma_{*,n_k})$, and as this holds true for each $k \in \mathbb{N}$ so w satisfies (2.3.5) for $C = C_0$ on $[1, \sigma)$. After this, the same arguments as in Case (3.1) will give $\sigma = \gamma_{*,0}$ and $w = v_0$ on $[1, \sigma)$ and $w \equiv 0$ on $[\sigma, m + 1]$, thereby giving $w = u_0$ on $[1, m + 1]$.

In both the Cases (3.1) and (3.2), $(u_{n_k}) \rightarrow u_0$ uniformly on $[1, m + 1]$ and $(\gamma_{*,n_k}) \rightarrow \gamma_{*,0}$.

Since our aim was to find only a subsequence of (u_n) which is uniformly convergent, the above Cases suffice. \square

Remark 3.1.3. The Cases in Theorem 3.1.2 have given us a hint that \mathbb{R} may be expressed as the disjoint set union of the set of all $C \in \mathbb{R}$ for which v_C exists on $[1, m + 1]$ with the set of all $C \in \mathbb{R}$ for which v_C breaks down at $\gamma_{*,C}$, and that the first set is an open interval and the second one is a closed interval.

3.2. Second part of the proof

We first do some calculations with the polynomial $p_C(\gamma)$ on $[1, m + 1]$ (in the following 3 results) which will be needed further. The first one is from Pingali [18].

Lemma 3.2.1 (Pingali). For each $C \in \mathbb{R}$ define $P_C(\gamma) := P(\gamma; C) := \int_1^\gamma p_C(y) y dy$, $\gamma \in [1, m + 1]$. Define $L := L(m)$ and $N := N(m)$ as follows:

$$\begin{aligned}
 L(m) &:= \frac{3}{10}(m+1)^2 - \frac{1}{20}(m+1)^4 - \frac{1}{4} - \frac{(m+1)^4 - 1}{20m} \left[1 - \frac{1}{(m+1)^2} \right] \\
 N(m) &:= \frac{1}{10}(m+1)^4 - \frac{2}{5}(m+1)^2 - \frac{1}{2} + \frac{(m+1)^4 - 1}{10m} \left[1 + \frac{1}{(m+1)^2} \right]
 \end{aligned}
 \tag{3.2.1}$$

Then $L < 0$ and $N > 0$, and $P_C(m + 1) = LC + N$ and $P_C(\gamma) \geq \min\{0, LC + N\}$.

Lemma 3.2.2. The polynomial $q(\gamma) := \frac{d}{dC}(p_C(\gamma)\gamma)$, $\gamma \in [1, m + 1]$ is independent of C , and further $q(\gamma) < 0$ for $\gamma \in (1, m + 1)$ and $q(1) = q(m + 1) = 0$.

Proof.

$$\frac{d}{dC}(A(C)) = \frac{3(m+2)}{(m+1)^2}, \quad \frac{d}{dC}(B(C)) = -\frac{2(m^2 + 3m + 3)}{(m+1)^2}
 \tag{3.2.2}$$

$$\begin{aligned}
 \frac{d}{dC}(p_C(\gamma)\gamma) &= \frac{m+2}{(m+1)^2}\gamma^4 - \frac{m^2 + 3m + 3}{(m+1)^2}\gamma^3 + \gamma \\
 &= \left(\gamma - \left(-\frac{m+1}{m+2} \right) \right) (\gamma - 0)(\gamma - 1)(\gamma - (m+1))
 \end{aligned}
 \tag{3.2.3}$$

So $q(\gamma) = \frac{d}{dC}(p_C(\gamma)\gamma)$ is independent of C , has its roots at 1 and $m + 1$, and does not change its sign in $(1, m + 1)$. Evaluating $q(\gamma)$ at $\gamma = \frac{m+2}{2} \in (1, m + 1)$:

$$\begin{aligned}
 q\left(\frac{m+2}{2}\right) &= \frac{(m+2)^5}{16(m+1)^2} - \frac{(m^2 + 3m + 3)(m+2)^3}{8(m+1)^2} + \frac{m+2}{2} \\
 &= -\frac{m^2(m+2)(m^2 + 6m + 6)}{16(m+1)^2}
 \end{aligned}
 \tag{3.2.4}$$

< 0

So for any $\gamma \in (1, m + 1)$, $q(\gamma) < 0$ and $q(1) = q(m + 1) = 0$. \square

Corollary 3.2.1. Define $Q(\gamma) := \int_1^\gamma q(y) dy$, $\gamma \in [1, m + 1]$. Then $Q(\gamma) < 0$ for all $\gamma \in (1, m + 1]$ and $Q(\gamma)$ is strictly decreasing on $[1, m + 1]$.

Motivation 3.2.1. The following calculations and estimates are going to give us that for each $\gamma > 1$, $\frac{d}{dC}(v_C(\gamma)) < 0$ on appropriate intervals of C and γ i.e. $v(\gamma; C)$ is strictly decreasing in C .

Theorem 3.2.1. Let V be a non-degenerate interval in \mathbb{R} and J be a non-degenerate subinterval of $[1, m + 1]$ containing 1 such that the smooth solution v_C to the ODE IVP (2.3.5) exists on J for every $C \in V$. Then for any $C, D \in V$ with $C < D$ we have $v(\gamma; C) \geq v(\gamma; D) - Q(\gamma)(D - C) \geq v(\gamma; D)$ for all $\gamma \in J$, with the second inequality being strict if $\gamma > 1$.

Proof. Consider the following operations performed on (2.3.5) with $'$ and $\frac{d}{dC}$ denoting derivatives w.r.t. γ and C respectively for $C \in V$ and $\gamma \in J$, and use $q(\gamma)$ and $Q(\gamma)$ from Lemma 3.2.2 and Corollary 3.2.1 respectively:

$$v'_C(\gamma) = 2\sqrt{2}\sqrt{v_C(\gamma)} + p_C(\gamma)\gamma, \quad v_C(1) = 2 \tag{3.2.5}$$

$$\frac{d}{dC}(v'_C(\gamma)) = \frac{\sqrt{2}}{\sqrt{v_C(\gamma)}} \frac{d}{dC}(v_C(\gamma)) + q(\gamma) \tag{3.2.6}$$

By Lemma 3.1.3, $\sqrt{v_C(\gamma)} > 0$ for all $\gamma \in J$. Multiplying by $e^{-\int_1^\gamma \frac{\sqrt{2}}{\sqrt{v(y;C)}} dy}$ and using the equality of second order mixed partial derivatives:

$$\left(\frac{d}{dC}(v_C(\gamma))\right)' e^{-\int_1^\gamma \frac{\sqrt{2}}{\sqrt{v(y;C)}} dy} + \frac{d}{dC}(v_C(\gamma)) \left(e^{-\int_1^\gamma \frac{\sqrt{2}}{\sqrt{v(y;C)}} dy}\right)' = q(\gamma) e^{-\int_1^\gamma \frac{\sqrt{2}}{\sqrt{v(y;C)}} dy} \tag{3.2.7}$$

For $\gamma \in J$ integrating on $[1, \gamma]$:

$$\frac{d}{dC}(v_C(\gamma)) e^{-\int_1^\gamma \frac{\sqrt{2}}{\sqrt{v(y;C)}} dy} = \int_1^\gamma q(y) e^{-\int_1^y \frac{\sqrt{2}}{\sqrt{v(x;C)}} dx} dy \tag{3.2.8}$$

$$\begin{aligned} \frac{d}{dC}(v_C(\gamma)) &= e^{\int_1^\gamma \frac{\sqrt{2}}{\sqrt{v(y;C)}} dy} \int_1^\gamma q(y) e^{-\int_1^y \frac{\sqrt{2}}{\sqrt{v(x;C)}} dx} dy \\ &\leq e^{\int_1^\gamma \frac{\sqrt{2}}{\sqrt{v(y;C)}} dy} \int_1^\gamma q(y) e^{-\int_1^\gamma \frac{\sqrt{2}}{\sqrt{v(x;C)}} dx} dy \\ &= Q(\gamma) \end{aligned} \tag{3.2.9}$$

Now for any $C, D \in V$ with $C < D$ and any $\gamma \in J$ we have (for some $E \in (C, D)$):

$$v(\gamma; D) - v(\gamma; C) = \frac{d}{dC}(v_E(\gamma))(D - C) \leq Q(\gamma)(D - C) \leq 0 \tag{3.2.10}$$

So $v(\gamma; C) \geq v(\gamma; D) - Q(\gamma)(D - C) \geq v(\gamma; D)$ for all $\gamma \in J$ and from Corollary 3.2.1, the second inequality here is strict if $\gamma > 1$. \square

Define $\mathcal{C} := \{C \in \mathbb{R} \mid v_C \text{ exists on } [1, m + 1]\} \subseteq \mathbb{R}$.

We have the following result of Pingali [18] which along with Lemmas 2.3.1, 3.1.1 and 3.2.1 was used by them in the proof of Theorem 2.3.1.

Lemma 3.2.3. $(-\infty, 2] \subseteq \mathcal{C}$ and $\lim_{C \rightarrow -\infty} v(m + 1; C) = \infty$.

Proof. We just mention the highlights of the proof of Lemma 3.2.3 and the detailed calculations are given in [18]. For any $C \leq 2$ by Lemma 3.1.2, the solution v_C to (2.3.5) a priori exists on some $[1, \tilde{\gamma})$ with $\tilde{\gamma} > 1$. By using L and N of Lemma 3.2.1, it can be checked that if $C \leq 2$ then $LC + N > 0$. Integrating (2.3.5) on $[1, \gamma]$ for $\gamma \in [1, \tilde{\gamma})$ and using Lemma 3.2.1 will give $v_C(\gamma) \geq 2 + P_C(\gamma) \geq 2$, and then by Lemma 3.1.1, v_C can be continued beyond $\tilde{\gamma}$, and this will be true for every $\tilde{\gamma} > 1$. So v_C exists on $[1, m + 1]$ if $C \leq 2$ i.e. $(-\infty, 2] \subseteq \mathcal{C}$. Now as $L < 0$ and $N > 0$ so $\lim_{C \rightarrow -\infty} (LC + N) = \infty$. So $\lim_{C \rightarrow -\infty} v(m + 1; C) \geq 2 + \lim_{C \rightarrow -\infty} P_C(m + 1) = \infty$ (by Lemma 3.2.1). An important point to be noted here is that if $C \leq 2$ then $LC + N > 0$ i.e. $C < -\frac{N}{L}$, and so $-\frac{N}{L} > 2$. \square

Motivation 3.2.2. In the remainder of Subsection 3.2 we will prove that $\mathcal{C} = (-\infty, M)$ and $\lim_{C \rightarrow M^-} v(m + 1; C) = 0$ which will give us a C satisfying the required final boundary condition viz. $v(m + 1; C) = 2(m + 1)^2$.

Theorem 3.2.2. *We have the following properties of \mathcal{C} :*

- (1) \mathcal{C} is an interval.
- (2) \mathcal{C} is open.
- (3) $\mathcal{C} \subsetneq \mathbb{R}$.
- (4) *There exists a unique $M = M(m) > 2$ such that $\mathcal{C} = (-\infty, M)$.*

Proof. (1) Let $D_1, D_2 \in \mathcal{C}$ with $D_1 < D_2$.

Claim. *There exists a $\tilde{\gamma} \in (1, m + 1]$ such that v_C exists at least on $[1, \tilde{\gamma})$ for all $C \in [D_1, D_2]$.*

If not true then there exists a sequence (C_n) in $[D_1, D_2]$ such that v_n exists maximally on $[1, \gamma_{*,n})$ and $(\gamma_{*,n}) \rightarrow 1$. Passing to a subsequence if necessary, assume $(C_n) \rightarrow C_0 \in [D_1, D_2]$. By Theorem 3.1.2 Case (3), there exists a subsequence $(u_{n_k}) \rightarrow u_0$ uniformly on $[1, m + 1]$ with $(\gamma_{*,n_k}) \rightarrow \gamma_{*,0}$ where $[1, \gamma_{*,0})$ is the maximal interval of existence of v_0 . By Lemma 3.1.2, $\gamma_{*,0} > 1$. So $(\gamma_{*,n_k}) \rightarrow \gamma_{*,0}$ and $(\gamma_{*,n}) \rightarrow 1$, a contradiction. Hence the Claim.

Take any $\tilde{\gamma} > 1$ with the property mentioned in the Claim above. Applying Theorem 3.2.1 with $V = [D_1, D_2]$ and $J = [1, \tilde{\gamma})$ we will get for any $C \in [D_1, D_2]$ and for all $\gamma \in [1, \tilde{\gamma})$, $v(\gamma; C) \geq v(\gamma; D_2)$. Since $D_2 \in \mathcal{C}$ so v_{D_2} exists on $[1, m + 1]$ and by

Theorem 3.1.1, there exists an $\epsilon > 0$ (depending only on D_2) such that $v(\gamma; D_2) \geq \epsilon$ for all $\gamma \in [1, m + 1]$ and hence in particular for all $\gamma \in [1, \tilde{\gamma}]$. So $v(\gamma; C) \geq \epsilon$ for all $\gamma \in [1, \tilde{\gamma})$ and so by Theorem 3.1.1, v_C can be continued beyond $\tilde{\gamma}$ for all $C \in [D_1, D_2]$. Since this holds true for any $\tilde{\gamma} > 1$ with the property mentioned in the Claim above and the lower bound $\epsilon > 0$ on v_C does not depend on $C \in [D_1, D_2]$ as well as on $\tilde{\gamma} > 1$ so v_C has to exist on $[1, m + 1]$ for all $C \in [D_1, D_2]$ i.e. $[D_1, D_2] \subseteq \mathcal{C}$. So \mathcal{C} is an interval.

(2) Take a sequence $(C_n) \rightarrow C_0 \in \mathbb{R}$ of points in $\mathcal{C}^c := \mathbb{R} \setminus \mathcal{C}$. Then v_n has maximal interval of existence $[1, \gamma_{*,n})$ and we are in Theorem 3.1.2 Case (3). So there exists a subsequence $(u_{n_k}) \rightarrow u_0$ uniformly on $[1, m + 1]$ with $(\gamma_{*,n_k}) \rightarrow \gamma_{*,0}$. So $[1, \gamma_{*,0})$ is the maximal interval of existence of v_0 and so by definition, $C_0 \in \mathcal{C}^c$. So \mathcal{C} is open.

(3) Let if possible v_C exist on $[1, m + 1]$ for all $C \in \mathbb{R}$. Taking $V = \mathbb{R}$ and $J = [1, m + 1]$ in Theorem 3.2.1 we get for a fixed $C_0 \in \mathbb{R}$ and for any $C > C_0$, and with $\gamma = m + 1$, $v(m + 1; C) \leq v(m + 1; C_0) + Q(m + 1)(C - C_0)$. Taking $C_n := C_0 + n$ for $n \in \mathbb{N}$ we get $v(m + 1; C_n) \leq v(m + 1; C_0) + nQ(m + 1)$. Since $v(m + 1; C_n), v(m + 1; C_0) > 0$ and $Q(m + 1) < 0$ (from Lemma 3.1.3 and Corollary 3.2.1 respectively) we have $n < \frac{-v(m+1;C_0)}{Q(m+1)}$ for all $n \in \mathbb{N}$, a contradiction. So there exists a $C \in \mathbb{R}$ such that v_C has maximal interval of existence $[1, \gamma_{*,C})$ for some $\gamma_{*,C} \in (1, m + 1]$ i.e. $C \in \mathcal{C}^c$. So $\mathcal{C} \subsetneq \mathbb{R}$.

(4) From Lemma 3.2.3 and (1), (2) and (3), there exists a unique $M = M(m) > 2$ such that $\mathcal{C} = (-\infty, M)$. \square

For each $C \in \mathbb{R}$ let $I_C \subseteq [1, m + 1]$ denote the maximal interval of existence of the solution v_C to the ODE IVP (2.3.5). Let $\mathcal{P}([1, m + 1])$ denote the power set of $[1, m + 1]$. By Lemma 3.1.2 and Theorem 3.1.1 we get the set map $\mathbb{R} \rightarrow \mathcal{P}([1, m + 1]), C \mapsto I_C$. Then using Theorems 3.1.2, 3.2.1 and 3.2.2 and the definitions of u_C, Φ and \mathcal{C} we get the following 2 results:

Corollary 3.2.2. *The set map $\mathbb{R} \rightarrow \mathcal{P}([1, m + 1]), C \mapsto I_C$ and Φ are monotone decreasing:*

- (1) *If $C_1, C_2 \in \mathcal{C}^c, C_1 < C_2$ then $[C_1, C_2] \subseteq \mathcal{C}^c$ and $\gamma_{*,C_2} < \gamma_{*,C_1}$ i.e. $I_{C_2} \subsetneq I_{C_1}$. In general if $C_1, C_2 \in \mathbb{R}, C_1 < C_2$ then $I_{C_2} \subseteq I_{C_1}$, with the set containment being strict if $C_2 \in \mathcal{C}^c$ and it being set equality otherwise.*
- (2) *If $C_1, C_2 \in \mathcal{C}, C_1 < C_2$ then $[C_1, C_2] \subseteq \mathcal{C}$ and $v(\gamma; C_2) < v(\gamma; C_1)$ i.e. $u(\gamma; C_2) < u(\gamma; C_1)$ for all $\gamma \in (1, m + 1]$. In general if $C_1, C_2 \in \mathbb{R}, C_1 < C_2$ then $u(\gamma; C_2) \leq u(\gamma; C_1)$ for all $\gamma \in (1, m + 1]$, with the inequality being strict if $\gamma \in I_{C_1}$ and it being equality otherwise.*

Proof. (1) Given $C_1, C_2 \in \mathcal{C}^c, C_1 < C_2$ then as $\mathcal{C}^c = [M, \infty)$ so clearly $[C_1, C_2] \subseteq \mathcal{C}^c$. So for every $C \in [C_1, C_2], I_C = [1, \gamma_{*,C})$.

Claim. *There exists a $\tilde{\gamma} \in (1, m + 1]$ such that v_C exists at least on $[1, \tilde{\gamma}]$ i.e. $[1, \tilde{\gamma}] \subseteq I_C$ for all $C \in [C_1, C_2]$.*

The proof of the above Claim is exactly the same as that of the Claim in Theorem 3.2.2 (1) with D_1, D_2 being replaced by C_1, C_2 .

Take any $\tilde{\gamma} > 1$ with the property mentioned in the above Claim. Then clearly $\tilde{\gamma} \leq \gamma_{*,C_2}$. Taking any such $\tilde{\gamma} < \gamma_{*,C_2}$ and applying Theorem 3.2.1 with $V = [C_1, C_2]$ and $J = [1, \tilde{\gamma})$ we get $v(\gamma; C) \geq v(\gamma; C_2)$ for all $\gamma \in [1, \tilde{\gamma})$ and for any $C \in [C_1, C_2]$. Since $[1, \tilde{\gamma}) \subsetneq I_{C_2}$ so by Lemma 3.1.3 and Theorem 3.1.1, there exists an $\epsilon_{\tilde{\gamma}} > 0$ (depending only on $\tilde{\gamma}$ and C_2) such that $v(\gamma; C_2) \geq \epsilon_{\tilde{\gamma}}$ for all $\gamma \in [1, \tilde{\gamma})$. So for each $C \in [C_1, C_2]$, $v(\gamma; C) \geq \epsilon_{\tilde{\gamma}}$ for all $\gamma \in [1, \tilde{\gamma})$ and so by Theorem 3.1.1, v_C can be continued beyond $\tilde{\gamma}$ for all $C \in [C_1, C_2]$. Since this holds true for any $1 < \tilde{\gamma} < \gamma_{*,C_2}$ with the property mentioned in the above Claim and the lower bound $\epsilon_{\tilde{\gamma}} > 0$ on v_C does not depend on $C \in [C_1, C_2]$ so v_C has to exist on $[1, \gamma_{*,C_2})$ i.e. $I_{C_2} \subseteq I_C$ for all $C \in [C_1, C_2]$.

So in particular $I_{C_2} \subseteq I_{C_1}$. Let if possible $I_{C_1} = I_{C_2}$ i.e. $\gamma_{*,C_1} = \gamma_{*,C_2}$. So by Lemma 3.1.3 we have $\lim_{\gamma \rightarrow \gamma_{*,C_2}} v(\gamma; C_2) = 0 = \lim_{\gamma \rightarrow \gamma_{*,C_1}} v(\gamma; C_1)$. Taking $V = [C_1, C_2]$ and $J = [1, \gamma_{*,C_2})$ in Theorem 3.2.1 we get $v(\gamma; C_1) \geq v(\gamma; C_2) - Q(\gamma)(C_2 - C_1)$ for all $\gamma \in [1, \gamma_{*,C_2})$. Applying limits as $\gamma \rightarrow \gamma_{*,C_2}$ we get $Q(\gamma_{*,C_2}) \geq 0$ where $\gamma_{*,C_2} > 1$ which is a contradiction to Corollary 3.2.1. So $I_{C_2} \subsetneq I_{C_1}$ i.e. $\gamma_{*,C_2} < \gamma_{*,C_1}$.

If $C_2 \in \mathcal{C}^c$ and $C_1 \in \mathcal{C}$ then $C_1 < C_2$ and $I_{C_2} = [1, \gamma_{*,C_2}) \subsetneq [1, m + 1] = I_{C_1}$. If $C_1, C_2 \in \mathcal{C}$, $C_1 < C_2$ then $I_{C_1} = [1, m + 1] = I_{C_2}$. Thus the general statement for $C_1, C_2 \in \mathbb{R}$, $C_1 < C_2$ holds true.

(2) Given $C_1, C_2 \in \mathcal{C}$, $C_1 < C_2$ then from Theorem 3.2.2 (1), $[C_1, C_2] \subseteq \mathcal{C}$. So for every $C \in [C_1, C_2]$, $I_C = [1, m + 1]$ and $u_C = v_C$ on $[1, m + 1]$. Applying Theorem 3.2.1 with $V = [C_1, C_2]$ and $J = [1, m + 1]$ we get $v(\gamma; C_2) < v(\gamma; C_1)$ i.e. $u(\gamma; C_2) < u(\gamma; C_1)$ for all $\gamma \in (1, m + 1]$.

If $C_1 \in \mathcal{C}$ and $C_2 \in \mathcal{C}^c$ then $C_1 < C_2$ and $u_{C_1} = v_{C_1}$ on $I_{C_1} = [1, m + 1]$ and $u_{C_2} = v_{C_2}$ on $I_{C_2} = [1, \gamma_{*,C_2})$ and $u_{C_2} \equiv 0$ on $[\gamma_{*,C_2}, m + 1]$. From (1), $I_{C_2} \subseteq I_C$ for all $C \in [C_1, C_2]$ and so by Theorem 3.2.1 with $V = [C_1, C_2]$ and $J = [1, \gamma_{*,C_2})$ we get $v(\gamma; C_2) < v(\gamma; C_1)$ for all $\gamma \in [1, \gamma_{*,C_2})$. On $[\gamma_{*,C_2}, m + 1]$ by Lemma 3.1.3, $u_{C_2} = 0 < v_{C_1} = u_{C_1}$.

If $C_1, C_2 \in \mathcal{C}^c$, $C_1 < C_2$ then by (1), $u_C = v_C$ on $I_C = [1, \gamma_{*,C})$ and $u_C \equiv 0$ on $[\gamma_{*,C}, m + 1]$ for all $C \in [C_1, C_2]$. Also by (1), $I_{C_2} \subseteq I_C$ for all $C \in [C_1, C_2]$ and so again using Theorem 3.2.1 with $V = [C_1, C_2]$ and $J = [1, \gamma_{*,C_2})$ we get $v(\gamma; C_2) < v(\gamma; C_1)$ for all $\gamma \in [1, \gamma_{*,C_2})$. As $\gamma_{*,C_2} < \gamma_{*,C_1}$ (again by (1)) so on $[\gamma_{*,C_2}, \gamma_{*,C_1})$ by Lemma 3.1.3 and Theorem 3.1.1, $u_{C_2} = 0 < v_{C_1} = u_{C_1}$. On $[\gamma_{*,C_1}, m + 1]$, $u_{C_1} = 0 = u_{C_2}$ as $I_{C_2} \subsetneq I_{C_1}$.

Thus the general statement for $C_1, C_2 \in \mathbb{R}$, $C_1 < C_2$ holds true. \square

Because of monotonicity we can get the uniform convergence of the whole sequence (u_n) instead of just a subsequence (u_{n_k}) in Theorem 3.1.2:

Corollary 3.2.3. *If $(C_n) \uparrow C_0$ then $(u_n) \downarrow u_0$ and more generally if $(C_n) \rightarrow C_0$ then $(u_n) \rightarrow u_0$ uniformly on $[1, m + 1]$. Thus Φ is continuous.*

We now have the final result of Subsections 3.1 and 3.2 which will prove Theorem 2.3.2 and as a consequence Corollary 2.3.1:

Corollary 3.2.4. $\lim_{C \rightarrow M^-} v(m+1; C) = 0$. There exists a unique $C = C(m) \in (-\infty, M)$ such that $v(m+1; C) = 2(m+1)^2$ and the C with this property has to be strictly greater than 2.

Proof. By Theorem 3.2.2, $\mathcal{C}^c = [M, \infty)$. Let $(C_n) \downarrow M$ be a sequence of points in \mathcal{C}^c then from Corollary 3.2.3, $(u_n) \uparrow u_M$ uniformly on $[1, m+1]$ and by definition, $I_n := I_{C_n} = [1, \gamma_{*,n})$ and so by Theorem 3.1.2 Case (3.2) and Corollaries 3.2.2 and 3.2.3, $I_M = [1, \gamma_{*,M})$ where $(\gamma_{*,n}) \uparrow \gamma_{*,M}$ i.e. (I_n) is monotone increasing in $\mathcal{P}([1, m+1])$ and $I_M = \bigcup_{n \in \mathbb{N}} I_n$.

Let $(E_n) \uparrow M \in \mathcal{C}^c$ be a sequence of points in $\mathcal{C} = (-\infty, M)$ then $(u_{E_n}) \downarrow u_M$ uniformly on $[1, m+1]$ and by definition, $I_{E_n} = [1, m+1]$ and $u_{E_n} = v_{E_n}$ on $[1, m+1]$ for all $n \in \mathbb{N}$. So we will land up in either of the Cases (1) or (2) in Theorem 3.1.2. In Case (1), $\inf_{\gamma \in [1, m+1], n \in \mathbb{N}} u_{E_n}(\gamma) = \epsilon > 0$ and hence the uniform limit $u_M = v_M \geq \epsilon > 0$ on $[1, m+1]$, thereby implying (by Lemma 3.1.1) that $M \in \mathcal{C}$, a contradiction. So we are in Theorem 3.1.2 Case (2) and hence $I_M = [1, m+1]$ (i.e. $\gamma_{*,M} = m+1$) and $u_M = v_M > 0$ (by Lemma 3.1.3) on $[1, m+1]$ and $u_M(m+1) = 0$. So by pointwise convergence, $(u_{E_n}(m+1)) \downarrow u_M(m+1)$ i.e. $(v_{E_n}(m+1)) \downarrow 0$ and so we get $\lim_{C \rightarrow M^-} v(m+1; C) = 0$.

From this and Lemma 3.2.3 there exists a $C = C(m) \in \mathcal{C} = (-\infty, M)$ such that $v(m+1; C) = 2(m+1)^2$ and by the strictness of the inequalities in Theorem 3.2.1 and Corollary 3.2.2, this C has to be unique.

Now to show this C is strictly greater than 2 we recollect from Subsection 2.3 that if there exists a smooth solution v_C to the ODE IVP (2.3.5) satisfying both the boundary conditions viz. $v_C(1) = 2$ and $v_C(m+1) = 2(m+1)^2$ then $v_C(\gamma) > 2\gamma^2$ for all $\gamma \in (1, m+1)$. Substituting all this and $P_C(m+1) = LC + N$ (from Lemma 3.2.1) in (2.3.5) and integrating it over $[1, m+1]$ we get $2(m+1)^2 - 2 > 2 \left((m+1)^2 - 1 \right) + LC + N$ which implies $LC + N < 0$ i.e. $C > -\frac{N}{L}$. As was noted in the proof of Lemma 3.2.3, $-\frac{N}{L} > 2$. So if $v_C(m+1) = 2(m+1)^2$ then $C > 2$. \square

4. Bando-Futaki invariants and hcscK metrics

Let M be a compact Kähler n -manifold. Given a Kähler form ω on M there exists a $\lambda \in C^\infty(M, \mathbb{R})$ such that $c_n(\omega) = \lambda \omega^n \in \Omega^{(n,n)}(M) = \Omega^{2n}(M)$ where $c_j(\omega)$ is the j^{th} Chern form of ω , $\Omega^{(i,j)}(M)$ is the set of all real (i, j) -forms on M and $\Omega^r(M)$ is the set of all real r -forms on M . By using Hodge Theory (see Aubin [2]) we get:

$$c_n(\omega) - \text{Hc}_n(\omega) = \sqrt{-1} \partial \bar{\partial} \varphi \tag{4.1}$$

where H denotes harmonic projection and $\varphi \in \Omega^{(n-1, n-1)}(M) \subseteq \Omega^{2n-2}(M)$. Furthermore $Hc_n(\omega) = \lambda_0 \omega^n$ for some $\lambda_0 \in \mathbb{R}$. So equation (4.1) becomes:

$$\sqrt{-1}\partial\bar{\partial}\varphi = (\lambda - \lambda_0)\omega^n \tag{4.2}$$

The n^{th} Bando-Futaki invariant for the Kähler class $[\omega]$ on M is defined as (by Bando [3]):

$$\mathcal{F}_n(Y, [\omega]) := \int_M \mathcal{L}_Y \varphi \wedge \omega, \quad Y \in \mathfrak{h}(M) \tag{4.3}$$

where \mathcal{L}_Y denotes Lie derivative w.r.t. Y and $\mathfrak{h}(M)$ is the set of all holomorphic vector fields on M . It has been proven by Bando [3] that \mathcal{F}_n is a function of the Kähler class $[\omega]$ alone and does not depend on the choice of the Kähler metric ω in $[\omega]$ and the $(n - 1, n - 1)$ -form φ satisfying equation (4.1), so that we have $\mathcal{F}_n : \mathfrak{h}(M) \times H^{(1,1)}(M, \mathbb{R})^+ \rightarrow \mathbb{R}$. The top Bando-Futaki invariant for a Kähler class provides an obstruction to the existence of hcscK metrics in it, as a Kähler metric is hcscK if and only if its top Chern form is harmonic ([3] and Pingali [18]).

Theorem 4.1 (Bando). *If ω is hcscK then $\mathcal{F}_n(\cdot, [\omega]) \equiv 0$ on $\mathfrak{h}(M)$.*

Now by using standard facts about Lie derivatives and harmonicity of differential forms, and equations (4.1), (4.2) and (4.3) we do the following computations for the top Bando-Futaki invariant of a general Kähler class $[\omega]$ on M :

$$\begin{aligned} \mathcal{F}_n(Y, [\omega]) &= \int_M \mathcal{L}_Y \varphi \wedge \omega \\ &= - \int_M \varphi \wedge \mathcal{L}_Y \omega \\ &= - \int_M \varphi \wedge (d(\iota_Y \omega) + \iota_Y(d\omega)) && \left(\text{Where } \iota_Y \text{ denotes} \right. \\ &= - \int_M \varphi \wedge d(\iota_Y \omega) && \left. \text{interior product w.r.t. } Y \right) \\ &= - \int_M \varphi \wedge \sqrt{-1}\partial\bar{\partial}\mathbf{f} && \text{(For some } \mathbf{f} \in C^\infty(M, \mathbb{R}) \text{)} \\ &= - \int_M \sqrt{-1}\partial\bar{\partial}\varphi \wedge \mathbf{f} \\ &= - \int_M \mathbf{f}(\lambda - \lambda_0)\omega^n \end{aligned} \tag{4.4}$$

Actually we can make one more observation from equation (4.2) viz. that $\lambda_0 = \frac{\int_M \lambda \omega^n}{\int_M \omega^n}$ is the average value of λ on M . We will do one more set of calculations before proceeding to prove our main result of this section:

Let J denote the almost complex structure induced by the complex structure on M , g denote the Kähler metric associated with the Kähler form ω , and $\flat : \mathfrak{X}^{(1,0)}(M) \rightarrow \Omega^{(0,1)}(M)$ and $\sharp : \Omega^{(0,1)}(M) \rightarrow \mathfrak{X}^{(1,0)}(M)$ be the ‘musical isomorphisms’ induced by g , where $\mathfrak{X}^{(1,0)}(M)$ denotes the set of all $(1, 0)$ -vector fields on M . For any $Y \in \mathfrak{h}(M) \subseteq \mathfrak{X}^{(1,0)}(M)$ we have:

$$\sqrt{-1}Y^\flat(Z) = J(Y)^\flat(Z) = g(J(Y), Z) = \omega(Y, Z) = \iota_Y \omega(Z) \tag{4.5}$$

for all $Z \in TM$, the tangent bundle of M , so that $\iota_Y \omega = \sqrt{-1}Y^\flat$. Finally we can see that in a Kähler class where higher extremal Kähler metrics exist, the top Bando-Futaki invariant characterizes the obstruction to the existence of hscK metrics:

Theorem 4.2. *Let ω be a higher extremal Kähler metric on M . Then ω is hscK if and only if $\mathcal{F}_n(\cdot, [\omega]) \equiv 0$ on $\mathfrak{h}(M)$. Furthermore let ω be hscK. Then every higher extremal Kähler metric in the Kähler class $[\omega]$ is hscK.*

Proof. Since ω is higher extremal Kähler, $\nabla^{1,0}\lambda = (\bar{\partial}\lambda)^\sharp \in \mathfrak{h}(X)$. Substituting $Y = (\bar{\partial}\lambda)^\sharp$ in the calculations (4.4) and (4.5) and using equation (4.2) we get $\mathbf{f} = (\lambda - \lambda_0)$ as follows:

$$\iota_Y \omega = \sqrt{-1}\bar{\partial}\lambda \tag{4.6}$$

$$\mathcal{L}_Y \omega = d(\iota_Y \omega) = d(\sqrt{-1}\bar{\partial}\lambda) = \sqrt{-1}\partial\bar{\partial}\lambda = \sqrt{-1}\partial\bar{\partial}(\lambda - \lambda_0) \tag{4.7}$$

$$\begin{aligned} \mathcal{F}_n(Y, [\omega]) &= - \int_M \varphi \wedge \mathcal{L}_Y \omega \\ &= - \int_M \varphi \wedge \sqrt{-1}\partial\bar{\partial}(\lambda - \lambda_0) \\ &= - \int_M \sqrt{-1}\partial\bar{\partial}\varphi \wedge (\lambda - \lambda_0) \\ &= - \int_M (\lambda - \lambda_0)^2 \omega^n \\ &=: -\|\lambda - \lambda_0\|_{\mathcal{L}^2(M, \omega)}^2 \end{aligned} \tag{4.8}$$

Now if ω is higher extremal Kähler and $\mathcal{F}_n(\cdot, [\omega]) \equiv 0$ then $0 = \mathcal{F}_n((\bar{\partial}\lambda)^\sharp, [\omega]) = -\|\lambda - \lambda_0\|_{\mathcal{L}^2(M, \omega)}^2$, and hence $\lambda = \lambda_0$ i.e. ω is hscK. The converse was already done by

Bando [3] in Theorem 4.1. The second assertion of Theorem 4.2 follows directly from the first assertion. \square

Now returning back to the example of our surface X we state the following result of Pingali [18] saying that the higher extremal Kähler metric on X , obtained by the momentum construction method in Subsections 2.2 and 2.3, cannot be hcscK:

Theorem 4.3 (Pingali). *For any $m > 0$ if there exists a higher extremal Kähler metric ω in the Kähler class $2\pi(C + mS_\infty)$ on X satisfying the ansatz (2.2.1) and the conditions (2.3.6) with its momentum profile $\phi(\gamma)$ satisfying the ODE (2.2.10) on $[1, m + 1]$ with the boundary conditions (2.2.11) and with $\phi(\gamma) > 0$ on $(1, m + 1)$, then $\nabla^{1,0}\lambda \neq 0$ i.e. ω is not hcscK.*

Remark 4.1. Note that even though the existence of a higher extremal Kähler metric by the momentum construction method in the Kähler class $2\pi(C + mS_\infty)$ was not proven by Pingali [18] for a general m , but the fact, that such a metric if it exists is not hcscK, was proven by them for a general m .

Now just like being a higher extremal Kähler metric is a scale-invariant property, in exactly the same way being an hcscK metric is also a scale-invariant property. Indeed from the rescaling arguments in Subsection 2.3, on a compact Kähler n -manifold M if ω satisfying equation (1.1.1) is an hcscK metric then so is $k\omega$ for any $k > 0$, as seen from equation (2.3.7).

Thus on our surface X for any $k, m > 0$ if $\omega \in 2\pi(C + mS_\infty)$ is higher extremal Kähler but not hcscK and satisfies the equations (2.3.6) then $\eta := \frac{k}{2\pi}\omega \in k(C + mS_\infty)$ is also higher extremal Kähler but not hcscK and satisfies the equation $c_2(\eta) = \frac{\lambda}{2k^2}\eta^2$. From this and from Theorem 4.3 and Corollaries 2.3.1 and 2.3.2 we get the following result:

Corollary 4.1. *For all $a, b > 0$ there exists a Kähler metric η on X , which is higher extremal Kähler but not hcscK, satisfying the following:*

$$[\eta] = aC + bS_\infty, \quad c_2(\eta) = \frac{\lambda}{2a^2}\eta^2, \quad \nabla^{1,0}\lambda \neq 0 \in \mathfrak{h}(X) \tag{4.9}$$

We finally conclude the non-existence of any hcscK metrics on X from Theorem 4.2 and Corollary 4.1:

Corollary 4.2. *For all $a, b > 0$ there does not exist an hcscK metric in the Kähler class $aC + bS_\infty$ on X . Thus there do not exist any hcscK metrics on X .*

5. Generalization of the results to all pseudo-Hirzebruch surfaces

5.1. The Kähler cone of a pseudo-Hirzebruch surface

Following the terminology of Tønnesen-Friedman [21] we define a *pseudo-Hirzebruch surface* to be a compact complex surface of the form $X := \mathbb{P}(L \oplus \mathcal{O})$ where L is a holomorphic line bundle of degree $d \neq 0$ on a (compact) Riemann surface Σ of genus $g \geq 2$. Note that X is a minimal ruled (complex) surface. We equip Σ with a Kähler metric ω_Σ of constant scalar curvature $S_\Sigma := -2(g - 1)$ and we equip L with a Hermitian metric h of curvature form $F(h) := d\omega_\Sigma$ so that the area of Σ w.r.t. ω_Σ and the Ricci curvature form of ω_Σ are respectively the following (where $\chi = -2(g - 1)$ is the Euler characteristic of Σ):

$$A_\Sigma = \int_\Sigma \omega_\Sigma = 2\pi \frac{\chi}{S_\Sigma} = 2\pi, \quad \rho_\Sigma = S_\Sigma \omega_\Sigma = -2(g - 1) \omega_\Sigma \tag{5.1.1}$$

As in Subsection 2.1 let C be the Poincaré dual of a typical fibre of X , $S_\infty = \mathbb{P}(L \oplus \{0\}) \subseteq \mathbb{P}(L \oplus \mathcal{O}) = X$ be the infinity divisor of X and $S_0 = \mathbb{P}(\{0\} \oplus \mathcal{O}) \subseteq \mathbb{P}(L \oplus \mathcal{O}) = X$ be the zero divisor of X (and Σ be identified with S_0 as a (complex) curve in X), and further let $c_1(L)$ be the first Chern class of L , $[\omega_\Sigma]$ be the Kähler class of ω_Σ and $[\Sigma]$ be the fundamental class of Σ (and $[\Sigma]$ be identified with S_0 in $H^2(X, \mathbb{R})$). In this general case we have the following intersection formulae similar to the intersection formulae (2.1.1) and (2.1.2) in the special case in Subsection 2.1 (Barth, Hulek et al. [4], Székelyhidi [19] and Tønnesen-Friedman [21]):

$$C^2 = 0, \quad S_\infty^2 = -d, \quad S_0^2 = d, \quad C \cdot S_\infty = 1, \quad C \cdot S_0 = 1, \quad S_\infty \cdot S_0 = 0 \tag{5.1.2}$$

$$c_1(L) \cdot [\Sigma] = d, \quad [\omega_\Sigma] \cdot [\Sigma] = 2\pi, \quad C \cdot [\Sigma] = 1, \quad S_\infty \cdot [\Sigma] = 0, \quad S_0 \cdot [\Sigma] = d \tag{5.1.3}$$

Now let us first assume that $d < 0$ which will be the case in the remainder of Subsection 5.1 as well as in Subsection 5.2. By using the Leray-Hirsch Theorem and the Nakai-Moishezon Criterion in the real cohomology case (Fujiki [8], Lamari [13], LeBrun-Singer [16] and Tønnesen-Friedman [21]) we compute the Kähler cone of X as follows (attributed to Fujiki [8] and given in Tønnesen-Friedman [21]):

$$\begin{aligned} H^2(X, \mathbb{R}) &= \mathbb{R}C \oplus \mathbb{R}S_\infty \\ H^{(1,1)}(X, \mathbb{R})^+ &= \{aC + bS_\infty \mid 2ab - db^2 > 0, \quad b > 0, \quad a - db > 0, \quad a > 0, \quad a > 0\} \\ &= \{aC + bS_\infty \mid a, b > 0\} \end{aligned} \tag{5.1.4}$$

where the inequalities characterizing a general Kähler class $aC + bS_\infty$ on X are obtained by substituting the intersection formulae (5.1.2) and (5.1.3) in Corollary 2.1.1 which

follows from Theorem 2.1.1, both of which are applicable in this general setting of the pseudo-Hirzebruch surface $X = \mathbb{P}(L \oplus \mathcal{O})$ as well (see [8] and [21]).

Over here also we consider the Kähler classes $2\pi(C + mS_\infty)$ where $m > 0$ only and obtain the expected results on higher extremal Kähler and hscK metrics in these Kähler classes, and since being a higher extremal Kähler metric (and an hscK metric respectively) is a scale-invariant property as seen in Subsection 2.3 (and in Section 4 respectively), we can rescale the constructed metrics and generalize the results to all the Kähler classes $aC + bS_\infty$ where $a, b > 0$.

5.2. The momentum construction method applied to a pseudo-Hirzebruch surface

We now use the momentum construction method attributed to Hwang-Singer [12] (described briefly in Subsection 2.2) to construct a higher extremal Kähler metric ω in the Kähler class $2\pi(C + mS_\infty)$ with $m > 0$ whose top Chern form $c_2(\omega)$ satisfies the following equation:

$$c_2(\omega) = \frac{d^2\lambda}{2(2\pi)^2}\omega^2 \tag{5.2.1}$$

where $\nabla^{1,0}\lambda$ is a holomorphic vector field.

We carefully go through the calculations involving holomorphic coordinates on the surface X done in Pingali [18] and Székelyhidi [19] by following all their conventions and observe where the factors containing the genus g and the degree d appear in the expressions of ω , ω^2 , the curvature form matrix $\Theta(\omega)$ and $c_2(\omega)$. Note that $d < 0$ throughout this discussion.

As in Subsection 2.2 let $p : X \rightarrow \Sigma$ be the fibre bundle projection, z be a local holomorphic coordinate on Σ , w be a holomorphic fibre coordinate on L corresponding to a local holomorphic trivialization around z , $s := \ln |(z, w)|_h^2 = \ln |w|^2 + \ln h(z)$ be the fibrewise coordinate on the total space of L minus the zero section, $f(s)$ be strictly convex and $s - df(s)$ be strictly increasing, and let ω satisfy the following ansatz (as in [18] and [19]):

$$\omega = p^*\omega_\Sigma + \sqrt{-1}\partial\bar{\partial}f(s) \tag{5.2.2}$$

We have the following coordinate equations ([18] and [19]):

$$\begin{aligned} \sqrt{-1}\partial\bar{\partial}s &= \sqrt{-1}\partial\bar{\partial}\ln h(z) \\ &= -dp^*\omega_\Sigma \end{aligned} \tag{5.2.3}$$

$$\begin{aligned} \sqrt{-1}\partial\bar{\partial}f(s) &= f'(s)\sqrt{-1}\partial\bar{\partial}\ln h(z) + f''(s)\sqrt{-1}\frac{dw \wedge d\bar{w}}{|w|^2} \\ &= -df'(s)p^*\omega_\Sigma + f''(s)\sqrt{-1}\frac{dw \wedge d\bar{w}}{|w|^2} \end{aligned} \tag{5.2.4}$$

where we are using the facts that $F(h) = -\sqrt{-1}\partial\bar{\partial}\ln h(z)$ and $F(h) = \mathbf{d}\mathbf{p}^*\omega_\Sigma$ (as defined in Subsection 5.1).

We write down the expression for ω as follows:

$$\omega = (1 - \mathbf{d}f'(s))\mathbf{p}^*\omega_\Sigma + f''(s)\sqrt{-1}\frac{dw \wedge d\bar{w}}{|w|^2} \tag{5.2.5}$$

We want $[\omega] = 2\pi(C + mS_\infty)$ and the intersection formulae (5.1.2) and (5.1.3) and equation (5.2.5) help us in computing the following integrals:

$$\begin{aligned} 2\pi m = [\omega] \cdot C &:= \int_C \omega = \int_{\mathbb{C}\setminus\{0\}} f''(s)\sqrt{-1}\frac{dw \wedge d\bar{w}}{|w|^2} \\ &= 2\pi \left(\lim_{s \rightarrow \infty} f'(s) - \lim_{s \rightarrow -\infty} f'(s) \right) \end{aligned} \tag{5.2.6}$$

$$\begin{aligned} 2\pi(1 - \mathbf{d}m) = [\omega] \cdot S_\infty &:= \int_{S_\infty} \omega = \int_\Sigma \lim_{s \rightarrow \infty} (1 - \mathbf{d}f'(s))\omega_\Sigma \\ &= 2\pi \left(1 - \mathbf{d} \lim_{s \rightarrow \infty} f'(s) \right) \end{aligned} \tag{5.2.7}$$

which give us $0 \leq f'(s) \leq m$.

We then compute the curvature form matrix $\Theta(\omega)$ as follows:

$$\omega^2 = 2(1 - \mathbf{d}f'(s))f''(s)\mathbf{p}^*\omega_\Sigma\sqrt{-1}\frac{dw \wedge d\bar{w}}{|w|^2} \tag{5.2.8}$$

$$\Theta(\omega) = \begin{bmatrix} -\partial\bar{\partial}\ln(1 - \mathbf{d}f'(s)) + 2(g-1)\sqrt{-1}\mathbf{p}^*\omega_\Sigma & 0 \\ 0 & -\partial\bar{\partial}\ln(f''(s)) \end{bmatrix} \tag{5.2.9}$$

where we are using $\rho_\Sigma = -2(g-1)\omega_\Sigma$ from the equations (5.1.1), ρ_Σ being the Ricci curvature form of ω_Σ .

Again as in Subsection 2.2 we take the Legendre Transform $F(\tau)$ in the variable $\tau := f'(s) \in [0, m]$ as $f(s) + F(\tau) = s\tau$. Then the momentum profile of ω is $\phi(\tau) := \frac{1}{F''(\tau)} = f''(s)$. We further have $f'''(s) = \phi'(\tau)\phi(\tau)$ ([18]). We take the new variable $\gamma := -\mathbf{d}\tau + 1 \in [1, -\mathbf{d}m + 1]$ after which we get $\gamma = 1 - \mathbf{d}f'(s)$, $\phi(\gamma) = f''(s)$ and $f'''(s) = -\mathbf{d}\phi'(\gamma)\phi(\gamma)$. We will also need the following coordinate equation ([18] and [19]):

$$\sqrt{-1}\partial s \bar{\partial} s = \sqrt{-1}\frac{dw \wedge d\bar{w}}{|w|^2} \tag{5.2.10}$$

Using the coordinate equations (5.2.3), (5.2.4) and (5.2.10) we write down the curvature form matrix $\sqrt{-1}\Theta(\omega)$ in terms of $\phi(\gamma)$ as follows ([18]):

$$\begin{aligned}
 \sqrt{-1}\Theta(\omega) &= \begin{bmatrix} -\sqrt{-1}\partial\bar{\partial}\ln(\gamma) - 2(g-1)\mathbf{p}^*\omega_\Sigma & 0 \\ 0 & -\sqrt{-1}\partial\bar{\partial}\ln(\phi) \end{bmatrix} \tag{5.2.11} \\
 &= \begin{bmatrix} \sqrt{-1}\frac{\partial\gamma\bar{\partial}\gamma}{\gamma^2} - \frac{1}{\gamma}\sqrt{-1}\partial\bar{\partial}\gamma - 2(g-1)\mathbf{p}^*\omega_\Sigma & 0 \\ 0 & \frac{(\phi')^2 - \phi\phi''}{\phi^2}\sqrt{-1}\partial\gamma\bar{\partial}\gamma - \frac{\phi'}{\phi}\sqrt{-1}\partial\bar{\partial}\gamma \end{bmatrix} \\
 &= \begin{bmatrix} \mathbf{d}^2\frac{\phi}{\gamma}\left(\frac{\phi}{\gamma} - \phi'\right)\sqrt{-1}\frac{dw\wedge d\bar{w}}{|w|^2} - \left(\mathbf{d}^2\frac{\phi}{\gamma} + 2(g-1)\right)\mathbf{p}^*\omega_\Sigma & 0 \\ 0 & -\mathbf{d}^2\phi''\phi\sqrt{-1}\frac{dw\wedge d\bar{w}}{|w|^2} - \mathbf{d}^2\phi'\mathbf{p}^*\omega_\Sigma \end{bmatrix}
 \end{aligned}$$

The top Chern form of ω is given in terms of $\phi(\gamma)$ by:

$$c_2(\omega) = \frac{1}{(2\pi)^2}\mathbf{p}^*\omega_\Sigma\sqrt{-1}\frac{dw\wedge d\bar{w}}{|w|^2}\mathbf{d}^2\frac{\phi}{\gamma^2}(\gamma(\mathbf{d}^2\phi + 2(g-1)\gamma)\phi'' + \mathbf{d}^2\phi'(\phi'\gamma - \phi)) \tag{5.2.12}$$

Comparing equations (5.2.1), (5.2.8) and (5.2.12) and since $\nabla^{1,0}\lambda = -\mathbf{d}\lambda'w\frac{\partial}{\partial w}$ (from equation (2.2.9)) is a holomorphic vector field if and only if $\lambda = A\gamma + B$ for some $A, B \in \mathbb{R}$, we derive the following ODE for $\phi(\gamma)$ on $[1, -dm + 1]$ (for some $C \in \mathbb{R}$):

$$\begin{aligned}
 \gamma(\mathbf{d}^2\phi + 2(g-1)\gamma)\phi'' + \mathbf{d}^2\phi'(\phi'\gamma - \phi) &= (A\gamma + B)\gamma^3 \\
 2(g-1)\phi'' + \mathbf{d}^2\left(\frac{\phi\phi'}{\gamma}\right)' &= A\gamma^2 + B\gamma \\
 2(g-1)\phi' + \mathbf{d}^2\frac{\phi\phi'}{\gamma} &= A\frac{\gamma^3}{3} + B\frac{\gamma^2}{2} + C \\
 (2(g-1)\gamma + \mathbf{d}^2\phi)\phi' &= A\frac{\gamma^4}{3} + B\frac{\gamma^3}{2} + C\gamma \tag{5.2.13}
 \end{aligned}$$

As explained in [19], for ω to extend smoothly across the zero and infinity divisors of X we should have the following boundary conditions on ϕ :

$$\begin{aligned}
 \phi(1) &= \lim_{\gamma \rightarrow 1} \phi(\gamma) = \lim_{s \rightarrow -\infty} f''(s) = 0 \\
 \phi'(1) &= -\frac{1}{\mathbf{d}}\lim_{\gamma \rightarrow 1} (-\mathbf{d}\phi'(\gamma)) = -\frac{1}{\mathbf{d}}\lim_{s \rightarrow -\infty} \frac{f'''(s)}{f''(s)} = -\frac{1}{\mathbf{d}} \tag{5.2.14}
 \end{aligned}$$

$$\begin{aligned}
 \phi(-dm + 1) &= \lim_{\gamma \rightarrow -dm+1} \phi(\gamma) = \lim_{s \rightarrow \infty} f''(s) = 0 \\
 \phi'(-dm + 1) &= -\frac{1}{\mathbf{d}}\lim_{\gamma \rightarrow -dm+1} (-\mathbf{d}\phi'(\gamma)) = -\frac{1}{\mathbf{d}}\lim_{s \rightarrow \infty} \frac{f'''(s)}{f''(s)} = \frac{1}{\mathbf{d}} \tag{5.2.15}
 \end{aligned}$$

where we are using the relations between $\phi(\gamma)$ and $f(s)$ ([18]). Also as f was required to be strictly convex we must have $\phi > 0$ on $(1, -dm + 1)$ in addition to the boundary conditions (5.2.14) and (5.2.15).

Now as in Subsection 2.3 we define $p(\gamma) := \mathbf{d}^2\left(A\frac{\gamma^3}{3} + B\frac{\gamma^2}{2} + C\right)$ and do the change of variables $v := \frac{(2(g-1)\gamma + \mathbf{d}^2\phi)^2}{2}$, $\gamma \in [1, -dm + 1]$ and obtain the following ODE BVP:

$$\begin{aligned}
 v' &= 2(g-1)\sqrt{2}\sqrt{v} + p(\gamma)\gamma \text{ on } [1, -dm+1] \\
 v(1) &= 2(g-1)^2, \quad v(-dm+1) = 2(g-1)^2(-dm+1)^2 \\
 v'(1) &= 2(g-1)[2(g-1)-d], \quad v'(-dm+1) = 2(g-1)(-dm+1)[2(g-1)+d] \\
 v(\gamma) &> 2(g-1)^2\gamma^2 \text{ on } (1, -dm+1)
 \end{aligned}
 \tag{5.2.16}$$

After this point the entire analysis developed in Subsections 2.3, 3.1 and 3.2 with all the bounds and estimates can be worked out in this general setting as well and in exactly the same way it can be proven that for each $m > 0$ there exist unique $A, B, C \in \mathbb{R}$ such that there exists a unique smooth solution v to the ODE BVP (5.2.16) on $[1, -dm+1]$ satisfying all the required conditions.

Hence given a pseudo-Hirzebruch surface X of genus $g \geq 2$ and degree $d < 0$ we can construct higher extremal Kähler metrics ω satisfying equation (5.2.1) in the Kähler classes $2\pi(C + mS_\infty)$ for all $m > 0$, and by rescaling these constructed metrics we will get higher extremal Kähler metrics in all the Kähler classes on X . Then we can verify that these constructed metrics ω cannot be hcscK i.e. $\nabla^{1,0}\lambda \neq 0$ just like Pingali [18] did in the proof of Theorem 4.3 and by the arguments in Section 4 it will follow that hcscK metrics do not exist on the surface X .

5.3. The case of positive degree

Now we will see what happens when $d > 0$. All the other things about $X = \mathbb{P}(L \oplus \mathcal{O})$ are exactly the same as in Subsections 5.1 and 5.2. Let us first observe that if L has degree d over Σ then the dual vector bundle L^* is a holomorphic line bundle of degree $-d$ over Σ and there is a canonical isomorphism of complex manifolds $\mathbb{P}(L \oplus \mathcal{O}) \cong \mathbb{P}(L^* \oplus \mathcal{O})$ given in local holomorphic coordinates as $(z, w) \mapsto (z, w^{-1})$ which maps the zero and infinity divisors of $\mathbb{P}(L \oplus \mathcal{O})$ to the infinity and zero divisors of $\mathbb{P}(L^* \oplus \mathcal{O})$ respectively. Just as in Subsection 5.1 we use Theorem 2.1.1 and Corollary 2.1.1 and the intersection formulae (5.1.2) and (5.1.3) and describe the Kähler cone of X as follows (Fujiki [8] and Tønnesen-Friedman [21]):

$$\begin{aligned}
 H^2(X, \mathbb{R}) &= \mathbb{R}C \oplus \mathbb{R}S_0 \\
 H^{(1,1)}(X, \mathbb{R})^+ &= \{aC + bS_0 \mid 2ab + db^2 > 0, \quad b > 0, \quad a + db > 0, \quad a > 0, \quad a + db > 0\} \\
 &= \{aC + bS_0 \mid a, b > 0\}
 \end{aligned}
 \tag{5.3.1}$$

Note that in the description of the Kähler cone over here we are using the zero divisor of X instead of the infinity divisor.

We want to construct a higher extremal Kähler metric ω in the Kähler class $2\pi(C + mS_0)$ with $m > 0$ satisfying the equation (5.2.1). Following the calculations done in Subsection 5.2 we write the ansatz (5.2.2), and then the expression (5.2.5) holds

true in this case as well along with all the coordinate equations. Only the computation of the integrals in equations (5.2.6) and (5.2.7) changes in the following way:

$$2\pi m = [\omega] \cdot C := \int_C \omega = \int_{\mathbb{C} \setminus \{0\}} f''(s) \sqrt{-1} \frac{dw \wedge d\bar{w}}{|w|^2} \tag{5.3.2}$$

$$= 2\pi \left(\lim_{s \rightarrow \infty} f'(s) - \lim_{s \rightarrow -\infty} f'(s) \right)$$

$$2\pi (1 + dm) = [\omega] \cdot S_0 := \int_{S_0} \omega = \int_{\Sigma} \lim_{s \rightarrow -\infty} (1 - df'(s)) \omega_{\Sigma} \tag{5.3.3}$$

$$= 2\pi \left(1 - d \lim_{s \rightarrow -\infty} f'(s) \right)$$

which gives us $-m \leq f'(s) \leq 0$. We take the variable of the Legendre Transform as $\tau := f'(s) \in [-m, 0]$ and the new variable as $\gamma := -d\tau + 1 \in [1, dm + 1]$ while F and ϕ remain the same as in Subsection 5.2. The expressions for $\Theta(\omega)$ and $c_2(\omega)$ also remain unchanged and hence we will obtain the following ODE for $\phi(\gamma)$, $\gamma \in [1, dm + 1]$:

$$(2(g - 1)\gamma + d^2\phi) \phi' = A \frac{\gamma^4}{3} + B \frac{\gamma^3}{2} + C\gamma \tag{5.3.4}$$

The appropriate boundary conditions for ϕ in this case are the following (Pingali [18] and Székelyhidi [19]):

$$\phi(1) = \lim_{s \rightarrow \infty} f''(s) = 0 \tag{5.3.5}$$

$$\phi'(1) = -\frac{1}{d} \lim_{s \rightarrow \infty} \frac{f'''(s)}{f''(s)} = \frac{1}{d}$$

$$\phi(dm + 1) = \lim_{s \rightarrow -\infty} f''(s) = 0 \tag{5.3.6}$$

$$\phi'(dm + 1) = -\frac{1}{d} \lim_{s \rightarrow -\infty} \frac{f'''(s)}{f''(s)} = -\frac{1}{d}$$

Again as in Subsection 5.2, $\phi > 0$ on $(1, dm + 1)$.

Comparing the equations (5.3.4), (5.3.5) and (5.3.6) with the equations (5.2.13), (5.2.14) and (5.2.15) we observe that in the case when $d > 0$ we get the same ODE BVP for the momentum profile ϕ of the Kähler metric ω as we would have gotten for a holomorphic line bundle of degree $-d$ over Σ , except that the to be constructed metric ω now belongs to the Kähler class $2\pi(C + mS_0)$ on $X = \mathbb{P}(L \oplus \mathcal{O})$.

So for this case as well we obtain the same results on the existence of higher extremal Kähler metrics and the non-existence of hscK metrics as those gotten in Subsection 5.2.

We finally summarize the work done in Subsections 5.1, 5.2 and 5.3 in the following result:

Theorem 5.3.1. *Given a pseudo-Hirzebruch surface $X := \mathbb{P}(L \oplus \mathcal{O})$ where L is a holomorphic line bundle of degree $d \neq 0$ on a compact Riemann surface Σ of genus $g \geq 2$, for all $a, b > 0$ there exists a Kähler metric η on X , which is higher extremal Kähler but not hcscK, satisfying the following:*

$$[\eta] = aC + bS_\infty, \quad c_2(\eta) = \frac{d^2\lambda}{2a^2}\eta^2, \quad \nabla^{1,0}\lambda \neq 0 \in \mathfrak{h}(X) \quad (\text{If } d < 0) \tag{5.3.7}$$

$$[\eta] = aC + bS_0, \quad c_2(\eta) = \frac{d^2\lambda}{2a^2}\eta^2, \quad \nabla^{1,0}\lambda \neq 0 \in \mathfrak{h}(X) \quad (\text{If } d > 0) \tag{5.3.8}$$

Further hcscK metrics do not exist in any Kähler class on X .

6. Analogy with the extremal Kähler setup

In Tønnesen-Friedman [21] and Hwang-Singer [12] where the usual extremal Kähler analogue of this problem was studied, the following 2 questions were asked regarding the existence and uniqueness of extremal Kähler metrics:

Question 6.1. Given a compact Kähler manifold having an extremal Kähler metric in a given Kähler class, does it have an extremal Kähler metric in each Kähler class?

Question 6.2. Given a compact Kähler manifold having an extremal Kähler metric in a given Kähler class, is this extremal Kähler metric unique in its Kähler class modulo the group action of the maximal connected group of automorphisms of the Kähler manifold?

Tønnesen-Friedman [21] had proven that for a general pseudo-Hirzebruch surface X of genus $g \geq 2$ and degree $d \neq 0$, both *Question 6.1* and *Question 6.2* cannot have an affirmative answer simultaneously i.e. if every Kähler class on X has an extremal Kähler metric then there exists a Kähler class in which the uniqueness condition of extremal Kähler metrics (even up to automorphisms of X) fails, and vice versa if every Kähler class on X , which can be represented by an extremal Kähler metric, has a unique such extremal Kähler representative (up to automorphisms of X) then there exists a Kähler class in which extremal Kähler metrics do not exist (see [21] for the details). Specifically from the work of Tønnesen-Friedman [21] it followed that the momentum construction method of Hwang-Singer [12] yields extremal Kähler metrics which are not cscK only in the Kähler classes $aC + bS_\infty$ if $d < 0$ (and $aC + bS_0$ if $d > 0$) with $0 < \frac{b}{a} < k_1$ for a unique $k_1 = k_1(g, d) = k_1(g, -d) \in \mathbb{R}_{>0}$. It later followed from the work of Apostolov, Calderbank et al. [1] that there do not exist extremal Kähler metrics (even without the symmetries of the momentum construction method) in the Kähler classes $aC + bS_\infty$ if $d < 0$ (and $aC + bS_0$ if $d > 0$) with $\frac{b}{a} \geq k_1$. In the special case where $g = 2$ and $d = -1$ whose exposition is given in Székelyhidi [19] (and whose higher extremal Kähler analogue is studied in Pingali [18] and in this paper), it is seen that $k_1 = k_1(2, -1) \approx 18.889$ ([19]). But in complete contrast to that, we have been able to construct higher extremal

Kähler metrics which are not hscK in all the Kähler classes $aC + bS_\infty$ if $d < 0$ (and $aC + bS_0$ if $d > 0$) with $a, b > 0$, which precisely constitute the Kähler cone of X (Fujiki [8] and Tønnesen-Friedman [21]). So *Question 6.1* has negative answer for all pseudo-Hirzebruch surfaces whereas we have answered the higher extremal Kähler version of *Question 6.1* affirmatively for all pseudo-Hirzebruch surfaces. Regarding *Question 6.2*, it was only recently proved by Berman-Berndtsson [5] that on any compact Kähler n -manifold M , cscK metrics (and even extremal Kähler metrics) in a given Kähler class are unique modulo the group action of $\text{Aut}_0(M)$ (if they exist at all) where $\text{Aut}_0(M)$ denotes the maximal connected group of automorphisms of M . We hope to explore the higher extremal Kähler version of *Question 6.2* at least for pseudo-Hirzebruch surfaces in our future works.

Besides these, Tønnesen-Friedman [21] had already proven that cscK metrics do not exist on a general pseudo-Hirzebruch surface by actually computing the Futaki invariant on an arbitrary Kähler class and for a specific holomorphic vector field using a certain formula proven by LeBrun-Simanca [14], and observing that it turns out to be non-zero. But for proving the non-existence of hscK metrics on pseudo-Hirzebruch surfaces we have instead proven a result about the top Bando-Futaki invariant (viz. Theorem 4.2) analogous to the following result about the Futaki invariant proven by Calabi [7] and LeBrun-Simanca [14] (and mentioned in [19] and [21]) and then used the fact that the momentum construction method in our higher extremal Kähler case yields a higher extremal Kähler metric which is not hscK in each Kähler class on the surface.

Theorem 6.1 (*Calabi, LeBrun-Simanca*). *Let M be a compact Kähler n -manifold. Let $\mathcal{F}_1 : \mathfrak{h}(M) \times H^{(1,1)}(M, \mathbb{R})^+ \rightarrow \mathbb{R}$ be the Futaki invariant on M . Let ω be an extremal Kähler metric on M . Then ω is cscK if and only if $\mathcal{F}_1(\cdot, [\omega]) \equiv 0$ on $\mathfrak{h}(M)$. Furthermore let ω be cscK. Then every extremal Kähler metric in the Kähler class $[\omega]$ is cscK.*

We hope to work out in the future a formula for the calculation of the top Bando-Futaki invariant on a compact Kähler surface with some nice properties which would be analogous to the formula of LeBrun-Simanca [14] (used by Tønnesen-Friedman [21]) for the calculation of the Futaki invariant on such a surface, as it seems to be an exercise of independent importance.

Finally coming to the case when the Riemann surface Σ has genus $g = 0$ i.e. $\Sigma \cong \mathbb{C}P^1$ and the holomorphic line bundle L has degree $d \neq 0$ i.e. $L \cong \mathcal{O}(d)$, then the ruled surface $X \cong \mathbb{P}(\mathcal{O}(d) \oplus \mathcal{O})$ i.e. X is a Hirzebruch surface. Calabi [6] had constructed an extremal Kähler metric which is not cscK in each Kähler class on X and then Theorem 6.1 gave the non-existence of any cscK metrics on X (see [21]). The analogous problem of constructing higher extremal Kähler metrics on Hirzebruch surfaces would be interesting but is completely out of the scope of this paper. In the case when $g = 1$ i.e. Σ is a complex elliptic curve representing an embedding of the complex torus $S^1 \times S^1$ into $\mathbb{C}P^2$, the questions about the existence of higher extremal Kähler and hscK metrics on $X := \mathbb{P}(L \oplus \mathcal{O})$ seem to be even intriguing and mysterious.

It would also be interesting to see in the future if there are some nice properties of the set of all Kähler classes of higher extremal Kähler metrics (along with the subset of all Kähler classes of hscK metrics) analogous to those of the set of all Kähler classes of extremal Kähler metrics (along with the subset of all Kähler classes of cscK metrics) proven by LeBrun-Simanca [14,15], and if a ‘Deformation Theory’ can be developed for higher extremal Kähler and hscK metrics analogous to the one for extremal Kähler and cscK metrics developed by LeBrun-Simanca [14,15]. In a different direction, writing down a ‘higher Calabi functional’ and a ‘higher Mabuchi functional’, which can characterize higher extremal Kähler and hscK metrics in ways analogous to the characterization of extremal Kähler and cscK metrics by the Calabi functional and the Mabuchi functional, will be an important area of exploration in our future works. Understanding the properties of these functionals on the space of all Kähler metrics in a given Kähler class on a compact Kähler manifold may possibly yield us some results about the uniqueness of hscK (and even higher extremal Kähler) metrics modulo automorphisms similar to those about the uniqueness of cscK (and also extremal Kähler) metrics modulo automorphisms recently obtained in Berman-Berndtsson [5]. More generally we would be interested in building a theory of higher extremal Kähler metrics in our future works by looking at analogous results from the theory of extremal Kähler metrics given in Calabi [6,7], Székelyhidi [19] and Tian [20] and the relatively recent work of Berman-Berndtsson [5].

Just like Pingali’s higher extremal Kähler and hscK metrics [18] are a generalization (or an extension) of extremal Kähler and cscK metrics to the level of the top cohomology of a compact Kähler manifold, along a similar way but in a different direction one can view Maschler’s ‘central Kähler metrics’ whose special cases are ‘constant central curvature Kähler metrics’ [17]. The definitions of these metrics mimic those of extremal Kähler and cscK metrics with scalar curvature being replaced by something called as ‘central curvature’ [17]. Even these metrics involve going up to the top cohomology and their theory was built by Maschler [17] by treating central curvature as analogous to the usual scalar curvature and then trying to find the counterparts of analogous results from the theory of the usual extremal Kähler and cscK metrics. Even we hope to do something along similar lines for higher extremal Kähler and hscK metrics.

Declaration of competing interest

The author has no competing interests to declare.

Data availability

Data will be made available on request.

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