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Pointwise *a posteriori* error analysis of a discontinuous Galerkin method for the elliptic obstacle problem

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We present *a posteriori* error analysis in the supremum norm for the symmetric interior penalty discontinuous Galerkin method for the elliptic obstacle problem. We construct discrete barrier functions based on appropriate corrections of the conforming part of the solution obtained via a constrained averaging operator. The corrector function accounts properly for the nonconformity of the approximation and it is estimated by direct use of the Green's function of the unconstrained elliptic problem. The use of the continuous maximum principle guarantees the validity of the analysis without mesh restrictions but with shape regularity. The proposed residual-type estimators are shown to be reliable and efficient. Numerical results in two dimensions are included to verify the theory and validate the performance of the error estimator.

Keywords: finite element; discontinuous Galerkin; *a posteriori* error estimate; obstacle problem; variational inequalities; Lagrange multiplier; pointwise.

1. Introduction

Several problems in a variety of scientific fields (physics, biology, industry, finance, geometry to mention a few) are often described by partial differential equations (PDEs) that exhibit *a priori* unknown interfaces or boundaries. They are typically called free boundary problems and constitute an important line of research for the mathematical and numerical analysis communities. Among these problems, the most classical is the elliptic obstacle problem that arises in diverse applications in elasticity, fluid dynamics and operations research, to name a few (Ciarlet, 1978; Kinderlehrer & Stampacchia, 2000; Glowinski, 2008). It is also a prototype of the elliptic variational inequalities of the first kind. Its variational formulation reads as follows: find $u \in \mathcal{K}$ such that

$$\int_{\Omega} \nabla u \cdot \nabla (u - v)\, \mathrm{d}x \le (f, u - v) \quad \forall\, v \in \mathcal{K}, \tag{1.1}$$

where $\Omega \subset \mathbb{R}^d$, $d = 2, 3$ is a bounded, polygonal ($d = 2$) or polyhedral ($d = 3$) domain with boundary $\partial\Omega$, $f \in L^\infty(\Omega)$ denotes the force and $(\cdot,\cdot)$ refers to the $L^2(\Omega)$ inner product. For any $1 \le r \le \infty$ and $D \subset \Omega$, we denote the $L^r(D)$ norm by $\|\cdot\|_{L^r(D)}$. The set D^o denotes the interior of D for any set $D \subset \Omega$. We further use standard notation for Sobolev spaces (Adams, 1975, Chapter 3). The associated norm on Sobolev space $W^{m,r}(\Omega)$ is denoted by $\|\cdot\|_{W^{m,r}}$ and seminorm by $|\cdot|_{W^{m,r}}$. The Sobolev space $W^{m,r}(\Omega)$ with $m = 0$ will denote the standard $L^r(\Omega)$ space. The set of admissible displacements $\mathcal{K}$ is a nonempty, closed and convex set defined by

$$\mathcal{K} := \{v \in V : v \ge \chi \ \text{ a.e. in } \ \Omega\},$$

with $V := H_0^1(\Omega)$ and $\chi \in H^1(\Omega) \cap C^{0,\alpha}(\bar{\Omega})$, $\quad 0 < \alpha \le 1$ denoting the obstacle which satisfies the compatibility condition $\chi \le 0$ on $\partial\Omega$. In (1.1), the solution u could be regarded as the equilibrium position of an elastic membrane subject to the load f whose boundary is held fixed ($u \in H_0^1(\Omega)$) and that is constrained to lie above the given obstacle χ. Such constraint results in nonlinearity inherent in the underlying PDE, the Poisson equation here. The classical theory of Stampacchia (Kinderlehrer & Stampacchia, 2000, Chapter 2, page 40; Glowinski, 2008, Chapter 1, page 4), guarantees the existence and uniqueness of the solution. As for the regularity of the solution, it is shown in Caffarelli & Kinderlehrer (1980), Frehse (1983) that under our assumptions on the domain and the data, $f \in L^\infty(\Omega)$ and $\chi \in H^1(\Omega) \cap C^{0,\alpha}(\overline{\Omega})$, $\ 0 < \alpha \le 1$, the solution to the variational inequality (1.1) is also Hölder continuous, i.e., we have $u \in H_0^1(\Omega) \cap C^{0,\beta}(\overline{\Omega})$ for $0 < \beta \le \alpha$.

The contact (or coincidence) and noncontact sets of the exact solution u are defined as

$$\mathbb{C} := \{x \in \Omega : u(x) = \chi(x)\}^o,$$
$$\mathbb{N} := \{x \in \Omega : u(x) > \chi(x)\}.$$

The free boundary here is the interior boundary of the coincidence set $\mathbb{C}$. The Hölder continuity of u guarantees that both $\mathbb{C}$ and the free boundary are closed sets.

From the Riesz representation theorem, associated to the solution u we introduce the linear functional $\sigma(u) \in H^{-1}(\Omega)$,

$$\langle \sigma(u), v\rangle = (f, v) - a(u, v) \quad \forall\, v \in H_0^1(\Omega), \tag{1.2}$$

where $\langle\cdot,\cdot\rangle$ refers to the duality pairing of $H^{-1}(\Omega)$ and $H_0^1(\Omega)$. From (1.2) and (1.1), it follows that

$$\langle \sigma(u), v - u\rangle \le 0 \quad \forall\, v \in \mathcal{K}. \tag{1.3}$$

In particular,[1] $\sigma(u) = 0$ on the noncontact set $\mathbb{N}$ and the support of $\sigma(u)$ is contained in the contact set $\mathbb{C}$. In fact (Kinderlehrer & Stampacchia, 2000, Chapter II, Section 6), the Riesz representation

[1] Choosing $0 \le \phi \in H_0^1(\Omega)$ and setting $v = u + \phi$ in (1.3), we get $\sigma(u) \le 0$. Further, if $u > \chi$ on some open set $D \subset \Omega$ then $\sigma(u) = 0$ on D (see Kinderlehrer & Stampacchia, 2000, Chapter 2, page 44).

theorem states that there is a finite non-negative radon measure $-\mathrm{d}\sigma$ supported on $\mathbb{C}$ satisfying

$$\langle \sigma(u), v\rangle = \int_{\Omega} v\,\mathrm{d}\sigma = (f, v) - a(u, v) \qquad \forall\, v \in C_0(\bar{\Omega}),$$

where $C_0(\bar{\Omega})$ denotes the space of continuous functions on $\bar{\Omega}$ with compact support. Often, $\sigma(u)$ is also referred to as the continuous Lagrange multiplier. We stress that the solution operator of (1.1) is not only nonlinear and nondifferentiable, but also strikingly not one-to-one in the sense that any variation in f within the contact set might or might not result in a variation in the solution u.

A priori error analysis in the energy norm for (1.1) is discussed in Falk (1974), Brezzi *et al.* (1978, 1977), Wang (2002), Gaddam & Gudi (2018a) for conforming methods and in Wang *et al.* (2010) for DG methods. In energy norm, several *a posteriori* error estimators are studied in Chen & Nochetto (2000), Veeser (2001), Bartels & Carstensen (2004), Braess (2005), Nochetto *et al.* (2010), Gudi & Porwal (2015), Gaddam & Gudi (2018b) for conforming approximation of variational inequalities while residual type *a posteriori* error estimators have been derived in Gudi & Porwal (2014a,b) for discontinuous Galerkin (DG) approximation. In this article, we study *a posteriori* error analysis in the supremum norm, for the piecewise linear symmetric interior penalty discontinuous Galerkin (SIPG) approximation of the elliptic obstacle problem (1.1). The rationality of our choice of norm comes from the fact that estimates in the maximum norm are relevant for variational inequalities since they provide further localized information on the approximation. Moreover, they also constitute a cornerstone for a future convergence analysis of the free boundary (Brezzi & Caffarelli, 1983; Nochetto, 1995).

A priori error analysis in L^∞ norm goes back to the seminal works Baiocchi (1977), Nitsche (1977) for conforming linear finite elements. As the proof uses a discrete maximum principle, the analysis requires the classical assumption on acute angles for mesh partitions (Ciarlet, 1978). Unlike for *a priori* analysis, Nochetto *et al.* (2003, 2005) accomplish L^∞ norm *a posteriori* analysis of the obstacle problem by using a *dual approach* to Baiocchi (1977) and making use of the *continuous* maximum principle, hence avoiding any mesh restriction except the standard shape regularity. Nochetto *et al.* (2003) use a positivity-preserving interpolation operator and Nochetto *et al.* (2005) refine the estimators and the analysis to ensure further localization of the estimators. The ideas in Nochetto *et al.* (2003, 2005) have been extended in Nochetto *et al.* (2006) to the conforming approximation of monotone semilinear equations and in Fierro & Veeser (2003) to the conforming approximation of a double obstacle problem arising from regularized minimization of a related functional, providing a uniform error estimator with respect to the regularization parameter that can be further used in the minimization of the discrete functional. We also refer to the nice survey Nochetto *et al.* (2015) for the approximation of classical and fractional obstacle problems.

In this work we follow the path laid down in Nochetto *et al.* (2003, 2005), extending it and adapting it for the SIPG method. We stress that the nonconformity of the approximation and the nonlinearity of the problem preclude the extension from being trivial. Also, the discontinuous nature of the finite element space allows for a slightly better localization of the error estimators. To be able to use the continuous maximum principle, which permits mesh restrictions to be avoided, we construct discrete barrier functions for the solution u that belong to H^1. This in turn forces us to construct the discrete sub- and supersolutions starting from the conforming part of the DG approximation and amend them with some corrector function that accounts for the nonconformity of our approximation. The conforming part of the DG approximation is obtained by using an averaging operator introduced in Gudi & Porwal (2014a) that ensures the obstacle constraint is preserved. The proof of the reliability of the estimator relies then on providing a pointwise estimate on the corrector function. Here, our approach varies from

Nochetto *et al.* (2003, 2005) where a regularized Green's function (for the unconstrained elliptic problem) is employed. Instead, we consider the direct use of Green's functions catering for the sharpened estimates (on the Green's function) proved in Demlow & Georgoulis (2012), Demlow & Kopteva (2016). As a result, the final estimators are improved in the exponent of the logarithmic factor with respect to Nochetto *et al.* (2003), but more significantly, this approach allows for accounting appropriately for the nonconformity in the approximation and evidences that the analysis performed for the SIPG method cannot be extended to any of the nonsymmetric methods of the interior penalty family. The local efficiency estimates of the error estimator are obtained using bubble functions techniques (Ainsworth & Oden, 2000). To prove the efficiency of the estimator term that measures the error in approximation of the obstacle in the discrete contact set and discrete free boundary set, unlike in Nochetto *et al.* (2003, 2005) we need to assume $\chi < 0$ on the part of $\partial\Omega$ that intersects the free boundary. This assumption is not very restrictive and is done in many research works. We accept this technical restriction in order to keep the locality of the definition of the discrete Lagrange multiplier granted by the DG construction, which is computationally simpler than those introduced in Nochetto *et al.* (2003, 2005).

We expect that the analysis presented could be helpful in understanding the convergence of the DG approximation to the free boundary, taking into account the important works Brezzi & Caffarelli (1983), Nochetto (1995). This would allow assessment of whether the often-acclaimed flexibility of DG methods could bring any real benefit with respect to the conforming approximation in this context. This will be the subject of future research. The ideas developed here might be used in the analysis of DG approximation of control problems as well as the *a posteriori* error analysis of DG methods for other variational inequalities e.g. the Signorini problem (Glowinski, 2008). The error estimator obtained in this article improves the error estimator obtained in Nochetto *et al.* (2003) by a logarithmic factor when applied to conforming finite element methods. Except for this logarithmic term, our error estimator is comparable to that of Nochetto *et al.* (2003). The theory presented in this article holds for first-order approximation. While the reliability analysis could be extended for higher-order methods, the analysis for the efficiency part hinges crucially on the use of piecewise linear elements. We have included though a numerical experiment exploring the performance of the estimators for piecewise quadratic approximation.

The article is organized as follows. In the next section we introduce basic notation and some auxiliary results that will be used in further analysis. The DG method for approximating the elliptic obstacle problem (1.1) and its main properties are discussed in Section 3. In Section 4 we introduce the error estimators that are the main tools for the analysis and prove the major result of the paper, namely the reliability of the estimator. The efficiency of the *a posteriori* error estimator is discussed in Section 5. Numerical experiments in two dimensions are presented illustrating the theoretical findings and validating the performance of the estimator in Section 6.

2. Notation and preliminary results

In this section, we introduce useful notation and some auxiliary results that will be used throughout this work.

$$\mathcal{T}_h = \text{ simplicial triangulations of } \Omega, \text{ which are assumed to be shape regular,}$$

$$T = \text{an element of } \mathcal{T}_h, \qquad |T| = \text{volume of the simplex } T,$$

$$h_T = \text{diameter of } T, \qquad h = \max\{h_T : T \in \mathcal{T}_h\}, \qquad h_{\min} = \min\{h_T : T \in \mathcal{T}_h\},$$

$\mathcal{V}_h^i$ = set of all vertices in $\mathcal{T}_h$ that are in Ω,

$\mathcal{V}_h^b$ = set of all vertices in $\mathcal{T}_h$ that are on $\partial\Omega$, $\quad \mathcal{V}_h = \mathcal{V}_h^i \cup \mathcal{V}_h^b$,

$\mathcal{V}_T$ = set of three vertices of simplex T,

$\mathcal{E}_h^o$ = set of all interior edges/faces of $\mathcal{T}_h$,

$\mathcal{E}_h^\partial$ = set of all boundary edges/faces of $\mathcal{T}_h$, $\quad \mathcal{E}_h = \mathcal{E}_h^o \cup \mathcal{E}_h^\partial$,

Λ_h = set of all midpoints of faces of $\mathcal{T}_h$,

h_e = diameter of any edge/face $e \in \mathcal{E}_h$,

$\mathcal{T}_p$ = set of all elements sharing the vertex p,

$\mathcal{T}_e$ = patch of two elements sharing the face $e \in \mathcal{E}_h^o$, i.e., $\mathcal{T}_e = \{T_+, T_-\}$, where T_+ and T_- are two elements sharing the face e

$\mathcal{E}_p$ = set of all edges/faces connected to the vertex p,

∇_h = piecewise (elementwise) gradient.

For any $T \in \mathcal{T}_h$, $p \in \mathcal{V}_h$ and $e \in \mathcal{E}_h^o$, we define

$$\omega_T := \cup\left\{T' \in \mathcal{T}_h \ : \ T' \cap T \neq \emptyset\right\}, \tag{2.1}$$

$$\omega_p := \cup\{T' \in \mathcal{T}_h \ : T' \in \mathcal{T}_p\}, \tag{2.2}$$

$$\omega_e := T_+ \cup T_-. \tag{2.3}$$

Further,

- for any function v, v^+ denotes its positive part, i.e., $v^+(x) = \max\{v(x), 0\}$,
- for any function v, $\mathrm{supp}(v)$ denotes support of the function v,
- $\|\cdot\|_{L^\infty(\mathcal{T}_h)} := \max_{T\in\mathcal{T}_h} \|\cdot\|_{L^\infty(T)}$ and $\|\cdot\|_{L^\infty(\mathcal{E}_h)} := \max_{e\in\mathcal{E}_h} \|\cdot\|_{L^\infty(e)}$,
- $X \lesssim Y$: $\exists\ C > 0$, independent of mesh parameters and the solution such that $X \leq CY$.

Throughout the article, C denotes a generic positive constant which is independent of the mesh size h. The shape regularity assumption on $\mathcal{T}_h$ (Ciarlet, 1978, Chapter 3, page 124) implies that $|\mathcal{E}_p| \lesssim 1$ for all $p \in \mathcal{V}_h$, where $|\mathcal{E}_p|$ denotes the cardinality of $\mathcal{E}_p$. To define the jump and average of discontinuous functions appropriately, we introduce the broken Sobolev space

$$H^1(\Omega, \mathcal{T}_h) = \{v \in L^2(\Omega) : \ v_T = v|_T \in H^1(T) \quad \forall\ T \in \mathcal{T}_h\}.$$

Trace operators: Let $e \in \mathcal{E}_h^o$ be shared by two elements T_+ and T_- such that $e = \partial T_+ \cap \partial T_-$. If n_+ is the unit normal of e pointing from T_+ to T_- then $n_- = -n_+$. We set $v_\pm = v\big|_{T_\pm}$ and define the jump and the mean of $v \in H^1(\Omega, \mathcal{T}_h)$ on e by

$$[\![v]\!] = v_+ n_+ + v_- n_- \quad \text{and} \quad \{\!\{v\}\!\} = \frac{1}{2}(v_+ + v_-), \quad \text{respectively.}$$

Similarly for a vector-valued function $q \in H^1(\Omega, \mathcal{T}_h)^d$, the jump and mean on $e \in \mathcal{E}_h^o$ are defined by

$$[\![q]\!] = q_+ \cdot n_+ + q_- \cdot n_- \quad \text{and} \quad \{\!\{q\}\!\} = \frac{1}{2}(q_+ + q_-) \text{ respectively.}$$

For a boundary face $e \in \mathcal{E}_h^\partial$, with n_e denoting the unit normal of e that points outward to T for $\partial T \cap \partial\Omega = e$, we set

$$[\![v]\!] = v n_e \quad \text{and} \quad \{\!\{v\}\!\} = v \quad \forall\, v \in H^1(T),$$
$$[\![q]\!] = q \cdot n_e \text{ and } \{\!\{q\}\!\} = q \quad \forall\, q \in H^1(T)^d.$$

Discrete spaces: Let $\mathbb{P}^k(T)$, $k \geq 0$ integer, be the space of polynomials defined on T of degree less than or equal to k. We define the first-order discontinuous finite element space as

$$V_h = \{v \in L^2(\Omega) : v|_T \in \mathbb{P}^1(T),\, T \in \mathcal{T}_h\}.$$

The conforming finite element space, denoted here by V_h^{conf}, is then defined as $V_h^{\text{conf}} = V_h \cap H_0^1(\Omega)$. Let $\chi_h \in V_h^{\text{conf}}$ be the nodal interpolation of χ. We define the discrete analogue of $\mathcal{K}$ by

$$\mathcal{K}_h := \{v_h \in V_h : v_h|_T(p) \geq \chi_h(p) \;\; \forall\, p \in \mathcal{V}_T, \;\; \forall\, T \in \mathcal{T}_h\},$$

which is a nonempty, closed and convex subset of V_h. Note however that, since χ is a general function, we have $\mathcal{K}_h \nsubseteq \mathcal{K}$. In the case that χ is an affine function, we have $\mathcal{K}_h \cap H_0^1(\Omega) \subseteq \mathcal{K}$.

2.1 *Some discrete operators*

We now introduce some discrete operators that will be used in the further analysis. Following Demlow & Georgoulis (2012), we define a local projection operator $\Pi_h : L^1(\Omega) \longrightarrow V_h$. For any $T \in \mathcal{T}_h$, let $(\phi_p^T)_{p \in \mathcal{V}_T}$ be associated Lagrange basis functions. Then, for any $v \in L^1(T)$, we define $\Pi_h^T : L^1(T) \longrightarrow V_h$ by

$$\Pi_h^T v(x) := \sum_{p \in \mathcal{V}_T} \phi_p^T(x) \int_T \psi_p^T(s) v(s)\, \mathrm{d}s, \tag{2.4}$$

where $(\psi_p^T)_{p \in \mathcal{V}_T}$ is the $L^2(T)$ dual basis of $(\phi_p^T)_{p \in \mathcal{V}_T}$, i.e., $\int_T \phi_p^T(x)\psi_q^T(x)\, \mathrm{d}x = \delta_{pq}$, $p, q \in \mathcal{V}_T$. Then Π_h is defined by setting $\Pi_h v|_T = \Pi_h^T v$ for each $T \in \mathcal{T}_h$. Observe that from (2.4) we do have $\Pi_h^T \phi_p^T = \phi_p^T$ which implies Π_h^T is indeed a projection, and so is Π_h. Hence we have $\Pi_h v_h = v_h$ for all $v_h \in V_h$.

The following lemma from Demlow & Georgoulis (2012) provides stability and approximation properties of Π_h^T. Its proof follows by using the Bramble Hilbert lemma and standard scaling arguments; see Demlow & Georgoulis (2012) for details.

LEMMA 2.1 Let $T \in \mathcal{T}_h$ and let $\Pi_h^T : L^1(T) \longrightarrow V_h$ be the local projection defined in (2.4). Let s be an integer such that $0 \le s \le 2$ and $\psi \in W^{s,1}(T)$. Then

$$|\Pi^T \psi|_{W^{s,1}(T)} \lesssim |\psi|_{W^{s,1}(T)}, \qquad 0 \le s \le 2, \tag{2.5}$$

$$\sum_{k=0}^{s} h_T^{k-s} |\psi - \Pi_h^T \psi|_{W^{k,1}(T)} \lesssim |\psi|_{W^{s,1}(T)}, \qquad 0 \le s \le 2. \tag{2.6}$$

We now revise the following inverse and trace inequalities, which will be frequently used in our later analysis.

Inverse inequalities (Ciarlet, 1978, Chapter 3, page 140). For any $v_h \in V_h$ and $1 \le p, q \le \infty$,

$$\|v_h\|_{W^{m,p}(T)} \lesssim h_T^{\ell-m} h_T^{d\left(\frac{1}{p}-\frac{1}{q}\right)} \|v_h\|_{W^{\ell,q}(T)} \qquad \forall\, T \in \mathcal{T}_h, \quad l \le m, \tag{2.7}$$

$$\|\nabla v_h\|_{L^p(T)} \lesssim h_T^{-1} \|v_h\|_{L^p(T)} \qquad \forall\, T \in \mathcal{T}_h, \tag{2.8}$$

$$\|v_h\|_{L^\infty(T)} \lesssim h_T^{-\frac{d}{2}} \|v_h\|_{L^2(T)} \qquad \forall\, T \in \mathcal{T}_h, \tag{2.9}$$

$$\|v_h\|_{L^\infty(e)} \lesssim h_e^{\frac{1-d}{2}} \|v_h\|_{L^2(e)} \qquad \forall\, e \in \mathcal{E}_h, \tag{2.10}$$

where h_e and h_T denote, respectively, the diameters of the face e and the element T.

Trace inequality Agmon (1965). Let $\psi \in W^{1,p}(T)$, $T \in \mathcal{T}_h$ and let $e \in \mathcal{E}_h$ be an edge/face of T. Then for any $1 \le p < \infty$, it holds that

$$\|\psi\|_{L^p(e)}^p \lesssim h_e^{-1}\left(\|\psi\|_{L^p(T)}^p + h_e^p \|\nabla\psi\|_{L^p(T)}^p\right), \tag{2.11}$$

where h_e denotes the diameter of the face e.

Averaging or enriching operator. We close the section by introducing an averaging (sometimes called enriching) operator $E_h : V_h \longrightarrow V_h^{conf}$ that plays an essential role in the analysis. This type of operator was first introduced in the seminal work Karakashian & Pascal (2003), where the first *a posteriori* error analysis for DG approximations of elliptic problems was presented. For the obstacle problem, it is necessary to ensure that E_h exhibits an additional constraint-preserving property, as proposed in Gudi & Porwal (2014a). This construction will be essential in the forthcoming analysis.

Bearing in mind that any function in linear conforming finite element space V_h^{conf} is uniquely determined by its nodal values at the vertices $\mathcal{V}_h$ of $\mathcal{T}_h$, for $v_h \in V_h$ we define $E_h v_h \in V_h^{conf}$ as

$$E_h v_h = \sum_{p \in \mathcal{V}_h} \alpha(p) \phi_p^h(x) \quad \text{with } \alpha(p) := \begin{cases} 0 & \text{if } p \in \mathcal{V}_h^b, \\ \min\{v_h|_T(p) : T \in \mathcal{T}_p\} & \text{if } p \in \mathcal{V}_h^i. \end{cases} \tag{2.12}$$

Observe that from its definition it follows immediately that

$$E_h v_h(p) \geq \chi_h(p) \quad \forall\, v_h \in \mathcal{K}_h, \quad \forall\, p \in \mathcal{V}_h,$$

and in particular $E_h v_h \in \mathcal{K}_h \cap V_h^{conf}$ for all $v_h \in \mathcal{K}_h$.

The next Lemma provides some approximation properties for the operator E_h. Similar estimates for general polynomial degree for the unconstrained version of the averaging operator introduced in Karakashian & Pascal (2003) can be found in Demlow & Georgoulis (2012).

LEMMA 2.2 Let $E_h : V_h \longrightarrow V_h^{conf}$ be the operator defined through (2.12). Then, for any $v \in V_h$, it holds,

$$\|E_h v - v\|_{L^\infty(\mathcal{T}_h)} \lesssim \|[\![v]\!]\|_{L^\infty(\mathcal{E}_h)}, \tag{2.13}$$

$$\max_{T\in\mathcal{T}_h} h_T \|\nabla(E_h v - v)\|_{L^\infty(T)} \lesssim \|[\![v]\!]\|_{L^\infty(\mathcal{E}_h)}. \tag{2.14}$$

Proof. Let $T \in \mathcal{T}_h$. Note that the Lagrange basis functions ϕ_p^T satisfy the property $\|\phi_p^T\|_{L^\infty(T)} = 1$ for all $p \in \mathcal{V}_h$. For any $p \in \mathcal{V}_T \cap \mathcal{V}_h^i$, using the definition of E_h we have the existence of some $T_* \in \mathcal{T}_p$ such that $E_h v(p) = \alpha(p) = v|_{T_*}(p)$ and $E_h v(p) = 0$ for $p \in \mathcal{V}_T \cap \mathcal{V}_h^b$. Therefore, for any $v \in V_h$, we have

$$\begin{aligned}\|E_h v - v\|_{L^\infty(T)} &= \Bigg\|\sum_{p\in\mathcal{V}_T}(\alpha(p) - v(p))\phi_p^T\Bigg\|_{L^\infty(T)} \leq \sum_{p\in\mathcal{V}_T\cap\mathcal{V}_h^i}|\alpha(p) - v(p)| + \sum_{p\in\mathcal{V}_T\cap\mathcal{V}_h^b}|\alpha(p) - v(p)| \\ &\leq \sum_{p\in\mathcal{V}_T\cap\mathcal{V}_h^i}|v|_{T_*}(p) - v(p)| + \sum_{p\in\mathcal{V}_T\cap\mathcal{V}_h^b}|v(p)| \lesssim \sum_{p\in\mathcal{V}_T}\|[\![v]\!]\|_{L^\infty(\mathcal{E}_p)},\end{aligned}$$

which leads to (2.13). Finally, estimate (2.14) follows from the inverse inequality (2.8). □

2.2 *Some results on Green's functions and regularity for the unconstrained problem*

To prove the reliability estimates we will make use of the Green's function for an unconstrained Poisson problem. We collect here several results that will be required further in our analysis. We start with an elementary lemma on the regularity of the solution of the unconstrained problem.

LEMMA 2.3 Let $g \in W^{-1,r}(\Omega)$ for some $r > d \geq 2$. Then there exists a unique weak solution $y \in W^{1,r}(\Omega) \cap H_0^1(\Omega)$ to the problem

$$-\Delta y = g \quad \text{in } \Omega, \qquad y = 0 \text{ on } \partial\Omega, \tag{2.15}$$

such that it satisfies the estimate

$$\|y\|_{W^{1,r}(\Omega)} \leq C_r \|g\|_{W^{-1,r}(\Omega)}, \tag{2.16}$$

with $C_r = Cr$. Furthermore, $y \in C^0(\Omega)$ by virtue of the Sobolev embedding.

The proof of estimate (2.16) is done using the Calderon–Zygmund theory of singular integral operators (see Gilbarg & Trudinger, 2001; the estimate holds for $r \geq 2$). The continuity of the solution is guaranteed by Sobolev embedding since $y \in W^{1,r}(\Omega),\ r > d$. We remark that, although not advocated here, *a priori* bounds on the L^∞-norm of the solution to (2.15) in terms of the $W^{-1,r}(\Omega)$-norm of g are given in Stampacchia (1965, Theorem 4.2 (a)) and (Casas, 1986, Lemma 1).[2]

The results in Stampacchia (1965, Section 9) guarantee that we can define a Green's function associated to problem (2.15). More precisely, for all $x \in \Omega$ there exists a Green's function $\mathcal{G}(x,\xi) : \Omega \times \Omega \longrightarrow \mathbb{R}$, defined as the solution (in the sense of distributions) to

$$\begin{cases} -\Delta_\xi \mathcal{G} = \delta_x(\xi), & \xi \in \Omega, \\ \mathcal{G}(x,\xi) = 0, & \xi \in \partial\Omega, \end{cases} \tag{2.17}$$

such that for any $y \in H_0^1(\Omega) \cap W^{1,q}(\Omega), q > d$, the following representation holds:

$$y(x) = (\nabla y, \nabla \mathcal{G}(x,\cdot))_{L^2(\Omega)}. \tag{2.18}$$

In (2.17), $\delta_x(\xi) = \delta(x-\xi)$ denotes the delta function around the point x. Notice that the Green's function $\mathcal{G}$ in (2.17) indeed has a singularity at x. The next proposition provides some estimates on the Green's function $\mathcal{G}$ which account for the different regularity of $\mathcal{G}$ near and far from x. For their proof, we refer to Demlow & Kopteva (2016), which relies heavily on the estimates shown in Demlow & Georgoulis (2012), improving them by a logarithmic factor. Both works make use of the fine estimates shown in Dong & Kim (2009), Grüter & Widman (1982), Hofmann & Kim (2007).

Proposition 2.4 Let $\mathcal{G}$ be the Green's function defined in (2.17). Then for any $x \in \Omega$, the following estimate holds:

$$\|\mathcal{G}(x,\cdot)\|_{L^1(\Omega)} + \|\mathcal{G}(x,\cdot)\|_{L^{d/(d-1)}(\Omega)} + |\mathcal{G}(x,\cdot)|_{W^{1,1}(\Omega)} \lesssim 1.$$

In addition, for the ball $B(x,\rho)$ of radius ρ centered at $x \in \Omega$, the following estimates hold true:

$$\|\mathcal{G}(x,\cdot)\|_{L^1(B(x,\rho)\cap\Omega)} \lesssim \rho^2 \log(2+\rho^{-1})^{\kappa_d},$$

$$|\mathcal{G}(x,\cdot)|_{W^{1,1}(B(x,\rho)\cap\Omega)} \lesssim \rho,$$

$$|\mathcal{G}(x,\cdot)|_{W^{2,1}(\Omega\setminus B(x,\rho)\cap\Omega)} \lesssim \log(2+\rho^{-1}),$$

$$|\mathcal{G}(x,\cdot)|_{W^{1,d/(d-1)}(\Omega\setminus B(x,\rho)\cap\Omega)} \lesssim \log(2+\rho^{-1}),$$

where $\kappa_2 = 1$ and $\kappa_3 = 0$.

[2] This proof, although valid for polyhedral domains, uses elliptic regularity of (2.15), which requires convexity of the domain and hence we do not use this result.

3. Interior penalty discontinuous Galerkin approximation

In this section we introduce the interior penalty (IP) discontinuous Galerkin method for approximating the obstacle problem (1.1), discuss its properties and comment on the possibility of using any of its nonsymmetric versions. The IP method reads as follows: find $u_h \in \mathcal{K}_h$ such that

$$\mathcal{A}_h(u_h, u_h - v_h) \le (f, u_h - v_h) \qquad \forall\, v_h \in \mathcal{K}_h, \tag{3.1}$$

where the bilinear form $\mathcal{A}_h(\cdot,\cdot)$ can be written as the sum of bilinear forms $a_h(\cdot,\cdot)$ and $b_h(\cdot,\cdot)$, i.e.,

$$\mathcal{A}_h(v_h, w_h) = a_h(v_h, w_h) + b_h(v_h, w_h) \quad \forall\, v_h, w_h \in V_h, \tag{3.2}$$

with

$$a_h(v_h, w_h) = (\nabla_h v_h, \nabla_h w_h),$$

and the bilinear form $b_h(\cdot,\cdot)$ gathers the consistency and the stability terms:

$$b_h(v_h, w_h) = -\sum_{e\in\mathcal{E}_h} \int_e \big(\{\!\{\nabla v_h\}\!\}[\![w_h]\!] + \{\!\{\nabla w_h\}\!\}[\![v_h]\!]\big)\,\mathrm{d}s + \sum_{e\in\mathcal{E}_h} \int_e \frac{\gamma}{h_e}[\![v_h]\!][\![w_h]\!]\,\mathrm{d}s,$$

where $\gamma \ge \gamma_0 > 0$ with γ_0 as a sufficiently large positive number to ensure coercivity of $\mathcal{A}_h(\cdot,\cdot)$. It can be checked, using standard theory (Wang *et al.*, 2010), that (3.1) has a unique solution $u_h \in \mathcal{K}_h$.

We stress that we focus on the symmetric IP method (SIPG). While it is typical in the DG framework to consider the whole IP family, considering both the symmetric and the two nonsymmetric methods, our pointwise *a posteriori* error analysis holds only for the symmetric one. The underlying reason is, as will be exhibited later on, the use of duality in the error analysis which entails symmetry of the method. This is highlighted in Remark 4.6.

For our analysis, we consider an extension of $\mathcal{A}_h(\cdot,\cdot)$ that allows for testing nondiscrete functions less regular than $H^1(\Omega)$. Let $p > d$ and $1 \le q < \frac{d}{d-1}$ be the conjugate of p, i.e., $q = \frac{p}{p-1}$. Set

$$\mathcal{M} = W_0^{1,q}(\Omega) + V_h. \tag{3.3}$$

Then we define $\tilde{\mathcal{A}}_h : \big(W_0^{1,p}(\Omega) + V_h\big) \times \big(W_0^{1,q}(\Omega) + V_h\big) \longrightarrow \mathbb{R}$ by

$$\tilde{\mathcal{A}}_h(v, w) := a_h(v, w) + \tilde{b}_h(v, w), \quad v \in W_0^{1,p}(\Omega) + V_h,\ w \in \mathcal{M}, \tag{3.4}$$

where

$$\tilde{b}_h(v, w) = -\sum_{e\in\mathcal{E}_h} \int_e \{\!\{\nabla \Pi_h(v)\}\!\}[\![w]\!]\,\mathrm{d}s - \sum_{e\in\mathcal{E}_h} \int_e \{\!\{\nabla \Pi_h w\}\!\}[\![v]\!]\,\mathrm{d}s + \sum_{e\in\mathcal{E}_h} \int_e \frac{\gamma}{h_e}[\![v]\!][\![w]\!]\,\mathrm{d}s. \tag{3.5}$$

Notice that since Π_h is a projection, $\tilde{\mathcal{A}}_h(v, w) = \mathcal{A}_h(v, w)$ for $v, w \in V_h$. Following Gudi & Porwal (2014a), Veeser (2001), we define the discrete Lagrange multiplier $\sigma_h \in V_h$, which can be thought of as an approximation to the functional $\sigma(u)$ defined in (1.2):

$$\langle \sigma_h, v_h \rangle_h := (f, v_h) - \mathcal{A}_h(u_h, v_h) \quad \forall\, v_h \in V_h, \tag{3.6}$$

where $\langle \cdot, \cdot \rangle_h$ is given by the following: for $w_h, v_h \in V_h$,

$$\langle w_h, v_h \rangle_h = \sum_{T \in \mathcal{T}_h} \int_T I_h(w_h|_T v_h|_T)\, \mathrm{d}x = \sum_{T \in \mathcal{T}_h} \sum_{p \in \mathcal{V}_T} w_h(p) v_h(p) \int_T \phi_p^h\, \mathrm{d}x = \sum_{T \in \mathcal{T}_h} \frac{|T|}{d+1} \sum_{p \in \mathcal{V}_T} w_h(p) v_h(p), \tag{3.7}$$

with $\phi_p^h \in V_h$ as the Lagrange local basis function associated with the vertex p (hence, it takes the value 1 at the node p and vanishes at all other nodes) and I_h is defined by setting $I_h v|_T = I_h^T v$ for each $T \in \mathcal{T}_h$ where I_h^T denotes the standard Lagrange nodal interpolation operator. The use of the $\langle \cdot, \cdot \rangle_h$ inner product in definition (3.6) of σ_h allows for localizing σ_h at the vertices of the partition, which will play a crucial role in the further analysis. This definition of σ_h also accounts for an easy implementation in the sense that the discrete Lagrange multiplier σ_h can be computed locally. By taking v_h as the local basis functions in the definition of σ_h (equation (3.6)), the computation of σ_h over an element simply requires inversion of the local mass matrix and matrix multiplications. The use of (3.6)–(3.7) was brought into play in Gudi & Porwal (2014a), Veeser (2001) for the *a posteriori* error analysis in the energy norm of conforming and DG approximations of the obstacle problem. While more refined definitions of the discrete Lagrange multiplier have been proposed in Nochetto *et al.* (2005) in the conforming framework, in this work we stick to this definition. The main reason is that here, the nature of the discontinuous spaces already grants us the possibility of localizing the discrete multiplier σ_h to its support which is outside the discrete noncontact set. Also, as we shall show, the use of (3.7) allows for a first simple analysis of the DG approximation. We define the discrete contact, noncontact and free boundary sets relative to the discrete solution u_h as

$$\mathbb{C}_h := \{T \in \mathcal{T}_h : u_h(p) = \chi_h(p)\ \forall\ p \in \mathcal{V}_T\}, \tag{3.8}$$

$$\mathbb{N}_h := \{T \in \mathcal{T}_h : u_h(p) > \chi_h(p)\ \forall\ p \in \mathcal{V}_T\}, \tag{3.9}$$

$$\mathbb{M}_h = \mathcal{T}_h \setminus (\mathbb{C}_h \cup \mathbb{N}_h). \tag{3.10}$$

Basically, elements in $\mathbb{M}_h$ have the property of having at least a vertex p with $u_h(p) = \chi_h(p)$ and at least a vertex p' with $u_h(p') > \chi_h(p')$, forming the discrete free boundary set. Using (3.6) and the discrete problem (3.2), we obtain

$$\langle \sigma_h, v_h - u_h \rangle_h \leq 0 \quad \forall\ v_h \in \mathcal{K}_h. \tag{3.11}$$

By setting $v_h = u_h + \phi_p^h$ in (3.11) we obtain $\sigma_h(p) \leq 0$ for all $p \in \mathcal{V}_T$. Consequently, since the choice of the vertex p is arbitrary and σ_h is linear on T, we conclude $\sigma_h \leq 0$ in T for all $T \in \mathcal{T}_h$. Next, let p be a vertex of an element $T \in \mathbb{N}_h$. Then, taking $v_h = u_h - \delta\phi_p^h$ for some $\delta > 0$ sufficiently small, and recalling that $u_h(p) > \chi_h(p)$ (see (3.9)), combined with the previous estimate $\sigma_h \leq 0$ in T, gives

$$\sigma_h(p) = 0 \qquad \forall\ p \in \mathcal{V}_h \text{ for which } u_h(p) > \chi_h(p), \tag{3.12}$$

in particular, $\sigma_h(p) = 0$ for all $p \in \mathcal{V}_T,\ \ T \subset \mathbb{N}_h$. Observe that, in view of (3.12), expression (3.7) with $w_h = \sigma_h$ is reduced to

$$\begin{aligned}\langle \sigma_h, v_h \rangle_h &= \sum_{T\in\mathcal{T}_h} \int_T I_h(\sigma_h|_T v_h|_T)\,\mathrm{d}x = \sum_{\substack{T\in\mathcal{T}_h\\ T\notin\mathbb{N}_h}} \sum_{p\in\mathcal{V}_T} \sigma_h(p) v_h(p) \int_T \phi_p \mathrm{d}x \\ &= \sum_{T\in\mathbb{C}_h\cup\mathbb{M}_h} \frac{|T|}{d+1} \sum_{p\in\mathcal{V}_T} \sigma_h(p) v_h(p). \end{aligned} \tag{3.13}$$

We now give an approximation property for σ_h that is proved along the lines of Veeser (2001, Lemma 3.3). For the sake of completeness we provide the proof.

LEMMA 3.1 For any $1 \leq p \leq \infty, v \in V_h$ and $T \in \mathcal{T}_h$,

$$\int_T (\sigma_h v_h - I_h(\sigma_h v_h))\,\mathrm{d}x \leq h_T^2 \|\nabla \sigma_h\|_{L^p(T)} \|\nabla v_h\|_{L^q(T)},$$

where q is the conjugate of p, i.e., $\frac{1}{p} + \frac{1}{q} = 1$.

Proof. Using a scaling argument we find that

$$\int_T (\sigma_h v_h - I_h(\sigma_h v_h))\,\mathrm{d}x \leq h_T^2 \int_T |D^2(\sigma_h v_h)|\,\mathrm{d}x = h_T^2 \int_T |\nabla \sigma_h \cdot \nabla v_h|\,\mathrm{d}x,$$

where in the last line we have used the fact that σ_h and v_h are linear functions on T. Finally, the estimate of the lemma follows using the Hölder inequality. □

4. *A posteriori* error analysis: reliability

In this section we lay down the scheme for the pointwise *a posteriori* analysis, introducing the error estimator and stating the first main result of the paper, namely, we present the reliability analysis of the error estimator. The analysis follows using the ideas introduced in Nochetto *et al.* (2003), with adequate modifications to account for the nonconformity of the approximation. Before we state the main theorem of the section and outline the path of the proof, we define the Galerkin functional, which plays the same role as the residual functional for unconstrained problems.

Similar to the extension of the discrete bilinear form in (3.4), we first introduce an extended definition of the continuous $\sigma(u)$ defined in (1.2).

Define $\tilde{\sigma}(u) \in \mathcal{M}^*$ as

$$\langle \tilde{\sigma}(u), v \rangle = (f, v) - \tilde{\mathcal{A}}_h(u, v) \quad \forall\, v \in \mathcal{M}\,, \tag{4.1}$$

where the space $\mathcal{M}$ is defined in (3.3). Observe that due to the definition of $\tilde{\mathcal{A}}_h(\cdot,\cdot)$, we plainly have

$$\langle \tilde{\sigma}(u), v\rangle \equiv \langle \sigma(u), v\rangle \quad \forall\, v \in V.$$

We now define the Galerkin or *residual functional* $G_h \in \mathcal{M}^*$ for the nonlinear problem under consideration. Let $G_h \in \mathcal{M}^*$ be defined as

$$\langle G_h, v\rangle = \tilde{\mathcal{A}}_h(u-u_h, v) + \langle \tilde{\sigma}(u) - \sigma_h, v\rangle \quad \forall\, v \in \mathcal{M}. \tag{4.2}$$

Note that since $\sigma_h \in V_h \subset L^2(\Omega) \subset H^{-1}(\Omega)$, we have $\langle \sigma_h, v\rangle = (\sigma_h, v)$ for all $v \in \mathcal{M}$.

Observe that by restricting the action of the functional G_h to the subspace $H^1_0(\Omega) \subset W^{1,q}_0(\Omega)$ (and respectively to conforming subspace $V_h^{conf} \subset V_h$) we have the quasi-orthogonality property and for any $v \in V$,

$$\begin{aligned}\langle G_h, v\rangle &= \tilde{\mathcal{A}}_h(u-u_h, v) + \langle \tilde{\sigma}(u) - \sigma_h, v\rangle = a(u,v) - \tilde{\mathcal{A}}_h(u_h, v) + \langle \sigma(u) - \sigma_h, v\rangle \\ &= (f, v) - \tilde{\mathcal{A}}_h(u_h, v) - \langle \sigma_h, v\rangle. \end{aligned} \tag{4.3}$$

The functional equations (4.2) and (4.3) will be essential for our analysis.

We now define the error estimators:

$$\eta_1 = \max_{T\in\mathcal{T}_h} \|h_T^2(f-\sigma_h)\|_{L^\infty(T)}, \tag{4.4}$$

$$\eta_2 = \max_{e\in\mathcal{E}_h^o} \|h_e[\![\nabla u_h]\!]\|_{L^\infty(e)}, \tag{4.5}$$

$$\eta_3 = \|[\![u_h]\!]\|_{L^\infty(\mathcal{E}_h)}, \tag{4.6}$$

$$\eta_4 = \max_{T\in\mathbb{C}_h\cup\mathbb{M}_h} \|h_T^2\nabla\sigma_h\|_{L^d(T)}, \tag{4.7}$$

$$\eta_5 = \|(\chi - u_h)^+\|_{L^\infty(\Omega)}, \tag{4.8}$$

$$\eta_6 = \|(u_h - \chi)^+\|_{L^\infty(\{\sigma_h<0\})}. \tag{4.9}$$

The full *a posteriori* error estimator η_h is then defined as

$$\eta_h = |\log h_{min}|\Big(\eta_1 + \eta_2 + \eta_3 + \eta_4\Big) + \eta_5 + \eta_6.$$

Therein, the estimator term η_4 can be controlled by the volume residual term up to the oscillation of f as follows.

Lemma 4.1 For any $T \in \mathbb{C}_h \cup \mathbb{M}_h$, it holds that

$$\|h_T^2\nabla\sigma_h\|_{L^d(T)} \le \|h_T^2(f-\sigma_h)\|_{L^\infty(T)} + \mathrm{Osc}(f, T),$$

where

$$\mathrm{Osc}(f,T) = \min_{\tilde{f}\in\mathbb{P}^0(T)} h_T^2\|f-\tilde{f}\|_{L^\infty(T)}. \tag{4.10}$$

Proof. Let $T \in \mathbb{C}_h \cup \mathbb{M}_h$ and $\bar{f} \in \mathbb{P}^0(T)$. A use of the inverse inequality (2.8) and the triangle inequality yields

$$\begin{aligned}\|h_T^2\nabla\sigma_h\|_{L^d(T)} &= \|h_T^2\nabla(\sigma_h-\bar{f})\|_{L^d(T)} \le \|h_T^2(\bar{f}-\sigma_h)\|_{L^\infty(T)}\\ &\le \|h_T^2(f-\sigma_h)\|_{L^\infty(T)} + \mathrm{Osc}(f,T).\end{aligned}$$

□

We now can state the main result of this section, namely the reliability of the error estimator η_h.

THEOREM 4.2 Let $u \in \mathcal{K}$ and $u_h \in \mathcal{K}_h$ be the solutions of (1.1) and (3.2), respectively. Then

$$\|u-u_h\|_{L^\infty(\Omega)} \lesssim \eta_h.$$

The proof of this theorem is done in several steps and will be carried out in the following subsections. As mentioned before, the analysis is motivated by the ideas laid out in Nochetto *et al.* (2003), modified accordingly to account for the nonconformity of the approximation. To avoid having undesirable restrictions on the mesh, such as weakly acute, we refrain from using any discrete maximum principle, and we use instead the stability of the continuous problem and the continuous maximum principle. To do so, we construct sub- and supersolutions for the continuous solution, starting from the conforming part of the approximate solution u_h, i.e., E_hu_h, and correcting it appropriately. Throughout the rest of the paper, u_h^{conf} always denotes E_hu_h. The rationality of basing the construction on the conforming part u_h^{conf} stems from the fact that we aim to apply the stability of the continuous problem, and so we need the barriers to belong to $H^1(\Omega)$. Once the sub- and supersolutions are constructed, the continuous maximum principle provides a first bound on the pointwise error. This is done in Section 4.1.

The construction of the lower and upper barriers involves a corrector function, denoted by w, that accounts for the consistency and the nonconformity errors. The analysis of the reliability of the estimator, namely the proof of Theorem 4.2, is then completed by providing a maximum norm estimate for the corrector function in terms of the local estimators $\eta_i, i = 1, 2, 3, 4$ given in (4.4), (4.5), (4.6) and (4.7). Such a bound is given in Section 4.2, and it is proved using fine regularity estimates of the Green's function associated to the unconstrained Poisson problem. Here, our approach differs from that presented in Nochetto *et al.* (2003), and simplifies somewhat the analysis.

4.1 *Sub- and supersolutions*

We introduce the following barrier functions of the exact solution. We define u^* and u_* as

$$u^* = u_h^{conf} + w + \|w\|_{L^\infty(\Omega)} + \|(\chi - u_h^{conf})^+\|_{L^\infty(\Omega)}, \tag{4.11}$$

$$u_* = u_h^{conf} + w - \|w\|_{L^\infty(\Omega)} - \|(u_h^{conf} - \chi)^+\|_{L^\infty(\{\sigma_h<0\})}, \tag{4.12}$$

where $w \in H_0^1(\Omega)$ is the weak solution of the Poisson problem with right-hand-side functional defined as

$$\mathcal{F}_h(v) = \langle G_h, v\rangle + \tilde{\mathcal{A}}_h(u_h - u_h^{conf}, v) \qquad \forall\, v \in W_0^{1,q}(\Omega),\ 1 \le q < (d/d-1),$$

i.e., $w \in H_0^1(\Omega)$ solves the following linear problem:

$$\int_\Omega \nabla w \cdot \nabla v \,\mathrm{d}x = \mathcal{F}_h(v) \quad \forall\, v \in H_0^1(\Omega). \tag{4.13}$$

The existence of w is guaranteed by the Riesz representation theorem. Notice that $\mathcal{F}_h$ is well defined as a functional in the dual of $W_0^{1,q}(\Omega)$. We also observe that when acting on more regular functions, say $v \in H_0^1(\Omega)$, $\mathcal{F}_h$ reduces to

$$\begin{aligned}\mathcal{F}_h(v) = \langle G_h, v\rangle + \tilde{\mathcal{A}}_h(u_h - u_h^{conf}, v) &= \tilde{\mathcal{A}}_h(u - u_h^{conf}, v) + \langle \sigma(u) - \sigma_h, v\rangle \\ &= \int_\Omega \nabla(u - u_h^{conf}) \cdot \nabla v \,\mathrm{d}x + \langle \sigma(u) - \sigma_h, v\rangle.\end{aligned} \tag{4.14}$$

Next, we prove that u_* and u^* are sub- and supersolutions of the solution of variational inequality (1.1). This proof will also establish the essence of having the conforming part of u_h in the definitions of the barrier functions, so as to use the stability of the continuous problem, which in turn forces us to use functions in $H_0^1(\Omega)$. We would like to remark that in Nochetto *et al.* (2003), the definition of Galerkin functional G_h entailed a more refined $\tilde{\sigma}_h$, which used the positivity-preserving operator from Chen & Nochetto (2000) and allowed for suitable cancellation, providing hence the localization of the error estimator. Here, thanks to the local nature of discontinuous space and the particular construction of the averaging operator E_h in (2.12), we can complete the proof of Lemma 4.3 below by sticking to the definition (3.6) of σ_h in the expression of Galerkin functional G_h.

LEMMA 4.3 Let u be the solution of (1.1) and u^*, u_* be the upper and lower barriers defined in equations (4.11) and (4.12), respectively. Then it holds that

$$u \le u^* \quad \text{and} \quad u_* \le u.$$

REMARK 4.4 The subsolution in (4.12) could be alternatively defined by

$$u_* = u_h^{conf} + w - \|w\|_{L^\infty(\Omega)} - \|(u_h - \chi)^+\|_{L^\infty(\Lambda_h^1)} - \|(u_h^{conf} - \chi)^+\|_{L^\infty(\Lambda_h^2)},$$

where $\Lambda_h^1 = \{T \in \mathcal{T}_h : T \cap \partial\Omega = \emptyset \text{ and } \sigma_h < 0 \text{ on } T\}$ and $\Lambda_h^2 = \{T \in \mathcal{T}_h : T \cap \partial\Omega \neq \emptyset \text{ and } \sigma_h < 0 \text{ on } T\}$ and Lemma 4.3 would still hold true.

Proof. First we show that $u \le u^*$. Set $v = (u - u^*)^+$ and we claim to prove that $v = 0$. Note that u^* is defined using the conforming part of u_h, i.e., u_h^{conf}, which guarantees that $v \in H^1(\Omega)$. We have

$$(u - u^*)|_{\partial\Omega} \le (u - u_h^{conf})|_{\partial\Omega} - \|(\chi - u_h^{conf})^+\|_{L^\infty(\Omega)} \le 0,$$

which implies $v|_{\partial\Omega} = 0$, therefore $v \in H_0^1(\Omega)$. It then suffices to show that $\|\nabla v\|_{L^2(\Omega)} = 0$. A use of (4.13) together with definitions of $\mathcal{F}_h$ and G_h yields

$$\begin{aligned}\|\nabla v\|^2_{L^2(\Omega)} &= \int_\Omega \nabla(u - u_h^{conf}) \cdot \nabla v \,\mathrm{d}x - \int_\Omega \nabla w \cdot \nabla v \,\mathrm{d}x = \int_\Omega \nabla(u - u_h^{conf}) \cdot \nabla v \,\mathrm{d}x - \mathcal{F}_h(v)\\ &= \int_\Omega \nabla(u - u_h^{conf}) \cdot \nabla v \,\mathrm{d}x - \langle G_h, v\rangle - \tilde{\mathcal{A}}_h(u_h - u_h^{conf}, v)\\ &= \tilde{\mathcal{A}}_h(u - u_h, v) - \langle G_h, v\rangle = -\langle \sigma(u) - \sigma_h, v\rangle \le -\langle \sigma(u), v\rangle,\end{aligned}$$

where in the last step we have used the property $\sigma_h \le 0$ in T for all $T \in \mathcal{T}_h$. We arrive at the desired claim if we prove that $\langle \sigma(u), v\rangle = 0$. Note that $\mathrm{supp}(\sigma(u)) \subset \{u = \chi\}$ and $\{v > 0\} \subset \{u > \chi\}$ which follows from below

$$\begin{aligned} v > 0 \implies u > u^* &\ge u_h^{conf} + \|(\chi - u_h^{conf})^+\|_{L^\infty(\Omega)}\\ &= \chi - (\chi - u_h^{conf}) + \|(\chi - u_h^{conf})^+\|_{L^\infty(\Omega)} \ge \chi.\end{aligned}$$

Hence we obtain $\langle \sigma(u), v\rangle = \int_\Omega v \,\mathrm{d}\sigma = 0$. Thus $\|\nabla v\|_{L^2(\Omega)} = 0$ and a use of the Poincaré inequality concludes $u \le u^*$.

Next we show that $u_* \le u$. Let $v = (u_* - u)^+$; then it suffices to prove that $v = 0$. Since $u, u_h^{conf} \in H_0^1(\Omega)$, we have

$$(u_* - u)|_{\partial\Omega} \le (u_h^{conf} - u)|_{\partial\Omega} - \|(u_h^{conf} - \chi)^+\|_{L^\infty(\{\sigma_h<0\})} \le 0.$$

Therefore $v|_{\partial\Omega} = 0$, thus $v \in H_0^1(\Omega)$. We obtain the desired result from the Poincaré inequality if we show that $\|\nabla v\|_{L^2(\Omega)} = 0$. Using definitions of u_*, w, $\mathcal{F}_h$ and G_h, we find

$$\begin{aligned}\|\nabla v\|^2_{L^2(\Omega)} &= \int_\Omega \nabla(u_h^{conf} - u) \cdot \nabla v \,\mathrm{d}x + \int_\Omega \nabla w \cdot \nabla v \,\mathrm{d}x = \int_\Omega \nabla(u_h^{conf} - u) \cdot \nabla v \,\mathrm{d}x + \mathcal{F}_h(v)\\ &= \langle \sigma(u) - \sigma_h, v\rangle \le -(\sigma_h, v) = -\sum_{T\in\mathcal{T}_h} \int_T \sigma_h v \,\mathrm{d}x \qquad (\text{since } \sigma(u) \le 0). \end{aligned} \tag{4.15}$$

Now, to obtain the desired claim, it suffices to show that $(\sigma_h, v) = 0$, which we prove in the following two steps.

Step 1. Herein, we show that if there exists $x \in \mathrm{int}(T)$ s.t. $v(x) > 0$, i.e., $u_*(x) > u(x)$, then $\sigma_h|_T = 0$. We prove it by contradiction. Suppose there exists $x \in \mathrm{int}(T)$ s.t. $u_*(x) > u(x)$ and $\sigma_h(x) < 0$. Then, using (4.12) and $u \ge \chi$, we find

$$\begin{aligned} u_h^{conf}(x) &> u(x) + (\|w\|_{L^\infty(\Omega)} - w(x)) + \|(u_h^{conf} - \chi)^+\|_{L^\infty(\{\sigma_h<0\})}\\ &\ge \chi(x) + \|(u_h^{conf} - \chi)^+\|_{L^\infty(\{\sigma_h<0\})} \ge u_h^{conf}(x),\end{aligned}$$

which is a contradiction. Thus $\sigma_h(x) = 0$. Since $\sigma_h \leq 0$ in T and it is linear in T, which implies that $\sigma_h = 0$ in T.

Step 2. We have

$$(\sigma_h, v) = \sum_{T \in \mathcal{T}_h} \int_T \sigma_h v \, \mathrm{d}x.$$

Let $T \subset \mathcal{T}_h$ be arbitrary. If $v = 0$ in T then $\int_T \sigma_h v \, \mathrm{d}x = 0$. If not, then there exists $x \in int(T)$ such that $v(x) > 0$. A use of Step 1 yields $\sigma_h|_T = 0$ and thus $\int_T \sigma_h v \, \mathrm{d}x = 0$. Therefore, from equation (4.15) we get $\|\nabla v\|_{L^2(\Omega)} = 0$ and the Poincaré inequality ensures $u_* \leq u$.
This completes the proof of this lemma. □

4.2 *Pointwise estimate on the corrector function* $\|w\|_{L^\infty(\Omega)}$

We now provide an estimate of the supremum norm of w in terms of the local error indicators (4.4)–(4.7). As is typical in the maximum-norm analysis of finite elements, we employ the Green's function of the unconstrained Poisson problem. Rather than using the approach in Nochetto *et al.* (2003) employing a regularized Green's function, we follow the path from Demlow & Georgoulis (2012), Demlow & Kopteva (2016) by considering the Green's function singular with respect to the point $x_0 \in \Omega$ for which w attains its maximum. This allows for a better account of the nonconformity of the method (see Remark 4.6) and a slight improvement in the estimate (by one power less in the logarithmic factor), since we benefit from the fine estimates collected in Proposition 2.4 that account for the different regularity (of the Green's function) near and far from the singularity.

Proposition 4.5 Let $w \in H_0^1(\Omega)$ be the function defined in (4.13). There holds

$$\|w\|_{L^\infty(\Omega)} \lesssim |\log h_{min}|(\eta_1 + \eta_2 + \eta_3 + \eta_4).$$

Proof. Let $x_0 \in \Omega \setminus \partial\Omega$ be the point at which the maximum of the function w is attained, i.e., $|w(x_0)| = \|w\|_{L^\infty(\Omega)}$. Then there is an element $T_0 \in \mathcal{T}_h$ such that its closure contains $x_0 \in \overline{T_0}$. Let $\mathcal{G}(x_0, \cdot)$ be the Green's function with singularity at x_0, as defined in (2.17) for $x = x_0$. First, notice that from the definition of w in (4.13)–(4.14) and having defined the functional $\mathcal{F}_h \in (W_0^{1,q}(\Omega))^*$ $1 \leq q < \frac{d}{d-1}$, it follows that w is indeed more regular, and so in view of (2.18) and the definition in (4.13)–(4.14) we have

$$w(x_0) = \int_\Omega w(\xi)\delta_{x_0}(\xi)\, \mathrm{d}\xi = (\nabla w, \nabla \mathcal{G}(x_0, \cdot)) = \mathcal{F}_h(\mathcal{G}(x_0, \cdot)) = \langle G_h, \mathcal{G}\rangle + \tilde{\mathcal{A}}_h(u_h - u_h^{conf}, \mathcal{G}), \tag{4.16}$$

where we have dropped the dependence on x_0 (which is now fixed) in the last terms above to simplify the notation. Therefore, to provide the estimate on $|w(x_0)| = \|w\|_{L^\infty(\Omega)}$, we need to bound the terms of the right-hand side of the last equation. Let $\omega_0 = \omega_{T_0}$ be the set of elements touching T_0 (as defined in (2.1)) and let ω_1 be the patch of elements touching ω_0, i.e., the set of elements with nonempty intersection with $T' \in \omega_0$ (we use the notation ω_1 to avoid the cumbersome $\omega_{\omega_{T_0}}$). In our estimates, we will consider a disjoint decomposition of the finite element partition $\mathcal{T}_h = (\mathcal{T}_h \cap \omega_0) \cup (\mathcal{T}_h \setminus \omega_0)$, which will allow us to use the localized regularity estimates for the Green's function stated in Proposition 2.4.

We now estimate the terms in the right-hand side of (4.16). For the first one, we set $\mathcal{G}_h = \Pi_h(\mathcal{G}) \in V_h$ and write

$$\langle G_h, \mathcal{G}\rangle = \langle G_h, \mathcal{G} - \Pi_h(\mathcal{G})\rangle + \langle G_h, \Pi_h(\mathcal{G})\rangle. \tag{4.17}$$

Now we handle the two terms on the right-hand side of above equation as follows: from the definitions of G_h, $\tilde{\sigma}(u)$, $\sigma(u)$ and σ_h, we find

$$\begin{aligned}\langle G_h, \Pi_h(\mathcal{G})\rangle &= \tilde{\mathcal{A}}_h(u-u_h, \Pi_h(\mathcal{G})) + \langle \tilde{\sigma}(u) - \sigma_h, \Pi_h(\mathcal{G})\rangle = (f, \Pi_h(\mathcal{G})) - \tilde{\mathcal{A}}_h(u_h, \Pi_h(\mathcal{G})) - (\sigma_h, \Pi_h(\mathcal{G}))\\ &= (f, \Pi_h(\mathcal{G})) - \mathcal{A}_h(u_h, \Pi_h(\mathcal{G})) - (\sigma_h, \Pi_h(\mathcal{G})) = \langle \sigma_h, \Pi_h(\mathcal{G})\rangle_h - (\sigma_h, \Pi_h(\mathcal{G})),\end{aligned} \tag{4.18}$$

where we have used that $\tilde{\mathcal{A}}_h(v,w) = \mathcal{A}_h(v,w)$ for all $v, w \in V_h$. Equation (4.2) together with the definition of $\tilde{\sigma}(u)$ gives

$$\begin{aligned}\langle G_h, \mathcal{G} - \Pi_h(\mathcal{G})\rangle &= \tilde{\mathcal{A}}_h(u-u_h, \mathcal{G} - \Pi_h(\mathcal{G})) + \langle \tilde{\sigma}(u) - \sigma_h, \mathcal{G} - \Pi_h(\mathcal{G})\rangle\\ &= (f, \mathcal{G} - \Pi_h(\mathcal{G})) - \tilde{\mathcal{A}}_h(u_h, \mathcal{G} - \Pi_h(\mathcal{G})) - (\sigma_h, \mathcal{G} - \Pi_h(\mathcal{G})).\end{aligned} \tag{4.19}$$

Therefore, plugging (4.18) and (4.19) into (4.17) and substituting the result into (4.16) we have

$$\begin{aligned}\mathcal{F}_h(\mathcal{G}) &= (f - \sigma_h, \mathcal{G} - \Pi_h(\mathcal{G})) + \langle \sigma_h, \Pi_h(\mathcal{G})\rangle_h - (\sigma_h, \Pi_h(\mathcal{G})) - \tilde{\mathcal{A}}_h(u_h, \mathcal{G} - \Pi_h(\mathcal{G}))\\ &\quad + \tilde{\mathcal{A}}_h(u_h - u_h^{conf}, \mathcal{G})\end{aligned} \tag{4.20}$$

For the first term above, the Hölder inequality gives

$$\left|(f - \sigma_h, \mathcal{G} - \Pi_h(\mathcal{G}))\right| \lesssim \max_{T\in\mathcal{T}_h} \|h_T^2(f - \sigma_h)\|_{L^\infty(T)} \sum_{T\in\mathcal{T}_h} \|h_T^{-2}(\mathcal{G} - \Pi_h\mathcal{G})\|_{L^1(T)}.$$

Now using the approximation estimate (2.6) from Lemma 2.1, we obtain

$$\begin{aligned}\sum_{T\in\mathcal{T}_h} \|h_T^{-2}(\mathcal{G} - \Pi_h\mathcal{G})\|_{L^1(T)} &= \sum_{T\in\mathcal{T}_h\cap\omega_0} \|h_T^{-2}(\mathcal{G} - \Pi_h\mathcal{G})\|_{L^1(T)} + \sum_{T\in\mathcal{T}_h\setminus\omega_0} \|h_T^{-2}(\mathcal{G} - \Pi_h\mathcal{G})\|_{L^1(T)}\\ &\lesssim \sum_{T\in\mathcal{T}_h\cap\omega_0} h_T^{-1}|\mathcal{G}|_{W^{1,1}(T)} + \sum_{T\in\mathcal{T}_h\setminus\omega_0} |\mathcal{G}|_{W^{2,1}(T)}.\end{aligned} \tag{4.21}$$

Hence, recalling the definition of η_1 from (4.4), we have

$$\left|(f - \sigma_h, \mathcal{G} - \Pi_h(\mathcal{G}))\right| \lesssim \eta_1 \left(\sum_{T\in\mathcal{T}_h\cap\omega_0} h_T^{-1}|\mathcal{G}|_{W^{1,1}(T)} + \sum_{T\in\mathcal{T}_h\setminus\omega_0} |\mathcal{G}|_{W^{2,1}(T)}\right). \tag{4.22}$$

For the second and third terms in (4.20), recalling the localization property observed in (3.13), we first set $\mathcal{T}_h^C = \mathbb{C}_h \cup \mathbb{M}_h$ and use the Hölder inequality but splitting the contributions in the disjoint decomposition of the partition $(\mathcal{T}_h^C \setminus \omega_0) \cup (\mathcal{T}_h^C \cap \omega_0)$. In particular, also using Lemma 3.1, Sobolev embedding (or even the inverse inequality (2.7)), together with the stability estimate (2.5) from Lemma 2.1, we obtain

$$
\begin{aligned}
\left|(\sigma_h, \Pi_h\mathcal{G}) - \langle \sigma_h, \Pi_h\mathcal{G}\rangle_h\right| &\lesssim \sum_{T\in\mathcal{T}_h^C\setminus\omega_0} \|h_T^2\nabla\sigma_h\|_{L^d(T)}\|\nabla\Pi_h\mathcal{G}\|_{L^{d/(d-1)}(T)} \\
&\quad + \sum_{T\subset\mathcal{T}_h^C\cap\omega_0} \|h_T^2\nabla\sigma_h\|_{L^d(T)} h_T^{-1}\|\nabla\Pi_h\mathcal{G}\|_{L^1(T)} \\
&\lesssim \max_{T\in\mathcal{T}_h^C}\|h_T^2\nabla\sigma_h\|_{L^d(T)}\left(\sum_{T\in\mathcal{T}_h^C\setminus\omega_0}\left(|\Pi_h\mathcal{G}|_{W^{1,1}(T)} + |\Pi_h\mathcal{G}|_{W^{2,1}(T)}\right) + \sum_{T\subset\mathcal{T}_h^C\cap\omega_0} h_T^{-1}\|\nabla\mathcal{G}\|_{L^1(T)}\right) \\
&\lesssim \eta_4\left(|\mathcal{G}|_{W^{1,1}(\mathcal{T}_h^C\setminus\omega_0)} + |\mathcal{G}|_{W^{2,1}(\mathcal{T}_h^C\setminus\omega_0)} + \sum_{T\subset\mathcal{T}_h^C\cap\omega_0} h_T^{-1}\|\nabla\mathcal{G}\|_{L^1(T)}\right),
\end{aligned}
\tag{4.23}
$$

where in the last step we have used the definition of η_4 from (4.7). For the fourth term in (4.20), using the fact that Π_h is a projection (hence $\Pi_h(\mathcal{G}-\Pi_h(\mathcal{G}))=0$) and integrating by parts, taking into account that $u_h\in\mathbb{P}^1(\mathcal{T}_h)$ we find

$$
\begin{aligned}
\tilde{\mathcal{A}}_h(u_h, \mathcal{G}-\Pi_h(\mathcal{G})) &= a_h(u_h, (\mathcal{G}-\Pi_h(\mathcal{G}))) + \tilde{b}_h(u_h, (\mathcal{G}-\Pi_h(\mathcal{G}))) \\
&= \sum_{T\in\mathcal{T}_h}\int_{\partial T}\frac{\partial u_h|_T}{\partial n_T}(\mathcal{G}-\Pi_h(\mathcal{G}))\,\mathrm{d}s + \tilde{b}_h(u_h, (\mathcal{G}-\Pi_h(\mathcal{G}))) \\
&= \sum_{e\in\mathcal{E}_h^o}\int_e [\![\nabla u_h]\!]\{\mathcal{G}-\Pi_h(\mathcal{G})\}\,\mathrm{d}s + \sum_{e\in\mathcal{E}_h}\int_e\frac{\gamma}{h_e}[\![u_h]\!][\![\mathcal{G}-\Pi_h(\mathcal{G})]\!]\,\mathrm{d}s.
\end{aligned}
$$

The Hölder inequality and the trace inequality (2.11), together with the localized estimates on $\mathcal{G}$, give

$$
\begin{aligned}
|\tilde{\mathcal{A}}_h(u_h, \mathcal{G}-\Pi_h(\mathcal{G}))| &\lesssim \sum_{e\in\mathcal{E}_h^o}\|h_e[\![\nabla u_h]\!]\|_{L^\infty(e)} h_e^{-1}\|\{\mathcal{G}-\Pi_h\mathcal{G}\}\|_{L^1(e)} \\
&\quad + \|[\![u_h]\!]\|_{L^\infty(\mathcal{E}_h)}\sum_{e\in\mathcal{E}_h} h_e^{-1}\|[\![\mathcal{G}-\Pi_h\mathcal{G}]\!]\|_{L^1(e)} \\
&\lesssim \left(\max_{e\in\mathcal{E}_h^o}\|h_e[\![\nabla u_h]\!]\|_{L^\infty(e)} + \|[\![u_h]\!]\|_{L^\infty(\mathcal{E}_h)}\right)\sum_{T\in\mathcal{T}_h}\left(h_T^{-2}\|\mathcal{G}-\Pi_h\mathcal{G}\|_{L^1(T)} + h_T^{-1}\|\nabla(\mathcal{G}-\Pi_h\mathcal{G})\|_{L^1(T)}\right).
\end{aligned}
$$

Arguing as in (4.21) and using the definitions from (4.5) and (4.6) we get

$$
|\tilde{\mathcal{A}}_h(u_h, \mathcal{G}-\Pi_h(\mathcal{G}))| \lesssim (\eta_2+\eta_3)\left(\sum_{T\in\mathcal{T}_h\cap\omega_0} h_T^{-1}|\mathcal{G}|_{W^{1,1}(T)} + \sum_{T\in\mathcal{T}_h\setminus\omega_0}|\mathcal{G}|_{W^{2,1}(T)}\right). \tag{4.24}
$$

For the very last term in (4.20), following Demlow & Georgoulis (2012), we also introduce a continuous but piecewise linear function $\xi \in V_h^{conf}$ that is identically 1 on ω_0 and is identically 0 at the nodes in $\mathcal{T}_h \setminus \overline{\omega_0}$, so that its support is contained in $\overline{\omega_1}$, i.e., $\mathrm{supp}(\xi) \subseteq \overline{\omega_1}$, while the complementary function $\mathrm{supp}(1-\xi) \subset \Omega \setminus \overline{\omega_0}$. This function will be used as a cut-off function and allow us to localize the terms and to integrate by parts in one of them. Notice that $\|\xi\|_{L^\infty(\mathcal{T}_h)} = 1$ and $\|\nabla\xi\|_{L^\infty(\mathcal{T}_h)} \lesssim \delta^{-1}$ with δ being the maximum diameter of the support of ξ.

We write $u_h - u_h^{conf} = \xi(u_h - u_h^{conf}) + (1-\xi)(u_h - u_h^{conf}) = \xi^0 + \xi^1$ and then

$$\tilde{\mathcal{A}}_h(u_h - u_h^{conf}, \mathcal{G}) = \tilde{\mathcal{A}}_h(\xi^0, \mathcal{G}) + \tilde{\mathcal{A}}_h(\xi^1, \mathcal{G}). \tag{4.25}$$

First, we consider the second term of the last equation. Integrating by parts and using the facts that on the support of $(1-\xi)$ the Green's function $\mathcal{G}$ is harmonic and satisfies $[\![\mathcal{G}]\!] = 0$ and $[\![\nabla\mathcal{G}]\!] = 0$ on any $e \in \mathcal{E}_h \cap (\Omega \setminus \omega_0)$, we find

$$\begin{aligned} \tilde{\mathcal{A}}_h(\xi^1, \mathcal{G}) &= (\xi^1, -\Delta\mathcal{G}) + \sum_{e\in\mathcal{E}_h} \int_e [\![\xi^1]\!] \{\!\{\nabla\mathcal{G}\}\!\} \,\mathrm{d}s - \sum_{e\in\mathcal{E}_h} \int_e \{\!\{\nabla\Pi_h(\mathcal{G})\}\!\} [\![\xi^1]\!] \,\mathrm{d}s \qquad (4.26) \\ &= \sum_{e\in\mathcal{E}_h\cap(\Omega\setminus\omega_0)} \int_e [\![\xi^1]\!] \{\!\{\nabla(\mathcal{G} - \Pi_h(\mathcal{G}))\}\!\} \,\mathrm{d}s, \end{aligned}$$

hence, the trace inequality (2.11) gives

$$\begin{aligned} \left|\tilde{\mathcal{A}}_h(\xi^1, \mathcal{G})\right| &\lesssim \|[\![u_h]\!]\|_{L^\infty(\mathcal{E}_h)} \sum_{T\in(\mathcal{T}_h\setminus\omega_0)} \left(h_T^{-1} \|\nabla(\mathcal{G} - \Pi_h\mathcal{G})\|_{L^1(T)} + |\mathcal{G} - \Pi_h\mathcal{G}|_{W^{2,1}(T)} \right) \\ &\lesssim \eta_3 |\mathcal{G}|_{W^{2,1}(\mathcal{T}_h\setminus\omega_0)}. \end{aligned} \tag{4.27}$$

Finally, for the term near the singularity, taking into account the properties of ξ and using inverse estimates, the Hölder inequality, the trace inequality (2.11), the approximation results (2.13) and Lemma 2.2, together with inverse inequality (2.8), we find

$$\begin{aligned} \left|\tilde{\mathcal{A}}_h(\xi^0, \mathcal{G})\right| &= \left| \int_{\omega_1} \nabla_h(\xi(u_h - u_h^{conf})) \cdot \nabla\mathcal{G} \,\mathrm{d}x - \sum_{e\in\mathcal{E}_h} \int_e \{\!\{\nabla\Pi_h(\mathcal{G})\}\!\} [\![\xi(u_h - u_h^{conf})]\!] \,\mathrm{d}s \right| \\ &\lesssim \|[\![u_h]\!]\|_{L^\infty(\mathcal{E}_h\cap\omega_1)} \left(\sum_{T\in\omega_1} (\delta^{-1} + h_T^{-1}) \|\nabla\mathcal{G}\|_{L^1(T)} + h_T^{-1} \|\nabla\Pi_h\mathcal{G}\|_{L^1(T)} + |\Pi_h\mathcal{G}|_{W^{2,1}(T)} \right). \end{aligned} \tag{4.28}$$

Plugging (4.27) and the last estimate (4.28) into (4.25), then using the stability estimates (2.5) from Lemma 2.1, we get

$$\left|\tilde{\mathcal{A}}_h(u_h - u_h^{conf}, \mathcal{G})\right| \lesssim \eta_3 \left(|\mathcal{G}|_{W^{2,1}(\mathcal{T}_h \setminus \omega_0)}\right) + \eta_3 \left(\sum_{T\in\omega_1} (\delta^{-1} + h_T^{-1}) \|\nabla \mathcal{G}\|_{L^1(T)}\right). \tag{4.29}$$

To conclude we now note that from the shape regularity assumption we can guarantee that there exists $C_0, C_1 > 0$ with $C_1 > C_0$ such that the balls centered at x_0 and with radius $C_0 h_0$ and $C_1 h_0$ satisfy

$$\begin{aligned} \mathcal{B}_0 &:= \mathcal{B}(x_0, C_0 h_0) \subset \omega_0 \quad \Longrightarrow \quad \Omega \setminus \omega_0 \subset \Omega \setminus \mathcal{B}_0, \\ \mathcal{B}_1 &:= \mathcal{B}(x_0, C_1 h_0) \supset \omega_1, \end{aligned}$$

and therefore the regularity estimates from Proposition 2.4, can be applied by taking the radius in the statement of Proposition 2.4 as $2\rho = C_1 h_0 = 2 C_0 h_0$. Then we have the estimates

$$\sum_{T\in\omega_1} (\delta^{-1} + h_T^{-1}) \|\nabla \mathcal{G}\|_{L^1(T)} \lesssim h_0^{-1} \|\nabla \mathcal{G}\|_{L^1(\Omega\cap\mathcal{B}_1)} \lesssim (1 + |\log(h_0)|),$$

$$|\nabla \mathcal{G}|_{W^{1,1}(\mathcal{T}_h\setminus\omega_0)} + |\mathcal{G}|_{W^{2,1}(\mathcal{T}_h\setminus\omega_0)} \lesssim |\mathcal{G}|_{W^{2,1}(\Omega\setminus\mathcal{B}_0)} \lesssim (1 + |\log(h_0)|),$$

$$\sum_{T\in\mathcal{T}_h\cap\omega_0} h_T^{-1} |\mathcal{G}|_{W^{1,1}(T)} \lesssim h_0^{-1} \|\nabla \mathcal{G}\|_{L^1(\Omega\cap\mathcal{B}_1)} \lesssim 1, \quad |\mathcal{G}|_{W^{1,1}(\mathcal{T}_h^C\setminus\omega_0)} \lesssim 1,$$

$$|\mathcal{G}|_{W^{2,1}(\mathcal{T}_h\setminus\omega_0)} \lesssim |\mathcal{G}|_{W^{2,1}(\Omega\setminus\mathcal{B}_0)} \lesssim (1 + \log(h_0)).$$

Combining these estimates on the Green's function together with equations (4.16), (4.20), (4.22), (4.23), (4.24) and (4.29), we get the desired result. □

REMARK 4.6 We stress that the proof of Proposition 4.5 uses duality and so the symmetry of the SIPG method is essential to it; it would break down for any of the nonsymmetric versions of the IP family. In particular, by considering a nonsymmetric method (either NIPG or IIPG), the last term in (4.26) (which is negative and has the correct sign) would be either positive (for NIPG) or zero (for IIPG). More precisely, if we had defined $\tilde{b}_h(v, w)$ by

$$\tilde{b}_h(v, w) = -\sum_{e\in\mathcal{E}_h} \int_e \{\nabla \Pi_h(v)\} [\![w]\!] \, \mathrm{d}s + \theta \sum_{e\in\mathcal{E}_h} \int_e \{\nabla \Pi_h w\} [\![v]\!] \, \mathrm{d}s + \sum_{e\in\mathcal{E}_h} \int_e \frac{\gamma}{h_e} [\![v]\!] [\![w]\!] \, \mathrm{d}s,$$

with $\theta = -1, 1$ and 0 corresponding to the SIPG, NIPG and IIPG methods, respectively, then while estimating the term $\tilde{\mathcal{A}}_h(\xi^1, \mathcal{G})$ in equation (4.26) in the proof of Proposition 4.5, we would have got

$$\tilde{\mathcal{A}}_h(\xi^1, \mathcal{G}) = a_h(\xi^1, \mathcal{G}) + \tilde{b}_h((\xi^1, \mathcal{G}) = (\nabla_h \xi^1, \nabla \mathcal{G}) + \theta \sum_{e \in \mathcal{E}_h} \int_e \{\nabla \Pi_h(\mathcal{G})\} [\![\xi^1]\!] \, \mathrm{d}s$$

$$= (\xi^1, -\Delta \mathcal{G}) + \sum_{e \in \mathcal{E}_h} \int_e [\![\xi^1]\!] \{\nabla \mathcal{G}\} \, \mathrm{d}s + \theta \sum_{e \in \mathcal{E}_h} \int_e \{\nabla \Pi_h(\mathcal{G})\} [\![\xi^1]\!] \, \mathrm{d}s.$$

This would preclude any further analysis for $\theta = 0, 1$ in the sense that optimal estimates cannot be obtained for these choices of θ. We further stress that a naive analysis without accounting for the nonconformity of the method will not reveal this issue (or show this drawback), but one would be inevitably providing only an estimator for the conforming part of the solution.

4.3 *Proof of Reliability Theorem* 4.2

We now have all ingredients to complete the proof of Theorem 4.2.

Proof of Theorem 4.2. We have

$$\|u - u_h\|_{L^\infty(\Omega)} \lesssim \|u - u_h^{conf}\|_{L^\infty(\Omega)} + \|u_h^{conf} - u_h\|_{L^\infty(\Omega)}.$$

From Lemma 4.3 we find

$$w - \|w\|_{L^\infty(\Omega)} - \|(u_h^{conf} - \chi)^+\|_{L^\infty(\{\sigma_h < 0\})} \lesssim u - u_h^{conf} \lesssim w + \|w\|_{L^\infty(\Omega)} + \|(\chi - u_h^{conf})^+\|_{L^\infty(\Omega)},$$

which implies

$$\|u - u_h^{conf}\|_{L^\infty(\Omega)} \lesssim 2\|w\|_{L^\infty(\Omega)} + \|(\chi - u_h^{conf})^+\|_{L^\infty(\Omega)} + \|(u_h^{conf} - \chi)^+\|_{L^\infty(\{\sigma_h < 0\})}. \qquad (4.30)$$

□

By combining (4.30) and (2.13) together with the observation that

$$\|(\chi - u_h^{conf})^+\|_{L^\infty(\Omega)} \lesssim \|(\chi - u_h)^+\|_{L^\infty(\Omega)} + \|u_h - u_h^{conf}\|_{L^\infty(\mathcal{T}_h)}$$
$$\lesssim \|(\chi - u_h)^+\|_{L^\infty(\Omega)} + \|[\![u_h]\!]\|_{L^\infty(\mathcal{E}_h)},$$

and

$$\|(u_h^{conf} - \chi)^+\|_{L^\infty(\{\sigma_h < 0\})} \lesssim \|(u_h^{conf} - u_h)\|_{L^\infty(\{\sigma_h < 0\})} + \|(u_h - \chi)^+\|_{L^\infty(\{\sigma_h < 0\})}$$
$$\lesssim \|(u_h - \chi)^+\|_{L^\infty(\{\sigma_h < 0\})} + \|[\![u_h]\!]\|_{L^\infty(\mathcal{E}_h)},$$

we obtain

$$\|u - u_h\|_{L^\infty(\Omega)} \lesssim \|w\|_{L^\infty(\Omega)} + \|(\chi - u_h)^+\|_{L^\infty(\Omega)} + \|(u_h - \chi)^+\|_{L^\infty(\{\sigma_h < 0\})} + \|[\![u_h]\!]\|_{L^\infty(\mathcal{E}_h)}.$$

Now using the estimates for $\|w\|_{L^\infty(\Omega)}$ from Proposition 4.5, the result follows. We remark here that a use of Lemma 4.1 leads to the following reliability estimates.

COROLLARY 4.7 Let $u \in \mathcal{K}$ and $u_h \in \mathcal{K}_h$ be the solutions of (1.1) and (3.2), respectively. Then

$$\|u - u_h\|_{L^\infty(\Omega)} \lesssim \tilde{\eta}_h,$$

where

$$\tilde{\eta}_h = |\log h_{min}|\Big(\eta_1 + \eta_2 + \eta_3\Big) + \|(\chi - u_h)^+\|_{L^\infty(\Omega)} + \|(u_h - \chi)^+\|_{L^\infty(\{\sigma_h<0\})} + \max_{T\in\mathcal{T}_h} \mathrm{Osc}(f, T).$$

REMARK 4.8 It can be observed directly from definitions (3.8), (3.9) and (3.10) of discrete contact, noncontact and free boundary sets that $\{T \in \mathcal{T}_h : \sigma_h < 0 \text{ on } T\}) \subset \mathbb{C}_h \cup \mathbb{M}_h$. Therefore, the term $\|(u_h - \chi)^+\|_{L^\infty(\{\sigma_h<0\})}$ measures the error in approximation of the obstacle in the discrete contact set and discrete free boundary set and detects the nonaffine situation $\chi \neq \chi_h$.

The term $\|(\chi - u_h)^+\|_{L^\infty(\Omega)}$ measures the error in the violation of the obstacle constraint and allows $\chi > u_h$ to be detected.

4.4 *Estimation of the error in the Lagrange multiplier*

We derive a bound for the error in the Lagrange multiplier in a dual norm in terms of the error estimator. Let $D \subseteq \Omega$ be any set. Denote $\mathcal{T}_D$ by the set of all $T \in \mathcal{T}_h$ such that $T \subset D$. Furthermore, let $\mathcal{E}_D$ be the set of all edges/faces in $\mathcal{E}_h$ that are in D.

We introduce the functional space

$$\mathcal{W}_D := W_0^{2,1}(D) = \{v \in W^{2,1}(D) \ : \ v = 0 \text{ and } \nabla v \cdot \mathbf{n} = 0 \text{ on } \partial D\}, \tag{4.31}$$

endowed with the norm $\|v\|_{\mathcal{W}_\mathcal{D}} := |v|_{W^{1,1}(D)} + |v|_{W^{2,1}(D)}$. For notational convenience we set $\mathcal{W} := \mathcal{W}_\Omega$, which is essentially the space $W_0^{2,1}(\Omega)$.

For any $\mathcal{F} \in \mathcal{W}_D^*$, the norm $\|\mathcal{F}\|_{-2,\infty,D}$ is defined by

$$\|\mathcal{F}\|_{-2,\infty,D} = \|\mathcal{F}\|_{\mathcal{W}_D^*} := \sup\{\langle \mathcal{F}, v\rangle \ : \ \tilde{v} \in \mathcal{W}_D \ , \ |\tilde{v}|_{\mathcal{W}_D} \leq 1\}.$$

The subsequent analysis requires the bound on $\|G_h\|_{-2,\infty,\Omega}$ which is estimated in next lemma.

LEMMA 4.9 It holds that

$$\|G_h\|_{-2,\infty,\Omega} \lesssim \eta_1 + \eta_2 + \eta_3 + \eta_4. \tag{4.32}$$

Proof. For any $v \in \mathcal{W}$ and $v_h = \Pi_h v$, using the definition of G_h from (4.3) and (3.6) we have

$$\begin{aligned}\langle G_h, v\rangle &= (f, v - v_h) - \tilde{\mathcal{A}}_h(u_h, v - v_h) - (\sigma_h, v - v_h) + (f, v_h) - \tilde{\mathcal{A}}_h(u_h, v_h) - (\sigma_h, v_h) \\ &= (f - \sigma_h, v - v_h) - \tilde{\mathcal{A}}_h(u_h, v - v_h) + \langle \sigma_h, v_h\rangle_h - (\sigma_h, v_h).\end{aligned} \tag{4.33}$$

We begin by estimating the terms on the right-hand side of (4.33) as follows: a use of the Hölder inequality and Lemma 2.1 yields

$$(f-\sigma_h, v-v_h) \lesssim \max_{T\in\mathcal{T}_h} \|h_T^2(f-\sigma_h)\|_{L^\infty(T)} \sum_{T\in\mathcal{T}_h} \|h_T^{-2}(v-v_h)\|_{L^1(T)} \lesssim \eta_1 |v|_{W^{2,1}(\Omega)}. \tag{4.34}$$

To estimate the second term in(4.33), integrating by parts taking into account that $u_h \in \mathbb{P}^1(\mathcal{T}_h)$ and the fact that Π_h is a projection (hence $\Pi_h(v-\Pi_h v)=0$) gives

$$\tilde{\mathcal{A}}_h(u_h, v-v_h) = \sum_{e\in\mathcal{E}_h^o} \int_e [\![\nabla u_h]\!] \{\!\!\{ v-v_h \}\!\!\}\, \mathrm{d}s + \sum_{e\in\mathcal{E}_h} \int_e \frac{\gamma}{h_e} [\![u_h]\!][\![v-v_h]\!]\, \mathrm{d}s.$$

Therefore, using the Hölder inequality, the trace inequality (2.11) and Lemma 2.1 we find

$$|\tilde{\mathcal{A}}_h(u_h, v-v_h)| \lesssim (\eta_2+\eta_3)\Big(\sum_{T\in\mathcal{T}_h} \Big(h_T^{-2}\|v-v_h\|_{L^1(T)} + h_T^{-1}\|\nabla(v-v_h)\|_{L^1(T)} \Big) \Big) \lesssim (\eta_2+\eta_3)|v|_{W^{2,1}(\Omega)}. \tag{4.35}$$

We estimate the last two terms of (4.33) using Lemma 3.1, and the inverse inequality (2.7), together with the stability estimate (2.5) from Lemma 2.1, as follows:

$$\begin{aligned}\langle \sigma_h, v_h\rangle_h - (\sigma_h, v_h) &\lesssim \sum_{T\in\mathcal{T}_h^C} \|h_T^2 \nabla\sigma_h\|_{L^d(T)} \|\nabla v_h\|_{L^{\frac{d}{(d-1)}}(T)} \lesssim \max_{T\in\mathcal{T}_h^C} \|h_T^2\nabla\sigma_h\|_{L^d(T)} \|\nabla v\|_{W^{1,1}(\mathcal{T}_h^C)} \\ &\lesssim \eta_4 |v|_{W^{2,1}(\Omega)}. \end{aligned} \tag{4.36}$$

Combining (4.34), (4.35) and (4.36) together with (4.33), we obtain (4.32). □

Now we show that the following bound on the error $\|\tilde{\sigma}(u)-\sigma_h\|_{-2,\infty,\Omega}$ holds.

PROPOSITION 4.10 Let $\tilde{\sigma}(u)$ and σ_h be as defined in (4.1) and (3.6), respectively. Then

$$\|\tilde{\sigma}(u)-\sigma_h\|_{-2,\infty,\Omega} \lesssim \eta_h.$$

Proof. For any $v \in \mathcal{W}$, from equation (4.2) we have

$$\langle \tilde{\sigma}(u)-\sigma_h, v\rangle = \langle G_h, v\rangle - \tilde{\mathcal{A}}_h(u-u_h, v). \tag{4.37}$$

Now using integration by parts, (3.5), the trace inequality (2.11) and Lemma 2.1 we obtain

$$|\tilde{\mathcal{A}}_h(u-u_h, v)| \lesssim \big(\|u-u_h\|_{L^\infty(\Omega)} + \|[\![u_h]\!]\|_{L^\infty(\mathcal{E}_h)} \big) |v|_{W^{2,1}(\Omega)}. \tag{4.38}$$

Combining (4.38) together with (4.37) we have

$$\|\tilde{\sigma}(u)-\sigma_h\|_{-2,\infty,\Omega} \lesssim \|u-u_h\|_{L^\infty(\Omega)} + \|[\![u_h]\!]\|_{L^\infty(\mathcal{E}_h)} + \|G_h\|_{-2,\infty,\Omega}. \tag{4.39}$$

Finally, in view of (4.39), Lemma 4.9 and Theorem 4.2 we have the desired reliability estimate for η_h for the error in the Lagrange multiplier. □

4.5 *Error estimator for the conforming finite element method*

The conforming finite element method for the model problem (1.1) is to find $\tilde{u}_h \in \mathcal{K}_h^{conf}$ such that

$$a(\tilde{u}_h, v_h - u_h) \geq (f, v_h - u_h) \quad \forall\, v_h \in \mathcal{K}_h^{conf}, \tag{4.40}$$

where

$$\mathcal{K}_h^{conf} := \{v_h \in V_h^{conf} : v_h(p) \geq \chi_h(p) \ \ \forall\, p \in \mathcal{V}_h^i\}.$$

In this case, our analysis will lead to the following reliability estimates.

COROLLARY 4.11 Let $u \in \mathcal{K}$ and $\tilde{u}_h \in \mathcal{K}_h^{conf}$ be the solutions of (1.1) and (4.40), respectively. Then

$$\|u - \tilde{u}_h\|_{L^\infty(\Omega)} \lesssim \zeta_h,$$

where

$$\zeta_h = |\log h_{min}|\Big(\eta_1 + \eta_2\Big) + \|(\chi - \tilde{u}_h)^+\|_{L^\infty(\Omega)} + \|(\tilde{u}_h - \chi)^+\|_{L^\infty(\{\sigma_h<0\})} + \max_{T\in\mathcal{T}_h} \mathrm{Osc}(f, T).$$

REMARK 4.12 This error estimator ζ_h is comparable with the error estimator reported in Nochetto *et al.* (2003). The slight improvement in the log term is obtained using the refined Green's function estimates.

5. Efficiency of the error estimator

In this section we derive the local efficiency estimates of *a posteriori* error estimator η_h obtained in the previous section. Therein, we require a bound on a negative norm of the residual in terms of the L^∞ error in the solution u and a negative norm of the error in the Lagrange multiplier. Let $D \subset \Omega$ be a set formed by some triangles in $\mathcal{T}_h$. Essentially, in what follows, D is either a simplex or the union of two simplices sharing a face/edge. Denote by $\mathcal{T}_D$ the set of all $T \in \mathcal{T}_h$ such that $T \subset D$. Furthermore, let $\mathcal{E}_D$ be the set of all edges/faces in $\mathcal{E}_h$ that are in D.

Note that for any $\tilde{v} \in \mathcal{W}_D$ (defined in (4.31)) extended to Ω by zero outside D, the resulting extended function denoted by v is in $\mathcal{W} = W_0^{2,1}(\Omega)$. For any $\tilde{v} \in \mathcal{W}_D$, using the definition of G_h we have

$$\langle G_h, \tilde{v}\rangle = \langle G_h, v\rangle = \tilde{\mathcal{A}}_h(u - u_h, v) + \langle \tilde{\sigma}(u) - \sigma_h, v\rangle.$$

Therein, we simplify the first term using integration by parts and (3.5) to get

$$\tilde{\mathcal{A}}_h(u - u_h, v) = -\sum_{T\in\mathcal{T}_h} \int_T (u - u_h)\Delta v \,\mathrm{d}x - \sum_{e\in\mathcal{E}_h} \int_e [\![u_h]\!]\{\nabla(v - \Pi_h v)\}\,\mathrm{d}s.$$

Therefore,

$$\langle G_h, \tilde{v}\rangle = -\int_D (u-u_h)\Delta\tilde{v}\,\mathrm{d}x - \sum_{e\in\mathcal{E}_D}\int_e [\![u_h]\!]\{\!\!\{\nabla(\tilde{v}-\Pi_h\tilde{v})\}\!\!\}\,\mathrm{d}s + \langle\tilde{\sigma}(u)-\sigma_h,\tilde{v}\rangle. \tag{5.1}$$

We begin by estimating the terms on the right-hand side of (5.1) as follows: first,

$$\left|\int_D (u-u_h)\Delta\tilde{v}\,\mathrm{d}x\right| \lesssim \|u-u_h\|_{L^\infty(D)}|\tilde{v}|_{\mathcal{W}},$$

and a use of the Hölder inequality, the trace inequality (2.11) and Lemma 2.1 yields

$$\sum_{e\in\mathcal{E}_D}\int_e [\![u_h]\!]\{\!\!\{\nabla(\tilde{v}-\Pi_h\tilde{v})\}\!\!\}\,\mathrm{d}s \lesssim \|[\![u_h]\!]\|_{L^\infty(\mathcal{E}_D)}\sum_{T\in\mathcal{T}_D}|\tilde{v}|_{W^{2,1}(T)} \lesssim \|[\![u_h]\!]\|_{L^\infty(\mathcal{E}_D)}|\tilde{v}|_{\mathcal{W}_D},$$

and

$$\langle\tilde{\sigma}(u)-\sigma_h, v\rangle \lesssim \|\tilde{\sigma}(u)-\sigma_h\|_{-2,\infty,D}|\tilde{v}|_{\mathcal{W}_D}.$$

The following bound on the Galerkin functional G_h, in the dual space of $\mathcal{W}_D$, is then immediate from equation (5.1):

$$\|G_h\|_{-2,\infty,D} \lesssim \|u-u_h\|_{L^\infty(D)} + \|[\![u_h]\!]\|_{L^\infty(\mathcal{E}_D)} + \|\tilde{\sigma}(u)-\sigma_h\|_{-2,\infty,D}. \tag{5.2}$$

This estimate on G_h will be useful in deriving local efficiency estimates.

The derivation of the following local efficiency estimates of the error estimator uses the standard bubble functions technique (Ainsworth & Oden, 2000, Chapter 2, page 23). We have discussed the main ideas involved in proving these efficiency estimates and skipped the standard details.

THEOREM 5.1 Let $u \in \mathcal{K}$ and $u_h \in \mathcal{K}_h$ be the solutions of (1.1) and (3.2), respectively. Then it holds that

$$\begin{aligned} h_T^2\|f-\sigma_h\|_{L^\infty(T)} + \|(\chi-u_h)^+\|_{L^\infty(T)} &\lesssim \|u-u_h\|_{L^\infty(T)} + \|\sigma(u)-\sigma_h\|_{-2,\infty,T} \\ &\quad + \mathrm{Osc}(f,T) \quad \forall\, T\in\mathcal{T}_h, \end{aligned} \tag{5.3}$$

$$\|[\![u_h]\!]\|_{L^\infty(e)} \lesssim \|u-u_h\|_{L^\infty(\omega_e)} \quad \forall\, e\in\mathcal{E}_h, \tag{5.4}$$

$$h_e\|[\![\nabla u_h]\!]\|_{L^\infty(e)} \lesssim \|u-u_h\|_{L^\infty(\omega_e)} + \|\sigma(u)-\sigma_h\|_{-2,\infty,\omega_e} + \mathrm{Osc}(f,\omega_e) \qquad \forall\, e\in\mathcal{E}_h^o, \tag{5.5}$$

$$h_T^2\|\nabla\sigma_h\|_{L^d(T)} \lesssim \|u-u_h\|_{L^\infty(T)} + \|\sigma(u)-\sigma_h\|_{-2,\infty,T} + \mathrm{Osc}(f,T) \qquad \forall\ T\in\mathbb{C}_h\cup\mathbb{M}_h \tag{5.6}$$

Further, assuming $\chi < 0$ on $T \cap \partial\Omega$ if $T \in \mathcal{M}_h$ and $T \cap \partial\Omega \neq \emptyset$, then for all elements $T \in \mathcal{T}_h$ such that $T \cap \{\sigma_h < 0\} \neq \emptyset$ we have

$$\|(u_h - \chi)^+\|_{L^\infty(T)} \lesssim \|u - u_h\|_{L^\infty(T)} + \|\sigma(u) - \sigma_h\|_{-2,\infty,T} + \|(\chi - \chi_h)^+\|_{L^\infty(T)} + \sum_{p \in \mathcal{V}_T, T \in \mathbb{M}_h} \sum_{e \in \mathcal{E}_p \cap \mathcal{E}_h^o} h_e(\|[\![\nabla u_h]\!]\|_{L^\infty(e)} + \|[\![\nabla \chi_h]\!]\|_{L^\infty(e)}). \tag{5.7}$$

Here, ω_e denotes the union of elements sharing the face e and $\mathrm{Osc}(f, T)$ is defined in (4.10).

REMARK 5.2 The proof of efficiency estimate (5.6) follows directly from Lemma 4.1 and estimate (5.3).

Proof. • We start by showing that for any $T \in \mathcal{T}_h$,

$$h_T^2 \|f - \sigma_h\|_{L^\infty(T)} + \|(\chi - u_h)^+\|_{L^\infty(T)} \lesssim \|u - u_h\|_{L^\infty(T)} + \|\sigma(u) - \sigma_h\|_{-2,\infty,T} + \mathrm{Osc}(f, T).$$

Note that the bound on the second term of the left-hand side in the above estimate follows immediately since $(\chi - u_h)^+ \leq (u - u_h)^+$ in view of $u \geq \chi$. We have

$$\|(\chi - u_h)^+\|_{L^\infty(T)} \lesssim \|u - u_h\|_{L^\infty(T)}. \tag{5.8}$$

To estimate $h_T^2 \|f - \sigma_h\|_{L^\infty(T)}$, let $b_T \in \mathbb{P}^{2(d+1)}(T) \cap H_0^2(T)$ be the polynomial bubble function which vanishes up to first-order derivative on ∂T, attains unity at the barycenter of T and $\|b_T\|_{L^\infty(T)} = 1$.

Let $\tilde{f} \in \mathbb{P}^0(T)$ be arbitrary. Set $\phi_T = b_T(\tilde{f} - \sigma_h)$ on T and let ϕ be the extension of ϕ_T to $\bar{\Omega}$ by zero. It is easy to see that ϕ is continuously differentiable on $\bar{\Omega}$ and $\phi \in H_0^2(\Omega)$. Using the equivalence of norms in finite-dimensional normed spaces and taking into consideration that u_h is linear on T, identity (4.3) and integration by parts yielding $a_h(u_h, \phi) = 0$, we have

$$\begin{aligned} \|\tilde{f} - \sigma_h\|_{L^2(T)}^2 &\lesssim \int_T (\tilde{f} - \sigma_h)\phi_T \,\mathrm{d}x = \int_T (f - \sigma_h)\phi_T \,\mathrm{d}x + \int_T (\tilde{f} - f)\phi_T \,\mathrm{d}x \\ &\lesssim a(u, \phi) + \langle \sigma(u) - \sigma_h, \phi \rangle + \int_T (\tilde{f} - f)\phi_T \,\mathrm{d}x \\ &\lesssim \langle G_h, \phi \rangle + \tilde{b}_h(u_h, \phi) + \int_T (\tilde{f} - f)\phi_T \,\mathrm{d}x. \end{aligned} \tag{5.9}$$

Therefore, using the inverse estimate (2.9) and (5.9) we have

$$h_T^4 \|\tilde{f} - \sigma_h\|_{L^\infty(T)}^2 \lesssim h_T^{4-d} \|\tilde{f} - \sigma_h\|_{L^2(T)}^2 \lesssim h_T^{4-d}\Big(\langle G_h, \phi \rangle + \tilde{b}_h(u_h, \phi) + \int_T (\tilde{f} - f)\phi_T \,\mathrm{d}x\Big). \tag{5.10}$$

To bound the terms of the right-hand side of the last equation, we first estimate the term $\tilde{b}_h(u_h, \phi)$ as follows: a use of the Hölder inequality, the trace inequality (2.11), the inverse inequality and Lemma 2.1

yields

$$\tilde{b}_h(u_h,\phi) = -\sum_{e\in\partial T}\int_e [\![u_h]\!]\{\nabla \Pi_h\phi\}\,\mathrm{d}s \lesssim \|[\![u_h]\!]\|_{L^\infty(\partial T)}(h_T^{-2}\|\phi_T\|_{L^1(T)} + |\phi_T|_{W^{2,1}(T)})$$
$$\lesssim h_T^{-2}\|[\![u_h]\!]\|_{L^\infty(\partial T)}\|\phi_T\|_{L^1(T)}. \tag{5.11}$$

Further, using inverse estimates (2.7), (2.8), the Hölder inequality and the structure of b_T we have

$$|\phi_T|_{W^{2,1}(T)} \lesssim h_T^{-2}\|\phi_T\|_{L^1(T)} \lesssim h_T^{d-2}\|\phi_T\|_{L^\infty(T)} \lesssim h_T^{d-2}\|\tilde{f}-\sigma_h\|_{L^\infty(T)}$$

and

$$\|\phi_T\|_{L^1(T)} \lesssim h_T^d\|\phi_T\|_{L^\infty(T)} \lesssim h_T^d\|\tilde{f}-\sigma_h\|_{L^\infty(T)}.$$

Therefore, from (5.10), (5.11) and the last estimates we have

$$h_T^4\|\tilde{f}-\sigma_h\|^2_{L^\infty(T)} \lesssim h_T^{4-d}h_T^{d-2}\Big(\|G_h\|_{-2,\infty,T} + \|[\![u_h]\!]\|_{L^\infty(\partial T)} + h_T^2\|\tilde{f}-f\|_{L^\infty(T)}\Big)\|\tilde{f}-\sigma_h\|_{L^\infty(T)}.$$

Finally, from (5.2), we obtain

$$h_T^2\|\tilde{f}-\sigma_h\|_{L^\infty(T)} \lesssim \|u-u_h\|_{L^\infty(T)} + \|[\![u_h]\!]\|_{L^\infty(\partial T)} + \|\sigma(u)-\sigma_h\|_{-2,\infty,T} + \mathrm{Osc}(f,T). \tag{5.12}$$

Combining estimates (5.8) and (5.12), taking into account (5.4), we obtain estimate (5.3).

• Estimate (5.4) can be obtained easily. Let $e\in\mathcal{E}_h^o$ and $\omega_e = \overline{T}_+\cup\overline{T}_-$, where T_+ and T_- are elements sharing the face e. Then, by taking into consideration $[\![u]\!]=0$ on e, we have

$$\|[\![u_h]\!]\|_{L^\infty(e)} = \|[\![u-u_h]\!]\|_{L^\infty(e)} \lesssim \|u-u_h\|_{L^\infty(\omega_e)}.$$

For $e\in\mathcal{E}_h^\partial$, similar arguments can be used to estimate $\|[\![u_h]\!]\|_{L^\infty(e)}$ taking into account $u=0$ on $\partial\Omega$.

• Next we provide the proof of estimate (5.5). Let $e\in\mathcal{E}_h^o$ and $\omega_e = \overline{T}_+\cup\overline{T}_-$, where T_+ and T_- are elements sharing the face e. Let $b_e\in\mathbb{P}^{4d}(\omega_e)\cap H_0^2(\omega_e)$ denote the edge bubble function, which takes unit value at the center of e and $\|b_e\|_{L^\infty(\omega_e)}=1$.

Define $\phi_e = [\![\nabla u_h]\!]b_e$ on ω_e and let ϕ be the extension of ϕ_e by zero to $\bar{\Omega}$; clearly $\phi\in H_0^2(\Omega)$.

The equivalence of norms in finite-dimensional spaces, integration by parts and (4.3) gives

$$\|[\![\nabla u_h]\!]\|^2_{L^2(e)} \lesssim \int_e [\![\nabla u_h]\!]\phi_e\,\mathrm{d}s = \int_{\omega_e}\nabla u_h\cdot\nabla\phi_e\,\mathrm{d}s = a_h(u_h,\phi) = (f-\sigma_h,\phi) - \langle G_h,\phi\rangle - \tilde{b}_h(u_h,\phi).$$

The last term of the last equation is estimated as in (5.11), yielding

$$\tilde{b}_h(u_h,\phi) \lesssim \|[\![u_h]\!]\|_{L^\infty(\partial\omega_e)}\Big(\sum_{T\in\omega_e} h_T^{-2}\|\phi_e\|_{L^1(T)}\Big).$$

Thus, using the inverse estimates (2.7)–(2.10), the Hölder inequality and the structure of the bubble function b_e we have

$$
\begin{aligned}
h_e^2\|[\![\nabla u_h]\!]\|_{L^\infty(e)}^2 &\lesssim h_e^{3-d}\|[\![\nabla u_h]\!]\|_{L^2(e)}^2 \lesssim h_e^{3-d}\Big((f-\sigma_h,\phi) - \langle G_h,\phi\rangle - \tilde{b}_h(u_h,\phi)\Big)\\
&\lesssim h_e^{3-d}\Big(\big(\|f-\sigma_h\|_{L^\infty(\omega_e)} + h_e^{-2}\|[\![u_h]\!]\|_{L^\infty(\partial\omega_e)}\big)\|\phi_e\|_{L^1(\omega_e)} + \|G_h\|_{-2,\infty,\omega_e}|\phi_e|_{W^{2,1}(\omega_e)}\Big)\\
&\lesssim h_e^{3-d}\Big(h_e^d\|f-\sigma_h\|_{L^\infty(\omega_e)} + h_e^{d-2}\|[\![u_h]\!]\|_{L^\infty(\partial\omega_e)} + h_e^{d-2}\|G_h\|_{-2,\infty,\omega_e}\Big)\|\phi_e\|_{L^\infty(\omega_e)}\\
&\lesssim h_e\Big(h_e^2\|f-\sigma_h\|_{L^\infty(\omega_e)} + \|G_h\|_{-2,\infty,\omega_e} + \|[\![u_h]\!]\|_{L^\infty(\partial\omega_e)}\Big)\|[\![\nabla u_h]\!]\|_{L^\infty(e)}.
\end{aligned}
$$

We then deduce estimate (5.5) using estimates (5.3), (5.4) and (5.2).

• Now we provide the proof of estimate (5.7). Let $T \in \mathcal{T}_h$ be such that $T \cap \{\sigma_h < 0\} \neq \emptyset$. Using the triangle inequality we find that

$$\|(u_h-\chi)^+\|_{L^\infty(T)} \leq \|u_h - u_h^{conf}\|_{L^\infty(T)} + \|(u_h^{conf}-\chi)^+\|_{L^\infty(T)}.$$

The first term on the right-hand side of the above estimate can be controlled by using (2.13). We estimate the second term as follows: in view of Remark 4.8, we have that either $T \in \mathbb{C}_h$ or $T \in \mathbb{M}_h$. If $T \in \mathbb{C}_h$ then using the definition of $E_h(\cdot)$, we have $u_h^{conf} = \chi_h$ on T. Therefore,

$$\|(u_h^{conf}-\chi)^+\|_{L^\infty(T)} = \|(\chi_h-\chi)^+\|_{L^\infty(T)}. \tag{5.13}$$

Otherwise, if $T \in \mathbb{M}_h$ then

$$\|(u_h^{conf}-\chi)^+\|_{L^\infty(T)} \leq \|(u_h^{conf}-\chi_h)^+\|_{L^\infty(T)} + \|(\chi_h-\chi)^+\|_{L^\infty(T)}. \tag{5.14}$$

In view of the definition of u_h^{conf}, we observe that $\|(u_h^{conf}-\chi_h)^+\|_{L^\infty(T)} = \|(u_h^{conf}-\chi_h)\|_{L^\infty(T)}$. Next we note that, since $T \in \mathbb{M}_h$, there exists a node $z \in \mathcal{V}_T$ such that $u_h(z) = \chi_h(z)$, which implies $u_h^{conf}(z) = \chi_h(z)$.

We claim that there exists an interior node $z \in \mathcal{V}_T$ such that $u_h^{conf}(z) = \chi_h(z)$. This claim trivially holds if $T \subset \Omega$. If $T \cap \partial\Omega \neq \emptyset$, we have the following two possibilities.

(i) $T \cap \partial\Omega$ is a node, say $T \cap \partial\Omega = \{p\}$. If $\chi_h(p) = \chi(p) < 0$ then $0 = u_h^{conf}(p) > \chi_h(p)$. Hence there exists an interior node $z \in \mathcal{V}_T$ such that $u_h^{conf}(z) = \chi_h(z)$.

(ii) $T \cap \partial\Omega$ is an edge/face, say $T \cap \partial\Omega = e$. If $\chi(p) < 0$ for all $p \in e$ then $u_h^{conf} > \chi_h$ on e since $u_h^{conf} = 0$ on $\partial\Omega$. And hence there exists an interior node $z \in \mathcal{V}_T$ such that $u_h^{conf}(z) = \chi_h(z)$.
Thus with the assumption that $\chi_h < 0$ on $T \cap \partial\Omega$, we have an interior node $p \in \mathcal{V}_T$ such that $u_h^{conf}(p) = \chi_h(p)$. This allows us to use the quadratic growth property of a non-negative discrete function

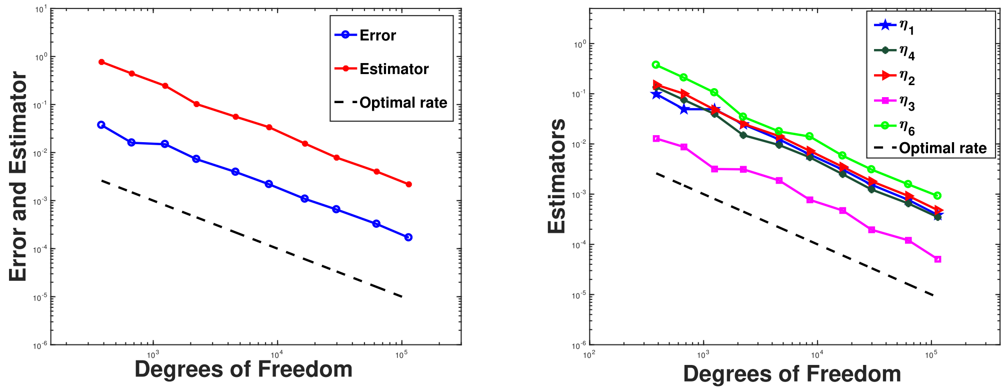

FIG. 1. Error and estimator for Example 1.

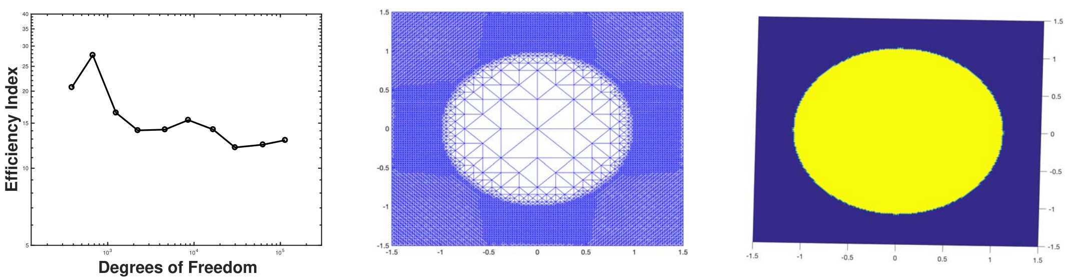

FIG. 2. Efficiency index, adaptive mesh and discrete active set (yellow part) for Example 1.

(see (Baiocchi, 1977, Section 2; Nochetto *et al.*, 2003, Lemma 6.4), which yields

$$\|(u_h^{conf} - \chi_h)^+\|_{L^\infty(T)} = \|(u_h^{conf} - \chi_h)\|_{L^\infty(T)} \lesssim \max_{e\in\mathcal{E}_h: e\cap w_T\neq\emptyset} \Big(\|h_e[\![\nabla u_h^{conf}]\!]\|_{L^\infty(e)} + \|h_e[\![\nabla \chi_h]\!]\|_{L^\infty(e)}\Big)$$

$$\lesssim \max_{e\in\mathcal{E}_h: e\cap w_T\neq\emptyset} \Big(\|h_e[\![\nabla (u_h^{conf} - u_h)]\!]\|_{L^\infty(e)} + \|h_e[\![\nabla u_h]\!]\|_{L^\infty(e)} + \|h_e[\![\nabla \chi_h]\!]\|_{L^\infty(e)}\Big). \quad (5.15)$$

Combining estimates (5.13), (5.14), (5.15) and using (5.5), (5.4) and (2.13), we deduce (5.7).

REMARK 5.3 The definition of σ_h in this article is very local to each simplex, thanks to the DG framework, which is computationally simpler. In Nochetto *et al.* (2003), the definition of σ_h at the boundary vertices is modified (see Nochetto *et al.*, 2003, page 169) by imposing full contact condition on the patches of those vertices, which in turn helps them to prove the efficiency of the term $\|(u_h - \chi)^+\|_{L^\infty(\{\sigma_h<0\})}$ whenever the discrete free-boundary set intersects with the simplices that touch the boundary $\partial\Omega$ (see Nochetto *et al.*, 2003, page 186). Particularly, it is observed that the difficulty arises when the obstacle $\chi = 0$ on $\partial\Omega$ (given that the solution $u = 0$ on $\partial\Omega$ and the compatibility condition $\chi \leq 0$ on $\partial\Omega$). In many research works the assumption $\chi < 0$ on $\partial\Omega$ is made, hence the condition used in this article in proving the efficiency of the term $\|(u_h - \chi)^+\|_{L^\infty(\{\sigma_h<0\})}$ is not restrictive.

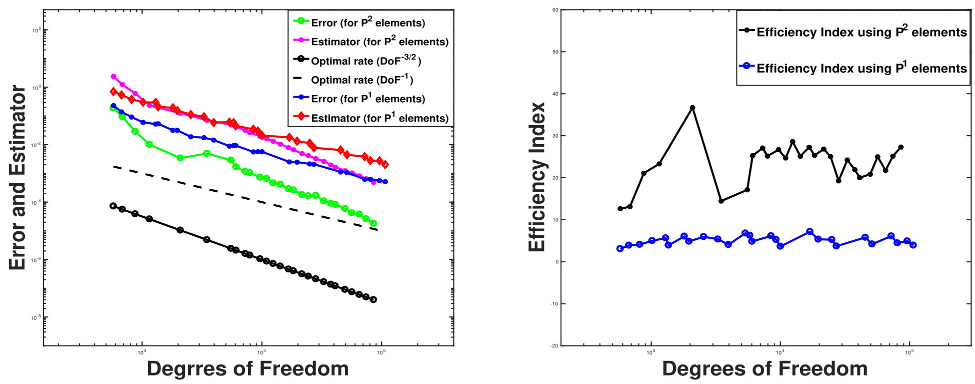

FIG. 3. Error, estimator and efficiency index for Example 2 using linear and quadratic finite elements.

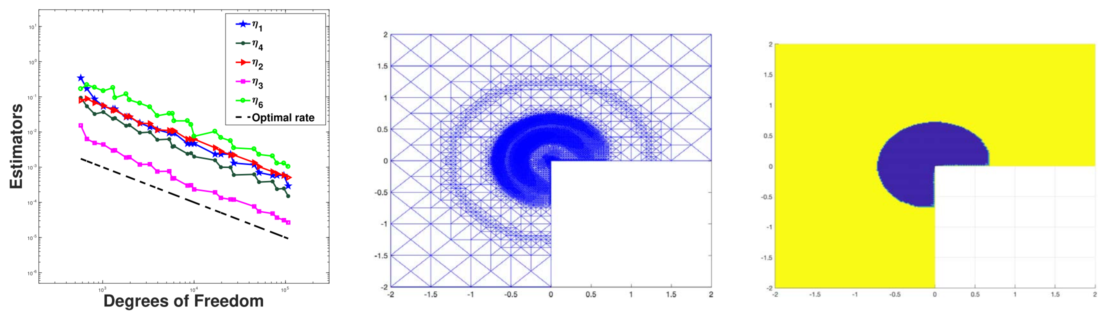

FIG. 4. Estimators, adaptive mesh and discrete active set (yellow part) for Example 2.

This also avoids fine tuning the definition of σ_h on the boundary nodes and makes the computations simpler. Further, it enabled us to prove an estimate for $\sigma - \sigma_h$ in a dual norm, unlike in Nochetto *et al.* (2003) estimate is provided for $\sigma - \tilde{\sigma}_h$, where $\tilde{\sigma}_h$ is a modified Lagrange multiplier which is not easily computable. □

6. Numerical experiments

In this section we present numerical results to demonstrate the performance of the *a posteriori* error estimator derived in Section 4. We have used the following algorithm for the adaptive refinement;

$$\textbf{SOLVE} \longrightarrow \textbf{ESTIMATE} \longrightarrow \textbf{MARK} \longrightarrow \textbf{REFINE}.$$

In the SOLVE step, the primal-dual active set strategy (Hintermüller *et al.*, 2003) is used to solve the discrete obstacle problem (3.1). The algebraic setting of the primal-dual active set method in accordance with the discrete problem (3.1) is discussed in detail in de Dios *et al.* (accepted), and more numerical experiments can be found therein. We compute the error estimator η_h on each element $T \in \mathcal{T}_h$ and use the maximum marking strategy with parameter $\theta = 0.3$ to mark the elements for refinement. Finally,

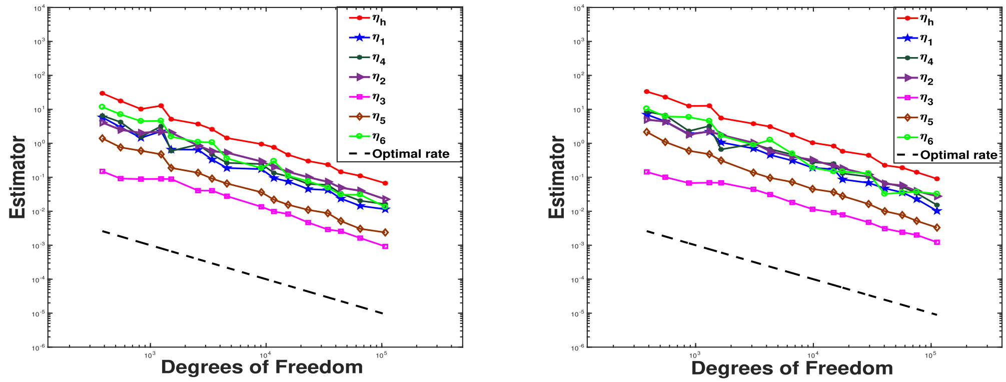

FIG. 5. Estimator for Example 3 for $f = 0$ and $f = -15$.

we refine the mesh using the newest vertex bisection algorithm and obtain a new adaptive mesh. For all the examples below, the penalty parameter $\gamma = 25$.

We discuss numerical results for the various test examples.

Example 1. In this example we consider $\Omega = (-1.5, 1.5)^2, f = -2, \chi = 0$. Set $r^2 = x^2 + y^2$ for $(x, y) \in \mathbb{R}^2$; the exact solution is given by

$$u := \begin{cases} r^2/2 - \ln(r) - 1/2, & r \geq 1, \\ 0 & \text{otherwise.} \end{cases}$$

Figure 1 illustrates the behavior of the pointwise error $\|u - u_h\|_{L^\infty(\Omega)}$ and *a posteriori* error estimator η_h with respect to degrees of freedom (DOFs) for the SIPG method. We observe that both error and estimator converge with the optimal rate DOFs^{-1}. This figure also depicts the convergence behavior of the individual estimators $\eta_i, i = 1, 2, 3, 4, 6$. Note that for this example η_5 is zero since $\chi = \chi_h = 0$. Figure 2 represents efficiency indices with respect to degrees of freedom (leftmost subfigure), adaptive mesh (center) and the discrete contact set (rightmost subfigure) at refinement level 10. In the contact region, the estimator should depend on the obstacle function χ while in the noncontact region it should be dictated by the load function f. Since the obstacle function is zero, we observe almost no mesh refinement in the contact zone.

Example 2. To illustrate the efficacy of adaptive refinement, we consider this example with nonconvex domain (Braess *et al.*, 2007). Therein, we have the following data:

$$\Omega = (-2, 2)^2 \setminus [0, 2) \times (-2, 0], \quad \chi = 0, \quad u = r^{2/3} \sin(2\theta/3)\gamma_1(r),$$

$$f = -r^{2/3} \sin(2\theta/3)\left(\frac{\gamma'_1(r)}{r} + \gamma''^{(}_1 r)\right) - \frac{4}{3} r^{-1/3} \sin(2\theta/3)\gamma'^{(}_1 r) - \gamma_2(r),$$

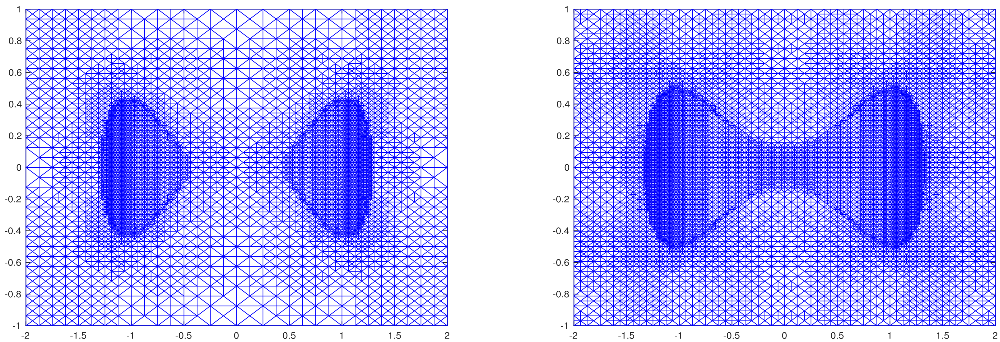

FIG. 6. Adaptive mesh for Example 3 for $f = 0$ and $f = -15$.

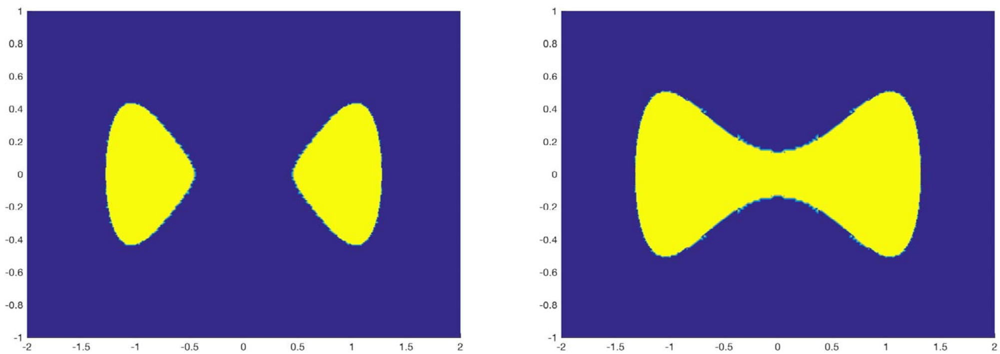

FIG. 7. Discrete active set (yellow part) for Example 3 for $f = 0$ and $f = -15$.

where

$$\gamma_1(r) = \begin{cases} 1, & \tilde{r} < 0, \\ -6\tilde{r}^5 + 15\tilde{r}^4 - 10\tilde{r}^3 + 1, & 0 \leq \tilde{r} < 1, \\ 0, & \tilde{r} \geq 1, \end{cases}$$

$$\gamma_2(r) = \begin{cases} 0, & r \leq \frac{5}{4}, \\ 1 & \text{otherwise}, \end{cases}$$

with $\quad \tilde{r} = 2(r - 1/4)$.

We report the numerical results for the linear as well as quadratic SIPG method for this example. For the quadratic finite element method, in the discrete admissible space, the constraints are imposed at the midpoints of the faces of the triangulation, i.e., $K_h^{(2)} = \{v_h \in V_h^{(2)} : v_h(m) \geq \chi(m) \;\; \forall\, m \in \varLambda_h\}$, where $V_h^{(2)}$ denotes the DG finite element space of the piecewise quadratic functions. The corresponding error estimator $\tilde{\zeta}_h$ is given by

$$\tilde{\zeta}_h^2 = |\log(h_{min})|\big(\tilde{\eta}_1 + \tilde{\eta}_2 + \tilde{\eta}_3\big) + \|(\chi - u_h^{(2)})^+\|_{L^\infty(\Omega)} + \|(u_h^{(2)} - \chi)^+\|_{L^\infty\{\bar{\sigma}_h<0\}},$$

where

$$\tilde{\eta}_1 = \max_{T\in\mathcal{T}_h} h_T^2\|\Delta u_h^{(2)} + f - \sigma_h\|_{L^\infty(T)}, \quad \tilde{\eta}_2 = \max_{e\in\mathcal{E}_h^i} h_e\|[\![\nabla u_h^{(2)}]\!]\|_{L^\infty(e)}, \quad \tilde{\eta}_3 = \|[\![u_h^{(2)}]\!]\|_{L^\infty(\mathcal{E}_h)},$$

with $u_h^{(2)}$ as the quadratic finite element approximation of u, and $\bar{\sigma}_h$ denotes the piece-wise constant approximation of σ_h. The behavior of the true error and the error estimators for the linear and quadratic SIPG method is depicted in the leftmost subfigure of Fig. 3. This figure ensures the optimal convergence (rate DOFs^{-1} using linear elements and DOFs$^{-3/2}$ using quadratic elements) of the error and the estimators together with the reliability of the estimator. The efficiency of the estimators is shown in the rightmost subfigure of Fig. 3. The convergence behavior of single estimators $\eta_i, i = 1, 2, 3, 4, 6$ for the linear SIPG method is depicted in Fig. 4 (leftmost subfigure). The adaptive mesh refinement and the discrete contact set at refinement level 26 are also reported in Fig. 4. From these figures we observe that the estimator captures the singular behavior of the solution very well. The mesh refinement near the free boundary is higher because of the large jump in gradients.

Example 3. In this example, we consider the following data from Nochetto *et al.* (2005):

$$\Omega = (-2, 2) \times (-1, 1), \quad \chi = 10 - 6(x^2 - 1)^2 - 20(r^2 - x^2), \quad r^2 = x^2 + y^2.$$

The exact solution is not known for this example. Figure 5 illustrates the behavior of the error estimator η_h, together with the individual estimators $\eta_i, i = 1, \ldots, 6$ for the load function $f = 0$ and $f = -15$. We clearly observe that in both cases the estimator converges with optimal rate (DOFs^{-1}). The adaptive mesh and discrete contact set at refinement level 13 for $f = 0$ and $f = -15$ are shown in Figs 6 and 7, respectively. The graph of the obstacle can be viewed as two hills connected by a saddle. As the load f increases, the change in the contact region can be observed from Fig. 7 and, as expected, we observe more refinement near the free boundaries.

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Conflict of interest

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