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Weighted estimates for bilinear fractional integral operator on the Heisenberg group

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ABSTRACT

In this article, we consider an analogue of Kenig and Stein's bilinear fractional integral operator on the Heisenberg group $\mathbb{H}^n$. We completely characterize exponents α, β and γ such that the operator is bounded from $L^p(\mathbb{H}^n, |x|^{\alpha p}) \times L^q(\mathbb{H}^n, |x|^{\beta q})$ to $L^r(\mathbb{H}^n, |x|^{-\gamma r})$. Also, analogous sharp results are obtained on the Euclidean space.

1. Introduction

Fractional integral operators are classical objects in analysis pertaining to the study of smoothness of functions, potential theory and embedding theorems. Recall that, for $0 < \lambda < n$, the fractional integral operator on Euclidean space, $\mathbb{R}^n$, is defined as follows

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$$I_\lambda f(x) = \int\limits_{\mathbb{R}^n} \frac{f(y)}{|x-y|^{n-\lambda}}\, dy, \quad x \in \mathbb{R}^n.$$

The operators I_λ are bounded off-diagonally, that is, $I_\lambda : L^p(\mathbb{R}^n) \to L^q(\mathbb{R}^n)$, where $1/q = 1/p - \lambda/n$, $1 < p < n/\lambda$. In 1958, Stein and Weiss introduced the following weighted Hardy–Littlewood–Sobolev inequality which is now commonly known as the Stein–Weiss inequality. We recall it here. The inequality

$$\left| \int\limits_{\mathbb{R}^n} \int\limits_{\mathbb{R}^n} \frac{\overline{f(x)} g(y)}{|x|^\alpha |x-y|^\lambda |y|^\beta}\, dx\, dy \right| \lesssim \|f\|_{L^p(\mathbb{R}^n)} \|g\|_{L^q(\mathbb{R}^n)}, \tag{1.1}$$

holds, where $1 < p, q < \infty$, $0 < \lambda < n$, $\alpha + \beta \geq 0$, $1/p + 1/q + (\alpha + \beta + \lambda)/n = 2$ with $\alpha < n/p'$, $\beta < n/q'$. Here, $p' := \frac{p}{p-1}$ and $q' := \frac{q}{q-1}$. Subsequently, Muckenhoupt and Wheeden in [13] extended the inequality (1.1) beyond power weights and characterized weights for which $I_\lambda : L^p(w^p) \to L^q(w^q)$, with $1/q = 1/p - \lambda/n$, $1 < p < n/\lambda$. The appropriate class of weights are denoted as $A_{p,q}$ weights.

The operator I_λ and its analogues are also investigated beyond the Euclidean setting. In this article, we are interested in bilinear analogue of I_λ on the Heisenberg group $\mathbb{H}^n$. Let us begin with the bilinear fractional integral operator BI_λ on $\mathbb{R}^n$ defined as

$$BI_\lambda(f,g)(x) = \int\limits_{\mathbb{R}^n} \frac{f(x-y)g(x+y)}{|y|^{n-\lambda}}\, dy, \ \ 0 < \lambda < n.$$

These operators are well studied, for example we refer to the works [4,9,12]. They also share similarities with the bilinear Hilbert transform of Lacey and Thiele (see [11]). It was proved in [9] that BI_λ is bounded from $L^p(\mathbb{R}^n) \times L^q(\mathbb{R}^n)$ to $L^r(\mathbb{R}^n)$ provided $1/r = 1/p + 1/q - \lambda/n > 0$ and $1 < p, q \leq \infty$, and also the expected weak type inequality holds if either p or q is 1. It is not difficult to see that using Hölder's inequality and weighted boundedness of I_λ, we can obtain that $BI_\lambda : L^p(w_1^p) \times L^q(w_2^q) \to L^r(w_1^r w_2^r)$ provided $1/r = 1/p + 1/q - \lambda/n$, $1 < r, s < \infty$ and $w_1^{p/s}, w_2^{q/s} \in A_{s,r}$, where $1/s = 1/p + 1/q$. However, the above approach is not useful when $r < 1$ and it was also pointed out in the influential work of Lerner et al. (see [10]) that linear Muckenhoupt classes are not the appropriate weights while studying these bilinear operators. In [10], multilinear $\mathcal{A}_{\vec{P}}$ weights are introduced in connection with the multilinear Hardy–Littlewood maximal operator and multilinear Calderón–Zygmund operators. Subsequently, Kabe Moen [12] has initiated the study of fractional multilinear weights and proved the following: BI_λ maps $L^p(w_1^p) \times L^q(w_2^q) \to L^r(w_1^r w_2^r)$ boundedly, when $1 < p, q < \infty$, $1/r = 1/p + 1/q - \lambda/n > 1$, and $\vec{w} \in \mathcal{A}_{p,q,r}$. Though it is not yet known whether the condition $\mathcal{A}_{p,q,r}$ is also necessary for the boundedness of BI_λ. But importantly, if we only consider power weights then the author in [8] has obtained both necessary and sufficient conditions on α and β such that BI_λ is bounded from $L^p(\mathbb{R}^n, |x|^{\alpha p}) \times L^q(\mathbb{R}^n, |x|^{\beta q})$ to $L^r(\mathbb{R}^n, |x|^{(\alpha+\beta)r})$. Let us state their result.

Theorem 1.1 *(Theorem 1, [8]). Let* $1 < p, q < \infty$, $0 < \lambda < n$, $\frac{1}{r} = \frac{1}{p} + \frac{1}{q} - \frac{\lambda}{n} > 0$, *and*

$$\alpha < \frac{n}{p'}, \quad \beta < \frac{n}{q'} \quad \text{and} \quad -\frac{n}{r} < \alpha + \beta. \tag{1.2}$$

Further, assume that

$$\text{(I) } \alpha \le n - \lambda, \quad \text{(II) } \beta \le n - \lambda, \quad \text{(III) } -n + \lambda \le \alpha + \beta. \tag{1.3}$$

Then, there exists a constant $K > 0$, *such that*

$$\| \, |x|^{(\alpha+\beta)} BI_\lambda(f, g)\|_{L^r(\mathbb{R}^n)} \le K \| |x|^\alpha f\|_{L^p(\mathbb{R}^n)} \| |x|^\beta g\|_{L^q(\mathbb{R}^n)}, \tag{1.4}$$

for all $f \in L^p(\mathbb{R}^n, |x|^{\alpha p})$ *and* $g \in L^q(\mathbb{R}^n, |x|^{\beta q})$.

It is also proved in [8] that the conditions (1.3) are necessary for (1.4) to hold. However, in the general case of exponents, the authors have only provided sufficient conditions on the exponents α, β, and γ such that the bilinear fractional operator BI_λ maps $L^p(\mathbb{R}^n, |x|^{\alpha p}) \times L^q(\mathbb{R}^n, |x|^{\beta q})$ to $L^r(\mathbb{R}^n, |x|^{-\gamma r})$.

Our primary goal in the present article is to obtain a complete characterization of α, β, and γ such that the bilinear fractional operator is bounded from $L^p(|x|^{\alpha p}) \times L^q(|x|^{\beta q})$ to $L^r(|x|^{-\gamma r})$ on the Heisenberg group, $\mathbb{H}^n$. Our methods also produce the analogous sharp characterization of exponents α, β and γ such that the bilinear fractional operator BI_λ is bounded from $L^p(\mathbb{R}^n, |x|^{\alpha p}) \times L^q(\mathbb{R}^n, |x|^{\beta q})$ to $L^r(\mathbb{R}^n, |x|^{-\gamma r})$. More precisely, we have the following.

Theorem 1.2 *(Characterization of power weights on Euclidean space). Let* $1 < p, q < \infty$, $0 < r < \infty$ *and* $\frac{1}{r} \le \frac{1}{p} + \frac{1}{q}$. *Let* $0 < \lambda < n$ *and*

$$\alpha < \frac{n}{p'}, \quad \beta < \frac{n}{q'} \quad \text{and} \quad \gamma < \frac{n}{r}. \tag{1.5}$$

Further, assume $\frac{1}{r} = \frac{1}{p} + \frac{1}{q} - \frac{\lambda - \alpha - \beta - \gamma}{n} > 0$.

Then, the following are equivalent:

(*a*) *There exists a constant* $K > 0$, *such that*

$$\| \, |x|^{-\gamma} BI_\lambda(f, g)\|_{L^r(\mathbb{R}^n)} \le K \| |x|^\alpha f\|_{L^p(\mathbb{R}^n)} \| |x|^\beta g\|_{L^q(\mathbb{R}^n)}, \tag{1.6}$$

for all $f \in L^p(\mathbb{R}^n, |x|^{\alpha p})$ *and* $g \in L^q(\mathbb{R}^n, |x|^{\beta q})$;

(*b*) *The exponents* α, β *and* γ *satisfy*

$$\text{(I) } -n+\lambda \le \beta+\gamma, \quad \text{(II) } -n+\lambda \le \gamma+\alpha, \quad \text{(III) } -n+\lambda \le \alpha+\beta, \quad \text{and} \quad \text{(IV) } \alpha+\beta+\gamma \ge 0. \tag{1.7}$$

We would like to remark that previously sufficient conditions on α, β, and γ for (1.6) were also obtained in the works [6] (Theorem 10.1) and in [1] (Theorem 4.4). Our result improves all of them.

We now prepare to state our results on the Heisenberg group $\mathbb{H}^n$, Theorem 1.3 and Theorem 1.4, below. Let us recall that the linear fractional integral operator $\mathcal{I}_\lambda$ on the Heisenberg group $\mathbb{H}^n$ is defined as

$$\mathcal{I}_\lambda f(x) := \int_{\mathbb{H}^n} f(xy^{-1}) \quad \frac{dy}{|y|^{Q-\lambda}}, \quad 0 < \lambda < Q,$$

where, $|y|$ denotes the Korányi norm of $y \in \mathbb{H}^n$ and Q is the homogeneous dimension of the group $\mathbb{H}^n$.

These operators have a long history, starting with the foundational work of Folland and Stein in [3], where the authors have proved Hardy–Littlewood–Sobolev inequality on the Heisenberg group, that is, $\mathcal{I}_\lambda : L^p(\mathbb{H}^n) \mapsto L^q(\mathbb{H}^n)$ with $1/q = 1/p - \lambda/Q, 1 < p < Q/\lambda$. The natural end-point boundedness $\mathcal{I}_\lambda : L^1(\mathbb{H}^n) \mapsto L^{Q/(Q-\lambda),\infty}(\mathbb{H}^n)$ was also obtained in [3]. In recent times, in [2], Frank and Lieb studied Hardy–Littlewood–Sobolev inequality on $\mathbb{H}^n$ and obtained the sharp constant and existence of unique optimizer in the Hardy–Littlewood–Sobolev inequality on $\mathbb{H}^n$. Also, we enlist the article [5] where analogues of Stein-Weiss inequality (1.1) are studied in the context of the Heisenberg group. It is thus very natural to study the aforementioned aspects for the bilinear analogues of $\mathcal{I}_\lambda$, and for that purpose, let us define bilinear fractional integral operator on $\mathbb{H}^n$.

Definition 1.1. For $0 < \lambda < Q$, the bilinear fractional integral operator B_λ on $\mathbb{H}^n$ is defined as follows

$$B_\lambda(f,g)(x) = \int_{\mathbb{H}^n} f(xy^{-1})\, g(xy)\, \frac{dy}{|y|^{Q-\lambda}}, \quad x \in \mathbb{H}^n.$$

On the Heisenberg group, our first result addresses the unweighted boundedness of the operator B_λ, therefore, we have the following extension of the result by Kenig and Stein to the Heisenberg group.

Theorem 1.3 *(Unweighted boundedness). Let* $0 < \lambda < Q$, $\frac{1}{r} = \frac{1}{p} + \frac{1}{q} - \frac{\lambda}{Q} > 0$, *and that* $f \in L^p(\mathbb{H}^n)$, $g \in L^q(\mathbb{H}^n)$, $1 \le p, q \le \infty$. *Then,*

(*a*) *If* $1 < p, q \le \infty$, *there exists a constant* $K > 0$ *such that*

$$\| B_\lambda(f,g)\|_{L^r(\mathbb{H}^n)} \le K \, \|f\|_{L^p(\mathbb{H}^n)} \|g\|_{L^q(\mathbb{H}^n)}, \tag{1.8}$$

for all $f \in L^p(\mathbb{H}^n)$ *and* $g \in L^q(\mathbb{H}^n)$.

(*b*) *If* $1 \leq p, q \leq \infty$ *with either* $p = 1$ *or* $q = 1$*, there exists a constant* $K > 0$ *such that*

$$\| B_\lambda(f, g)\|_{L^{r,\infty}(\mathbb{H}^n)} \leq K \|f\|_{L^p(\mathbb{H}^n)} \|g\|_{L^q(\mathbb{H}^n)}, \tag{1.9}$$

for all $f \in L^p(\mathbb{H}^n)$ *and* $g \in L^q(\mathbb{H}^n)$.

Now, we present the main result of this article concerning the sharp characterization of power weights for the boundedness of B_λ. Theorem 1.4 can also be realized as an extension of the Stein-Weiss inequality in the bilinear setting on the Heisenberg group. Precisely, we obtain the following:

Theorem 1.4 *(Characterization of power weights on Heisenberg group). Let* $1 < p, q < \infty$, $0 < r < \infty$ *and* $\frac{1}{r} \leq \frac{1}{p} + \frac{1}{q}$. *Let* $0 < \lambda < Q$ *and*

$$\alpha < \frac{Q}{p'}, \quad \beta < \frac{Q}{q'} \quad \text{and} \quad \gamma < \frac{Q}{r}. \tag{1.10}$$

Further, assume $\frac{1}{r} = \frac{1}{p} + \frac{1}{q} - \frac{\lambda-\alpha-\beta-\gamma}{Q} > 0$.

Then, the following are equivalent:

(*a*) *There exists a constant* $K > 0$*, such that*

$$\| \, |x|^{-\gamma} B_\lambda(f, g)\|_{L^r(\mathbb{H}^n)} \leq K \| |x|^\alpha f\|_{L^p(\mathbb{H}^n)} \| |x|^\beta g\|_{L^q(\mathbb{H}^n)}, \tag{1.11}$$

for all $f \in L^p(\mathbb{H}^n, |x|^{\alpha p})$ *and* $g \in L^q(\mathbb{H}^n, |x|^{\beta q})$;

(*b*) *The exponents* α, β *and* γ *satisfy*

$$\text{(I)} \ -Q+\lambda \leq \beta+\gamma, \quad \text{(II)} \ -Q+\lambda \leq \gamma+\alpha, \quad \text{(III)} \ -Q+\lambda \leq \alpha+\beta, \ \text{and} \ \text{(IV)} \ \alpha+\beta+\gamma \geq 0. \tag{1.12}$$

The conditions (1.10) are necessary for (1.11) to hold. The conditions $\alpha < Q/p'$ and $\beta < Q/q'$ ensure that $f, g \in L^1_{\text{loc}}(\mathbb{H}^n)$, whereas $\gamma < Q/r$ guarantees the local integrability of the weight $|x|^{-\gamma r}$. The necessity of $\frac{1}{r} \leq \frac{1}{p} + \frac{1}{q}$ will be addressed in Subsection 4.2.

We are not aware of any work, even in the linear case, where counterexamples are created in order to have a complete characterization of exponents for Stein-Weiss type inequalities on $\mathbb{H}^n$. As far as we know, this is the first time where counterexamples are constructed on $\mathbb{H}^n$ to obtain necessary conditions on the exponents so that B_λ is bounded from $L^p(\mathbb{H}^n, |x|^{\alpha p}) \times L^q(\mathbb{H}^n, |x|^{\beta q})$ to $L^r(\mathbb{H}^n, |x|^{-\gamma r})$. In the linear case, Theorem 1.1 in [5] addresses only sufficient conditions on the power weights for $\mathcal{I}_\lambda$. Moreover, our counterexamples can be modified for the linear case but we left that for interested readers.

There are essentially two major difficulties in proving Theorem 1.4. Firstly, the singularity of the kernel, namely $|y|^{-Q+\lambda}\delta(x \cdot y)$, is spread along the diagonal. On the other hand, while constructing counterexamples, it requires a better understanding of the precise description of the sets of the form $x \cdot supp(f) \cap supp(g) \cdot x^{-1}$, which demands delicate analysis due to the non-commutative nature of the Heisenberg group. We will provide the proof of Theorem 1.4 and skip the proof of Theorem 1.2, since the proof of the latter is relatively simpler and follows from the proof of Theorem 1.4 with some obvious changes.

In the next section, we briefly recall the necessary preliminaries to prove our main results. Section 4 is dedicated to the proof of our main result, Theorem 1.4. Throughout this article, we write $A \lesssim B$ and $B \gtrsim A$ to abbreviate $A \leq CB$ for some constant C independent of A and B, and $A \simeq B$ means both $A \lesssim B$ and $A \gtrsim B$. For $E, F \subset \mathbb{H}^n$, the notation $E \cdot F$ represents the set $\{x \cdot y : x \in E, y \in F\}$, and $E^2 = E \cdot E$. As usual, $L^p(X, w)$ denotes the space of all measurable functions on X such that $\int_X |f|^p w < \infty$.

2. Preliminaries

The Heisenberg group, $\mathbb{H}^n$, is the two step nilpotent Lie group with underlying manifold $\mathbb{C}^n \times \mathbb{R}$ associated with the group law

$$(z,t) \cdot (w,s) := \left(z + w, t + s + \frac{1}{2}\Im(z.\bar{w}) \right), \quad \text{for all} \quad (z,t), (w,s) \in \mathbb{H}^n. \tag{2.1}$$

We have a family of non-isotropic dilations defined by $\delta_r(z,t) := (rz, r^2 t)$, for all $(z,t) \in \mathbb{H}^n$, for every $r > 0$. The Korányi norm on $\mathbb{H}^n$ is defined by

$$|(z,t)| := \left(\|z\|^4 + t^2 \right)^{\frac{1}{4}}, \quad (z,t) \in \mathbb{H}^n,$$

which is homogeneous of degree 1, that is $|\delta_r(z,t)| = r\,|(z,t)|$. Here, $\|z\|$ denotes the Euclidean norm of $z \in \mathbb{C}^n$. The Haar measure on $\mathbb{H}^n$ coincides with the Lebesgue measure $dzdt$. Let $B(0,r) := \{(z,t) \in \mathbb{H}^n : |(z,t)| < r\}$ be the ball of radius r with respect to Korányi norm. One has its measure $|B(0,r)| = C_Q\, r^Q$, where $Q = (2n+2)$ is known as the homogeneous dimension of $\mathbb{H}^n$. The convolution of f with g on $\mathbb{H}^n$ is defined by

$$f * g\,(x) = \int_{\mathbb{H}^n} f(xy^{-1}) g(y) dy, \quad x \in \mathbb{H}^n.$$

We denote the Euclidean convolution of f and g on $\mathbb{R}^{2n+1}$ by

$$f *_e g(x) := \int_{\mathbb{R}^{2n+1}} f(y) g(x-y)\, dy, \quad x \in \mathbb{R}^{2n+1}.$$

Most of these notions are very standard and we refer to [14] for more details. The following simple observations will be important for our purpose.

Lemma 2.1. *Let $x = (z, t)$ and $y = (w, s) \in \mathbb{H}^n$ with $2|y| \leq |x|$. Then there is a universal constant κ such that*

$$|(z + w, s + t)| \geq \kappa |x|. \tag{2.2}$$

Proof. Let $\|w\|$ denote the Euclidean norm of $w \in \mathbb{C}^n$. Since $2|y| \leq |x|$ so $\|w\| \leq |x|/2$ and $|s| \leq (|x|/2)^2$. If $\|z\| \geq |x|/2^{1/4}$ then

$$\begin{aligned}
|(z + w, t + s)|^4 &= \|z + w\|^4 + (t + s)^2 \\
&\geq (\|z\| - \|w\|)^4 + (t + s)^2 \\
&\geq (2^{-1/4}|x| - \|w\|)^4 \\
&\geq \left(\frac{2^{3/4} - 1}{2}\right)^4 |x|^4,
\end{aligned}$$

whereas if $|t| \geq |x|^2/2^{1/2}$ then

$$|(z + w, t + s)|^4 \geq (t + s)^2 \geq \left(\frac{2^{1/2} - 1/2}{2}\right)^2 |x|^4.$$

Therefore, (2.2) holds with $\kappa = (2^{3/4} - 1)/2$. □

The following bilinear interpolation theorem will be very useful for our purpose.

Theorem 2.2 *([7], Theorem 3 [9]). Suppose that a bilinear operator $T : L^{p_i,1} \times L^{q_i,1} \to L^{r_i,\infty}$, where $0 < p_i, q_i \leq \infty$, $0 < r_i \leq \infty$, for three points $\left(\frac{1}{p_i}, \frac{1}{q_i}\right)$, $i = 1, 2, 3$ in $\mathbb{R}^2$, that are non-collinear. Suppose, further, that there are $\theta_0, \theta_1, \theta_2 \in \mathbb{R}$ with $\theta_1, \theta_2 > 0$ so that $\frac{1}{r_i} = \theta_0 + \frac{\theta_1}{p_i} + \frac{\theta_2}{q_i}$, $i = 1, 2, 3$. Then,*

$$T : L^p \times L^q \to L^r,$$

provided $1 \leq p, q \leq \infty$, $\frac{1}{p} + \frac{1}{q} \geq \frac{1}{r}$ and $\left(\frac{1}{p}, \frac{1}{q}, \frac{1}{r}\right)$ lies in the open convex hull of $\left(\frac{1}{p_i}, \frac{1}{q_i}, \frac{1}{r_i}\right)$.

3. Proof of Theorem 1.3

We first prove part (b) in Theorem 1.3 when $p = q = 1$. This is the key estimate for proving Theorem 1.3. Let us introduce the following operators which are pieces of the operator B_λ.

$$B(f, g)(x) = \int\limits_{|y| \simeq 1} f(xy^{-1}) g(xy) dy,$$

and

$$B_k(f,g)(x) = \int\limits_{|y|\simeq 2^{-k}} f(xy^{-1})g(xy)dy.$$

Our primary ingredients to prove Theorem 1.3 are the following bounds for the pieces B_k.

Lemma 3.1. *The following statements hold:*

(*i*) $\|B(f,g)\|_{L^{1/2}(\mathbb{H}^n)} \lesssim \|f\|_{L^1(\mathbb{H}^n)} \|g\|_{L^1(\mathbb{H}^n)}$.
(*ii*) $\|B(f,g)\|_{L^1(\mathbb{H}^n)} \lesssim \|f\|_{L^1(\mathbb{H}^n)} \|g\|_{L^1(\mathbb{H}^n)}$.
(*iii*) $\|B_k(f,g)\|_{L^{1/2}(\mathbb{H}^n)} \lesssim 2^{-Qk} \|f\|_{L^1(\mathbb{H}^n)} \|g\|_{L^1(\mathbb{H}^n)}$.
(*iv*) $\|B_k(f,g)\|_{L^1(\mathbb{H}^n)} \lesssim \|f\|_{L^1(\mathbb{H}^n)} \|g\|_{L^1(\mathbb{H}^n)}$.

Proof of Lemma 3.1. The statements (iii) and (iv) follow from (i) and (ii), respectively, by scaling: Set $(\delta_r f)(x) = f(\delta_r x)$. Let $r = 2^{-k}$, then

$$\delta_{r^{-1}} \left[B(\delta_r f, \delta_r g)\right] = r^{-Q}\, B_k(f,g), \quad Q = 2n+2.$$

We assume, without loss of generality, that $f \geq 0$, $g \geq 0$. We begin with proving (i). For $a \in \mathbb{Z}^{2n+1}$, let $Q_a = a \cdot Q_0$, where $Q_0 = [0,1)^{2n+1}$. Then,

$$\begin{aligned}
&\|B(f,g)\,\chi_{Q_a}\|_{L^{1/2}(\mathbb{H}^n)} \\
&\qquad \leq \int\limits_{Q_a} B(f,g)(x)dx \leq \int\limits_{x\in Q_a} \int\limits_{y\in B(0,1)} f(xy^{-1})\, g(xy)\, dy\, dx \\
&\qquad \overset{y\to y^{-1}\cdot x}{=} \int\limits_{Q_a} \int\limits_{y\in x\cdot B(0,1)} f(y)\, g(xy^{-1}x)\, dy\, dx \\
&\qquad \leq \int\limits_{Q_a} \int\limits_{y\in Q_a\cdot B(0,1)} f(y)\, g(xy^{-1}x)\, dy\, dx \\
&\qquad = \int\limits_{y\in Q_a\cdot B(0,1)} f(y) \int\limits_{x\in Q_a} g(xy^{-1}x)\, dx\, dy \\
&\qquad \overset{x\to x\cdot y}{=} \int\limits_{y\in Q_a\cdot B(0,1)} f(y) \int\limits_{x\in Q_a\cdot y^{-1}} g(x^2y)\, dx\, dy \qquad (3.1) \\
&\qquad = \int\limits_{y\in Q_a\cdot B(0,1)} f(y) \int\limits_{x\in Q_a\cdot y^{-1}} g((2x)\cdot y)\, dx\, dy \quad \left(\text{since}, x^2 = x\cdot x = 2x\right)
\end{aligned}$$

$$\overset{x\to x/2}{=} 2^{-2n-1} \int\limits_{y\,\in\, Q_a\cdot B(0,1)} f(y) \int\limits_{x\in 2(Q_a\cdot y^{-1})} g(xy)\,dx\,dy,$$

$$\leq 2^{-2n-1} \int\limits_{y\,\in\, Q_a\cdot B(0,1)} f(y) \int\limits_{x\in (Q_a\cdot y^{-1})^2} g(xy)\,dx\,dy,$$

$$\overset{x\to x\cdot y^{-1}}{=} 2^{-2n-1} \int\limits_{y\,\in\, Q_a\cdot B(0,1)} f(y) \int\limits_{x\in (Q_a\cdot y^{-1})^2\cdot y} g(x)\,dx\,dy$$

$$\leq 2^{-2n-1} \int\limits_{y\,\in\, Q_a^*} f(y)\,dy \int\limits_{x\in Q_a^*} g(x)\,dx,$$

where $Q_a^* := a\cdot Q_0\cdot B(0,1)\cdot Q_0^{-1}.Q_0 \subset a\cdot\big([-4,4]^{2n}\times[-16,16]\big)$.

Observe that Q_a and Q_a^* have bounded overlapping and covers whole of $\mathbb{H}^n$. Indeed, let $(z,t)=(x+iy,t)\in\mathbb{H}^n$. Choose an $a'\in\mathbb{Z}^{2n}$ such that $a'\leq(x,y)<a'+(1,\cdots,1)$, component wise. Having chosen a', choose integer, say a_{2n+1} such that $t-\frac{1}{2}\Im(a'.\bar{z})\in[a_{2n+1},a_{2n+1}+1)$. Then we have an $a=(a',a_{2n+1})\in\mathbb{Z}^{2n+1}$ such $a^{-1}\cdot(z,t)\in Q_0$. So $\mathbb{H}^n=\cup_{a\in\mathbb{Z}^{2n+1}}Q_a$. For bounded overlapping of Q_a, let us fix an $a\in\mathbb{Z}^{2n+1}$ and consider $\tilde{a}\in\mathbb{Z}^{2n+1}$ such that

$$a\cdot Q_0\cap\tilde{a}\cdot Q_0\neq\emptyset.$$

Equivalently, $Q_0\cap a^{-1}\tilde{a}\cdot Q_0\neq\emptyset$. Let $(z,t)=a^{-1}\cdot\tilde{a}$ and $(w,s)\in Q_0$. Let $\|z\|$ denote the Euclidean norm of $z\in\mathbb{C}^n$. Then, if $\|z\|>2\sqrt{n}$, then $(z,t)\cdot(w,s)=(z+w,t+s+\frac{1}{2}\Im(z.\bar{w}))\notin Q_0$. If $\|z\|\leq 2\sqrt{n}$ but $|t|>2n+2$, then $|t+s+\frac{1}{2}\Im(z.\bar{w})|\geq 2(n+1)-(n+1)=n+1$. So, again $(z,t)\cdot(w,s)\notin Q_0$. If $\|z\|\leq 2\sqrt{n}$ and $|t|\leq 2n+2$, then for fixed a, we are, at most, counting the number of lattice points $\tilde{a}\in\mathbb{Z}^{2n+1}$ such that $\tilde{a}\in a\cdot B(0,4\sqrt{n})$ which is, clearly, $\simeq n^{Q/2}$. Similarly, we can argue for the sets Q_a^*.

So,

$$\begin{aligned}
\|B(f,g)\|_{L^{1/2}(\mathbb{H}^n)}^{1/2} &\simeq \sum_{a\in\mathbb{Z}^{2n+1}} \|B(f,g)\,\chi_{Q_a}\|_{L^{1/2}(\mathbb{H}^n)}^{1/2}\\
&\lesssim \sum_{a\in\mathbb{Z}^{2n+1}} \|f\,\chi_{Q_a^*}\|_{L^1(\mathbb{H}^n)}^{1/2}\,\|g\,\chi_{Q_a^*}\|_{L^1(\mathbb{H}^n)}^{1/2}\\
&\leq \left(\sum_{a\in\mathbb{Z}^{2n+1}} \|f\,\chi_{Q_a^*}\|_{L^1(\mathbb{H}^n)}\right)^{1/2} \left(\sum_{a\in\mathbb{Z}^{2n+1}} \|g\,\chi_{Q_a^*}\|_{L^1(\mathbb{H}^n)}\right)^{1/2}\\
&\simeq \|f\|_{L^1(\mathbb{H}^n)}\,\|g\|_{L^1(\mathbb{H}^n)},
\end{aligned}$$

establishing (i).

Next, (ii) follows from using the same set of change of variables and Fubini's theorem as in (3.1). Thus, completing the proof of Lemma 3.1. □

Returning to the proof of part (b) in Theorem 1.3, when $p = q = 1$, $\frac{1}{r} = 2 - \frac{\lambda}{Q}$. Let $\|f\|_{L^1(\mathbb{H}^n)} = \|g\|_{L^1(\mathbb{H}^n)} = 1$. Let us decompose the operator B_λ as

$$\begin{aligned} B_\lambda(f,g)(x) &\simeq \sum_{k\in\mathbb{Z}} 2^{k(Q-\lambda)} B_k(f,g)(x) \\ &= \sum_{k\le k_0} + \sum_{k\ge k_0} =: F_1 + F_2. \end{aligned}$$

For F_1 and F_2 we have, using (iii) and (iv) in Lemma 3.1, the following

$$\|F_1\|_{L^1(\mathbb{H}^n)} \le \sum_{k\le k_0} 2^{k(Q-\lambda)} \|B_k(f,g)\|_{L^1(\mathbb{H}^n)} \lesssim \sum_{k\le k_0} 2^{k(Q-\lambda)} \simeq 2^{k_0(Q-\lambda)}$$

and

$$\|F_2\|_{L^{1/2}(\mathbb{H}^n)}^{1/2} \le \sum_{k>k_0} 2^{\frac{k(Q-\lambda)}{2}} \|B_k(f,g)\|_{L^{1/2}(\mathbb{H}^n)}^{1/2} \lesssim \sum_{k>k_0} 2^{\frac{k(Q-\lambda)}{2}} 2^{-\frac{kQ}{2}} \simeq 2^{-\frac{\lambda}{2}k_0}.$$

Then, for all $t > 0$,

$$\begin{aligned} |\{B_\lambda(f,g) > t\}| &\le \left|\left\{F_1 > \frac{Ct}{2}\right\}\right| + \left|\left\{F_2 > \frac{Ct}{2}\right\}\right| \\ &\lesssim \frac{\|F_1\|_{L^1(\mathbb{H}^n)}}{t} + \frac{\|F_2\|_{L^{1/2}(\mathbb{H}^n)}^{1/2}}{t^{1/2}} \\ &\lesssim \frac{2^{k_0(Q-\lambda)}}{t} + \frac{2^{-\frac{\lambda}{2}k_0}}{t^{1/2}}. \end{aligned}$$

Optimising the right hand side of the above with respect to k_0, that is, choosing k_0 such that $2^{k_0(Q-\lambda)}/t = 2^{-\frac{\lambda}{2}k_0}/t^{1/2}$, yields the desired estimate

$$|\{B_\lambda(f,g) > t\}| \lesssim \frac{1}{t^r}, \quad \frac{1}{r} = 2 - \frac{\lambda}{Q},$$

which settles the proof of part (b), in Theorem 1.3 when $p = q = 1$. To finish part (b), observe that, if $g \in L^\infty(\mathbb{H}^n)$, we have

$$B_\lambda(f,g)(x) \le \|g\|_{L^\infty(\mathbb{H}^n)} \left(f * \frac{1}{|y|^{Q-\lambda}}\right)(x), \quad x \in \mathbb{H}^n.$$

So, from linear fractional integration on $\mathbb{H}^n$,

$$\begin{aligned} \|B_\lambda(f,g)\|_{L^{r,\infty}(\mathbb{H}^n)} &\le \|g\|_{L^\infty(\mathbb{H}^n)} \left\| f * \frac{1}{|y|^{Q-\lambda}} \right\|_{L^{r,\infty}(\mathbb{H}^n)} \\ &\lesssim \|g\|_{L^\infty(\mathbb{H}^n)} \|f\|_{L^1(\mathbb{H}^n)}, \end{aligned}$$

if $\frac{1}{r} = 1 - \frac{\lambda}{Q}$ which is, indeed, the situation when $p = 1, q = \infty$. If $g \in L^q(\mathbb{H}^n)$, $1 < q < \infty$, then (b) follows from linear interpolation, by fixing $f \in L^1(\mathbb{H}^n)$, and using the boundedness of $B_\lambda : L^1(\mathbb{H}^n) \times L^1(\mathbb{H}^n) \to L^{r,\infty}(\mathbb{H}^n)$ (the key estimate) and that of $B_\lambda : L^1(\mathbb{H}^n) \times L^\infty(\mathbb{H}^n) \to L^{r,\infty}(\mathbb{H}^n)$ (the sub case discussed above).

The part (a) is obtained from (b), by applying bilinear interpolation, Theorem 2.2. For the sake of completeness, we briefly explain this point from [9]. Consider the open convex set in $\mathbb{R}^2$,

$$C := \left\{(x, y) \in \mathbb{R}^2 : x + y > \frac{\lambda}{Q}, 0 < x, y < 1\right\}, \quad 0 < \lambda < Q.$$

Observe that C is precisely the union of interior of triangles whose vertices lie on different sides of the square $[0, 1]^2$ intersected with the closure of C. Thus, by symmetry and part (b), it suffices to establish the "weak-type" inequality for $p = \infty$ and $1 < q < \infty$. For $f \in L^\infty(\mathbb{H}^n)$, we have

$$B_\lambda(f, g)(x) \leq \|f\|_{L^\infty(\mathbb{H}^n)} \left(g * \frac{1}{|y|^{Q-\lambda}}\right)(x), \quad x \in \mathbb{H}^n.$$

Therefore,

$$\begin{aligned} \| B_\lambda(f, g)\|_{L^r(\mathbb{H}^n)} &\leq \|f\|_{L^\infty(\mathbb{H}^n)} \left\|g * \frac{1}{|y|^{Q-\lambda}}\right\|_{L^r(\mathbb{H}^n)} \\ &\lesssim \|f\|_{L^\infty(\mathbb{H}^n)} \|g\|_{L^q(\mathbb{H}^n)}, \end{aligned}$$

where in the last inequality we have used the strong type boundedness of $\mathcal{I}_\lambda$ on $\mathbb{H}^n$, since in this case we have $\frac{1}{r} = \frac{1}{q} - \frac{\lambda}{Q} > 0$, $1 < q < \infty$.

4. Characterization of power weights

In this section, we provide the proof of Theorem 1.4. Let us start with proving the sufficient part. In contrast with the proof of [8], we provide a unified approach to deal with the operator B_λ irrespective of the signs of α and β.

4.1. Proof of the sufficient part

Proof of (1.12) $\implies$ (1.11): Since $|B_\lambda(f, g)| \leq B_\lambda(|f|, |g|)$, throughout this proof we will assume that f, g are non-negative functions. First we will prove the following weak type estimate

$$\|S(f, g)\|_{L^{r,\infty}(\mathbb{H}^n)} \leq K\|f\|_{L^p(\mathbb{H}^n)}\|g\|_{L^q(\mathbb{H}^n)}, \tag{4.1}$$

where,

$$S(f,g) = S_{\alpha,\beta,\gamma}(f,g)(x) := |x|^{-\gamma} \int\limits_{\mathbb{H}^n} \frac{f(xy^{-1})}{|xy^{-1}|^{\alpha}} \frac{g(xy)}{|xy|^{\beta}} \frac{dy}{|y|^{Q-\lambda}}.$$

Once the proof of (4.1) is complete, as an application of bilinear interpolation, we can conclude the strong type estimate $\|S(f,g)\|_{L^r(\mathbb{H}^n)} \leq K\|f\|_{L^p(\mathbb{H}^n)}\|g\|_{L^q(\mathbb{H}^n)}$, which is equivalent to estimate (1.11).

Our proof involves delicate analysis of singularities of the operator S. We decompose the operator S into three parts, namely, $S(f,g)(x) = \sum_{i=1}^{3} \mathcal{J}_i(x)$, where

$$\mathcal{J}_1(x) := |x|^{-\gamma} \int\limits_{y\in B(0,\frac{|x|}{2})\cdot x} \frac{f(xy^{-1})}{|xy^{-1}|^{\alpha}} \frac{g(xy)}{|xy|^{\beta}} \frac{dy}{|y|^{Q-\lambda}},$$

$$\mathcal{J}_2(x) := |x|^{-\gamma} \int\limits_{y\in x^{-1}\cdot B(0,\frac{|x|}{2})} \frac{f(xy^{-1})}{|xy^{-1}|^{\alpha}} \frac{g(xy)}{|xy|^{\beta}} \frac{dy}{|y|^{Q-\lambda}},$$

$$\mathcal{J}_3(x) := |x|^{-\gamma} \int\limits_{\mathbb{H}^n\setminus\left[B(0,\frac{|x|}{2})\cdot x \,\bigcup\, x^{-1}\cdot B(0,\frac{|x|}{2})\right]} \frac{f(xy^{-1})}{|xy^{-1}|^{\alpha}} \frac{g(xy)}{|xy|^{\beta}} \frac{dy}{|y|^{Q-\lambda}}.$$

Estimate for $\mathcal{J}_1$: Let $y \in B(0,\frac{|x|}{2})\cdot x$, then $|xy| \simeq |x|$. Indeed, for such y, one has $y = \xi\cdot x$ for some $\xi \in B(0,\frac{|x|}{2})$. Further, observe that $xy = x\cdot\xi\cdot x = 2x+\xi$. So, in view of Lemma 2.1, $|xy| = |2x+\xi| \gtrsim |x|$. Moreover, $|xy| \leq \frac{5}{2}|x|$. Again, observe that $|y| \simeq |x|$. Incorporating these estimates, we obtain

$$\mathcal{J}_1(x) \simeq |x|^{-\gamma-\beta-Q+\lambda} \int\limits_{y\in B(0,\frac{|x|}{2})\cdot x} \frac{f(xy^{-1})g(xy)}{|xy^{-1}|^{\alpha}}\,dy. \tag{4.2}$$

Let us consider the case when $-Q+\lambda < \beta+\gamma$. This together with $\alpha < Q/p'$ ensures that there exists $\mu > 0$ such that $Q(1-\frac{1}{p}-\frac{1}{q}) < \mu < Q(1-\frac{1}{p})$ and $\alpha < \mu < \alpha+\beta+\gamma+Q-\lambda$. Indeed, observe that $Q(1-\frac{1}{p}) > \alpha$ and $\alpha+\beta+\gamma+Q-\lambda > Q\left(1-\frac{1}{p}-\frac{1}{q}\right)$. The latter inequality being equivalent to $\frac{Q}{r} > 0$, so the two intervals intersect.

Therefore,

$$\begin{aligned}\mathcal{J}_1(x) &\simeq |x|^{-\gamma-\beta-Q+\lambda} \int\limits_{y\in B(0,\frac{|x|}{2})\cdot x} |xy^{-1}|^{\mu-\alpha}\frac{f(xy^{-1})g(xy)}{|xy^{-1}|^{\mu}}\,dy \\ &\lesssim |x|^{-\gamma-\beta-Q+\lambda+\mu-\alpha} \int\limits_{y\in B(0,\frac{|x|}{2})\cdot x} \frac{f(xy^{-1})g(xy)}{|xy^{-1}|^{\mu}}\,dy \\ &\overset{y\to y^{-1}\cdot x}{\lesssim} |x|^{-\gamma-\beta-Q+\lambda+\mu-\alpha} \int\limits_{\mathbb{R}^{2n+1}} \frac{f(y)g(xy^{-1}x)}{|y|^{\mu}}\,dy\end{aligned}$$

$$= |x|^{-\gamma-\beta-Q+\lambda+\mu-\alpha} \int\limits_{\mathbb{R}^{2n+1}} \frac{f(y)g(2x+y^{-1})}{|y|^{\mu}}\, dy, \quad (\text{since } x\, y^{-1} x = 2x + y^{-1}),$$

$$= |x|^{-\gamma-\beta-Q+\lambda+\mu-\alpha} \left(\frac{f}{|\cdot|^{\mu}} *_e g \right)(2x).$$

Set $\frac{1}{s} = \frac{1}{p} + \frac{1}{q} - 1 + \frac{\mu}{Q}$, then $\frac{1}{r} = \frac{1}{s} + \frac{\alpha+\beta+\gamma+Q-\mu-\lambda}{Q}$. This implies

$$\|\mathcal{J}_1\|_{L^{r,\infty}(\mathbb{H}^n)} \lesssim \| |x|^{-\gamma-\beta-Q+\lambda+\mu-\alpha} \|_{L^{\frac{Q}{\alpha+\beta+\gamma+Q-\lambda-\mu},\infty}(\mathbb{R}^{2n+1})} \left\| \left(\frac{f}{|\cdot|^{\mu}} *_e g \right) \right\|_{L^{s,\infty}(\mathbb{R}^{2n+1})}.$$

Define $\frac{1}{t} := \frac{1}{p} + \frac{\mu}{Q}$, then $\frac{1}{s} = \frac{1}{t} + \frac{1}{q} - 1$. By Young's inequality, we obtain

$$\begin{aligned} \|\mathcal{J}_1\|_{L^{r,\infty}(\mathbb{H}^n)} &\lesssim \left\| \frac{f}{|\cdot|^{\mu}} \right\|_{L^{t,\infty}(\mathbb{R}^{2n+1})} \|g\|_{L^q(\mathbb{R}^{2n+1})} \\ &\lesssim \|f\|_{L^p(\mathbb{H}^n)} \|g\|_{L^q(\mathbb{H}^n)}, \end{aligned}$$

where the last inequality follows from Hölder's inequality for the weak-type spaces. This completes the case $-Q + \lambda < \beta + \gamma$.

We are left with the case when $\beta + \gamma = -Q + \lambda$. As a consequence of the condition $\alpha + \beta + \gamma \geq 0$, we obtain $\alpha \geq Q - \lambda > 0$. Now, (4.2) implies

$$\mathcal{J}_1(x) \lesssim \int\limits_{y \in B(0, \frac{|x|}{2}) \cdot x} \frac{f(xy^{-1})g(xy)}{|xy^{-1}|^{\alpha}}\, dy \lesssim \left(\frac{f}{|\cdot|^{\alpha}} *_e g \right)(2x).$$

Observe that in this case $\frac{1}{r} = \frac{1}{p} + \frac{1}{q} - 1 + \frac{\alpha}{Q}$. Define $\frac{1}{t} := \frac{1}{p} + \frac{\alpha}{Q}$, therefore $\frac{1}{r} = \frac{1}{t} + \frac{1}{q} - 1$. Again, by Young's inequality and Hölder's inequality for the weak-type spaces, we obtain

$$\begin{aligned} \|\mathcal{J}_1\|_{L^{r,\infty}(\mathbb{R}^{2n+1})} &\lesssim \left\| \left(\frac{f}{|\cdot|^{\alpha}} *_e g \right) \right\|_{L^{r,\infty}(\mathbb{R}^{2n+1})} \\ &\lesssim \left\| \frac{f}{|\cdot|^{\alpha}} \right\|_{L^{t,\infty}(\mathbb{R}^{2n+1})} \|g\|_{L^q(\mathbb{R}^{2n+1})} \lesssim \|f\|_{L^p(\mathbb{H}^n)} \|g\|_{L^q(\mathbb{H}^n)}. \end{aligned}$$

Estimate for $\mathcal{J}_2$: This case is similar to $\mathcal{J}_1$, so we skip it.

Estimate for $\mathcal{J}_3$: Let us denote the set $\mathbb{H}^n \setminus \left[B(0, \frac{|x|}{2}) \cdot x \bigcup x^{-1} \cdot B(0, \frac{|x|}{2}) \right]$ by G_x. One can decompose $\mathcal{J}_3$ as follows:

$$\begin{aligned} &\mathcal{J}_3(x) \leq \mathcal{J}_{31}(x) + \mathcal{J}_{32}(x), \text{ where,} \\ &\qquad \mathcal{J}_{31}(x) := |x|^{-\gamma} \int\limits_{\{y : |y| \geq 2|x|\}} \frac{f(xy^{-1})}{|xy^{-1}|^{\alpha}} \frac{g(xy)}{|xy|^{\beta}} \frac{dy}{|y|^{Q-\lambda}}, \end{aligned}$$

$$\mathcal{J}_{32}(x) := |x|^{-\gamma} \int\limits_{\{y\in G_x : |y|<2|x|\}} \frac{f(xy^{-1})}{|xy^{-1}|^{\alpha}} \frac{g(xy)}{|xy|^{\beta}} \frac{dy}{|y|^{Q-\lambda}}.$$

Observe that for y such that $|y| \geq 2|x|$, we have $|xy^{-1}| \simeq |y|$ and $|xy| \simeq |y|$. First, let us consider the case when $-Q+\lambda = \alpha+\beta$. Therefore,

$$\mathcal{J}_{31}(x) \lesssim |x|^{-\gamma}(f *_e g)(2x).$$

We also have $\gamma \geq -\alpha-\beta = Q-\lambda > 0$. Moreover, $\frac{1}{r} = \frac{1}{p} + \frac{1}{q} - 1 + \frac{\gamma}{Q}$. So, using Hölder's inequality for the weak-type spaces and Young's convolution inequality subsequently, we obtain $\|\mathcal{J}_{31}\|_{L^{r,\infty}(\mathbb{H}^n)} \lesssim \|f\|_{L^p(\mathbb{H}^n)}\|g\|_{L^q(\mathbb{H}^n)}$, which is the required estimate.

Next, when $-Q+\lambda < \alpha+\beta$ then from the condition $\gamma < Q/r$, we can ensure that $\lambda-\alpha-\beta < Q(\frac{1}{p}+\frac{1}{q})$. Hence, choosing $\mu > 0$ such that $\mu \in (\lambda-\alpha-\beta-\gamma, Q(\frac{1}{p}+\frac{1}{q})) \cap (\lambda-\alpha-\beta, Q)$, we conclude the following

$$\begin{aligned}
\mathcal{J}_{31}(x) &\simeq |x|^{-\gamma} \int\limits_{\{y:|y|\geq 2|x|\}} \frac{f(xy^{-1})}{|y|^{\alpha}} \frac{g(xy)}{|y|^{\beta}} \frac{dy}{|y|^{Q-\lambda}}, \\
&= |x|^{-\gamma} \int\limits_{\{y:|y|\geq 2|x|\}} \frac{|y|^{-\alpha-\beta+\lambda-\mu}}{|y|^{Q-\mu}} f(xy^{-1})g(xy)\,dy \\
&\lesssim |x|^{-\gamma-\alpha-\beta+\lambda-\mu} B_{\mu}(f,g)(x). \qquad (4.3)
\end{aligned}$$

Define $\frac{1}{s} := \frac{1}{p} + \frac{1}{q} - \frac{\mu}{Q}$, then it is trivial to see that $\frac{1}{r} = \frac{1}{s} + \frac{\alpha+\beta+\gamma+\mu-\lambda}{Q}$. Denote $h(x) = |x|^{-\gamma-\alpha-\beta-\mu+\lambda}$. Using Hölder's inequality for weak-type spaces, we obtain

$$\|\mathcal{J}_{31}\|_{L^{r,\infty}(\mathbb{H}^n)} \lesssim \|h\|_{L^{\frac{Q}{\alpha+\beta+\gamma+\mu-\lambda},\infty}} \|B_{\mu}(f,g)\|_{L^{s,\infty}} \lesssim K_{\alpha,\beta,\gamma,Q,\lambda}\|f\|_{L^p(\mathbb{H}^n)}\|g\|_{L^q(\mathbb{H}^n)},$$

where we have used Theorem 1.3 in the last inequality. This completes the estimate for $\mathcal{J}_{31}$.

To estimate $\mathcal{J}_{32}$, observe that for points $y \in G_x$ with $|y| < 2|x|$, we have $|xy^{-1}| = |yx^{-1}| \simeq |x|$ and $|xy| \simeq |x|$. Assuming that $\alpha+\beta+\gamma > 0$, one can choose $\mu_1 \in (0, Q(\frac{1}{p}+\frac{1}{q}))$ such that $-\alpha-\beta-\gamma+\lambda < \mu_1 < \lambda$. Therefore,

$$\begin{aligned}
\mathcal{J}_{32}(x) &\lesssim |x|^{-\gamma-\alpha-\beta} \int\limits_{\{y:|y|<2|x|\}} \frac{|y|^{\lambda-\mu_1}}{|y|^{Q-\mu_1}} f(xy^{-1})g(xy)\,dy \\
&\lesssim |x|^{-\gamma-\alpha-\beta+\lambda-\mu_1} B_{\mu_1}(f,g)(x).
\end{aligned}$$

At this point, we follow the argument provided after inequality (4.3) to conclude that $\|\mathcal{J}_{32}\|_{L^{r,\infty}(\mathbb{H}^n)} \lesssim \|f\|_{L^p(\mathbb{H}^n)}\|g\|_{L^q(\mathbb{H}^n)}$. Similarly, if $\alpha+\beta+\gamma = 0$, we have $\mathcal{J}_{32}(x) \lesssim$

$B_\lambda(f,g)(x)$, then also we have the required estimate, by invoking Theorem 1.3. This completes the proof of the "weak-type" estimate (4.1).

Once we have the week type inequalities, achieving the strong type inequality just uses the multilinear interpolation, Theorem 2.2. We explain it here. For fixed α, β and γ in Theorem 1.4, we have

$$0 < \frac{1}{p} < \min\left[1,\ 1-\frac{\alpha}{Q}\right], \quad \text{and} \quad 0 < \frac{1}{q} < \min\left[1,\ 1-\frac{\beta}{Q}\right]. \tag{4.4}$$

The condition $\frac{1}{r} \leq \frac{1}{p}+\frac{1}{q}$ is equivalent to $\alpha+\beta+\gamma \leq \lambda$, which combined with $0 \leq \alpha+\beta+\gamma$, gives $0 \leq \alpha+\beta+\gamma \leq \lambda$. Further, the conditions $r < \infty$ and $\gamma < \frac{Q}{r}$ being, respectively, equivalent to $\frac{\lambda-(\alpha+\beta+\gamma)}{Q} < \frac{1}{p}+\frac{1}{q}$ and $\frac{-\alpha-\beta+\lambda}{Q} < \frac{1}{p}+\frac{1}{q}$, lead to

$$\frac{-\alpha-\beta+\lambda}{Q} + \max\left[-\frac{\gamma}{Q},\ 0\right] < \frac{1}{p}+\frac{1}{q}.$$

Therefore, consider the open convex set in $\mathbb{R}^2$,

$$C_{\alpha,\beta,\gamma} := \Big\{(x,y) \in (0,1)^2 \ : \ x+y > \frac{-\alpha-\beta+\lambda}{Q} + \max\left[-\frac{\gamma}{Q},\ 0\right], \\ 0 < x < \min\left[1,\ 1-\frac{\alpha}{Q}\right],\ 0 < y < \min\left[1,\ 1-\frac{\beta}{Q}\right]\Big\}.$$

Depending on the sign of α, β and γ, the set $C_{\alpha,\beta,\gamma} \subseteq (0,1)^2$ changes. But, in all cases, for each point $(\frac{1}{p}, \frac{1}{q})$ in $C_{\alpha,\beta,\gamma}$ one can always choose three non-collinear points inside $C_{\alpha,\beta,\gamma}$ such that $(\frac{1}{p}, \frac{1}{q})$ is contained in the interior of the solid triangle inside $C_{\alpha,\beta,\gamma}$, determined by these three points. Therefore, in view of Theorem 2.2, it suffices to show the "weak-type" inequality for $(\frac{1}{p}, \frac{1}{q})$ in $C_{\alpha,\beta,\gamma}$. □

4.2. Proof of the necessary part

This subsection is dedicated to constructing counterexamples on the Heisenberg group which imply the necessity of the conditions (1.12) for the boundedness of B_λ. In [8], some counterexamples were constructed to conclude necessary conditions for the boundedness of BI_λ on the real line. Here, we construct them on the Heisenberg group $\mathbb{H}^n$ of any dimension.

Proof of (1.11) $\implies$ (1.12)**:** Recall that the inequality (1.11) is equivalent to the following unweighted boundedness

$$\|S_{\alpha,\beta,\gamma}(f,g)\|_{L^r(\mathbb{H}^n)} \leq K\|f\|_{L^p(\mathbb{H}^n)}\|g\|_{L^q(\mathbb{H}^n)}, \tag{4.5}$$

where the operator $S_{\alpha,\beta,\gamma}(f,g)$ is defined as follows

$$S_{\alpha,\beta,\gamma}(f,g)(x) = |x|^{-\gamma} \int_{\mathbb{H}^n} \frac{f(xy^{-1})}{|xy^{-1}|^{\alpha}} \frac{g(xy)}{|xy|^{\beta}} \frac{dy}{|y|^{Q-\lambda}}. \tag{4.6}$$

Necessity of $-Q+\lambda \leq \alpha+\beta$ in Theorem 1.4: On the contrary, suppose that $\alpha+\beta < -Q+\lambda$. Since $\gamma < \frac{Q}{r}$, whence

$$(-\alpha-\beta-Q+\lambda), \quad \frac{Q}{r} - \gamma > 0. \tag{4.7}$$

Also, recall that

$$\frac{1}{r} = \frac{1}{p} + \frac{1}{q} - \frac{\lambda}{Q} + \frac{\alpha+\beta+\gamma}{Q}, \tag{4.8}$$

which implies that $\frac{1}{p} + \frac{1}{q} > 1$.

Let $N, M \gg 1$ to be specified later. For $a \in \mathbb{Z}^{2n+1} \setminus \{0\}$, consider sets

$$E_a := a \cdot Q(0, r_a) \cdot a.$$

Here, $r_a := |a|^{-N-1}$, with $|a|$ denoting the Korányi norm of $a \in \mathbb{Z}^{2n+1} \setminus \{0\} \subseteq \mathbb{H}^n$ and $Q(0,s) := [0,s]^{2n} \times [0,s^2]$, $0 < s < \infty$. Observe that

$$E_a = 2a + Q(0, r_a).$$

Here, "+" denotes usual addition in $\mathbb{R}^{2n+1}$. Consider the functions

$$f(\xi) := \sum_{a \in \mathbb{Z}^{2n+1} \setminus \{0\}} |a|^{M/p} \chi_{E_a}(\xi), \quad \xi \in \mathbb{H}^n,$$

and

$$g(\xi) := \sum_{a \in \mathbb{Z}^{2n+1} \setminus \{0\}} |a|^{M/q} \chi_{E_a}(\xi), \quad \xi \in \mathbb{H}^n.$$

Since, E_a's are disjoint sets, so

$$\begin{aligned} \|f\|^p_{L^p(\mathbb{H}^n)} &= \sum_{a \in \mathbb{Z}^{2n+1} \setminus \{0\}} |a|^M |E_a| = \sum_{a \in \mathbb{Z}^{2n+1} \setminus \{0\}} |a|^M |Q(0, r_a)| \\ &\simeq \sum_{a \in \mathbb{Z}^{2n+1} \setminus \{0\}} \frac{1}{|a|^{Q(N+1)-M}} \simeq \int_{\{(z,t) \in \mathbb{H}^n : \, |(z,t)| \geq 1\}} \frac{1}{|(z,t)|^{Q(N+1)-M}} dz\, dt < \infty \end{aligned}$$

if $Q(N+1) - M > Q$.

Therefore, the functions $f \in L^p(\mathbb{H}^n)$ and $g \in L^q(\mathbb{H}^n)$ if

$$M < QN. \tag{4.9}$$

We will show that for these choices of functions, $S_{\alpha,\beta,\gamma}(f,g)(x)\,\chi_{|x|\ll 1} \notin L^r(\mathbb{H}^n)$.

Fix $x \in \mathbb{R}_+^{2n+1} := (0,\infty)^{2n+1}$ such that $|x| \ll 1$ and choose $\Lambda \gg 1$ such that $(\Lambda + 1)^{-N-1} \leq |x| < \Lambda^{-N-1}$. For $a \in \mathbb{Z}^{2n+1} \setminus \{0\}$ such that $|a| < \Lambda/2$, consider sets

$$\widetilde{E}_{a,x} := E_a \cdot x \,\cap\, x^{-1} \cdot E_a.$$

By definition, whenever $y \in \widetilde{E}_{a,x}$ then yx^{-1}, $xy \in E_a$.

Since $|a| < \Lambda/2$, we see that $|\widetilde{E}_{a,x}| \gtrsim r_a^Q = |a|^{-QN-Q}$. Indeed, we observe that

$$\widetilde{E}_{a,x} = x^{-1} \cdot (x \cdot E_a \cdot x \,\cap\, E_a) = x^{-1} \cdot \Big([2x + 2a + Q(0, r_a)] \bigcap [2a + Q(0, r_a)] \Big).$$

Thus, $|\widetilde{E}_{a,x}| = \left[\prod_{j=1}^{2n}(r_a - 2x_j)\right]\,(r_a^2 - 2x_{2n+1})$. From our choice of Λ, we have $x_{2n+1} \leq \Lambda^{-2(N+1)}$, whence $r_a^2 - 2\,x_{2n+1} \geq |a|^{-2(N+1)} - 2\Lambda^{-2(N+1)} \gtrsim |a|^{-2(N+1)} = r_a^2$, since $|a| < \Lambda/2$. Similarly, $r_a - 2x_j \gtrsim r_a$, $j = 1, \ldots, 2n$. Altogether, we have $|\widetilde{E}_{a,x}| \gtrsim r_a^Q$.

For $y \in \widetilde{E}_{a,x}$, we have $|xy|$, $|yx^{-1}|$, $|y| \simeq |a|$. Indeed, since $xy \in E_a$, so $xy = a \cdot \xi \cdot a = 2a + \xi$ for some $\xi \in Q(0, r_a)$. Thus, $|xy|^4 = \left[\sum_{j=1}^{2n}(2a_j + \xi_j)^2\right]^2 + (2a_{2n+1} + \xi_{2n+1})^2 \geq |a|^4$. So, we have $|a| \leq |xy| \leq |x| + |y|$ which gives $|y| \geq |a| - |x| \gtrsim |a|$. Since $y \in E_a \cdot x$, so $|y| \lesssim |a|$ which in turn implies $|xy|, |y| \simeq |a|$. Similarly, $|yx^{-1}| \simeq |a|$.

Further, for fixed $|x| \ll 1$, the collection $\{\widetilde{E}_{a,x}\}_{a \in \mathbb{Z}^{2n+1}}$ is a disjoint family of sets. Indeed, $\widetilde{E}_{a,x}$'s are disjoint if and only if the sets $x \cdot E_a \cdot x \,\cap\, E_a = [2x + 2a + Q(0,r)] \bigcap [2a + Q(0,r)]$ are disjoint, which is true since $|x| \ll 1$.

Therefore, for $|x| \ll 1$,

$$\begin{aligned} S_{\alpha,\beta,\gamma}(f,g)(x) &\gtrsim |x|^{-\gamma} \sum_{a \in \mathbb{Z}^{2n+1}\setminus\{0\}: |a| < \Lambda/2} \int_{y \in \widetilde{E}_{a,x}} \frac{f(yx^{-1})}{|yx^{-1}|^{\alpha}} \frac{g(xy)}{|xy|^{\beta}} \frac{dy}{|y|^{Q-\lambda}} \\ &\gtrsim |x|^{-\gamma} \sum_{a \in \mathbb{Z}^{2n+1}\setminus\{0\}: |a| < \Lambda/2} |a|^{(-\alpha-\beta-Q+\lambda) + M\left(\frac{1}{p}+\frac{1}{q}\right)} |a|^{-QN-Q}. \end{aligned}$$

Assuming,

$$M\left(\frac{1}{p} + \frac{1}{q}\right) - QN > 0, \tag{4.10}$$

we are dealing with the sum of the form

$$\sum_{a = (a', a_{2n+1}) \in \mathbb{Z}^{2n+1}\setminus\{0\} \,:\, \left(|a'|^4 + a_{2n+1}^2\right)^{1/4} \leq \Lambda/2} |a|^{R-Q} \simeq \Lambda^R, \quad R > 0.$$

Hence,

$$S_{\alpha,\beta,\gamma}(f,g)(x) \gtrsim |x|^{-\gamma - \frac{(-\alpha-\beta-Q+\lambda) + M\left(\frac{1}{p}+\frac{1}{q}\right) - QN}{N+1}} \chi_{|x| \ll 1},$$

which implies $\|S_{\alpha,\beta,\gamma}(f,g)\|_{L^r(\mathbb{H}^n)}$ will diverge if

$$\begin{aligned}
&\gamma + \frac{(-\alpha-\beta-Q+\lambda)+M\left(\frac{1}{p}+\frac{1}{q}\right)-QN}{N+1} \geq \frac{Q}{r} \\
&\iff \gamma(N+1)-Q(N+1)+\lambda-(\alpha+\beta)+M\left(\frac{1}{p}+\frac{1}{q}\right) \\
&\geq \frac{Q}{r}+\frac{QN}{r}=\frac{Q}{r}+QN\left(\frac{1}{p}+\frac{1}{q}\right)-N\lambda+(\alpha+\beta+\gamma)N \\
&\iff (-\alpha-\beta-Q+\lambda)(N+1) > \left(\frac{Q}{r}-\gamma\right)+(QN-M)\left(\frac{1}{p}+\frac{1}{q}\right).
\end{aligned}$$

Here, we have used (4.8). First pick out $N \gg 1$. Since $\frac{1}{p}+\frac{1}{q}>1$, we can choose M close to QN such that (4.9) and (4.10) are satisfied. Subsequently, the last inequality holds true for large N because of (4.7). Thus, we arrive at a contradiction. Therefore, we must have $\alpha+\beta \geq -Q+\lambda$.

Necessity of $-Q+\lambda \leq \beta+\gamma$ in Theorem 1.4: On the contrary, assume $-\beta-\gamma-Q+\lambda > 0$. For $x \in \mathbb{R}^{2n+1}_+$ such that $|x| \gg 1$, consider the following portion of $S_{\alpha,\beta,\gamma}(f,g)(x)$:

$$\int_{y\in Q(0,|x|/2\sqrt{n})\cdot x} \frac{|x|^{-\gamma}}{|xy|^{\beta}\,|y|^{Q-\lambda}} f(xy^{-1})g(xy)\,\frac{dy}{|xy^{-1}|^{\alpha}}. \tag{4.11}$$

Arguing as in the previous example, we see that if $y \in Q(0,|x|/2\sqrt{n})\cdot x$ then $|xy|,\ |y| \simeq |x|$.

Therefore, (4.11) is bounded below by a constant times of the following

$$|x|^{-\beta-\gamma+\lambda-Q}\int_{y\in Q(0,|x|/2\sqrt{n})\cdot x} f(xy^{-1})g(xy)\,\frac{dy}{|xy^{-1}|^{\alpha}}. \tag{4.12}$$

Take $f(y)=|y|^{-s}\,\chi_{Q(0,1)}(y)$ with $s<\frac{Q}{p}$ so that $f \in L^p(\mathbb{H}^n)$. Substituting this choice of f and performing the change of variable $y \to y\cdot x$, (4.12) reduces to

$$|x|^{-\beta-\gamma+\lambda-Q}\int_{y\in Q(0,|x|/2\sqrt{n})} g(x\cdot y\cdot x)\,\frac{dy}{|y|^{\alpha+s}}. \tag{4.13}$$

Let N be a large positive real number to be specified later. Next, we choose the function g as follows:

$$g(\xi) := \sum_{a\in\mathbb{Z}^{2n+1}:|a|>e} |a|^{Q(N-1)/q}\,(\log|a|)^{-2/q}\,\chi_{E_a}(\xi),\quad \xi\in\mathbb{H}^n,$$

where $E_a := a \cdot Q(0, r_a) \cdot a$, with $r_a = \frac{1}{|a|^N}$. The function $g \in L^q(\mathbb{H}^n)$, which follows from disjointness of the sets $E_a = 2a + Q(0, r_a)$, $a \in \mathbb{Z}^{2n+1} \setminus \{0\}$ and from the fact that $|E_a| = |Q(0, r_a)| \simeq |a|^{-QN}$.

Setting $\tilde{E_a} := a^{1/2} \cdot Q(0, r_a/4) \cdot a^{1/2}$, where the notation $(z,t)^{1/2}$ means $(z/2, t/2)$ for $(z,t) \in \mathbb{H}^n$.

If $x \in \tilde{E_a} = a^{1/2} \cdot Q(0, r_a/4) \cdot a^{1/2}$ and $y \in Q(0, r_a/2)$, then $xyx \in E_a = a \cdot Q(0, r_a) \cdot a$ and $|x| \simeq |a|$. Indeed, for such x and y, we have $xyx = 2x + y \in 2\left(a^{1/2} \cdot Q(0, r_a/4) \cdot a^{1/2}\right) + Q(0, r_a/2) = 2\left[a + Q(0, r_a/4)\right] + Q(0, r_a/2) \subset 2a + 2Q(0, r_a/4) + Q(0, r_a/2) \subset 2a + Q(0, r_a) = a \cdot Q(0, r_a) \cdot a = E_a$. Also, $x = a + \xi$, for some $\xi = (\xi', \xi_{2n+1}) \in [0, \frac{r_a}{4}]^{2n} \times [0, (\frac{r_a}{4})^2]$. Writing $a = (a', a_{2n+1})$, we have $|x|^4 = \|a' + \xi'\|^4 + (a_{2n+1} + \xi_{2n+1})^2 \geq |a|^4$, where $\|a'\|$ is the Euclidean norm of $a' \in \mathbb{R}^{2n}$.

Further, since $\tilde{E_a} = a^{1/2} \cdot Q(0, r_a/4) \cdot a^{1/2} = a + Q(0, r_a/4)$, so clearly $|\tilde{E_a}| = |Q(0, r_a/4)| \simeq r_a^Q = |a|^{-QN}$ and $\{\tilde{E_a}\}_{a \in \mathbb{Z}^{2n+1}}$ is a disjoint collection.

Incorporating the above, (4.13) implies

$$\begin{aligned}\|S_{\alpha,\beta,\gamma}(f,g)\|^r_{L^r(\mathbb{H}^n)} &\gtrsim \sum_{|a|>e} \int_{x \in \tilde{E_a}} |x|^{(-\beta-\gamma+\lambda-Q)r} \left| \int_{y \in Q(0, r_a/2)} g(x \cdot y \cdot x) \frac{dy}{|y|^{\alpha+s}} \right|^r dx \\ &\gtrsim \sum_{|a|>e} |a|^{(-\beta-\gamma+\lambda-Q)r} |a|^{\frac{rQ(N-1)}{q}} (\log|a|)^{-\frac{2r}{q}} |a|^{(s+\alpha-Q)Nr} |a|^{-QN}.\end{aligned}$$

Therefore, $\|S_{\alpha,\beta,\gamma}(f,g)\|_{L^r(\mathbb{H}^n)}$ diverges provided

$$\begin{aligned}&(-\beta-\gamma+\lambda-Q)r + \frac{rQ(N-1)}{q} + (s+\alpha-Q)Nr - QN > -Q \\ &\iff -\beta-\gamma+\lambda-Q + \frac{Q(N-1)}{q} + (s+\alpha-Q)N > \frac{Q(N-1)}{r} \\ &\iff -\beta-\gamma+\lambda-Q + \frac{Q(N-1)}{q} + (s+\alpha-Q)N \\ &> \left(\frac{QN}{p} - \frac{Q}{p}\right) + \frac{Q(N-1)}{q} + (N-1)(\alpha+\beta+\gamma-\lambda) \\ &\iff N\left((-\beta-\gamma-Q+\lambda) - \left(\frac{Q}{p} - s\right)\right) > \frac{Q}{p'} - \alpha.\end{aligned}$$

Since $-\beta-\gamma-Q+\lambda > 0$, we choose $s < \frac{Q}{p}$ sufficiently close to $\frac{Q}{p}$ so that $(-\beta-\gamma-Q+\lambda) - \left(\frac{Q}{p} - s\right) > 0$ and then, taking N large, we have that the last inequality holds true.

Necessity of $\alpha+\beta+\gamma \geq 0$ in Theorem 1.4: Contrarily, suppose $\gamma_0 := \alpha+\beta+\gamma < 0$. Then, the homogeneity condition takes the form of $\frac{1}{r} = \frac{1}{p} + \frac{1}{q} - \frac{\lambda-\gamma_0}{Q} > 0$.

On the set $\{y \in \mathbb{H}^n : |y| \ll |x|\}$, one has $|x| \gtrsim |xy|, |xy^{-1}| \geq \left||x| - |y|\right| \gtrsim |x|$. Therefore,

$$S_{\alpha,\beta,\gamma}(f,g)(x) \gtrsim |x|^{-\gamma_0} \int\limits_{\{y\in\mathbb{H}^n |y|\ll|x|\}} f(xy^{-1})g(xy)\,\frac{dy}{|y|^{Q-\lambda}}. \tag{4.14}$$

Take $N \gg 1$, to be specified later, and consider functions

$$f(y) = \sum_{a\in\mathbb{Z}^{2n+1}:|a|>e} |a|^{Q(N-1)/p}\,(\log|a|)^{-2/p}\,\chi_{E_a}(y),$$

and

$$g(y) = \sum_{a\in\mathbb{Z}^{2n+1}:|a|>e} |a|^{Q(N-1)/q}\,(\log|a|)^{-2/q}\,\chi_{E_a}(y), \quad y\in\mathbb{H}^n,$$

with $E_a := a\cdot B(0,r_a)^2$, $r_a = \frac{1}{|a|^N}$. Observe that $|E_a| = |B(0,r_a)^2| \sim r_a^Q = \frac{1}{|a|^{NQ}}$ and that $\{E_a : a\in\mathbb{Z}^{2n+1}\setminus\{0\}\}$ is a disjoint collection of sets. Therefore, $f\in L^p(\mathbb{H}^n)$ and $g\in L^q(\mathbb{H}^n)$.

Define sets $\tilde{E_a} := a\cdot B(0,r_a)$. For $x\in\tilde{E_a}$ and $y\in B(0,r_a)$, we have xy^{-1}, $xy\in a\cdot B(0,r_a)^2 = E_a$. Therefore, from (4.14),

$$\begin{aligned}
\|S_{\alpha,\beta,\gamma}(f,g)\|^r_{L^r(\mathbb{H}^n)} &\gtrsim \sum_{|a|>e}\int\limits_{x\in\tilde{E_a}} \left| |x|^{-\gamma_0} \int\limits_{|y|<r_a} f(xy^{-1})g(xy)\,\frac{dy}{|y|^{Q-\lambda}}\right|^r dx\\
&\gtrsim \sum_{|a|>e}\int\limits_{x\in\tilde{E_a}} |x|^{-\gamma_0\, r}|a|^{rQ(N-1)(\frac{1}{p}+\frac{1}{q})}\,(\log|a|)^{-2r(\frac{1}{p}+\frac{1}{q})}\,\frac{1}{|a|^{N\lambda r}}dx\\
&\gtrsim \sum_{|a|>e} |a|^{-\gamma_0\, r+rQ(N-1)(\frac{1}{p}+\frac{1}{q})-rN\lambda}\,(\log|a|)^{-2r(\frac{1}{p}+\frac{1}{q})}\,\frac{1}{|a|^{NQ}}\\
&=: \sum_{|a|>e} |a|^R\,(\log|a|)^{-2r(\frac{1}{p}+\frac{1}{q})},
\end{aligned}$$

which diverges if $R>-Q$, wherein we have used that the family $\{\tilde{E_a} : a\in\mathbb{Z}^{2n+1}\setminus\{0\}\}$ is a disjoint collection of sets, and that for $x\in\tilde{E_a} = a\cdot B(0,r_a)$, $|x|\sim|a|$.

Therefore, in view of the homogeneity condition, it suffices to check whether

$$\begin{aligned}
&-\gamma_0\, r + rQ(N-1)\left(\frac{1}{p}+\frac{1}{q}\right) - rN\lambda - NQ > -Q\\
\iff\ & -\gamma_0 + Q(N-1)\left(\frac{1}{p}+\frac{1}{q}\right) - N\lambda > \frac{(N-1)Q}{r}\\
\iff\ & -\gamma_0 - Q\left(\frac{1}{p}+\frac{1}{q}\right) + QN\left(\frac{1}{p}+\frac{1}{q}-\frac{\lambda}{Q}-\frac{1}{r}\right) + \frac{Q}{r} > 0\\
\iff\ & -\gamma_0 - Q\left(\frac{1}{p}+\frac{1}{q}\right) - \gamma_0\, N + \frac{Q}{r} > 0,
\end{aligned}$$

which is true for N sufficiently large, since $\gamma_0 < 0$.

Necessity of $\frac{1}{r} \leq \frac{1}{p} + \frac{1}{q}$ in Theorem 1.4:

On the contrary, let us assume $\frac{1}{r} > \frac{1}{p} + \frac{1}{q}$. Take $f(x) := |x|^{-Q/p}(\log|x|)^{-\tau_1}\chi_{\{|x|>16\}}$ and $g(x) := |x|^{-Q/q}(\log|x|)^{-\tau_2}\chi_{\{|x|>16\}}$. It is not hard to see that $f \in L^p(\mathbb{H}^n)$ and $g \in L^q(\mathbb{H}^n)$ provided $\tau_1 > 1/p$ and $\tau_2 > 1/q$, respectively. For $|x| \gg 1$, we see that

$$\begin{aligned} S_{\alpha,\beta,\gamma}(f,g)(x) &\gtrsim |x|^{-\gamma} \int\limits_{B(0,\frac{|x|}{2})} \frac{f(xy^{-1})}{|xy^{-1}|^{\alpha}} \frac{g(xy)}{|xy|^{\beta}} \frac{dy}{|y|^{Q-\lambda}} \\ &\gtrsim |x|^{-(\alpha+\beta+\gamma)} \int\limits_{B(0,\frac{|x|}{2})} f(xy^{-1})g(xy) \frac{dy}{|y|^{Q-\lambda}} \\ &\gtrsim |x|^{-(\alpha+\beta+\gamma)-Q(\frac{1}{p}+\frac{1}{q})+\lambda}(\log|x|)^{-\tau_1-\tau_2} = |x|^{-Q/r}(\log|x|)^{-\tau_1-\tau_2}. \end{aligned}$$

The above implies that $\|S_{\alpha,\beta,\gamma}(f,g)\|_{L^r(\mathbb{H}^n)} = \infty$ if we choose $\tau_1 > 1/p$ and $\tau_2 > 1/q$ such that $(\tau_1 + \tau_2) \leq \frac{1}{r}$ and which is possible thanks to the assumption $\frac{1}{r} > \frac{1}{p} + \frac{1}{q}$. For example, one can choose $\tau_1 = \frac{1}{p} + \epsilon$ and $\tau_2 = \frac{1}{q} + \epsilon$ with $\epsilon > 0$ such that $0 < 2\epsilon < \frac{1}{r} - \frac{1}{p} - \frac{1}{q}$. Thus, we arrive at a contradiction. $\square$

Declaration of competing interest

There is no conflict of interest.

Data availability

No data was used for the research described in the article.

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