



Carleman estimates for sub-Laplacians on Carnot groups

Vedansh Arya¹ · Dharmendra Kumar²

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Abstract

In this note, we establish a new Carleman estimate with singular weights for the sub-Laplacian on a Carnot group \mathbb{G} for functions satisfying the discrepancy assumption in (2.16) below. We use such an estimate to derive a sharp vanishing order estimate for solutions to stationary Schrödinger equations.

Keywords Carleman estimate · Carnot groups · Unique continuation

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Contents

1	Introduction
2	Notations, preliminaries and statements of the main results
2.1	Statement of the main results
3	Proof of Theorem 2.1 and 2.2
	References

1 Introduction

In this note, we give an elementary proof of an $L^2 - L^2$ type Carleman estimate with singular weights for the sub-Laplacian on Carnot groups. Using such an estimate, we present a new application to an upper bound on the maximal order of vanishing for solutions to stationary Schrödinger equations (2.19). Such a result as in Theorem 2.2

✉ Vedansh Arya
vedansh.v.arya@jyu.fi
Dharmendra Kumar
dharmendrak@iisc.ac.in

¹ Department of Mathematics and Statistics, University of Jyväskylä, Jyväskylä, Finland

² Department of Mathematics, Indian Institute of Science, Bangalore, India

below constitutes a quantitative version of the strong unique continuation property and can be thought of as a subelliptic generalization of a similar quantitative uniqueness result due to Bourgain and Kenig in [11] (see Proposition 2.4).

Concerning the question of interest in this note, the unique continuation property, we mention that for general uniformly elliptic equations there are essentially two known methods for proving it. The former is based on Carleman inequalities, which are appropriate weighted versions of Sobolev–Poincaré inequalities. This method was first introduced by T. Carleman in his fundamental work [12] in which he showed that strong unique continuation holds for equations of the type $-\Delta u + Vu = 0$, with $V \in L^\infty_{loc}(\mathbb{R}^2)$. Subsequently, his estimates were generalised in [2] and [3] to uniformly elliptic operators with $C^{2,\alpha}_{loc}$ and $C^{0,1}_{loc}$ principal part respectively in all dimensions. We recall that unique continuation fails in general when the coefficients of the principal part are only Hölder continuous, see [26]. The second approach came up in the works of Lin and Garofalo, see [20, 21]. Their method is based on the almost monotonicity of a generalisation of the frequency function, first introduced by Almgren in [1] for harmonic functions. Using this approach, they were able to obtain new quantitative information for the solutions to divergence form elliptic equations with Lipschitz coefficients which in particular encompass and improve on those in [3].

The unique continuation in subelliptic setting of a Carnot group is however much subtler in the sense that strong unique continuation property is in general not true for solutions to (2.19). This follows from some interesting work of Bahouri ([4]) where the author showed that unique continuation is not true for even smooth and compactly supported perturbations of the sub-Laplacian. Subsequently in the setting of the Heisenberg group \mathbb{H}^n , it is shown by Garofalo and Lanconelli in [19] that if the solutions to (2.19) additionally satisfy the discrepancy assumption of the type (2.20), then the strong unique continuation holds. Such a result has been generalized to Carnot groups of arbitrary step in [22]. We also refer to the recent work [18] where it is shown that in general, the Almgren type monotonicity fails even when $\mathbb{G} = \mathbb{H}^n$. It is to be noted that the discrepancy condition (2.20) trivially holds in the Euclidean case. See Sect. 2 below.

The purpose of this note is to establish a new Carleman estimate in the framework of [22] where the strong unique continuation is known so far using which we prove the vanishing order estimate in Theorem 2.2 below. Our main results Theorem 2.1 and Theorem 2.2 can be regarded as subelliptic generalizations of the ones in [3] and [11].

We mention that the proof of our Carleman estimate in Theorem 2.1 is based on elementary arguments using integration by parts and an appropriate Rellich type identity and is inspired by the recent work [9] where a similar Carleman estimate has been established for Baouendi–Grushin operators. Our proof however additionally exploits the discrepancy condition in (2.16) below in a very crucial way. The reader will see that proof of our Carleman estimate relies on some non-trivial geometric facts in the subelliptic setting that beautifully combine.

The paper is organized as follows. In Sect. 2, we introduce some basic notations, state our main results and also gather some known results that are relevant to our work. In Sect. 3, we prove our main results.

2 Notations, preliminaries and statements of the main results

In this section we introduce the relevant notation, state our main results and gather some auxiliary results that will be useful in the rest of the paper. We will follow the same notations as in [22] and [7]. For detail, we refer the reader to the book [10]. We now recall that a Carnot group of step h is a simply connected Lie group \mathbb{G} whose Lie algebra \mathfrak{g} admits a stratification $\mathfrak{g} = V_1 \oplus \dots \oplus V_h$ which is h nilpotent, i.e., $[V_1, V_i] = V_{i+1}$ for $i = 1, \dots, h - 1$ and $[V_i, V_h] = 0$ for $i = 1, \dots, h$. We will denote an arbitrary element of \mathbb{G} by g and e will denote the identity of the group \mathbb{G} . For any open subset Ω of \mathbb{G} , We indicate with $C_0^k(\Omega)$ the set of compactly supported C^k functions in Ω . We will assume that \mathfrak{g} is equipped with an inner product $\langle \cdot, \cdot \rangle_{\mathfrak{g}}$ such that V_i 's are mutually orthogonal.

By the assumptions on the Lie algebra \mathfrak{g} , any basis of *horizontal layer* V_1 generates the whole \mathfrak{g} . Let $\{e_1, \dots, e_m\}$ be an orthonormal basis of the first layer V_1 of the Lie algebra. We then define the corresponding left invariant smooth vector fields by

$$X_i(g) = dL_g(e_i), \quad i = 1, \dots, m \tag{2.1}$$

where L_g denotes the left-translation operator given by $L_g(g') = gg'$ and dL_g denote its differential. Further, we assume that \mathbb{G} is equipped with a left invariant Riemannian metric with respect to which $\{X_1, \dots, X_m\}$ is an orthonormal set of vector fields. The sub-Laplacian corresponding to the basis $\{e_1, \dots, e_m\}$ is given by the formula

$$\Delta_H u = \sum_{i=1}^m X_i^2 u. \tag{2.2}$$

We note that by Hormander's theorem, Δ_H is hypoelliptic. We will denote the horizontal gradient of u by

$$\nabla_H u = \sum_{i=1}^m X_i u X_i \tag{2.3}$$

and we let

$$|\nabla_H u|^2 = \sum_{i=1}^m (X_i u)^2. \tag{2.4}$$

We now define the non-isotropic dilations δ_λ on \mathbb{G} by

$$\delta_\lambda(g) = \exp \circ \tilde{\delta}_\lambda \circ \exp^{-1} g, \tag{2.5}$$

where the exponential mapping $\exp : \mathfrak{g} \rightarrow \mathbb{G}$ defines an analytic diffeomorphism onto \mathbb{G} and for $\xi = \xi_1 + \xi_2 + \dots + \xi_h$, where $\xi_i \in V_i$, we define

$$\tilde{\delta}_\lambda \xi = \lambda \xi_1 + \dots + \lambda^h \xi_h, \tag{2.6}$$

where we have assigned the formal degree i to the each element of the layer V_i . We will denote the infinitesimal generator of the non-isotropic dilations (2.5) by Z , note

that such smooth vector fields is characterized by the following property

$$\frac{d}{dr}u(\delta_r(g)) = \frac{1}{r}Zu(\delta_r(g)). \tag{2.7}$$

Hence, $u \in C^1(\mathbb{G})$ is a homogeneous function of degree k with respect to (2.5), i.e., $u(\delta_r(g)) = r^k u(g)$ if and only if

$$Zu = ku.$$

We will denote the bi-invariant Haar measure on \mathbb{G} , which is obtained by lifting via the exponential map exp the Lebesgue measure on \mathfrak{g} by dg . Let m_i denote the dimension of V_i . We then have

$$(d \circ \delta_\lambda)(g) = \lambda^Q dg, \tag{2.8}$$

where $Q = \sum_{i=1}^h im_i$ is referred as the homogeneous dimension of \mathbb{G} .

Let $\Gamma(g, g') = \Gamma(g', g)$ be the positive unique fundamental solution of $-\Delta_H$. We have that Γ is left-translation invariant, i.e.,

$$\Gamma(g, g') = \tilde{\Gamma}(g^{-1} \circ g') \tag{2.9}$$

for some $\tilde{\Gamma} \in C^\infty(\mathbb{G} \setminus \{e\})$. For every $r > 0$, we define

$$B_r := \left\{ g \in \mathbb{G} \mid \Gamma(g, e) > \frac{1}{r^{Q-2}} \right\}. \tag{2.10}$$

In [17], Folland has proved that $\tilde{\Gamma}(g)$ is homogeneous function of degree $2 - Q$ with respect to the non-isotropic dilations (2.5). Therefore, if we define

$$\rho(g) = \tilde{\Gamma}(g)^{\frac{-1}{Q-2}}, \tag{2.11}$$

then ρ is homogeneous of degree 1. Hence B_r can be equivalently defined as

$$B_r = \{g : \rho(g) < r\}. \tag{2.12}$$

We now let

$$\psi \stackrel{def}{=} |\nabla_H \rho|^2. \tag{2.13}$$

Since ρ is a homogeneous function of degree 1, $\nabla_H \rho$ is a homogeneous function of degree 0. Hence we have

$$Z\psi = 0. \tag{2.14}$$

Like in [22], for a function f , we define the discrepancy E_f at e by

$$E_f \stackrel{def}{=} \langle \nabla_H f, \nabla_H \rho \rangle - \frac{Zf}{\rho} |\nabla_H \rho|^2. \tag{2.15}$$

We now state our main results.

2.1 Statement of the main results

Our first result is the subelliptic analogue of the well known Carleman estimate in [11]. See also [5, 16].

Theorem 2.1 *Let $w \in C_0^2(B_R \setminus \{e\})$ satisfy $(\Delta_H w + Vw)^2 \leq C_1 \psi$ for some $C_1 > 0$ and the following discrepancy assumption*

$$|E_w| \leq \frac{C_E}{\rho^{1-\delta}} |w| |\nabla_H \rho|^2 \tag{2.16}$$

for some $\delta \in (0, 1)$, where ρ is as in (2.11) and E_w denotes the discrepancy of w as defined in (2.15) above. Also assume that the function $V : \mathbb{G} \rightarrow \mathbb{R}$ satisfies the following growth condition

$$|V| \leq K \psi, \tag{2.17}$$

where K is a non-negative constant and ψ is as in (2.13) above. Then there exist universal constants $C, R_0 > 0$ depending on δ, C_E and Q such that for all $R \leq R_0$ and $\alpha > CK^{2/3} + Q$, the following estimate holds

$$\alpha^3 \int \rho^{-2\alpha-4+\varepsilon} e^{2\alpha\rho^\varepsilon} w^2 \psi dg \leq C \int \rho^{-2\alpha} e^{2\alpha\rho^\varepsilon} (\Delta_H w + Vw)^2 \psi^{-1} dg, \tag{2.18}$$

for $\varepsilon = \delta/2$ and where dg is the bi-invariant Haar measure on \mathbb{G} .

Using the Carleman estimate in Theorem 2.1 above, we derive the following quantitative uniqueness result for solutions to

$$-\Delta_H u = Vu \text{ in } B_{R_0}, \tag{2.19}$$

where R_0 is as in the Theorem 2.1 and V satisfies the growth condition as in (2.17) above.

Since the regularity issues are not our main concern, we will assume apriori that $u, X_i u, X_i X_j u, Zu$ are in $L^2(B_1)$ with respect to the Haar measure dg .

Theorem 2.2 *Let u be a non-trivial solution to (2.19) where V satisfies (2.17). Furthermore assume that for some $\delta \in (0, 1)$*

$$|E_u| \leq \frac{C_E}{\rho^{1-\delta}} |u| |\nabla_H \rho|^2. \tag{2.20}$$

Then there exists a constant $C = C(Q, C_E, \delta) > 0$ such that for all $r < R_0/8$, we have

$$\|u\psi^{1/2}\|_{L^2(B_r)} > Cr^A, \tag{2.21}$$

where $A = CK^{2/3} + C + C \left(\left(1 + \|u\psi^{1/2}\|_{L^2(B_{R_0})}\right) / \|u\psi^{1/2}\|_{L^2(B_{R_0/4})} \right)^{4/3}$ and R_0 is as in the Theorem 2.1.

We first make a remark regarding the dependence of maximal vanishing order on the solution u .

Remark 2.3 If we consider $u = \text{Re}(z^k)$ in R^2 then $\Delta u = 0$ i.e., u satisfies the equation $\Delta u + Vu = 0$ for $V = 0$ and has vanishing order k which corresponds to its homogeneity. Since $k \in \mathbb{N}$ can be arbitrarily large, this suggests that the maximal vanishing order has to depend on u as well.

It is worth emphasizing that, when nilpotency step of the group is 1, i.e., $h = 1$, from (2.13) we have $\psi \equiv 1$. In this case the constant K in (2.17) can be taken to be $\|V\|_{L^\infty}$ and the discrepancy condition (2.20) trivially holds, and therefore Theorem 2.2 reduces to the following Euclidean result in [24], which is a consequence of [11, Lemma 3.15]:

Proposition 2.4 *Let u be a solution of $\Delta u = Vu$ in $B(0, 10) \subset \mathbb{R}^n$. Then, there exist constants a_1, a_2 depending u, n such that*

$$\max_{|x| \leq r} |u(x)| \geq a_1 r^{a_2(\|V\|_{L^\infty}^{2/3} + 1)}$$

for all $r > 0$ small enough.

Note that Proposition 2.4 is sharp in view of Meshov’s counterexample in [25]. We also note that when V satisfies the additional hypothesis

$$|ZV| \leq K\psi,$$

then, using a variant of the frequency function approach, the following sharper estimate was established in [7] for solutions to (2.19),

$$\|u\|_{L^\infty(B_r)} \geq C_1 \left(\frac{r}{R_0} \right)^{C_2(\sqrt{K} + 1)}. \tag{2.22}$$

We now make a remark regarding the discrepancy condition (2.20).

Remark 2.5 We would like to mention over here that there is a fairly detailed discussion on the validity of the discrepancy assumption (2.20) in [22, Section 6] under various symmetry assumptions. For instance, if G is a group of Heisenberg type and u has a cylindrical symmetry, then $E_u = 0$. See for instance [22, Proposition 6.11] for a proof of this fact. Moreover in the case of Heisenberg group \mathbb{H}^n , it turns out that polyradial

functions have zero discrepancy. Furthermore for a general Carnot group, it is easily seen that any radial function has zero discrepancy. Therefore in such settings, if we take $\Omega = B_1$ and consider potentials V as well boundary values g which satisfy similar symmetry conditions, then by energy methods (when the norm of V is small enough) or Fredholm alternative (in the general case), one can obtain solutions to the following Dirichlet problem

$$\begin{cases} \Delta_H u = Vu \text{ in } B_1, \\ u = g \text{ on } \partial B_1, \end{cases} \tag{2.23}$$

which satisfy similar symmetry conditions (by uniqueness) and consequently (2.20). The existence of such symmetric solutions appear in the work of Garofalo and Vassilev in [23, Section 6]. See also [28].

The reader should note that for Laplacian on a compact manifold the counterpart of (2.22) was first obtained using Carleman estimates by Bakri in [5]. This generalises the sharp vanishing order estimate of Donnelly and Fefferman in [14, 15] for eigenfunctions of the Laplacian. We also mention that, for the standard Laplacian, the result of Bakri was subsequently obtained by Zhu [29], using a variant of the frequency function approach in [20, 21]. This was extended in [8] to more general elliptic equations with Lipschitz principal part where the authors also established a certain boundary version of the vanishing order estimate.

We now gather some known results that will be needed in the present work. The following proposition below concerns the action of the sub-Laplacian on radial functions (see [22]). This will be needed in the proof of Theorem 2.1.

Proposition 2.6 *Let $f : (0, \infty) \rightarrow \mathbb{R}$ be a C^2 function, and define $w(g) = f(\rho(g))$. Then, one has*

$$\Delta_H w = |\nabla_H \rho|^2 \left\{ f''(\rho) + \frac{Q-1}{\rho} f'(\rho) \right\}, \quad \text{in } \mathbb{G} \setminus \{e\}.$$

We then collect the following elementary facts from [13] and [23].

Lemma 2.7 *In a Carnot group \mathbb{G} , the infinitesimal generator of group dilations Z enjoys the following properties:*

- (i) *One has $[X_i, Z] = X_i, \quad i = 1, \dots, m$.*
- (ii) *$\operatorname{div}_{\mathbb{G}}(\rho^{-l}Z) = (Q - l)\rho^{-l}$.*

We also need the following Rellich type identity in the proof of Theorem 2.1, which corresponds to Theorem 3.1 in [23]. This can be seen as the sub-elliptic analogue of Rellich type identity in [27].

Lemma 2.8 *For a C^1 vector field F and $v \in C^2(\mathbb{G})$, the following holds*

$$\int_{B_r} \operatorname{div}_{\mathbb{G}} F |\nabla_H v|^2 - 2 \sum_{i=1}^m \int_{B_r} X_i v [X_i, F] v - 2 \int_{B_r} F v \Delta_H v$$

$$= \int_{\partial B_r} |\nabla_H v|^2 \langle F, v \rangle - 2 \sum_{i=1}^m \int_{\partial B_r} F v X_i v \langle X_i, v \rangle. \tag{2.24}$$

We now state a Caccioppoli type energy inequality which will be used in the proof of Theorem 2.2. The proof of such an energy inequality is identical to that of [9, Lemma 4.1] and we therefore skip the details.

Lemma 2.9 *Let u be a solution to (2.19) with V satisfying (2.17). Then, there exists a universal constant $C = C(Q) > 0$ such that for any $0 < a < 1$, we have*

$$\int_{B_{(1-a)R}} |\nabla_H u|^2 dg \leq \frac{C}{a^2 R^2} \int_{B_R} (1 + K) u^2 \psi dg. \tag{2.25}$$

3 Proof of Theorem 2.1 and 2.2

Proof of Theorem 2.1 For $R \leq R_0$, let $w \in C_0^2(B_R \setminus \{e\})$ be as in Theorem 2.1. We now set $v = \rho^{-\beta} e^{\alpha\rho^\varepsilon} w$, where ε and β will be chosen later depending on δ and α respectively. Then $w = \rho^\beta e^{-\alpha\rho^\varepsilon} v$ and it is easy to see that

$$\Delta_H w = v \Delta_H (\rho^\beta e^{-\alpha\rho^\varepsilon}) + 2 \langle \nabla_H (\rho^\beta e^{-\alpha\rho^\varepsilon}), \nabla_H v \rangle + \rho^\beta e^{-\alpha\rho^\varepsilon} \Delta_H v. \tag{3.1}$$

Now we use Proposition 2.6 and recall $|\nabla_H \rho|^2 = \psi$ to obtain

$$\begin{aligned} \Delta_H (\rho^\beta e^{-\alpha\rho^\varepsilon}) &= \left(\beta(\beta + Q - 2) \rho^{\beta-2} + \alpha^2 \varepsilon^2 \rho^{\beta+2\varepsilon-2} \right. \\ &\quad \left. - \alpha \varepsilon (2\beta + \varepsilon + Q - 2) \rho^{\beta+\varepsilon-2} \right) e^{-\alpha\rho^\varepsilon} \psi. \end{aligned} \tag{3.2}$$

Also, it is easy to check that

$$2 \langle \nabla_H (\rho^\beta e^{-\alpha\rho^\varepsilon}), \nabla_H v \rangle = \left(2\beta \rho^{\beta-1} - 2\varepsilon \alpha \rho^{\beta+\varepsilon-1} \right) \langle \nabla_H \rho, \nabla_H v \rangle e^{-\alpha\rho^\varepsilon}. \tag{3.3}$$

Now we use (3.2) and (3.3) in (3.1) to get

$$\begin{aligned} \Delta_H w &= v \left(\beta(\beta + Q - 2) \rho^{\beta-2} + \alpha^2 \varepsilon^2 \rho^{\beta+2\varepsilon-2} \right. \\ &\quad \left. - \alpha \varepsilon (2\beta + \varepsilon + Q - 2) \rho^{\beta+\varepsilon-2} \right) e^{-\alpha\rho^\varepsilon} \psi \\ &\quad + \left(2\beta \rho^{\beta-1} - 2\varepsilon \alpha \rho^{\beta+\varepsilon-1} \right) \langle \nabla_H \rho, \nabla_H v \rangle e^{-\alpha\rho^\varepsilon} + \rho^\beta e^{-\alpha\rho^\varepsilon} \Delta_H v. \end{aligned} \tag{3.4}$$

From definition (2.15) for E_v , it is easy to see that (3.4) can be equivalently written as

$$\begin{aligned} \Delta_H w + V w &= \left(\beta(\beta + Q - 2)\rho^{\beta-2} + \alpha^2 \varepsilon^2 \rho^{\beta+2\varepsilon-2} \right. \\ &\quad \left. - \alpha \varepsilon (2\beta + \varepsilon + Q - 2) \rho^{\beta+\varepsilon-2} \right) e^{-\alpha\rho^\varepsilon} \psi v \\ &+ \left(2\beta\rho^{\beta-1} - 2\varepsilon\alpha\rho^{\beta+\varepsilon-1} \right) e^{-\alpha\rho^\varepsilon} \left(\frac{Zv}{\rho} \psi + E_v \right) + \rho^\beta e^{-\alpha\rho^\varepsilon} \Delta_H v \\ &+ \rho^\beta e^{-\alpha\rho^\varepsilon} V v. \end{aligned} \quad (3.5)$$

We now use the inequality $(a + b)^2 \geq a^2 + 2ab$, with $a = 2\beta\rho^{\beta-2}e^{-\alpha\rho^\varepsilon} \psi Zv$ and $b = \Delta_H w + V w - a$, where the expression for $\Delta_H w + V w$ is given by (3.5), to find

$$\begin{aligned} &\int \rho^{-2\alpha} e^{2\alpha\rho^\varepsilon} (\Delta_H w + V w)^2 \psi^{-1} \\ &\geq 4\beta^2 \int \rho^{2\beta-2\alpha-4} \psi (Zv)^2 + 4\beta^2 (\beta + Q - 2) \int \rho^{2\beta-2\alpha-4} \psi v Zv \\ &\quad + 4\alpha^2 \beta \varepsilon^2 \int \rho^{2\beta-2\alpha-4+2\varepsilon} \psi v Zv - 4\alpha\beta\varepsilon (2\beta + \varepsilon + Q - 2) \int \rho^{2\beta-2\alpha-4+\varepsilon} \psi v Zv \\ &\quad - 8\alpha\beta\varepsilon \int \rho^{2\beta-2\alpha-4+\varepsilon} \psi (Zv)^2 + 8\beta \int (\beta\rho^{2\beta-2\alpha-3} - \varepsilon\alpha\rho^{2\beta-2\alpha-3+\varepsilon}) E_v Zv \\ &\quad + 4\beta \int \rho^{2\beta-2\alpha-2} \Delta_H v Zv + 4\beta \int \rho^{2\beta-2\alpha-2} V v Zv \\ &= I_1 + I_2 + I_3 + I_4 + I_5 + I_6 + I_7 + I_8. \end{aligned} \quad (3.6)$$

We now estimate each of the integrals individually. In order to estimate the I_2 , I_3 and I_4 , first note that (2.14) and (ii) in Lemma 2.7 gives

$$\operatorname{div}(\rho^{-l} \psi v^2 Z) = \psi v^2 \operatorname{div}(\rho^{-l} Z) + \rho^{-l} Z(\psi v^2) = (Q - l)\rho^{-l} \psi v^2 + \rho^{-l} \psi Z(v^2), \quad (3.7)$$

Also, $\operatorname{supp}(u) \subset (B_R \setminus \{e\})$ Hence, (3.7) gives

$$\int \rho^{-l} \psi Z(v^2) = -(Q - l) \int \rho^{-l} \psi v^2. \quad (3.8)$$

Thus using $2vZv = Z(v^2)$ and (3.8), I_2 becomes

$$\begin{aligned} 4\beta^2 (\beta + Q - 2) \int \rho^{2\beta-2\alpha-4} \psi v Zv &= 2\beta^2 (\beta + Q - 2) \int \rho^{2\beta-2\alpha-4} \psi Z(v^2) \\ &= -2\beta^2 (\beta + Q - 2)(Q + 2\beta - 2\alpha - 4) \int \rho^{2\beta-2\alpha-4} \psi v^2. \end{aligned} \quad (3.9)$$

Observe that in order to equate I_2 to zero, we need the following relation between α and β

$$2\beta - 2\alpha - 4 + Q = 0. \quad (3.10)$$

Hence

$$I_2 = 0. \quad (3.11)$$

Again using $2vZv = Z(v^2)$, (3.8) and (3.10) we get

$$I_3 + I_4 = -4\alpha^2\beta\varepsilon^3 \int \rho^{-Q+2\varepsilon}\psi v^2 + 2\alpha\beta\varepsilon^2(2\beta + \varepsilon + Q - 2) \int \rho^{-Q+2\varepsilon}\psi v^2. \quad (3.12)$$

We now estimate the integral I_6 . First note that using the relation (3.10) and $\rho^\varepsilon \leq R_0^\varepsilon < 1$, we find

$$\begin{aligned} |I_6| &= 8\beta \left| \int \left(\beta\rho^{2\beta-2\alpha-3} - \varepsilon\alpha\rho^{2\beta-2\alpha-3+\varepsilon} \right) E_v Zv \right| \\ &\leq 8\beta(\beta + \varepsilon\alpha) \int \rho^{-Q+1} |Zv| |E_v|. \end{aligned} \quad (3.13)$$

In order to simplify (3.13) we make use of the assumption (2.16) on discrepancy. Now since $E_{f(\rho)} = 0$, we get

$$E_v = \rho^{-\beta} e^{\alpha\rho^\varepsilon} E_w. \quad (3.14)$$

Consequently, using (3.14), (2.16) and recalling $v = \rho^{-\beta} e^{\alpha\rho^\varepsilon} w$, we deduce from (3.13)

$$|I_6| \leq 8\beta(\beta + \varepsilon\alpha) C_E \int \rho^{-Q+\delta} |Zv| |v| \psi. \quad (3.15)$$

From (3.10), it is easy to see that for $\alpha > Q-4$, we have $2\beta > \alpha$. Also, we have $\varepsilon < 1$. Therefore we get $8\beta(\beta + \varepsilon\alpha) \leq 24\beta^2$. Subsequently, we apply Young's equality in (3.15) to find

$$\begin{aligned} |I_6| &\leq 24\beta^2 C_E \int \rho^{-Q+\delta} |Zv| |v| \psi \leq 12\beta^2 C_E \int \rho^{-Q+\delta} |Zv|^2 \psi \\ &\quad + 12\beta^2 C_E \int \rho^{-Q+\delta} |v|^2 \psi. \end{aligned}$$

Thus, we obtain

$$I_6 \geq -12\beta^2 C_E \int \rho^{-Q+\delta} |Zv|^2 \psi - 12\beta^2 C_E \int \rho^{-Q+\delta} |v|^2 \psi. \quad (3.16)$$

Next, we simplify I_7 . Note that from (3.10), we have

$$I_7 = 4\beta \int \rho^{2\beta-2\alpha-2} Zv \Delta_H v = 4\beta \int \rho^{-Q+2} Zv \Delta_H v. \quad (3.17)$$

We now apply the Rellich type identity (2.24) to the vector field $F = \rho^{-Q+2}Z$. Also, note that since v is compactly supported in $(B_R \setminus \{e\})$, the boundary terms become zero. Therefore, (3.17) becomes

$$4\beta \int \rho^{-Q+2}Zv \Delta_H v = 2\beta \int \operatorname{div}(\rho^{-Q+2}Z)|\nabla_H v|^2 - 4\beta \sum_{i=1}^m \int X_i v [X_i, \rho^{-Q+2}Z]v. \quad (3.18)$$

To simplify integrals in right-hand side of (3.18), recall that from (ii) of Lemma 2.7 we have

$$\operatorname{div}(\rho^{-Q+2}Z) = 2\rho^{-Q+2} \quad (3.19)$$

and using (i) of Lemma 2.7, it is easy to obtain

$$\begin{aligned} [X_i, \rho^{-Q+2}Z]v &= \rho^{-Q+2}[X_i, Z]v + X_i(\rho^{-Q+2})Zv \\ &= \rho^{-Q+2}X_i v + (2-Q)\rho^{-Q+1}X_i \rho Zv. \end{aligned} \quad (3.20)$$

Consequently, using (3.19) and (3.20) in (3.18) we find

$$\begin{aligned} 4\beta \int \rho^{-Q+2}Zv \Delta_H v &= 4\beta \int \rho^{-Q+2}|\nabla_H v|^2 - 4\beta \int \rho^{-Q+2}|X_i v|^2 \\ &\quad + 4\beta(Q-2) \int \rho^{-Q+1}X_i \rho X_i v Zv. \end{aligned} \quad (3.21)$$

Since $|\nabla_H v|^2 = \sum_{i=1}^m |X_i v|^2$ and $\{X_1, X_2, \dots, X_m\}$ is an orthonormal set, we can rewrite (3.21) as follows

$$4\beta \int \rho^{-Q+2}Zv \Delta_H v = 4\beta(Q-2) \int \rho^{-Q+1}\langle \nabla_H v, \nabla_H \rho \rangle Zv. \quad (3.22)$$

Now, we use the definition (2.15) for E_v in (3.22) to get

$$\begin{aligned} 4\beta \int \rho^{-Q+2}Zv \Delta_H v &= 4\beta(Q-2) \int \rho^{-Q+1} \left(E_v + \frac{Zv}{\rho} \psi \right) Zv \\ &= 4\beta(Q-2) \int \rho^{-Q+1} E_v Zv + 4\beta(Q-2) \int \rho^{-Q} \psi (Zv)^2. \end{aligned} \quad (3.23)$$

We now use (3.14) and (2.16) in first integral of right-hand side of (3.23) to obtain

$$4\beta \int \rho^{-Q+2} Zv \Delta_H v \geq -4\beta(Q-2)C_E \int \rho^{-Q+\delta} |v| |Zv| \psi \\ + 4\beta(Q-2) \int \rho^{-Q} \psi(Zv)^2.$$

Subsequently we apply Young's inequality to get

$$4\beta \int \rho^{-Q+2} Zv \Delta_H v \\ \geq -2\beta(Q-2)C_E \int \rho^{-Q+\delta} \psi |v|^2 - 2\beta(Q-2)C_E \int \rho^{-Q+\delta} \psi(Zv)^2 \\ + 4\beta(Q-2) \int \rho^{-Q} \psi(Zv)^2. \quad (3.24)$$

We now choose R_0 small enough such that $C_E R_0^\delta \leq 1$, consequently, $C_E \rho^\delta < 1$. Hence (3.24) becomes

$$4\beta \int \rho^{-Q+2} Zv \Delta_H v \geq -2\beta(Q-2)C_E \int \rho^{-Q+\delta} \psi v^2 - 2\beta(Q-2) \\ \int \rho^{-Q} \psi(Zv)^2 + 4\beta(Q-2) \int \rho^{-Q} \psi(Zv)^2 \\ \geq -2\beta(Q-2)C_E \int \rho^{-Q+\delta} \psi v^2 + 2\beta(Q-2) \int \rho^{-Q} \psi(Zv)^2 \\ \geq -2\beta(Q-2)C_E \int \rho^{-Q+\delta} \psi v^2, \quad (3.25)$$

where the last inequality is a consequence of the fact that $Q \geq 2$.

We now simplify I_8 . We use the assumption (2.17) followed by Young's inequality ($2AB \leq A^2 + B^2$) with $A = Kv$ and $B = \beta Zv$ to get

$$|I_8| \leq 4\beta \int \rho^{2\beta-2\alpha-2} |V| |v| |Zv| \leq 4\beta K \int \rho^{2\beta-2\alpha-2} \psi |v| |Zv| \leq 2K^2 \\ \int \rho^{2\beta-2\alpha-2} \psi v^2 + 2\beta^2 \int \rho^{2\beta-2\alpha-2} \psi |Zv|^2. \quad (3.26)$$

Subsequently, we use the (3.10) to find

$$I_8 \geq -2K^2 \int \rho^{-Q+2} \psi v^2 - 2\beta^2 \int \rho^{-Q+2} \psi |Zv|^2. \quad (3.27)$$

Therefore using (3.11), (3.12), (3.16), (3.25), (3.27), (3.10) and (3.6), for $\alpha > Q$ and R_0 small enough we have we obtain

$$\begin{aligned}
& \int \rho^{-2\alpha} e^{2\alpha\rho^\varepsilon} (\Delta_H w + Vw)^2 \psi^{-1} \\
& \geq 4\beta^2 \int \rho^{-Q} |Zv|^2 \psi - 4\alpha^2 \beta \varepsilon^3 \int \rho^{-Q+2\varepsilon} \psi v^2 + 2\alpha\beta\varepsilon^2 (2\beta + \varepsilon + Q - 2) \\
& \quad \int \rho^{-Q+\varepsilon} \psi v^2 \\
& \quad - 8\alpha\beta\varepsilon \int \rho^{-Q+\varepsilon} (Zv)^2 \psi - 12\beta^2 C_E \int \rho^{-Q+\delta} |Zv|^2 \psi - 12\beta^2 C_E \int \rho^{-Q+\delta} |v|^2 \psi \\
& \quad - 2\beta(Q-2)C_E \int \rho^{-Q+\delta} \psi v^2 - 2K^2 \int \rho^{-Q+2} \psi v^2 - 2\beta^2 \int \rho^{-Q+2} \psi |Zv|^2. \quad (3.28)
\end{aligned}$$

Now we use $\varepsilon < 1$, $2\alpha > \beta$, $2\beta > \alpha$, which are consequences of (3.10) and $\alpha > Q$ respectively, and rearrange the terms in right-hand side of (3.28) to get

$$\begin{aligned}
& \int \rho^{-2\alpha} e^{2\alpha\rho^\varepsilon} (\Delta_H w + Vw)^2 \psi^{-1} \\
& \geq 4\beta^2 \int \rho^{-Q} (Zv)^2 \psi - 16\beta^2 \int \rho^{-Q+\varepsilon} (Zv)^2 \psi - 12\beta^2 C_E \int \rho^{-Q+\delta} |Zv|^2 \psi \\
& \quad - 2\beta^2 \int \rho^{-Q+2} \psi |Zv|^2 + 2\beta^3 \varepsilon^2 \int \rho^{-Q+\varepsilon} \psi v^2 - 16\beta^3 \int \rho^{-Q+2\varepsilon} \psi v^2 \\
& \quad - 12\beta^2 C_E \int \rho^{-Q+\delta} |v|^2 \psi - 4\beta^2 C_E \int \rho^{-Q+\delta} \psi v^2 - 2K^2 \int \rho^{-Q+2} \psi v^2 \quad (3.29)
\end{aligned}$$

At this point we would like to make the crucial observation that $-16\beta^2 C_E \int \rho^{-Q+\delta} \psi v^2$ can be absorbed in the term $2\beta^3 \varepsilon^2 \int \rho^{-Q+\varepsilon} \psi v^2$ provided that $\varepsilon < \delta$ and R_0 is chosen small enough. Thus we now choose $\varepsilon = \frac{\delta}{2}$ and R_0 small enough such that

$$(18 + 12C_E)R_0^{\delta/2} < 1 \text{ and } (16 + 16C_E)R_0^{\delta/2} < \varepsilon^2 \quad (3.30)$$

therefore we find

$$\begin{aligned}
& 4\beta^2 \int \rho^{-Q} (Zv)^2 \psi - (16 + 12C_E + 2)\beta^2 R_0^{\delta/2} \int \rho^{-Q} (Zv)^2 \psi \geq 3\beta^2 \\
& \quad \int \rho^{-Q} (Zv)^2 \psi \quad (3.31)
\end{aligned}$$

and

$$\begin{aligned}
& 2\beta^3 \varepsilon^2 \int \rho^{-Q+\varepsilon} \psi v^2 - (16 + 12C_E + 4C_E)\beta^3 R_0^{\delta/2} \int \rho^{-Q+\varepsilon} \psi v^2 \geq \beta^3 \varepsilon^2 \\
& \quad \int \rho^{-Q+\varepsilon} \psi v^2. \quad (3.32)
\end{aligned}$$

Hence using (3.31) and (3.32) in (3.29), we obtain

$$\int \rho^{-2\alpha} e^{2\alpha\rho^\varepsilon} (\Delta_H w + Vw)^2 \psi^{-1} \geq 3\beta^2 \int \rho^{-Q} (Zv)^2 \psi + \beta^3 \varepsilon^2 \int \rho^{-Q+\varepsilon} \psi v^2 - 2K^2 \int \rho^{-Q+2} \psi v^2. \tag{3.33}$$

Subsequently, if we choose

$$\alpha > \frac{2}{\varepsilon^{2/3}} K^{2/3} + Q$$

then from (3.10), we get $\beta^3 \varepsilon^2 \geq 8K^2$. Hence (3.33) becomes

$$\int \rho^{-2\alpha} e^{2\alpha\rho^\varepsilon} (\Delta_H w + Vw)^2 \psi^{-1} \geq \frac{\alpha^3 \varepsilon^2}{16} \int \rho^{-Q+\varepsilon} \psi v^2. \tag{3.34}$$

We now substitute $v = \rho^{-\beta} e^{\alpha\rho^\varepsilon} w$ and use (3.10) to get the desired estimate (2.18). This completes the proof of Theorem 2.1. \square

Proof of Theorem 2.2 We adapt arguments from [6, 9]. For a given $R_1 < R_2$, A_{R_1, R_2} will denote the annulus $B_{R_2} \setminus B_{R_1}$. We will denote an all purpose constant by letter C which might vary from line to line, and will depend only on C_E , Q and δ . Let R_0 be as in the Theorem 2.1 and let $0 < R_1 < 2R_1 < R_2 = R_0/4$. Also, we take a radial function $\phi \in C_0^\infty(B_{2R_2})$, i.e., $\phi(g) = f(\rho(g))$ for some f , such that

$$\begin{cases} \phi \equiv 0 \text{ if } \rho < R_1 \text{ and } \rho > 2R_2 \\ \phi \equiv 1 \text{ in } A_{2R_1, R_2} \end{cases} \tag{3.35}$$

and in the region $A_{R_1, 2R_1} \cup A_{R_2, 2R_2}$, the following bounds hold,

$$|\nabla_H \phi(g)| \leq \frac{C\psi^{1/2}}{\rho(g)}, \quad |\Delta_H \phi(g)| \leq \frac{C\psi}{\rho(g)^2}. \tag{3.36}$$

Note that we can assume that

$$\|u\psi^{1/2}\|_{L^2(B_{R_2})} \neq 0. \tag{3.37}$$

Otherwise by the arguments that follow we could conclude $u \equiv 0$ in B_{R_0} , which is a contradiction to the assumption that u is a non-trivial solution to (2.19). Since u satisfies $-\Delta_H u = Vu$, $w = u\phi$ satisfies

$$\Delta_H w + Vw = u\Delta_H \phi + 2\langle \nabla_H \phi, \nabla_H u \rangle. \tag{3.38}$$

As $\rho(g) \geq R_1$, we use (3.36) to obtain

$$(\Delta_H w + Vw)^2 \leq 2(u\Delta_H \phi)^2 + 8|\nabla_H \phi|^2 |\nabla_H u|^2 \leq C_1 \psi$$

for some $C_1 > 0$. Since ϕ is radial, we have $E_\phi = 0$ and consequently we get $E_w = \phi E_u$. Also since $0 \leq \phi \leq 1$, using (2.20) we find that E_w satisfies (2.16) and moreover by a standard limiting argument via approximation with smooth functions, we can apply the Carleman estimate in (2.18) to w . We thus obtain

$$\alpha^3 \int \rho^{-2\alpha-4+\varepsilon} e^{2\alpha\rho^\varepsilon} u^2 \phi^2 \psi \leq C \int \rho^{-2\alpha} e^{2\alpha\rho^\varepsilon} (u \Delta_H \phi + 2 \langle \nabla_H \phi, \nabla_H u \rangle)^2 \psi^{-1}. \quad (3.39)$$

Now we use the inequality $(a + b)^2 \leq 2a^2 + 2b^2$ and Cauchy-Schwarz inequality in the right-hand side of (3.39) to obtain

$$\begin{aligned} \alpha^3 \int \rho^{-2\alpha-4+\varepsilon} e^{2\alpha\rho^\varepsilon} u^2 \phi^2 \psi &\leq 2C \int \rho^{-2\alpha} e^{2\alpha\rho^\varepsilon} (u^2 (\Delta_H \phi)^2 \psi^{-1} \\ &\quad + |\nabla_H u|^2 |\nabla_H \phi|^2 \psi^{-1}). \end{aligned} \quad (3.40)$$

For convenience, we will denote L^2 norm of f in B_R and A_{R_1, R_2} by $\|f\|_R$ and $\|f\|_{R_1, R_2}$ respectively. Note that from (3.35), the functions $\nabla_H \phi$ and $\Delta_H \phi$ are supported in $A_{R_1, 2R_1} \cup A_{R_2, 2R_2}$. Further using (3.35) and (3.36) in (3.40), there exists a universal constant C such that

$$\begin{aligned} &\alpha^{3/2} \|\rho^{-\alpha-2+\varepsilon/2} e^{\alpha\rho^\varepsilon} u \psi^{1/2}\|_{2R_1, R_2} \\ &\leq C \left(\|\rho^{-\alpha-2} e^{\alpha\rho^\varepsilon} u \psi^{1/2}\|_{R_1, 2R_1} + \|\rho^{-\alpha-2} e^{\alpha\rho^\varepsilon} u \psi^{1/2}\|_{R_2, 2R_2} \right) \\ &\quad + C \left(R_1 \|\rho^{-\alpha-2} e^{\alpha\rho^\varepsilon} |\nabla_H u|\|_{R_1, 2R_1} + R_2 \|\rho^{-\alpha-2} e^{\alpha\rho^\varepsilon} |\nabla_H u|\|_{R_2, 2R_2} \right). \end{aligned} \quad (3.41)$$

We observe that the functions

$$r \rightarrow r^{-\alpha-2+\varepsilon/2} e^{\alpha r^\varepsilon}, \quad r \rightarrow r^{-\alpha-2} e^{\alpha r^\varepsilon}$$

are decreasing in $(0, 1)$, therefore (3.41) gives

$$\begin{aligned} &\alpha^{3/2} R_2^{-\alpha-2+\varepsilon/2} e^{\alpha R_2^\varepsilon} \|u \psi^{1/2}\|_{2R_1, R_2} \\ &\leq C \left(R_1^{-\alpha-2} e^{\alpha R_1^\varepsilon} \|u \psi^{1/2}\|_{R_1, 2R_1} + R_2^{-\alpha-2} e^{\alpha R_2^\varepsilon} \|u \psi^{1/2}\|_{R_2, 2R_2} \right) \\ &\quad + C \left(R_1 R_1^{-\alpha-2} e^{\alpha R_1^\varepsilon} \|\nabla_H u\|_{R_1, 2R_1} + R_2 R_2^{-\alpha-2} e^{\alpha R_2^\varepsilon} \|\nabla_H u\|_{R_2, 2R_2} \right). \end{aligned} \quad (3.42)$$

From the Caccioppoli estimate in Lemma 2.9, we have

$$\begin{cases} R_1 \|\nabla_H u\|_{R_1, 2R_1} \leq C(1 + K^{1/2}) \|u \psi^{1/2}\|_{4R_1}, \\ R_2 \|\nabla_H u\|_{R_2, 2R_2} \leq C(1 + K^{1/2}) \|u \psi^{1/2}\|_{R_0}. \end{cases} \quad (3.43)$$

We now use (3.43) in (3.42) and with possibly some large universal constant C , get

$$\begin{aligned} & \alpha^{3/2} R_2^{-\alpha-2+\varepsilon/2} e^{\alpha R_2^\varepsilon} \|u\psi^{1/2}\|_{2R_1, R_2} \\ & \leq C (R_1^{-\alpha-2} e^{\alpha R_1^\varepsilon} \|u\psi^{1/2}\|_{R_1, 2R_1} + R_2^{-\alpha-2} e^{\alpha R_2^\varepsilon} \|u\psi^{1/2}\|_{R_2, 2R_2} \\ & \quad + R_1^{-\alpha-2} e^{\alpha R_1^\varepsilon} (1 + K^{1/2}) \|u\psi^{1/2}\|_{4R_1} + R_2^{-\alpha-2} e^{\alpha R_2^\varepsilon} (1 + K^{1/2}) \|u\psi^{1/2}\|_{R_0}). \end{aligned} \quad (3.44)$$

At this point, we can repeat the arguments as in the proof of Theorem 1.3 in [9] to get to the desired conclusion. \square

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Declarations

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