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Entwined Modules Over Representations of Categories

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Abstract
We introduce a theory of modules over a representation of a small category taking values in entwining structures over a semiperfect coalgebra. This takes forward the aim of developing categories of entwined modules to the same extent as that of module categories as well as the philosophy of Mitchell of working with rings with several objects. The representations are motivated by work of Estrada and Virili, who developed a theory of modules over a representation taking values in small preadditive categories, which were then studied in the same spirit as sheaves of modules over a scheme. We also describe, by means of Frobenius and separable functors, how our theory relates to that of modules over the underlying representation taking values in small K-linear categories.

Keywords Rings with several objects · Entwined modules · Separable functors · Frobenius pairs

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1 Introduction

The purpose of this paper is to study modules over representations of a small category taking values in spaces that behave like quotients of categorified fiber bundles. Let H be a Hopf algebra having a coaction $\rho : A \longrightarrow A \otimes H$ on an algebra A such that A becomes an H-comodule algebra. Let B denote the algebra of coinvariants of this coaction. Suppose that the inclusion $B \hookrightarrow A$ is faithfully flat and the canonical morphism

$$can : A \otimes_B A \longrightarrow A \otimes H \qquad x \otimes y \mapsto x \cdot \rho(y)$$

is an isomorphism. This datum is the algebraic counterpart of a principal fiber bundle given by the quotient of an affine algebraic group scheme acting freely on an affine scheme over

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a field K (see, for instance, [25, 29]). If H has bijective antipode, then modules over the algebra B of coinvariants may be recovered as "(A, H)-Hopf modules" (see Schneider [29]).

These (A, H)-Hopf modules may be rolled into the more general concept of modules over an 'entwining structure' consisting of an algebra R, a coalgebra C and a morphism $\psi : C \otimes R \longrightarrow R \otimes C$ satisfying certain conditions. Entwining structures were introduced by Brzeziński and Majid [10]. It was soon realized (see Brzeziński [5]) that entwining structures provide a single formalism that unifies relative Hopf modules, Doi-Hopf modules, Yetter-Drinfeld modules and several other concepts such as coalgebra Galois extensions. As pointed out in Brzeziński [7], an entwining structure (R, C, ψ) behaves like a single bialgebra, or more generally a comodule algebra over a bialgebra. Accordingly, the investigation of entwining structures as well as the modules over them has emerged as an object of study in its own right (see, for instance, [1, 3–5, 8, 12–14, 21, 22, 28]).

We consider an entwining structure consisting of a small K-linear category $\mathcal{R}$, a coalgebra C and a family of morphisms

$$\psi = \{\psi_{rs} : C \otimes \mathcal{R}(r, s) \longrightarrow \mathcal{R}(r, s) \otimes C\}_{r,s \in \mathcal{R}}$$

satisfying certain conditions (see Definition 2.1). This is in keeping with the general philosophy of Mitchell [24], where a small K-linear category is viewed as a K-algebra with several objects. In fact, we consider the category $\mathscr{E}nt$ of such entwining structures. When the coalgebra C is fixed, we have the subcategory $\mathscr{E}nt_C$. Given an entwining structure $(\mathcal{R}, C, \psi)$, we have a category $\mathbf{M}^C_{\mathcal{R}}(\psi)$ of modules over it (see our earlier work in [4]). These entwined modules over $(\mathcal{R}, C, \psi)$ may be seen as modules over a certain categorical quotient space of $\mathcal{R}$, which need not exist in an explicit sense, but is studied only through its category of modules.

We work with representations $\mathscr{R} : \mathscr{X} \longrightarrow \mathscr{E}nt_C$ of a small category $\mathscr{X}$ taking values in $\mathscr{E}nt_C$, where C is a fixed coalgebra. This is motivated by the work of Estrada and Virili [19], who introduced a theory of modules over a representation $\mathscr{A} : \mathscr{X} \longrightarrow Add$, where Add is the category of small preadditive categories. The modules over $\mathscr{A} : \mathscr{X} \longrightarrow Add$ were studied in the spirit of sheaves of modules over a scheme, or more generally, a ringed space. By considering small preaditive categories, the authors in [19] also intended to take Mitchell's idea one step forward: from replacing rings with small preadditive categories to replacing ring representations by representations taking values in small preadditive categories. In this paper, we develop a theory of modules over a representation $\mathscr{R} : \mathscr{X} \longrightarrow \mathscr{E}nt_C$ taking values in entwining structures. We also describe, by means of Frobenius and separable functors, how this theory relates to that of modules over the underlying representation taking values in small K-linear categories.

This paper has two parts. In the first part, we introduce and develop the properties of the category $Mod^C - \mathscr{R}$ of modules over $\mathscr{R} : \mathscr{X} \longrightarrow \mathscr{E}nt_C$. For this, we have to combine techniques on comodules along with adapting the methods of Estrada and Virili [19]. When $\mathscr{R} : \mathscr{X} \longrightarrow \mathscr{E}nt_C$ is a flat representation (see Section 6), we also consider the subcategory $Cart - \mathscr{R}$ of cartesian entwined modules over $\mathscr{R}$. In the analogy with sheaves of modules over a scheme, the cartesian objects may be seen as similar to quasi-coherent sheaves.

Let $\mathscr{L}in$ be the category of small K-linear categories. In the second part, we consider the underlying representation $\mathscr{R} : \mathscr{X} \longrightarrow \mathscr{E}nt_C \longrightarrow \mathscr{L}in$, which we continue to denote by $\mathscr{R}$. Accordingly, we have a category $Mod - \mathscr{R}$ of modules over $\mathscr{R} : \mathscr{X} \longrightarrow \mathscr{E}nt_C \longrightarrow \mathscr{L}in$ in the sense of Estrada and Virili [19]. We study the relation between $Mod^C - \mathscr{R}$ and$Mod - \mathscr{R}$

by describing Frobenius and separability conditions for a pair of adjoint functors between them (see Section 7)

$$\mathscr{F} : Mod^C - \mathscr{R} \longrightarrow Mod - \mathscr{R} \qquad\qquad \mathscr{G} : Mod - \mathscr{R} \longrightarrow Mod^C - \mathscr{R}$$

Here, the left adjoint $\mathscr{F}$ may be thought of as an 'extension of scalars' and the right adjoint $\mathscr{G}$ as a 'restriction of scalars.'

The idea is as follows: as mentioned before, modules over an entwining structure $(\mathcal{R}, C, \psi)$ may be seen as modules over a certain categorical quotient space of $\mathcal{R}$, which behaves like a subcategory of $\mathcal{R}$. Again, this "subcategory" of $\mathcal{R}$ need not exist in an explicit sense, but is studied only through the category of modules $\mathbf{M}^C_{\mathcal{R}}(\psi)$. Accordingly, a representation $\mathscr{R} : \mathscr{X} \longrightarrow \mathscr{E}nt_C$ taking values in $\mathscr{E}nt_C$ may be thought of as a subfunctor of the underlying representation $\mathscr{R} : \mathscr{X} \longrightarrow \mathscr{E}nt_C \longrightarrow \mathscr{L}in$. We want to understand the properties of the inclusion of this "subfunctor": in particular, whether it behaves like a separable, split or Frobenius extension of rings. We recall here (see [9, Theorem 1.2]) that if $R \longrightarrow S$ is an extension of rings, these properties may be expressed in terms of the functors $F : Mod - R \longrightarrow Mod - S$ (extension of scalars) and $G : Mod - S \longrightarrow Mod - R$ (restriction of scalars) as follows

$R \longrightarrow S$ split extension	$\Leftrightarrow$	$F : Mod - R \longrightarrow Mod - S$ separable
$R \longrightarrow S$ separable extension	$\Leftrightarrow$	$G : Mod - S \longrightarrow Mod - R$ separable
$R \longrightarrow S$ Frobenius extension	$\Leftrightarrow$	(F, G)Frobenius pair of functors

We now describe the paper in more detail. Throughout,we let K be a field. We begin in Section 2 by describing the categories of entwining structures and entwined modules. For a morphism $(\alpha, \gamma) : (\mathcal{R}, C, \psi) \longrightarrow (\mathcal{S}, D, \psi')$ of entwining structures, we describe 'extension of scalars' and 'restriction of scalars' on categories of entwined modules. Our first result is as follows.

Theorem 1 *(see Propositions 2.3, 2.4 and Theorem 2.5) Let* $(\alpha, \gamma) : (\mathcal{R}, C, \psi) \longrightarrow (\mathcal{S}, D, \psi')$ *be a morphism of entwining structures.*

(1) There is a functor $(\alpha, \gamma)^* : \mathbf{M}^C_{\mathcal{R}}(\psi) \longrightarrow \mathbf{M}^D_{\mathcal{S}}(\psi')$ *of extension of scalars.*
(2) Suppose that the coalgebra map $\gamma : C \longrightarrow D$ *is also a monomorphism of vector spaces. Then, there is a functor* $(\alpha, \gamma)_* : \mathbf{M}^D_{\mathcal{S}}(\psi') \longrightarrow \mathbf{M}^C_{\mathcal{R}}(\psi)$ *of restriction of scalars. Further, there is an adjunction of functors which is given by natural isomorphisms*

$$\mathbf{M}^D_{\mathcal{S}}(\psi')((\alpha, \gamma)^*\mathcal{M}, \mathcal{N}) = \mathbf{M}^C_{\mathcal{R}}(\psi)(\mathcal{M}, (\alpha, \gamma)_*\mathcal{N})$$

for any $\mathcal{M} \in \mathbf{M}^C_{\mathcal{R}}(\psi)$ *and* $\mathcal{N} \in \mathbf{M}^D_{\mathcal{S}}(\psi')$.

In Section 3, we give conditions for the category $\mathbf{M}^C_{\mathcal{R}}(\psi)$ of modules over an entwining structure $(\mathcal{R}, C, \psi)$ to have projective generators. We recall that a K-coalgebra C is said to be right semiperfect if the category of right C-comodules has enough projectives.

Theorem 2 *(see Theorem 3.5) Let* $(\mathcal{R}, C, \psi)$ *be an entwining structure and let C be a right semiperfect K-coalgebra. Then, the category* $\mathbf{M}^C_{\mathcal{R}}(\psi)$ *of entwined modules is a Grothendieck category with a set of projective generators.*

In Section 4, we fix a coalgebra C. We introduce the category $Mod^C - \mathscr{R}$ of modules over a representation $\mathscr{R} : \mathscr{X} \longrightarrow \mathscr{E}nt_C$, which is our main object of study. Our first purpose is to show that $Mod^C - \mathscr{R}$ is a Grothendieck category.

Theorem 3 *(see Theorem 4.9) Let C be a right semiperfect coalgebra over a field K. Let $\mathscr{R} : \mathscr{X} \longrightarrow \mathscr{E}nt_C$ be an entwined C-representation of a small category $\mathscr{X}$. Then, the category $Mod^C - \mathscr{R}$ of entwined modules over $\mathscr{R}$ is a Grothendieck category.*

Given $\mathscr{R} : \mathscr{X} \longrightarrow \mathscr{E}nt_C$, we have an entwining structure $(\mathscr{R}_x, C, \psi_x)$ for each $x \in \mathscr{X}$. Our next aim is to give conditions for $Mod^C - \mathscr{R}$ to have projective generators. For this, we will construct an extension functor ex_x^C and an evaluation functor ev_x^C relating the categories $Mod^C - \mathscr{R}$ and $\mathbf{M}^C_{\mathscr{R}_x}(\psi_x)$ at each $x \in \mathscr{X}$.

Theorem 4 *(see Proposition 5.3 and Theorem 5.5) Let C be a right semiperfect coalgebra over a field K. Let $\mathscr{X}$ be a poset and let $\mathscr{R} : \mathscr{X} \longrightarrow \mathscr{E}nt_C$ be an entwined C-representation of $\mathscr{X}$.*

(1) For each $x \in \mathscr{X}$, there is an extension functor $ex_x^C : \mathbf{M}^C_{\mathscr{R}_x} \longrightarrow Mod^C - \mathscr{R}$ which is left adjoint to an evaluation functor $ev_x^C : Mod^C - \mathscr{R} \longrightarrow \mathbf{M}^C_{\mathscr{R}_x}(\psi_x)$.

(2) The family $\{ex_x^C(V \otimes H_r) | x \in \mathscr{X}, r \in \mathscr{R}_x, V \in Proj^f(C)\}$ is a set of projective generators for $Mod^C - \mathscr{R}$, where $Proj^f(C)$ is the set of isomorphism classes of finite dimensional projective C-comodules.

We introduce the category of cartesian entwined modules in Section 6. Here, we will assume that $\mathscr{X}$ is a poset and $\mathscr{R} : \mathscr{X} \longrightarrow \mathscr{E}nt_C$ is a flat representation, i.e., for any morphism $\alpha : x \longrightarrow y$ in $\mathscr{X}$, the functor $\alpha^* := \mathscr{R}^*_\alpha : \mathbf{M}^C_{\mathscr{R}_x}(\psi_x) \longrightarrow \mathbf{M}^C_{\mathscr{R}_y}(\psi_y)$ is exact. We then apply induction on $\mathbb{N} \times Mor(\mathscr{X})$ to show that any cartesian entwined module may be expressed as a sum of submodules whose cardinality is $\leq \kappa := sup\{|\mathbb{N}|, |C|, |K|, |Mor(\mathscr{X})|, |Mor(\mathscr{R}_x)|, x \in \mathscr{X}\}$.

Theorem 5 *(see Theorem 6.1) Let C be a right semiperfect coalgebra over a field K. Let $\mathscr{X}$ be a poset and let $\mathscr{R} : \mathscr{X} \longrightarrow \mathscr{E}nt_C$ be an entwined C-representation of $\mathscr{X}$. Suppose that $\mathscr{R}$ is flat. Then, $Cart^C - \mathscr{R}$ is a Grothendieck category.*

In the next three sections, we study separability and Frobenius conditions for functors relating $Mod^C - \mathscr{R}$ to the category $Mod - \mathscr{R}$ of modules over the underlying representation $\mathscr{R} : \mathscr{X} \longrightarrow \mathscr{E}nt_C \longrightarrow \mathscr{L}in$. For this, we have to adapt the techniques from [9] as well as our earlier work in [4]. For more on Frobenius and separability conditions for Doi-Hopf modules and modules over entwining structures of algebras, we refer the reader to [6, 15–17].

At each $x \in \mathscr{X}$, we have functors $\mathscr{F}_x : \mathbf{M}^C_{\mathscr{R}_x}(\psi_x) \longrightarrow \mathbf{M}_{\mathscr{R}_x}$ and $\mathscr{G}_x : \mathbf{M}_{\mathscr{R}_x} \longrightarrow \mathbf{M}^C_{\mathscr{R}_x}(\psi_x)$ which combine to give functors $\mathscr{F} : Mod^C - \mathscr{R} \longrightarrow Mod - \mathscr{R}$ and $\mathscr{G} : Mod - \mathscr{R} \longrightarrow Mod^C - \mathscr{R}$ respectively. We will also need to consider a space V_1 of elements $\theta = \{\theta_x(r) : C \otimes C \longrightarrow \mathscr{R}_x(r, r)\}_{x \in \mathscr{X}, r \in \mathscr{R}_x}$ and a space W_1 of elements $\eta = \{\eta_x(s, r) : \mathscr{R}_x(s, r) \longrightarrow \mathscr{R}_x(s, r) \otimes C\}_{x \in \mathscr{X}, r, s \in \mathscr{R}_x}$ satisfying certain conditions (see Sections 7 and 8).

Theorem 6 *(see Propositions 7.2, 7.3 and 7.7) Let $\mathscr{X}$ be a poset, C be a right semiperfect K-coalgebra and $\mathscr{R} : \mathscr{X} \longrightarrow \mathscr{E}nt_C$ be an entwined C-representation.*

(1) The forgetful functor $\mathscr{F} : Mod^C - \mathscr{R} \longrightarrow Mod - \mathscr{R}$ has a right adjoint $\mathscr{G} : Mod - \mathscr{R} \longrightarrow Mod^C - \mathscr{R}$.

(2) A natural transformation $\upsilon \in Nat(\mathscr{G}\mathscr{F}, 1_{Mod^C - \mathscr{R}})$ corresponds to a collection of natural transformations $\{\upsilon_x \in Nat(\mathscr{G}_x\mathscr{F}_x, 1_{\mathbf{M}^C_{\mathscr{R}_x}(\psi_x)})\}_{x \in \mathscr{X}}$ such that for any $\alpha : x \longrightarrow y$ in $\mathscr{X}$ and object $\mathscr{M} \in Mod^C - \mathscr{R}$, we have $\mathscr{M}_\alpha \circ \upsilon_x(\mathscr{M}_x) = \alpha_* \upsilon_y(\mathscr{M}_y) \circ \mathscr{G}_x\mathscr{F}_x(\mathscr{M}_\alpha)$.

(3) The space $Nat(\mathscr{G}\mathscr{F}, 1_{Mod^C - \mathscr{R}})$ is isomorphic to V_1.

The main results in Sections 7 and 8 give necessary and sufficient conditions for the forgetful functor $\mathscr{F} : Mod^C - \mathscr{R} \longrightarrow Mod - \mathscr{R}$ and its right adjoint $\mathscr{G} : Mod - \mathscr{R} \longrightarrow Mod^C - \mathscr{R}$ to be separable. In Section 9, we give necessary and sufficient conditions for $(\mathscr{F}, \mathscr{G})$ to be a Frobenius pair, i.e., $\mathscr{G}$ is both a left and a right adjoint of $\mathscr{F}$.

Theorem 7 *(see Theorem 7.8, Propositions 7.9 and 7.10) Let $\mathscr{X}$ be a partially ordered set. Let C be a right semiperfect K-coalgebra and let $\mathscr{R} : \mathscr{X} \longrightarrow \mathscr{E}nt_C$ be an entwined C-representation.*

(1) The functor $\mathscr{F} : Mod^C - \mathscr{R} \longrightarrow Mod - \mathscr{R}$ is separable if and only if there exists $\theta \in V_1$ such that $\theta_x(r)(c_1 \otimes c_2) = \varepsilon_C(c) \cdot id_r$ for every $x \in \mathscr{X}$, $r \in \mathscr{R}_x$ and $c \in C$.
(2) Suppose additionally that the representation $\mathscr{R} : \mathscr{X} \longrightarrow \mathscr{E}nt_C$ is flat. Then, we have

(a) The functor $\mathscr{F} : Mod^C - \mathscr{R} \longrightarrow Mod - \mathscr{R}$ restricts to a functor $\mathscr{F}^c : Cart^C - \mathscr{R} \longrightarrow Cart - \mathscr{R}$. Moreover, $\mathscr{F}^c$ has a right adjoint $\mathscr{G}^c : Cart - \mathscr{R} \longrightarrow Cart^C - \mathscr{R}$.
(b) Suppose there exists $\theta \in V_1$ such that $\theta_x(r)(c_1 \otimes c_2) = \varepsilon_C(c) \cdot id_r$ for every $x \in \mathscr{X}$, $r \in \mathscr{R}_x$ and $c \in C$. Then, $\mathscr{F}^c : Cart^C - \mathscr{R} \longrightarrow Cart - \mathscr{R}$ is separable.

Theorem 8 *(see Proposition 8.2 and Theorem 8.3) Let $\mathscr{X}$ be a partially ordered set, C be a right semiperfect K-coalgebra and let $\mathscr{R} : \mathscr{X} \longrightarrow \mathscr{E}nt_C$ be an entwined C-representation.*

(1) The spaces $Nat(1_{Mod-\mathscr{R}}, \mathscr{F}\mathscr{G})$ and W_1 are isomorphic.
(2) The functor $\mathscr{G} : Mod - \mathscr{R} \longrightarrow Mod^C - \mathscr{R}$ is separable if and only if there exists $\eta \in W_1$ such that $id = (id \otimes \varepsilon_C) \circ \eta_x(s, r)$ for each $x \in \mathscr{X}$ and $s, r \in \mathscr{R}_x$.

Theorem 9 *(see Theorem 9.1, Proposition 9.4, Corollary 9.5) Let $\mathscr{X}$ be a partially ordered set, C be a right semiperfect K-coalgebra and let $\mathscr{R} : \mathscr{X} \longrightarrow \mathscr{E}nt_C$ be an entwined C-representation.*

(1) $(\mathscr{F}, \mathscr{G})$ is a Frobenius pair if and only if there exist $\theta \in V_1$ and $\eta \in W_1$ such that $\varepsilon_C(d) f = \sum \widehat{f} \circ \theta_x(r)(c_f \otimes d)$ and $\varepsilon_C(d) f = \sum \widehat{f_{\psi_x}} \circ \theta_x(r)(d^{\psi_x} \otimes c_f)$ for every $x \in \mathscr{X}$, $r \in \mathscr{R}_x$, $f \in \mathscr{R}_x(r, s)$ and $d \in C$, where $\eta_x(r, s)(f) = \widehat{f} \otimes c_f$.
(2) Suppose additionally that the representation $\mathscr{R} : \mathscr{X} \longrightarrow \mathscr{E}nt_C$ is flat. Then, $\mathscr{G} : Mod - \mathscr{R} \longrightarrow Mod^C - \mathscr{R}$ restricts to a functor $\mathscr{G}^c : Cart - \mathscr{R} \longrightarrow Cart^C - \mathscr{R}$. Further, $(\mathscr{F}^c, \mathscr{G}^c)$ is a Frobenius pair of adjoint functors between $Cart^C - \mathscr{R}$ and $Cart - \mathscr{R}$.

We conclude in Section 10 by giving examples of how to construct entwined representations and describe modules over them. In particular, we show how to construct entwined representations using B-comodule categories, where B is a bialgebra.

2 Category of Entwining Structures

Let K be a field and let $Vect_K$ be the category of vector spaces over K. Let $\mathcal{R}$ be a small K-linear category. The category of right $\mathcal{R}$-modules will be denoted by $\mathbf{M}_{\mathcal{R}}$. For any object $r \in \mathcal{R}$, we denote by $H_r : \mathcal{R}^{op} \longrightarrow Vect_K$ the right $\mathcal{R}$-module represented by r and by ${}_rH : \mathcal{R} \longrightarrow Vect_K$ the left $\mathcal{R}$-module represented by r. Given a K-coalgebra C, the category of right C-comodules will be denoted by $Comod - C$.

Definition 2.1 (see [4, §2]) Let $\mathcal{R}$ be a small K-linear category and C be a K-coalgebra. An entwining structure $(\mathcal{R}, C, \psi)$ over K is a collection of K-linear morphisms

$$\psi = \{\psi_{rs} : C \otimes \mathcal{R}(r, s) \longrightarrow \mathcal{R}(r, s) \otimes C\}_{r,s\in\mathcal{R}}$$

satisfying the following conditions

$$(gf)_\psi \otimes c^\psi = g_\psi f_\psi \otimes c^{\psi\psi} \qquad \varepsilon_C(c^\psi)(f_\psi) = \varepsilon_C(c)f$$
$$f_\psi \otimes \Delta_C(c^\psi) = f_{\psi_\psi} \otimes c_1{}^\psi \otimes c_2{}^\psi \qquad \psi(c \otimes id_r) = id_r \otimes c \tag{2.1}$$

for each $f \in \mathcal{R}(r,s)$, $g \in \mathcal{R}(s,t)$ and $c \in C$. Here, we have suppressed the summation and written $\psi(c \otimes f)$ simply as $f_\psi \otimes c^\psi$.

A morphism $(\alpha,\gamma) : (\mathcal{R}, C, \psi) \longrightarrow (\mathcal{S}, D, \psi')$ of entwining structures consists of a functor $\alpha : \mathcal{R} \longrightarrow \mathcal{S}$ and a counital coalgebra map $\gamma : C \longrightarrow D$ such that $\alpha(f_\psi)\otimes\gamma(c^\psi) = \alpha(f)_{\psi'} \otimes \gamma(c)^{\psi'}$ for any $c \otimes f \in C \otimes \mathcal{R}(r,s)$, where $r, s \in \mathcal{R}$.

We will denote by $\mathscr{E}nt$ the category of entwining structures over K.

If $\mathcal{M}$ is a right $\mathcal{R}$-module, $m \in \mathcal{M}(r)$ and $f \in \mathcal{R}(s,r)$, the element $\mathcal{M}(f)(r) \in \mathcal{M}(s)$ will often be denoted by mf.

If $\alpha : \mathcal{R} \longrightarrow \mathcal{S}$ is a functor of small K-linear categories, there is an obvious functor $\alpha_* : \mathbf{M}_\mathcal{S} \longrightarrow \mathbf{M}_\mathcal{R}$ of restriction of scalars. For the sake of convenience, we briefly recall here the well known extension of scalars $\alpha^* : \mathbf{M}_\mathcal{R} \longrightarrow \mathbf{M}_\mathcal{S}$. For $\mathcal{M} \in \mathbf{M}_\mathcal{R}$, the module $\alpha^*(\mathcal{M}) \in \mathbf{M}_\mathcal{S}$ is determined by setting

$$\alpha^*(\mathcal{M})(s) := \left(\bigoplus_{r\in\mathcal{R}} \mathcal{M}(r) \otimes \mathcal{S}(s,\alpha(r))\right)/V \tag{2.2}$$

for $s \in \mathcal{S}$, where V is the subspace generated by elements of the form

$$(m' \otimes \alpha(g)f) - (m'g \otimes f) \tag{2.3}$$

for $m' \in \mathcal{M}(r')$, $g \in \mathcal{R}(r,r')$, $f \in \mathcal{S}(s,\alpha(r))$ and $r, r' \in \mathcal{R}$.

On the other hand, if $\gamma : C \longrightarrow D$ is a morphism of coalgebras and N is a right C-comodule, there is an obvious corestriction of scalars $\gamma^* : Comod - C \longrightarrow Comod - D$. The functor γ^* has a well known right adjoint $\gamma_* : Comod - D \longrightarrow Comod - C$, known as the coinduction functor, given by the cotensor product $N \mapsto N\square_D C$ (see, for instance, [11, §11.10]). In general, we recall that the cotensor product $N\square_D N'$ of a right D-comodule $(N, \rho : N \longrightarrow N \otimes D)$ with a left D-comodule $(N', \rho' : N' \longrightarrow D \otimes N')$ is given by the equalizer

$$N \square_D N' := Eq\left(N \otimes N' \underset{id\otimes\rho'}{\overset{\rho\otimes id}{\rightrightarrows}} N \otimes D \otimes N'\right) \tag{2.4}$$

In other words, an element $\sum n_i \otimes n_i' \in N \otimes N'$ lies in $N\square_D N'$ if and only if $\sum n_{i0}\otimes n_{i1} \otimes n_i' = \sum n_i \otimes n'_{i0} \otimes n'_{i1}$. However, we will continue to suppress the summation and write an element of $N\square_D N'$ simply as $n \otimes n'$. We will now consider modules over an entwining structure $(\mathcal{R}, C, \psi)$.

Definition 2.2 (see [4, Definition 2.2]) Let $\mathcal{M}$ be a right $\mathcal{R}$-module with a given right C-comodule structure $\rho_{\mathcal{M}(s)} : \mathcal{M}(s) \longrightarrow \mathcal{M}(s) \otimes C$ on $\mathcal{M}(s)$ for each $s \in \mathcal{R}$. Then, $\mathcal{M}$ is said to be an entwined module over $(\mathcal{R}, C, \psi)$ if the following compatibility condition holds:

$$\rho_{\mathcal{M}(s)}(mf) = (mf)_0 \otimes (mf)_1 = m_0 f_\psi \otimes m_1{}^\psi \tag{2.5}$$

for every $f \in \mathcal{R}(s,r)$ and $m \in \mathcal{M}(r)$.

A morphism $\eta : \mathcal{M} \longrightarrow \mathcal{N}$ of entwined modules is a morphism $\eta : \mathcal{M} \longrightarrow \mathcal{N}$ in $\mathbf{M}_\mathcal{R}$ such that $\eta(r) : \mathcal{M}(r) \longrightarrow \mathcal{N}(r)$ is C-colinear for each $r \in \mathcal{R}$. The category of entwined modules over $(\mathcal{R}, C, \psi)$ will be denoted by $\mathbf{M}^C_\mathcal{R}(\psi)$.

Proposition 2.3 *Let* $(\alpha,\gamma):(\mathcal{R},C,\psi)\longrightarrow(\mathcal{S},D,\psi')$ *be a morphism of entwining structures. Then, there is a functor* $(\alpha,\gamma)^*:\mathbf{M}_{\mathcal{R}}^C(\psi)\longrightarrow\mathbf{M}_{\mathcal{S}}^D(\psi')$.

Proof We take $\mathcal{M}\in\mathbf{M}_{\mathcal{R}}^C(\psi)$. Then, $\mathcal{M}\in\mathbf{M}_{\mathcal{R}}$ and we consider $\mathcal{N}:=\alpha^*(\mathcal{M})\in\mathbf{M}_{\mathcal{S}}$. For $s\in S$, we consider an element $m\otimes f\in\mathcal{N}(s)$, where $m\in\mathcal{M}(r)$ and $f\in\mathcal{S}(s,\alpha(r))$ for some $r\in\mathcal{R}$. We claim that the morphism

$$\rho_{\mathcal{N}(s)}:\mathcal{N}(s)\longrightarrow\mathcal{N}(s)\otimes D\qquad (m\otimes f)\mapsto(m\otimes f)_0\otimes(m\otimes f)_1:=(m_0\otimes f_{\psi'})\otimes\gamma(m_1)^{\psi'} \tag{2.6}$$

makes $\mathcal{N}(s)$ a right D-comodule. Here, the association $m\mapsto m_0\otimes m_1$ comes from the C-comodule structure $\rho_{\mathcal{M}(r)}:\mathcal{M}(r)\longrightarrow\mathcal{M}(r)\otimes C$ of $\mathcal{M}(r)$.

First, we show that $\rho_{\mathcal{N}(s)}$ is well defined. For this, we consider $m'\in\mathcal{M}(r')$, $g\in\mathcal{R}(r,r')$ and $f\in\mathcal{S}(s,\alpha(r))$. We have

$$\begin{aligned}(m'g\otimes f)_0\otimes(m'g\otimes f)_1&=((m'g)_0\otimes f_{\psi'})\otimes\gamma((m'g)_1)^{\psi'}\\&=(m'_0g_\psi\otimes f_{\psi'})\otimes\gamma(m_1'^{\psi})^{\psi'}\\&=(m'_0\otimes\alpha(g_\psi)f_{\psi'})\otimes\gamma(m_1'^{\psi})^{\psi'}\\&=(m'_0\otimes\alpha(g)_{\psi'}f_{\psi'})\otimes\gamma(m'_1)^{\psi'\psi'}\\&=(m'_0\otimes(\alpha(g)f)_{\psi'})\otimes\gamma(m'_1)^{\psi'}\end{aligned} \tag{2.7}$$

From the properties of entwining structures, it may be easily verified that the structure maps in (2.6) are coassociative and counital, giving a right D-comodule structure on $\mathcal{N}(s)$. We now consider $f'\in\mathcal{S}(s',s)$. Then, we have

$$\begin{aligned}(m\otimes ff')_0\otimes(m\otimes ff')_1&=(m_0\otimes(ff')_{\psi'})\otimes\gamma(m_1)^{\psi'}\\&=(m_0\otimes f_{\psi'})f'_{\psi'}\otimes\gamma(m_1)^{\psi'\psi'}\\&=(m\otimes f)_0f'_{\psi'}\otimes(m\otimes f)_1^{\psi'}\end{aligned} \tag{2.8}$$

This shows that $\mathcal{N}\in\mathbf{M}_{\mathcal{S}}^D(\psi')$. □

Proposition 2.4 *Let* $(\alpha,\gamma):(\mathcal{R},C,\psi)\longrightarrow(\mathcal{S},D,\psi')$ *be a morphism of entwining structures. Suppose additionally that* $\gamma:C\longrightarrow D$ *is a monomorphism of vector spaces. Then, there is a functor* $(\alpha,\gamma)_*:\mathbf{M}_{\mathcal{S}}^D(\psi')\longrightarrow\mathbf{M}_{\mathcal{R}}^C(\psi)$.

Proof We take $\mathcal{N}\in\mathbf{M}_{\mathcal{S}}^D(\psi')$ and set $\mathcal{M}(r):=\mathcal{N}(\alpha(r))\square_D C$ for each $r\in\mathcal{R}$. For $f\in\mathcal{R}(r',r)$, we define

$$\mathcal{M}(f):\mathcal{M}(r)\longrightarrow\mathcal{M}(r')\qquad n\otimes c\mapsto(n\otimes c)\cdot f:=n\alpha(f_\psi)\otimes c^\psi \tag{2.9}$$

To show that this morphism is well defined, we need to check that $\mathcal{M}(f)(n\otimes c)\in\mathcal{M}(r')=\mathcal{N}(\alpha(r'))\square_D C$. Since $n\otimes c\in\mathcal{N}(\alpha(r))\square_D C$, we know that

$$n_0\otimes n_1\otimes c=n\otimes\gamma(c_1)\otimes c_2 \tag{2.10}$$

In particular, it follows that

$$n \otimes c \otimes f \in Eq \left(\begin{array}{c} \mathcal{N}(\alpha(r)) \otimes C \otimes \mathcal{R}(r', r) \\ \downarrow\downarrow \\ \mathcal{N}(\alpha(r)) \otimes D \otimes C \otimes \mathcal{R}(r', r) \\ \\ \Big\downarrow{\scriptstyle id\otimes id\otimes\psi} \\ \mathcal{N}(\alpha(r)) \otimes D \otimes \mathcal{R}(r', r) \otimes C \\ \Big\downarrow{\scriptstyle id\otimes id\otimes\alpha\otimes id} \\ \mathcal{N}(\alpha(r)) \otimes D \otimes \mathcal{S}(\alpha(r'), \alpha(r)) \otimes C \\ \Big\downarrow{\scriptstyle id\otimes\psi'\otimes id} \\ \mathcal{N}(\alpha(r)) \otimes \mathcal{S}(\alpha(r'), \alpha(r)) \otimes D \otimes C \\ \Big\downarrow \\ \mathcal{N}(\alpha(r')) \otimes D \otimes C \end{array} \right) \tag{2.11}$$

From (2.11), it follows that

$$n_0\alpha(f_\psi)_{\psi'} \otimes n_1^{\psi'} \otimes c^\psi = n\alpha(f_\psi)_{\psi'} \otimes \gamma(c_1)^{\psi'} \otimes {c_2}^\psi \tag{2.12}$$

Applying (2.10) and (2.12), we now see that

$$\begin{aligned} (n\alpha(f_\psi))_0 \otimes (n\alpha(f_\psi))_1 \otimes c^\psi &= n_0\alpha(f_\psi)_{\psi'} \otimes n_1^{\psi'} \otimes c^\psi \\ &= n\alpha(f_\psi)_{\psi'} \otimes \gamma(c_1)^{\psi'} \otimes {c_2}^\psi \\ &= n\alpha(f_{\psi\psi}) \otimes \gamma({c_1}^\psi) \otimes {c_2}^\psi \\ &= n\alpha(f_\psi) \otimes \gamma({c^\psi}_1) \otimes {c^\psi}_2 \end{aligned} \tag{2.13}$$

From the definition, we may easily verify that the structure maps in (2.9) make $\mathcal{M}$ into a right $\mathcal{R}$-module. To show that $\mathcal{M}$ is entwined, it remains to check that

$$\begin{aligned} n\alpha(f_\psi) \otimes {c^\psi}_1 \otimes {c^\psi}_2 = ((n \otimes c) \cdot f)_0 \otimes ((n \otimes c) \cdot f)_1 = (n \otimes c)_0 \cdot f_\psi \otimes (n \otimes c)_1^\psi \\ = n\alpha(f_{\psi\psi}) \otimes c_1^\psi \otimes c_2^\psi \end{aligned} \tag{2.14}$$

in $\mathcal{N}(\alpha(r')) \otimes C \otimes C$. Since $\gamma : C \longrightarrow D$ is a monomorphism and all tensor products are taken over the field K, it suffices to show that

$$n\alpha(f_\psi) \otimes \gamma({c^\psi}_1) \otimes {c^\psi}_2 = n\alpha(f_{\psi\psi}) \otimes \gamma({c_1}^\psi) \otimes {c_2}^\psi \in \mathcal{N}(\alpha(r')) \otimes D \otimes C \tag{2.15}$$

Using (2.13) and the fact that (α, γ) is a morphism of entwining structures, the right hand side of (2.15) becomes

$$n\alpha(f_{\psi\psi}) \otimes \gamma({c_1}^\psi) \otimes {c_2}^\psi = n\alpha(f_\psi)_{\psi'} \otimes \gamma(c_1)^{\psi'} \otimes {c_2}^\psi = n_0\alpha(f_\psi)_{\psi'} \otimes n_1^{\psi'} \otimes c^\psi \tag{2.16}$$

From (2.13), we already know that $n\alpha(f_\psi) \otimes c^\psi \in \mathcal{N}(\alpha(r'))\Box_D C$. As such, we have

$$n\alpha(f_\psi) \otimes \gamma({c^\psi}_1) \otimes {c^\psi}_2 = (n\alpha(f_\psi))_0 \otimes (n\alpha(f_\psi))_1 \otimes c^\psi = n_0\alpha(f_\psi)_{\psi'} \otimes n_1^{\psi'} \otimes c^\psi \tag{2.17}$$

where the second equality follows from (2.13). From (2.16) and (2.17), the result of (2.15) is now clear. □

Theorem 2.5 *Let $(\alpha, \gamma) : (\mathcal{R}, C, \psi) \longrightarrow (\mathcal{S}, D, \psi')$ be a morphism of entwining structures such that $\gamma : C \longrightarrow D$ is a monomorphism of vector spaces. Then, there is an adjuction of functors*

$$\mathbf{M}_{\mathcal{S}}^{D}(\psi')((\alpha, \gamma)^{*}\mathcal{M}, \mathcal{N}) = \mathbf{M}_{\mathcal{R}}^{C}(\psi)(\mathcal{M}, (\alpha, \gamma)_{*}\mathcal{N}) \tag{2.18}$$

for $\mathcal{M} \in \mathbf{M}_{\mathcal{R}}^{C}(\psi)$ and $\mathcal{N} \in \mathbf{M}_{\mathcal{S}}^{D}(\psi')$.

Proof We consider a morphism $\eta : (\alpha, \gamma)^{*}\mathcal{M} \longrightarrow \mathcal{N}$ in $\mathbf{M}_{\mathcal{S}}^{D}(\psi')$. Then, η corresponds to a morphism $\eta : \alpha^{*}\mathcal{M} \longrightarrow \mathcal{N}$ in $\mathbf{M}_{\mathcal{S}}$ such that $\eta(s) : \alpha^{*}\mathcal{M}(s) \longrightarrow \mathcal{N}(s)$ is D-colinear for each $s \in S$. Accordingly, we have $\eta' : \mathcal{M} \longrightarrow \mathcal{N}$ in $\mathbf{M}_{\mathcal{R}}$ such that $\eta'(r) : \mathcal{M}(r) \longrightarrow \mathcal{N}(\alpha(r))$ is D-colinear for each $r \in \mathcal{R}$. Here, $\mathcal{M}(r)$ is treated as a D-comodule via corestriction of scalars. Therefore, we have morphisms $\eta''(r) : \mathcal{M}(r) \longrightarrow \mathcal{N}(\alpha(r))\Box_D C$ of C-comodules for each $r \in \mathcal{R}$. Together, these determine a morphism $\mathcal{M} \longrightarrow (\alpha, \gamma)_{*}\mathcal{N}$ in $\mathbf{M}_{\mathcal{R}}$. These arguments can be easily reversed and hence the result. □

3 Projective Generators and Entwined Modules

Let $(\mathcal{R}, C, \psi)$ be an entwining structure. In [4, Proposition 2.9], it was shown that the category $\mathbf{M}_{\mathcal{R}}^{C}(\psi)$ of entwined modules is a Grothendieck category. In this section, we will refine this result to give conditions for $\mathbf{M}_{\mathcal{R}}^{C}(\psi)$ to have a collection of projective generators.

Lemma 3.1 *Let $\mathcal{G}$ be a Grothendieck category. Fix a set of generators $\{G_k\}_{k\in K}$ for $\mathcal{G}$. Let $Z \in \mathcal{G}$ be an object. Let $i_X : X \hookrightarrow Z$, $i_Y : Y \hookrightarrow Z$ be two subobjects of Z such that for any $k \in K$ and any morphism $f_k : G_k \longrightarrow X$, there exists $g_k : G_k \longrightarrow Y$ such that $i_Y \circ g_k = i_X \circ f_k$. Then, $i_X : X \hookrightarrow Z$ factors through $i_Y : Y \hookrightarrow Z$, i.e., X is a subobject of Y.*

Proof Since $\{G_k\}_{k\in K}$ is a set of generators for $\mathcal{G}$, we can choose (see [20, Proposition 1.9.1]) an epimorphism $f : \bigoplus\limits_{j\in J} G_j \longrightarrow X$, corresponding to a collection of maps $f_j : G_j \longrightarrow X$, with each G_j a generator from the collection $\{G_k\}_{k\in K}$. Accordingly, we can choose morphisms $g_j : G_j \longrightarrow Y$ such that $i_Y \circ g_j = i_X \circ f_j$ for each $j \in J$. Together, these $\{g_j\}_{j\in J}$ determine a morphism $g : \bigoplus\limits_{j\in J} G_j \longrightarrow Y$ satisfying $i_Y \circ g = i_X \circ f$. Since i_X, i_Y are monomorphisms and f is an epimorphism, we have

$$X = Im(i_X) = Im(i_X \circ f) = Im(i_Y \circ g) = Im(i_Y|Im(g)) \subseteq Im(i_Y) = Y \tag{3.1}$$

□

Lemma 3.2 *Let $\mathcal{G}$ be a Grothendieck category having a set of projective generators $\{G_k\}_{k\in K}$. Let $f : X \longrightarrow Y$ be a morphism in $\mathcal{G}$. Let $i : X' \hookrightarrow X$ and $j : Y' \hookrightarrow Y$ be monomorphisms. Suppose that for any $k \in K$ and any morphism $f_k : G_k \longrightarrow X'$, there exists a morphism $g_k : G_k \longrightarrow Y'$ such that $f \circ i \circ f_k = j \circ g_k : G_k \longrightarrow Y$. Then, there exists $f' : X' \longrightarrow Y'$ such that $j \circ f' = f \circ i$.*

Proof It suffices to show that $Im(f \circ i) \subseteq Y'$. We choose any $k \in K$ and a morphism $h_k : G_k \longrightarrow Im(f \circ i) \hookrightarrow Y$. Since G_k is projective, we can choose $f_k : G_k \longrightarrow X'$

such that $f \circ i \circ f_k = h_k$. By assumption, we can now find $g_k : G_k \longrightarrow Y'$ such that $f \circ i \circ f_k = j \circ g_k : G_k \longrightarrow Y$. In particular, $j \circ g_k = h_k$. Applying Lemma 3.1, we obtain $Im(f \circ i) \subseteq Y'$. □

Lemma 3.3 *Let $(\mathcal{R}, C, \psi)$ be an entwining structure. Let V be a right C-comodule. Then, for any $r \in \mathcal{R}$, the module $V \otimes H_r$ given by*

$$\begin{array}{c} (V \otimes H_r)(r') = V \otimes \mathcal{R}(r', r) \\ (V \otimes H_r)(f) : (V \otimes H_r)(r') \longrightarrow (V \otimes H_r)(r'') \qquad v \otimes g \mapsto v \otimes gf \end{array} \tag{3.2}$$

for $r' \in \mathcal{R}$, $f \in \mathcal{R}(r'', r')$ is an entwined module in $\mathbf{M}_{\mathcal{R}}^C(\psi)$. Here, the right C-comodule structure on $(V \otimes H_r)(r')$ is given by taking $v \otimes g$ to $v_0 \otimes g_\psi \otimes v_1^\psi$.

Proof See [4, Lemma 2.5]. □

For the rest of this section, we will assume that the coalgebra C is such that the category $Comod-C$ of right C-comodules has enough projective objects. In other words, the coalgebra C is right semiperfect (see [18, Definition 3.2.4]).

Proposition 3.4 *Let $(\mathcal{R}, C, \psi)$ be an entwining structure with C a right semiperfect coalgebra. Let V be a projective right C-comodule. Then, for any $r \in \mathcal{R}$, the module $V \otimes H_r$ is a projective object of $\mathbf{M}_{\mathcal{R}}^C(\psi)$.*

Proof We begin with a morphism $\zeta : V \otimes H_r \longrightarrow \mathcal{M}$ and an epimorphism $\eta : \mathcal{N} \longrightarrow \mathcal{M}$ in $\mathbf{M}_{\mathcal{R}}^C(\psi)$. In particular, we consider the composition

$$V \longrightarrow V \otimes H_r(r) \longrightarrow \mathcal{M}(r) \qquad v \mapsto v \otimes id_r \mapsto \zeta(r)(v \otimes id_r) \tag{3.3}$$

which is a morphism in $Comod - C$. Since V is projective, we can lift the map in (3.3) to a map $T : V \longrightarrow \mathcal{N}(r)$ in $Comod - C$ such that $(\eta(r)(T(v)) = \zeta(r)(v \otimes id_r)$ for each $v \in V$.

We now define $\xi : V \otimes H_r \longrightarrow \mathcal{N}$ by setting for each $s \in \mathcal{R}$

$$\xi(s) : V \otimes H_r(s) \longrightarrow \mathcal{N}(s) \qquad v \otimes g \mapsto \mathcal{N}(g)(T(v)) \tag{3.4}$$

We first check that $\xi : V \otimes H_r \longrightarrow \mathcal{N}$ is a morphism in $\mathbf{M}_{\mathcal{R}}$. Given $g' \in \mathcal{R}(s', s)$, we have

$$\mathcal{N}(g')(\xi(s)(v \otimes g)) = \mathcal{N}(gg')(T(v)) = \xi(s')(v \otimes gg') = \xi(s')((V \otimes H_r)(g')(v \otimes g)) \tag{3.5}$$

We also have, for $v \otimes g \in V \otimes H_r(s)$,

$$\begin{aligned} \xi(s)(v \otimes g)_0 \otimes \xi(s)(v \otimes g)_1 &= \mathcal{N}(g)(T(v))_0 \otimes \mathcal{N}(g)(T(v))_1 \\ &= T(v)_0 g_\psi \otimes T(v)_1^\psi \\ &= T(v_0) g_\psi \otimes v_1^\psi \\ &= \mathcal{N}(g_\psi)(T(v_0)) \otimes v_1^\psi = (\xi(s) \otimes id_C)(v_0 \otimes g_\psi \otimes v_1^\psi) \end{aligned} \tag{3.6}$$

This shows that $\xi(s) : V \otimes H_r(s) \longrightarrow \mathcal{N}(s)$ is a morphism in $Comod - C$. Together with (3.5), it follows that $\xi : V \otimes H_r \longrightarrow \mathcal{N}$ is a morphism in $\mathbf{M}_{\mathcal{R}}^C(\psi)$. Finally, we see that for $v \otimes g \in V \otimes H_r(s)$, we have

$$\begin{aligned} (\eta(s) \circ \xi(s))(v \otimes g) &= \eta(s)(\mathcal{N}(g)(T(v))) \\ &= \mathcal{M}(g)(\eta(r)(T(v))) \\ &= \mathcal{M}(g)(\zeta(r)(v \otimes id_r)) \\ &= \zeta(s)((V \otimes H_r)(g)(v \otimes id_r)) \\ &= \zeta(s)(v \otimes g) \end{aligned} \tag{3.7}$$

This gives us $\eta \circ \xi = \zeta : V \otimes H_r \longrightarrow \mathcal{M}$. Hence the result. □

Theorem 3.1 *Let $(\mathcal{R}, C, \psi)$ be an entwining structure and let C be a right semiperfect K-coalgebra. Then, the category $\mathbf{M}_{\mathcal{R}}^C(\psi)$ of entwined modules is a Grothendieck category with a set of projective generators.*

Proof From [4, Proposition 2.9], we know that $\mathbf{M}_{\mathcal{R}}^C(\psi)$ is a Grothendieck category. Let $\mathcal{M}$ be an object of $\mathbf{M}_{\mathcal{R}}^C(\psi)$. From the proof of [4, Proposition 2.9], we know that there exists an epimorphism

$$\eta' : \bigoplus_{i \in I} V_i' \otimes H_{r_i} \longrightarrow \mathcal{M} \tag{3.8}$$

where each $r_i \in \mathcal{R}$ and each V_i' is a finite dimensional C-comodule. Since $Comod - C$ has enough projectives, it follows from [18, Corollary 2.4.21] that we can choose for each V_i' an epimorphism $V_i \longrightarrow V_i'$ in $Comod - C$ such that V_i is a finite dimensional projective in $Comod - C$. This induces an epimorphism

$$\eta : \bigoplus_{i \in I} V_i \otimes H_{r_i} \longrightarrow \mathcal{M} \tag{3.9}$$

The collection $\{V \otimes H_r\}$ now gives a set of projective generators for $\mathbf{M}_{\mathcal{R}}^C(\psi)$, where $r \in \mathcal{R}$ and V ranges over (isomorphism classes of) finite dimensional projective C-comodules. □

4 Modules Over an Entwined Representation

We fix a K-coalgebra C which is right semiperfect. We consider the category $\mathscr{E}nt_C$ whose objects are entwining structures $(\mathcal{R}, C, \psi)$. A morphism in $\mathscr{E}nt_C$ is a map (α, id) : $(\mathcal{R}, C, \psi) \longrightarrow (\mathcal{R}', C, \psi')$ of entwining structures, which we will denote simply by α. From Section 2, it follows that we have adjoint functors

$$\alpha^* = (\alpha, id_C)^* : \mathbf{M}_{\mathcal{R}}^C(\psi) \longrightarrow \mathbf{M}_{\mathcal{R}'}^C(\psi') \qquad \alpha_* = (\alpha, id_C)_* : \mathbf{M}_{\mathcal{R}'}^C(\psi') \longrightarrow \mathbf{M}_{\mathcal{R}}^C(\psi) \tag{4.1}$$

We note in particular that the functors $\alpha_* = (\alpha, id_C)_*$ are exact. In fact, the functors α_* preserve both limits and colimits.

Definition 4.1 Let $\mathscr{X}$ be a small category. Let C be a right semiperfect coalgebra over the field K. By an entwined C-representation of a small category, we will mean a functor $\mathscr{R} : \mathscr{X} \longrightarrow \mathscr{E}nt_C$.

In particular, for each object $x \in \mathscr{X}$, we have an entwining structure $(\mathscr{R}_x, C, \psi_x)$. Given a morphism $\alpha : x \longrightarrow y$ in $\mathscr{X}$, we have a morphism $\mathscr{R}_\alpha = (\mathscr{R}_\alpha, id_C) : (\mathscr{R}_x, C, \psi_x) \longrightarrow (\mathscr{R}_y, C, \psi_y)$ of entwining structures.

By abuse of notation, if $\mathscr{R} : \mathscr{X} \longrightarrow \mathscr{E}nt_C$ is an entwined C-representation, we will write

$$\alpha^* = \mathscr{R}_\alpha^* : \mathbf{M}_{\mathscr{R}_x}^C(\psi_x) \longrightarrow \mathbf{M}_{\mathscr{R}_y}^C(\psi_y) \qquad \alpha_* = \mathscr{R}_{\alpha *} : \mathbf{M}_{\mathscr{R}_y}^C(\psi_y) \longrightarrow \mathbf{M}_{\mathscr{R}_x}^C(\psi_x) \tag{4.2}$$

for any morphism $\alpha : x \longrightarrow y$ in $\mathscr{X}$. Also by abuse of notation, if $f : r' \longrightarrow r$ is a morphism in $\mathscr{R}_x$, we will often denote $\mathscr{R}_\alpha(f) : \mathscr{R}_\alpha(r') \longrightarrow \mathscr{R}_\alpha(r)$ in $\mathscr{R}_y$ simply as $\alpha(f) : \alpha(r') \longrightarrow \alpha(r)$. We will now consider modules over an entwined C-representation.

Definition 4.2 Let $\mathscr{R} : \mathscr{X} \longrightarrow \mathscr{E}nt_C$ be an entwined C-representation of a small category $\mathscr{X}$. An entwined module $\mathscr{M}$ over $\mathscr{R}$ will consist of the following data

(1) For each object $x \in \mathscr{X}$, an entwined module $\mathscr{M}_x \in \mathbf{M}^C_{\mathscr{R}_x}(\psi_x)$.
(2) For each morphism $\alpha : x \longrightarrow y$ in $\mathscr{X}$, a morphism $\mathscr{M}_\alpha : \mathscr{M}_x \longrightarrow \alpha_*\mathscr{M}_y$ in $\mathbf{M}^C_{\mathscr{R}_x}(\psi_x)$ (equivalently, a morphism $\mathscr{M}^\alpha : \alpha^*\mathscr{M}_x \longrightarrow \mathscr{M}_y$ in $\mathbf{M}^C_{\mathscr{R}_y}(\psi_y)$).

Further, we suppose that $\mathscr{M}_{id_x} = id_{\mathscr{M}_x}$ for each $x \in \mathscr{X}$ and that for any composable morphisms $x \xrightarrow{\alpha} y \xrightarrow{\beta} z$, we have $\alpha_*(\mathscr{M}_\beta) \circ \mathscr{M}_\alpha = \mathscr{M}_{\beta\alpha} : \mathscr{M}_x \longrightarrow \alpha_*\mathscr{M}_y \longrightarrow \alpha_*\beta_*\mathscr{M}_z = (\beta\alpha)_*\mathscr{M}_z$. The latter condition may be expressed in any of two equivalent ways

$$\mathscr{M}_{\beta\alpha} = \alpha_*(\mathscr{M}_\beta) \circ \mathscr{M}_\alpha \qquad \Leftrightarrow \qquad \mathscr{M}^{\beta\alpha} = \mathscr{M}^\beta \circ \beta^*(\mathscr{M}^\alpha) \tag{4.3}$$

A morphism $\eta : \mathscr{M} \longrightarrow \mathscr{N}$ of entwined modules over $\mathscr{R}$ consists of morphisms $\eta_x : \mathscr{M}_x \longrightarrow \mathscr{N}_x$ in each $\mathbf{M}^C_{\mathscr{R}_x}(\psi_x)$ such that the following diagram commutes

$$\begin{array}{ccc} \mathscr{M}_x & \xrightarrow{\eta_x} & \mathscr{N}_x \\ {\scriptstyle \mathscr{M}_\alpha}\downarrow & & \downarrow{\scriptstyle \mathscr{N}_\alpha} \\ \alpha_*\mathscr{M}_y & \xrightarrow{\alpha_*\eta_y} & \alpha_*\mathscr{N}_y \end{array} \tag{4.4}$$

for each $\alpha : x \longrightarrow y$ in $\mathscr{R}$. The category of entwined modules over $\mathscr{R}$ will be denoted by $Mod^C - \mathscr{R}$.

Proposition 4.3 *Let $\mathscr{R} : \mathscr{X} \longrightarrow \mathscr{E}nt_C$ be an entwined C-representation of a small category $\mathscr{X}$. Then, $Mod^C - \mathscr{R}$ is an abelian category.*

Proof Let $\eta : \mathscr{M} \longrightarrow \mathscr{N}$ be a morphism $Mod^C - \mathscr{R}$. We define the kernel and cokernel of η by setting

$$Ker(\eta)_x := Ker(\eta_x : \mathscr{M}_x \longrightarrow \mathscr{N}_x) \qquad Cok(\eta)_x := Cok(\eta_x : \mathscr{M}_x \longrightarrow \mathscr{N}_x) \tag{4.5}$$

for each $x \in \mathscr{X}$. For $\alpha : x \longrightarrow y$ in $\mathscr{X}$, the morphisms $Ker(\eta)_\alpha$ and $Cok(\eta)_\alpha$ are induced in the obvious manner, using the fact that $\alpha_* : \mathbf{M}^C_{\mathscr{R}_y}(\psi_y) \longrightarrow \mathbf{M}^C_{\mathscr{R}_x}(\psi_x)$ is exact. From this, it is also clear that $Cok(Ker(\eta) \hookrightarrow \mathscr{M}) = Ker(\mathscr{N} \twoheadrightarrow Cok(\eta))$. □

We now let $\mathscr{R} : \mathscr{X} \longrightarrow \mathscr{E}nt_C$ be an entwined C-representation of a small category $\mathscr{X}$ and let $\mathscr{M}$ be an entwined module over $\mathscr{R}$. We consider some $x \in \mathscr{X}$ and a morphism

$$\eta : V \otimes H_r \longrightarrow \mathscr{M}_x \tag{4.6}$$

in $\mathbf{M}^C_{\mathscr{R}_x}(\psi_x)$, where V is a finite dimensional projective in $Comod - C$ and $r \in \mathscr{R}_x$. For each $y \in \mathscr{X}$, we now set $\mathscr{N}_y \subseteq \mathscr{M}_y$ to be the image of the family of maps

$$\begin{aligned} \mathscr{N}_y &= Im\left(\bigoplus_{\beta \in \mathscr{X}(x,y)} \beta^*(V \otimes H_r) \xrightarrow{\oplus \beta^*\eta} \beta^*\mathscr{M}_x \xrightarrow{\mathscr{M}^\beta} \mathscr{M}_y \right) \\ &= \sum_{\beta \in \mathscr{X}(x,y)} Im\left(\beta^*(V \otimes H_r) \xrightarrow{\beta^*\eta} \beta^*\mathscr{M}_x \xrightarrow{\mathscr{M}^\beta} \mathscr{M}_y \right) \end{aligned} \tag{4.7}$$

We denote by ι_y the inclusion $\iota_y : \mathscr{N}_y \hookrightarrow \mathscr{M}_y$. For each $\beta \in \mathscr{X}(x, y)$, we denote by $\eta'_\beta : \beta^*(V \otimes H_r) \longrightarrow \mathscr{N}_y$ the canonical morphism induced from (4.7).

Lemma 4.4 *For any $\alpha \in \mathscr{X}(y, z)$, $\beta \in \mathscr{X}(x, y)$, the following composition*

$$\beta^*(V \otimes H_r) \xrightarrow{\eta'_\beta} \mathscr{N}_y \xrightarrow{\iota_y} \mathscr{M}_y \xrightarrow{\mathscr{M}_\alpha} \alpha_*\mathscr{M}_z \tag{4.8}$$

factors through $\alpha_*(\iota_z) : \alpha_*\mathscr{N}_z \longrightarrow \alpha_*\mathscr{M}_z$.

Proof Since (α^*, α_*) is an adjoint pair, it suffices to show that the composition

$$\alpha^*\beta^*(V \otimes H_r) \xrightarrow{\alpha^*(\eta'_\beta)} \alpha^*\mathscr{N}_y \xrightarrow{\alpha^*(\iota_y)} \alpha^*\mathscr{M}_y \xrightarrow{\mathscr{M}^\alpha} \mathscr{M}_z \tag{4.9}$$

factors through $\iota_z : \mathscr{N}_z \longrightarrow \mathscr{M}_z$. By definition, we know that the composition $\beta^*(V \otimes H_r) \xrightarrow{\eta'_\beta} \mathscr{N}_y \xrightarrow{\iota_y} \mathscr{M}_y$ factors through $\beta^*\mathscr{M}_x$, i.e., we have

$$\iota_y \circ \eta'_\beta = \mathscr{M}^\beta \circ \beta^*\eta \tag{4.10}$$

Applying α^*, composing with $\mathscr{M}^\alpha$ and using (4.3), we get

$$\mathscr{M}^\alpha \circ \alpha^*(\iota_y) \circ \alpha^*(\eta'_\beta) = \mathscr{M}^\alpha \circ \alpha^*(\mathscr{M}^\beta) \circ \alpha^*(\beta^*\eta) = \mathscr{M}^{\alpha\beta} \circ \alpha^*\beta^*\eta \tag{4.11}$$

From the definition in (4.7), it is now clear that the composition $\mathscr{M}^\alpha \circ \alpha^*(\iota_y) \circ \alpha^*(\eta'_\beta) = \mathscr{M}^{\alpha\beta} \circ \alpha^*\beta^*\eta$ factors through $\iota_z : \mathscr{N}_z \longrightarrow \mathscr{M}_z$ as $\mathscr{M}^\alpha \circ \alpha^*(\iota_y) \circ \alpha^*(\eta'_\beta) = \iota_z \circ \eta'_{\alpha\beta}$. □

Proposition 4.5 *For any $\alpha \in \mathscr{X}(y, z)$, the morphism $\mathscr{M}_\alpha : \mathscr{M}_y \longrightarrow \alpha_*\mathscr{M}_z$ restricts to a morphism $\mathscr{N}_\alpha : \mathscr{N}_y \longrightarrow \alpha_*\mathscr{N}_z$, giving us a commutative diagram*

$$\begin{array}{ccc} \mathscr{M}_y & \xrightarrow{\mathscr{M}_\alpha} & \alpha_*\mathscr{M}_z \\ \uparrow{\scriptstyle \iota_y} & & \uparrow{\scriptstyle \alpha_*(\iota_z)} \\ \mathscr{N}_y & \xrightarrow{\mathscr{N}_\alpha} & \alpha_*\mathscr{N}_z \end{array} \tag{4.12}$$

Proof We already know that $\iota_z : \mathscr{N}_z \longrightarrow \mathscr{M}_z$ is a monomorphism. Since α_* is a right adjoint, it follows that $\alpha_*(\iota_z)$ is also a monomorphism. Since C is right semiperfect, we know from Theorem 3.5 that $\mathbf{M}^C_{\mathscr{R}_y}(\psi_y)$ is a Grothendieck category with projective generators $\{G_k\}_{k\in K}$.Using Lemma 3.2, it suffices to show that for any $k \in K$ and any morphism $\xi_k : G_k \longrightarrow \mathscr{N}_y$, there exists $\xi'_k : G_k \longrightarrow \alpha_*\mathscr{N}_z$ such that $\alpha_*(\iota_z) \circ \xi'_k = \mathscr{M}_\alpha \circ \iota_y \circ \xi_k$.

From (4.7), we have an epimorphism

$$\bigoplus_{\beta\in\mathscr{X}(x,y)} \eta'_\beta : \bigoplus_{\beta\in\mathscr{X}(x,y)} \beta^*(V \otimes H_r) \longrightarrow \mathscr{N}_y \tag{4.13}$$

Since G_k is projective, we can lift $\xi_k : G_k \longrightarrow \mathscr{N}_y$ to a morphism $\xi''_k : G_k \longrightarrow \bigoplus_{\beta\in\mathscr{X}(x,y)} \beta^*(V \otimes H_r)$ such that

$$\xi_k = \left(\bigoplus_{\beta\in\mathscr{X}(x,y)} \eta'_\beta \right) \circ \xi''_k \tag{4.14}$$

From Lemma 4.4, we know that $\mathscr{M}_\alpha \circ \iota_y \circ \eta'_\beta$ factors through $\alpha_*(\iota_z) : \alpha_*\mathscr{N}_z \longrightarrow \alpha_*\mathscr{M}_z$ for each $\beta \in \mathscr{X}(x, y)$. The result is now clear. □

Using the adjointness of (α^*, α_*), we can also obtain a morphism $\mathscr{N}^\alpha : \alpha^*\mathscr{N}_y \longrightarrow \mathscr{N}_z$ for each $\alpha \in \mathscr{X}(y, z)$, corresponding to the morphism $\mathscr{N}_\alpha : \mathscr{N}_y \longrightarrow \alpha_*\mathscr{N}_z$ in (4.12). The objects $\{\mathscr{N}_y \in \mathbf{M}^C_{\mathscr{R}_y}(\psi_y)\}_{y\in\mathscr{X}}$, together with the morphisms $\{\mathscr{N}_\alpha\}_{\alpha\in Mor(\mathscr{X})}$ determine an object of $Mod^C - \mathscr{R}$ that we denote by $\mathscr{N}$. Additionally, Proposition 4.5 shows that we have an inclusion $\iota : \mathscr{N} \hookrightarrow \mathscr{M}$ in $Mod^C - \mathscr{R}$. Before we proceed further, we will describe the object $\mathscr{N}$ in a few more ways.

Lemma 4.6 *Let $\eta_1' : V \otimes H_r \longrightarrow \mathscr{N}_x$ be the canonical morphism corresponding to the identity map in $\mathscr{X}(x,x)$. Then, for any $y \in \mathscr{X}$, we have*

$$\mathscr{N}_y = Im\left(\bigoplus_{\beta\in\mathscr{X}(x,y)} \beta^*(V\otimes H_r) \xrightarrow{\oplus\beta^*\eta_1'} \beta^*\mathscr{N}_x \xrightarrow{\mathscr{N}^\beta} \mathscr{N}_y\right) \tag{4.15}$$

Proof For any $\beta \in \mathscr{X}(x,y)$, we consider the commutative diagram

$$\begin{array}{ccccc} \beta^*(V\otimes H_r) & \xrightarrow{\beta^*\eta_1'} & \beta^*\mathscr{N}_x & \xrightarrow{\mathscr{N}^\beta} & \mathscr{N}_y \\ & & {\scriptstyle\beta^*(\iota_x)}\downarrow & & \downarrow{\scriptstyle\iota_y} \\ & & \beta^*\mathscr{M}_x & \xrightarrow{\mathscr{M}^\beta} & \mathscr{M}_y \end{array} \tag{4.16}$$

By definition, we know that $\iota_x \circ \eta_1' = \eta$, which gives $\beta^*(\iota_x)\circ\beta^*(\eta_1') = \beta^*(\eta)$. Composing with $\mathscr{M}^\beta$, we get

$$Im(\mathscr{M}^\beta\circ\beta^*(\eta)) = Im(\mathscr{M}^\beta\circ\beta^*(\iota_x)\circ\beta^*(\eta_1')) = Im(\iota_y\circ\mathscr{N}^\beta\circ\beta^*\eta_1') \cong Im(\mathscr{N}^\beta\circ\beta^*\eta_1') \tag{4.17}$$

where the last isomorphism follows from the fact that ι_y is monic. The result is now clear from the definition in (4.7). □

Lemma 4.7 *For any $y \in \mathscr{X}$, we have*

$$\mathscr{N}_y = \sum_{\beta\in\mathscr{X}(x,y)} Im\left(\beta^*\mathscr{N}_x \xrightarrow{\beta^*(\iota_x)} \beta^*\mathscr{M}_x \xrightarrow{\mathscr{M}^\beta} \mathscr{M}_y\right) \tag{4.18}$$

Proof For the sake of convenience, we set

$$\mathscr{N}_y' := \sum_{\beta\in\mathscr{X}(x,y)} Im\left(\beta^*\mathscr{N}_x \xrightarrow{\beta^*(\iota_x)} \beta^*\mathscr{M}_x \xrightarrow{\mathscr{M}^\beta} \mathscr{M}_y\right)$$

From the commutative diagram in (4.16), we see that each of the morphisms $\beta^*\mathscr{N}_x \xrightarrow{\beta^*(\iota_x)} \beta^*\mathscr{M}_x \xrightarrow{\mathscr{M}^\beta} \mathscr{M}_y$ factors through the subobject $\mathscr{N}_y \subseteq \mathscr{M}_y$. Hence, $\mathscr{N}_y' \subseteq \mathscr{N}_y$. On the other hand, it is clear that

$$Im\left(\beta^*(V\otimes H_r) \xrightarrow{\beta^*\eta_1'} \beta^*\mathscr{N}_x \xrightarrow{\beta^*(\iota_x)} \beta^*\mathscr{M}_x \xrightarrow{\mathscr{M}^\beta} \mathscr{M}_y\right) \subseteq Im\left(\beta^*\mathscr{N}_x \xrightarrow{\beta^*(\iota_x)} \beta^*\mathscr{M}_x \xrightarrow{\mathscr{M}^\beta} \mathscr{M}_y\right)$$

Applying Lemma 4.6, it is now clear that $\mathscr{N}_y \subseteq \mathscr{N}_y'$. This proves the result. □

We now make a few conventions : if $\mathcal{M}$ is a module over a small K-linear category $\mathcal{R}$, we denote by $el(\mathcal{M})$ the union $\bigcup\limits_{r\in\mathcal{R}} \mathcal{M}(r)$. The cardinality of $el(\mathcal{M})$ will be denoted by $|\mathcal{M}|$. If $\mathscr{M}$ is a module over an entwined C-representation $\mathscr{R} : \mathscr{X} \longrightarrow \mathscr{E}nt_C$, we denote by $el_{\mathscr{X}}(\mathscr{M})$ the union $\bigcup\limits_{x\in\mathscr{X}} el(\mathscr{M}_x)$. The cardinality of $el_{\mathscr{X}}(\mathscr{M})$ will be denoted by $|\mathscr{M}|$. It is evident that if $\mathscr{M} \in Mod^C - \mathscr{R}$ and $\mathscr{N}$ is either a quotient or a subobject of $\mathscr{M}$, then $|\mathscr{N}| \le |\mathscr{M}|$.

We now define the following cardinality

$$\kappa = sup\{|\mathbb{N}|, |C|, |K|, |Mor(\mathscr{X})|, |Mor(\mathscr{R}_x)|, x \in \mathscr{X}\} \tag{4.19}$$

We observe that $|\beta^*(V\otimes H_r)| \le \kappa$, where V is any finite dimensional C-comodule and $\beta \in \mathscr{X}(x,y)$.

Lemma 4.8 *We have* $|\mathscr{N}| \leq \kappa$.

Proof We choose $y \in \mathscr{X}$. From Lemma 4.6, we have

$$\mathscr{N}_y = Im\left(\bigoplus_{\beta\in\mathscr{X}(x,y)} \beta^*(V\otimes H_r) \xrightarrow{\oplus\beta^*\eta_1'} \beta^*\mathscr{N}_x \xrightarrow{\mathscr{N}^\beta} \mathscr{N}_y\right) \tag{4.20}$$

Since $\mathscr{N}_y$ is an epimorphic image of $\bigoplus_{\beta\in\mathscr{X}(x,y)} \beta^*(V\otimes H_r)$, we have

$$|\mathscr{N}_y| \leq |\bigoplus_{\beta\in\mathscr{X}(x,y)} \beta^*(V\otimes H_r)| \leq \kappa \tag{4.21}$$

It follows that $|\mathscr{N}| = \sum_{y\in\mathscr{X}} |\mathscr{N}_y| \leq \kappa$. □

Theorem 4.9 *Let* C *be a right semiperfect coalgebra over a field* K. *Let* $\mathscr{R}:\mathscr{X}\longrightarrow\mathscr{E}nt_C$ *be an entwined* C*-representation of a small category* $\mathscr{X}$. *Then, the category* $Mod^C-\mathscr{R}$ *of entwined modules over* $\mathscr{R}$ *is a Grothendieck category.*

Proof Since filtered colimits and finite limits in $Mod^C-\mathscr{R}$ are computed pointwise, it is clear that they commute with each other.

We now consider an object $\mathscr{M}$ in $Mod^C-\mathscr{R}$ and an element $m\in el_{\mathscr{X}}(\mathscr{M})$. Then, $m\in\mathscr{M}_x(r)$ for some $x\in\mathscr{X}$ and $r\in\mathscr{R}_x$. By [4, Lemma 2.8], we can find a finite dimensional C-subcomodule $V'\subseteq\mathscr{M}_x(r)$ containing m and a morphism $\eta':V'\otimes H_r\longrightarrow\mathscr{M}_x$ in $\mathbf{M}^C_{\mathscr{R}_x}(\psi_x)$ such that $\eta'(r)(m\otimes id_r)=m$. Since C is semiperfect, we can choose a finite dimensional projective V in $Comod-C$ along with an epimorphism $V\longrightarrow V'$. This induces a morphism $\eta:V\otimes H_r\longrightarrow\mathscr{M}_x$ in $\mathbf{M}^C_{\mathscr{R}_x}(\psi_x)$. Corresponding to η, we now define the subobject $\mathscr{N}\subseteq\mathscr{M}$ as in (4.7). It is clear that $m\in el_{\mathscr{X}}(\mathscr{N})$. By Lemma L4.8, we know that $|\mathscr{N}|\leq\kappa$.

We now consider the set of isomorphism classes of objects in $Mod^C-\mathscr{R}$ having cardinality $\leq\kappa$. From the above, it is clear that any object in $Mod^C-\mathscr{R}$ may be expressed as a sum of such objects. By choosing one object from each such isomorphism class, we obtain a set of generators for $Mod^C-\mathscr{R}$. □

5 Entwined Representations of a Poset and Projective Generators

In this section, the small category $\mathscr{X}$ will always be a partially ordered set. If $x\leq y$ in $\mathscr{X}$, we will say that there is a single morphism $x\longrightarrow y$ in $\mathscr{X}$. We continue with C being a right semiperfect coalgebra over the field K and $\mathscr{R}:\mathscr{X}\longrightarrow\mathscr{E}nt_C$ being an entwined C-representation of $\mathscr{X}$. From Theorem 4.9, we know that $Mod^C-\mathscr{R}$ is a Grothendieck category.

In this section, we will show that $Mod^C-\mathscr{R}$ has projective generators. For this, we will construct a pair of adjoint functors

$$ex^C_x:\mathbf{M}^C_{\mathscr{R}_x}(\psi_x)\longrightarrow Mod^C-\mathscr{R} \qquad ev^C_x:Mod^C-\mathscr{R}\longrightarrow\mathbf{M}^C_{\mathscr{R}_x}(\psi_x) \tag{5.1}$$

for each $x\in\mathscr{X}$.

Lemma 5.1 *Let $\mathscr{X}$ be a poset. Fix $x \in \mathscr{X}$. Then, there is a functor $ex_x^C : \mathbf{M}^C_{\mathscr{R}_x}(\psi_x) \longrightarrow Mod^C - \mathscr{R}$ defined by setting*

$$ex_x^C(\mathcal{M})_y := \begin{cases} \alpha^*\mathcal{M} \; if \alpha \in \mathscr{X}(x, y) \\ 0 \qquad if \mathscr{X}(x, y) = \phi \end{cases} \tag{5.2}$$

for each $y \in \mathscr{X}$.

Proof It is immediate that each $ex_x^C(\mathcal{M})_y \in \mathbf{M}^C_{\mathscr{R}_y}(\psi_y)$. We consider $\beta : y \longrightarrow y'$ in $\mathscr{X}$. If $x \not\leq y$, we have $0 = ex_x^C(\mathcal{M})^\beta : 0 = \beta^* ex_x^C(\mathcal{M})_y \longrightarrow ex_x^C(\mathcal{M})_{y'}$ in $\mathbf{M}^C_{\mathscr{R}_{y'}}(\psi_{y'})$. Otherwise, we consider $\alpha : x \longrightarrow y$ and $\alpha' : x \longrightarrow y'$. Then, we have

$$id = ex_x^C(\mathcal{M})^\beta : \beta^* ex_x^C(\mathcal{M})_y = \beta^*\alpha^*\mathcal{M} \longrightarrow \alpha'^*\mathcal{M} = ex_x^C(\mathcal{M})_{y'}$$

which follows from the fact that $\beta \circ \alpha = \alpha'$. Given composable morphisms β, γ in $\mathscr{X}$, it is now clear from the definitions that $ex_x^C(\mathcal{M})^{\gamma\beta} = ex_x^C(\mathcal{M})^\gamma \circ \gamma^*(ex_x^C(\mathcal{M})^\beta)$. □

Lemma 5.2 *Let $\mathscr{X}$ be a poset. Fix $x \in \mathscr{X}$. Then, there is a functor*

$$ev_x^C : Mod^C - \mathscr{R} \longrightarrow \mathbf{M}^C_{\mathscr{R}_x}(\psi_x) \qquad \mathscr{M} \mapsto \mathscr{M}_x \tag{5.3}$$

Additionally, ev_x^C is exact.

Proof It is immediate that ev_x^C is a functor. Since finite limits and finite colimits in $Mod^C - \mathscr{R}$ are computed pointwise, it follows that ev_x^C is exact.

Proposition 5.3 *Let $\mathscr{X}$ be a poset. Fix $x \in \mathscr{X}$. Then, (ex_x^C, ev_x^C) is a pair of adjoint functors.*

Proof For any $\mathcal{M} \in \mathbf{M}^C_{\mathscr{R}_x}(\psi_x)$ and $\mathscr{N} \in Mod^C - \mathscr{R}$, we will show that

$$Mod^C - \mathscr{R}(ex_x^C(\mathcal{M}), \mathscr{N}) \cong \mathbf{M}^C_{\mathscr{R}_x}(\psi_x)(\mathcal{M}, ev_x^C(\mathscr{N})) \tag{5.4}$$

We begin with a morphism $f : \mathcal{M} \longrightarrow \mathscr{N}_x$ in $\mathbf{M}^C_{\mathscr{R}_x}(\psi_x)$. Corresponding to f, we define $\eta^f : ex_x^C(\mathcal{M}) \longrightarrow \mathscr{N}$ in $Mod^C - \mathscr{R}$ by setting

$$\eta_y^f : ex_x^C(\mathcal{M})_y = \alpha^*\mathcal{M} \xrightarrow{\alpha^* f} \alpha^*\mathscr{N}_x \xrightarrow{\mathscr{N}^\alpha} \mathscr{N}_y \tag{5.5}$$

whenever $x \leq y$ and $\alpha \in \mathscr{X}(x, y)$. Otherwise, we set $0 = \eta_y^f : 0 = ex_x^C(\mathcal{M})_y \longrightarrow \mathscr{N}_y$. For $\beta : y \longrightarrow y'$ in $\mathscr{X}$, we have to show that the following diagram is commutative.

$$\begin{array}{ccc} \beta^* ex_x^C(\mathcal{M})_y & \xrightarrow{\beta^*\eta_y^f} & \beta^*\mathscr{N}_y \\ {\scriptstyle ex_x^C(\mathcal{M})^\beta}\downarrow & & \downarrow{\scriptstyle \mathscr{N}^\beta} \\ ex_x^C(\mathcal{M})_{y'} & \xrightarrow{\eta_{y'}^f} & \mathscr{N}_{y'} \end{array} \tag{5.6}$$

If $x \not\leq y$, then $ex_x^C(\mathcal{M})_y = 0$ and the diagram commutes. Otherwise, we consider $\alpha : x \longrightarrow y$ and $\alpha' = \beta \circ \alpha : x \longrightarrow y'$. Then, (5.6) reduces to the commutative diagram

$$\begin{array}{ccc} \beta^*\alpha^*\mathcal{M} & \xrightarrow{\beta^*(\mathscr{N}^\alpha \circ \alpha^* f)} & \beta^*\mathscr{N}_y \\ {\scriptstyle id}\downarrow & & \downarrow{\scriptstyle \mathscr{N}^\beta} \\ \beta^*\alpha^*\mathcal{M} = \alpha'^*\mathcal{M} & \xrightarrow{\mathscr{N}^{\alpha'} \circ \alpha'^*(f) = \mathscr{N}^\beta \circ \beta^*(\mathscr{N}^\alpha) \circ \beta^*\alpha^* f} & \mathscr{N}_{y'} \end{array} \tag{5.7}$$

Conversely, we take $\eta : ex_x^C(\mathcal{M}) \longrightarrow \mathscr{N}$ in $Mod^C - \mathscr{R}$. In particular, this determines $f^\eta = \eta_x : \mathcal{M} \longrightarrow \mathscr{N}_x$ in $\mathbf{M}_{\mathscr{R}_x}^C(\psi_x)$. It may be easily verified that these two associations are inverse to each other. This proves the result. □

Corollary 5.4 *The functor* $ex_x^C : \mathbf{M}_{\mathscr{R}_x}^C(\psi_x) \longrightarrow Mod^C - \mathscr{R}$ *preserves projectives.*

Proof From Proposition 5.3, we know that (ex_x^C, ev_x^C) is a pair of adjoint functors. From Lemma 5.2, we know that the right adjoint ev_x^C is exact. It follows therefore that its left adjoint ex_x^C preserves projective objects. □

Theorem 5.1 *Let* C *be a right semiperfect coalgebra over a field* K. *Let* $\mathscr{X}$ *be a poset and let* $\mathscr{R} : \mathscr{X} \longrightarrow \mathscr{E}nt_C$ *be an entwined* C*-representation of* $\mathscr{X}$. *Then,* $Mod^C - \mathscr{R}$ *has projective generators.*

Proof We denote by $Proj^f(C)$ the set of isomorphism classes of finite dimensional projective C-comodules. We will show that the family

$$\mathcal{G} = \{ex_x^C(V \otimes H_r) | x \in \mathscr{X}, r \in \mathscr{R}_x, V \in Proj^f(C)\} \tag{5.8}$$

is a set of projective generators for $Mod^C - \mathscr{R}$. From Proposition 3.4, we know that $V \otimes H_r$ is projective in $\mathbf{M}_{\mathscr{R}_x}^C(\psi_x)$, where $r \in \mathscr{R}_x$ and $V \in Proj^f(C)$. It now follows from Corollary 5.4 that each $ex_x^C(V \otimes H_r)$ is projective in $Mod^C - \mathscr{R}$.

It remains to show that $\mathcal{G}$ is a set of generators for $Mod^C - \mathscr{R}$. For this, we consider a monomorphism $\iota : \mathscr{N} \hookrightarrow \mathscr{M}$ in $Mod^C - \mathscr{R}$ such that $\mathscr{N} \subsetneq \mathscr{M}$. Since kernels and cokernels in $Mod^C - \mathscr{R}$ are taken pointwise, it follows that there is some $x \in \mathscr{X}$ such that $\iota_x : \mathscr{N}_x \hookrightarrow \mathscr{M}_x$ is a monomorphism with $\mathscr{N}_x \subsetneq \mathscr{M}_x$.

From the proof of Theorem 5.4, we know that $\{V \otimes H_r\}_{r \in \mathscr{R}_x, V \in Proj^f(C)}$ is a set of generators for $\mathbf{M}_{\mathscr{R}_x}^C(\psi_x)$. Accordingly, we can choose a morphism $f : V \otimes H_r \longrightarrow \mathscr{M}_x$ with $r \in \mathscr{R}_x$ and $V \in Proj^f(C)$ such that f does not factor through $ev_x^C(\iota) = \iota_x : \mathscr{N}_x \hookrightarrow \mathscr{M}_x$. Applying the adjunction (ex_x^C, ev_x^C), we now obtain a morphism $\eta : ex_x^C(V \otimes H_r) \longrightarrow \mathscr{M}$ corresponding to f, which does not factor through $\iota : \mathscr{N} \longrightarrow \mathscr{M}$. It now follows (see, for instance, [20, §1.9]) that the family $\mathcal{G}$ is a set of generators for $Mod^C - \mathscr{R}$. □

6 Cartesian Modules over Entwined Representations

We continue with $\mathscr{X}$ being a poset, C being a right semiperfect K-coalgebra and $\mathscr{R} : \mathscr{X} \longrightarrow \mathscr{E}nt_C$ being an entwined C-representation of $\mathscr{X}$. In this section, we will introduce the category of cartesian modules over $\mathscr{R}$.

Given a morphism $\alpha : (\mathcal{R}, C, \psi) \longrightarrow (\mathcal{S}, C, \psi')$ in $\mathscr{E}nt_C$, we already know that the left adjoint α^* is right exact. We will say that $\alpha : (\mathcal{R}, C, \psi) \longrightarrow (\mathcal{S}, C, \psi')$ is flat if $\alpha^* : \mathbf{M}_{\mathcal{R}}^C(\psi) \longrightarrow \mathbf{M}_{\mathcal{S}}^C(\psi')$ is exact. Accordingly, we will say that an entwined C-representation $\mathscr{R} : \mathscr{X} \longrightarrow \mathscr{E}nt_C$ is flat if $\alpha^* = \mathscr{R}_\alpha^* : \mathbf{M}_{\mathscr{R}_x}^C(\psi_x) \longrightarrow \mathbf{M}_{\mathscr{R}_y}^C(\psi_y)$ is exact for each $\alpha : x \longrightarrow y$ in $\mathscr{X}$.

Definition 6.1 Let $\mathscr{R} : \mathscr{X} \longrightarrow \mathscr{E}nt_C$ be an entwined C-representation of $\mathscr{X}$. Suppose that $\mathscr{R}$ is flat. Let $\mathscr{M}$ be an entwined module over $\mathscr{R}$. We will say that $\mathscr{M}$ is cartesian if for each $\alpha : x \longrightarrow y$ in $\mathscr{X}$, the morphism $\mathscr{M}^\alpha : \alpha^*\mathscr{M}_x \longrightarrow \mathscr{M}_y$ in $\mathbf{M}_{\mathscr{R}_y}^C(\psi_y)$ is an isomorphism.

We will denote by $Cart^C - \mathscr{R}$ the full subcategory of $Mod^C - \mathscr{R}$ consisting of cartesian modules.

It is clear that $Cart^C - \mathscr{R}$ is an abelian category, with filtered colimits and finite limits coming from $Mod^C - \mathscr{R}$.

We will now give conditions so that $Cart - \mathscr{R}$ is a Grothendieck category. For this, we will need some intermediate results. First, we recall (see, for instance, [2]) that an object M in a Grothendieck category $\mathcal{A}$ is said to be finitely generated if the functor $\mathcal{A}(M, \_\_)$ satisfies

$$\varinjlim_{i\in I} \mathcal{A}(M, M_i) = \mathcal{A}(M, \varinjlim_{i\in I} M_i) \tag{6.1}$$

where $\{M_i\}_{i\in I}$ is any filtered system of objects in $\mathcal{A}$ connected by monomorphisms.

Proposition 6.2 *Let $(\mathcal{R}, C, \psi)$ be an entwining structure with C a right semiperfect coalgebra. Let V be a finite dimensional projective right C-comodule. Then, for any $r \in \mathcal{R}$, the module $V \otimes H_r$ is a finitely generated projective object in $\mathbf{M}_{\mathcal{R}}^C(\psi)$.*

Proof From Proposition 3.4, we already know that $V \otimes H_r$ is a projective object in $\mathbf{M}_{\mathcal{R}}^C(\psi)$. To show that it is finitely generated, we consider a filtered system $\{\mathcal{M}_i\}_{i\in I}$ of objects in $\mathbf{M}_{\mathcal{R}}^C(\psi)$ connected by monomorphisms and set $\mathcal{M} := \varinjlim_{i\in I} \mathcal{M}_i$. Since $\mathbf{M}_{\mathcal{R}}^C(\psi)$ is a Grothendieck category, we note that we have an inclusion $\eta_i : \mathcal{M}_i \hookrightarrow \mathcal{M}$ for each $i \in I$.

We now take a morphism $\zeta : V \otimes H_r \longrightarrow \mathcal{M}$ in $\mathbf{M}_{\mathcal{R}}^C(\psi)$. We choose a basis $\{v_1, ..., v_n\}$ for V. For each $1 \leq k \leq n$, we now have a morphism in $\mathbf{M}_{\mathcal{R}}$ given by

$$\zeta_k : H_r \longrightarrow V \otimes H_r \qquad H_r(s) = \mathcal{R}(s, r) \ni f \mapsto v_k \otimes f \in (V \otimes H_r)(s) \tag{6.2}$$

Then, each composition $\zeta \circ \zeta_k : H_r \longrightarrow \mathcal{M}$ is a morphism in $\mathbf{M}_{\mathcal{R}}$. Since H_r is a finitely generated object in $\mathbf{M}_{\mathcal{R}}$, we can now choose $j \in I$ such that every $\zeta \circ \zeta_k$ factors through $\eta_j : \mathcal{M}_j \hookrightarrow \mathcal{M}$. We now construct the following pullback diagram in $\mathbf{M}_{\mathcal{R}}^C(\psi)$

$$\begin{array}{ccc} \mathcal{N} & \longrightarrow & \mathcal{M}_j \\ {\scriptstyle\iota}\downarrow & & \downarrow{\scriptstyle\eta_j} \\ V \otimes H_r & \xrightarrow{\zeta} & \mathcal{M} \end{array} \tag{6.3}$$

Then, $\iota : \mathcal{N} \longrightarrow \mathcal{M}$ is a monomorphism in $\mathbf{M}_{\mathcal{R}}^C(\psi)$. From the construction of finite limits in $\mathbf{M}_{\mathcal{R}}^C(\psi)$, it follows that for each $s \in \mathcal{R}$, we have a pullback diagram in $Vect_K$

$$\begin{array}{ccc} \mathcal{N}(s) & \longrightarrow & \mathcal{M}_j(s) \\ {\scriptstyle\iota(s)}\downarrow & & \downarrow{\scriptstyle\eta_j(s)} \\ (V \otimes H_r)(s) & \xrightarrow{\zeta(s)} & \mathcal{M}(s) \end{array} \tag{6.4}$$

By assumption, we know that $\zeta(s)(v_k \otimes f) \in Im(\eta_j(s))$ for any basis element v_k and any $f \in H_r(s)$. It follows that $Im(\zeta(s)) \subseteq Im(\eta_j(s))$ and hence the pullback $\mathcal{N}(s) = (V \otimes H_r)(s)$. In other words, $\mathcal{N} = V \otimes H_r$. The result is now clear. □

Lemma 6.3 *Let $\alpha : (\mathcal{R}, C, \psi) \longrightarrow (\mathcal{S}, C, \psi')$ be a flat morphism in $\mathscr{E}nt_C$. Let $\mathcal{M} \in \mathbf{M}_{\mathcal{R}}^C(\psi)$.*

(a) There exists a family $\{r_i\}_{i\in I}$ of objects of $\mathcal{R}$ and a family $\{V_i\}_{i\in I}$ of finite dimensional projective C-comodules such that there is an epimorphism in $\mathbf{M}_{\mathcal{S}}^C(\psi')$

$$\eta : \bigoplus_{i\in I} (V_i \otimes H_{\alpha(r_i)}) \longrightarrow \alpha^*\mathcal{M} \tag{6.5}$$

(b) *Let $s \in \mathcal{S}$ and let W be a finite dimensional projective in $Comod-C$. Let $\zeta : W \otimes H_s \longrightarrow \alpha^*\mathcal{M}$ be a morphism in $\mathbf{M}_{\mathcal{S}}^C(\psi')$. Then, there exists a finite set $\{r_1, ..., r_n\}$ of objects of $\mathcal{R}$, a finite family $\{V_1, ..., V_n\}$ of finite dimensional projective C-comodules and a morphism $\eta'' : \bigoplus_{k=1}^{n} V_k \otimes H_{r_k} \longrightarrow \mathcal{M}$ in $\mathbf{M}_{\mathcal{R}}^C(\psi)$ such that ζ factors through $\alpha^*\eta''$.*

Proof (a) From the proof of Theorem 3.5, we know that there exists an epimorphism in $\mathbf{M}_{\mathcal{R}}^C(\psi)$

$$\eta' : \bigoplus_{i \in I} V_i \otimes H_{r_i} \longrightarrow \mathcal{M} \tag{6.6}$$

where each $r_i \in \mathcal{R}$ and each V_i is a finite dimensional projective C-comodule. Since $\alpha^* : \mathbf{M}_{\mathcal{R}}^C(\psi) \longrightarrow \mathbf{M}_{\mathcal{S}}^C(\psi')$ is a left adjoint, it induces an epimorphism $\alpha^*(\eta')$ in $\mathbf{M}_{\mathcal{S}}^C(\psi')$. From the definition in (2.2) and the construction in Proposition 2.3, it is clear that $\alpha^*(V_i \otimes H_{r_i}) = V_i \otimes \alpha^* H_{r_i} = V_i \otimes H_{\alpha(r_i)}$. This proves (a).

(b) We consider the epimorphism $\alpha^*\eta' = \eta : \bigoplus_{i \in I} (V_i \otimes H_{\alpha(r_i)}) \longrightarrow \alpha^*\mathcal{M}$ constructed in (a). From Proposition 6.2, we know that $W \otimes H_s$ is a finitely generated projective object in $\mathbf{M}_{\mathcal{S}}^C(\psi')$. As such $\zeta : W \otimes H_s \longrightarrow \alpha^*\mathcal{M}$ can be lifted to a morphism $\zeta' : W \otimes H_s \longrightarrow \bigoplus_{i \in I} (V_i \otimes H_{\alpha(r_i)})$ and ζ' factors through a finite direct sum of objects from the family $\{V_i \otimes H_{\alpha(r_i)}\}_{i \in I}$. The result is now clear.

□

Lemma 6.4 *Let $\alpha : (\mathcal{R}, C, \psi) \longrightarrow (\mathcal{S}, C, \psi')$ be a flat morphism in $\mathscr{E}nt_C$. Let κ_1 be any cardinal such that*

$$\kappa_1 \geq \max\{\mathbb{N}, |Mor(\mathcal{R})|, |C|, |K|\} \tag{6.7}$$

Let $\mathcal{M} \in \mathbf{M}_{\mathcal{R}}^C(\psi)$ and let $A \subseteq el(\alpha^*\mathcal{M})$ be a set of elements such that $|A| \leq \kappa_1$. Then, there is a submodule $\mathcal{N} \hookrightarrow \mathcal{M}$ in $\mathbf{M}_{\mathcal{R}}^C(\psi)$ with $|\mathcal{N}| \leq \kappa_1$ such that $A \subseteq el(\alpha^*\mathcal{N})$.

Proof We consider some element $a \in A \subseteq el(\alpha^*\mathcal{M})$. Then, we can choose a morphism $\zeta^a : W^a \otimes H_{s^a} \longrightarrow \alpha^*\mathcal{M}$ in $\mathbf{M}_{\mathcal{S}}^C(\psi')$ such that $a \in el(Im(\zeta^a))$, where $s^a \in \mathcal{S}$ and W^a is a finite dimensional projective in $Comod - C$. Using Lemma 6.3(b), we can now choose a finite set $\{r_1^a, ..., r_{n^a}^a\}$ of objects of $\mathcal{R}$, a finite family $\{V_1^a, ..., V_{n^a}^a\}$ of finite dimensional projective C-comodules and a morphism $\eta^{a''} : \bigoplus_{k=1}^{n^a} V_k^a \otimes H_{r_k^a} \longrightarrow \mathcal{M}$ in $\mathbf{M}_{\mathcal{R}}^C(\psi)$ such that ζ^a factors through $\alpha^*\eta^{a''}$. We now set

$$\mathcal{N} := Im\left(\eta'' := \bigoplus_{a \in A} \eta^{a''} : \bigoplus_{a \in A} \bigoplus_{k=1}^{n^a} V_k^a \otimes H_{r_k^a} \longrightarrow \mathcal{M}\right) \tag{6.8}$$

Since α is flat and α^* is a left adjoint, we obtain

$$\alpha^*\mathcal{N} = Im\left(\alpha^*\eta'' = \bigoplus_{a \in A} \alpha^*\eta^{a''} : \bigoplus_{a \in A} \bigoplus_{k=1}^{n^a} V_k^a \otimes H_{\alpha(r_k^a)} \longrightarrow \alpha^*\mathcal{M}\right) \tag{6.9}$$

Since each $a \in el(Im(\zeta^a))$ and ζ^a factors through $\alpha^*\eta^{a''}$, we get $A \subseteq el(\alpha^*\mathcal{N})$.

It remains to show that $|\mathcal{N}| \leq \kappa_1$. Since $\mathcal{N}$ is a quotient of $\bigoplus_{a \in A} \bigoplus_{k=1}^{n^a} V_k^a \otimes H_{r_k^a}$ and $|A| \leq \kappa_1$, it suffices to show that each $|V_k^a \otimes H_{r_k^a}| \leq \kappa_1$. This is clear from the definition of κ_1, using the fact that each V_k^a is finite dimensional. □

Remark 6.5 By considering $\alpha = id$ in Lemma 6.4, we obtain the following simple consequence: if $A \subseteq el(\mathcal{M})$ is any subset with $|A| \leq \kappa_1$, there is a submodule $\mathcal{N} \hookrightarrow \mathcal{M}$ in $\mathbf{M}^C_{\mathcal{R}}(\psi)$ with $|\mathcal{N}| \leq \kappa_1$ such that $A \subseteq el(\mathcal{N})$.

Lemma 6.6 *Let $\alpha : (\mathcal{R}, C, \psi) \longrightarrow (\mathcal{S}, C, \psi')$ be a flat morphism in $\mathscr{E}nt_C$ and let $\mathcal{M} \in \mathbf{M}^C_{\mathcal{R}}(\psi)$. Let κ_2 be any cardinal such that $\kappa_2 \geq max\{\mathbb{N}, |Mor(\mathcal{R})|, |Mor(\mathcal{S})|, |C|, |K|\}$ and let $A \subseteq el(\mathcal{M})$ and $B \subseteq el(\alpha^*\mathcal{M})$ be subsets with $|A|$, $|B| \leq \kappa_2$. Then, there exists a submodule $\mathcal{N} \subseteq \mathcal{M}$ in $\mathbf{M}^C_{\mathcal{R}}(\psi)$ such that*

(1) $|\mathcal{N}| \leq \kappa_2$, $|\alpha^*\mathcal{N}| \leq \kappa_2$
(2) $A \subseteq el(\mathcal{N})$ and $B \subseteq el(\alpha^*\mathcal{N})$.

Proof Applying Lemma 6.4 and Remark 6.5, we obtain submodules $\mathcal{N}_1, \mathcal{N}_2 \subseteq \mathcal{M}$ such that

(1) $|\mathcal{N}_1|, |\mathcal{N}_2| \leq \kappa_2$
(2) $A \subseteq el(\mathcal{N}_1)$, $B \subseteq el(\alpha^*\mathcal{N}_2)$.

We set $\mathcal{N} := (\mathcal{N}_1 + \mathcal{N}_2) \subseteq \mathcal{M}$. Then, $(\mathcal{N}_1 + \mathcal{N}_2)$ is a quotient of $\mathcal{N}_1 \oplus \mathcal{N}_2$ and hence $|\mathcal{N}| \leq \kappa_2$. Also, it is clear that $A \subseteq el(\mathcal{N}_1) \subseteq el(\mathcal{N})$. Since α is flat, we get $B \subseteq el(\alpha^*\mathcal{N}_2) \subseteq el(\alpha^*\mathcal{N})$.

It remains to show that $|\alpha^*\mathcal{N}| \leq \kappa_2$. By the definition in (2.2), we know that $\alpha^*(\mathcal{N})(s)$ is a quotient of

$$\left(\bigoplus_{r \in \mathcal{R}} \mathcal{N}(r) \otimes \mathcal{S}(s, \alpha(r))\right) \tag{6.10}$$

for each $s \in \mathcal{S}$. Since $\kappa_2 \geq |Mor(\mathcal{R})|, |Mor(\mathcal{S})|$, it follows from (6.10) that $|\alpha^*(\mathcal{N})(s)| \leq \kappa_2$. Again since $\kappa_2 \geq |Mor(\mathcal{S})|$, we get $|\alpha^*\mathcal{N}| \leq \kappa_2$. □

We will now show that $Cart^C - \mathscr{R}$ has a generator when $\mathscr{R} : \mathscr{X} \longrightarrow \mathscr{E}nt_C$ is a flat representation of the poset $\mathscr{X}$. This will be done using induction on $\mathbb{N} \times Mor(\mathscr{X})$ in a manner similar to the proof of [19, Proposition 3.25]. As in Section 4, we set

$$\kappa = sup\{|\mathbb{N}|, |C|, |K|, |Mor(\mathscr{X})|, |Mor(\mathscr{R}_x)|, x \in \mathscr{X}\} \tag{6.11}$$

Let $\mathscr{M}$ be a cartesian module over $\mathscr{R} : \mathscr{X} \longrightarrow \mathscr{E}nt_C$. We now consider an element $m \in el_{\mathscr{X}}(\mathscr{M})$. Suppose that $m \in \mathscr{M}_x(r)$ for some $x \in \mathscr{X}$ and $r \in \mathscr{R}_x$. As in the proof of Theorem 4.9, we fix a finite dimensional projective C-comodule V and a morphism $\eta : V \otimes H_r \longrightarrow \mathscr{M}_x$ in $\mathbf{M}^C_{\mathscr{R}_x}(\psi_x)$ such that m is an element of the image of η. Corresponding to η, we define $\mathscr{N} \subseteq \mathscr{M}$ as in (4.7). It is clear that $m \in el_{\mathscr{X}}(\mathscr{N})$. By Lemma 4.8, we know that $|\mathscr{N}| \leq \kappa$.

Next, we choose a well ordering of the set $Mor(\mathscr{X})$ and consider the induced lexicographic ordering of $\mathbb{N} \times Mor(\mathscr{X})$. Corresponding to each pair $(n, \alpha : y \longrightarrow z) \in \mathbb{N} \times Mor(\mathscr{X})$, we will now define a subobject $\mathscr{P}(n, \alpha) \hookrightarrow \mathscr{M}$ in $Mod^C - \mathscr{R}$ satisfying the following conditions.

(1) $m \in el_{\mathscr{X}}(\mathscr{P}(1, \alpha_0))$, where α_0 is the least element of $Mor(\mathscr{X})$.
(2) $\mathscr{P}(n, \alpha) \subseteq \mathscr{P}(m, \beta)$, whenever $(n, \alpha) \leq (m, \beta)$ in $\mathbb{N} \times Mor(\mathscr{X})$
(3) For each $(n, \alpha : y \longrightarrow z) \in \mathbb{N} \times Mor(\mathscr{X})$, the morphism $\mathscr{P}(n, \alpha)^\alpha : \alpha^*\mathscr{P}(n, \alpha)_y \longrightarrow \mathscr{P}(n, \alpha)_z$ is an isomorphism in $\mathbf{M}^C_{\mathscr{R}_z}(\psi_z)$.
(4) $|\mathscr{P}(n, \alpha)| \leq \kappa$.

For $(n, \alpha : y \longrightarrow z) \in \mathbb{N} \times Mor(\mathscr{X})$, we start the process of constructing the module $\mathscr{P}(n, \alpha)$ as follows: we set

$$A_0^0(w) := \begin{cases} \mathscr{N}_w & \text{if } n = 1 \text{ and } \alpha = \alpha_0 \\ \bigcup\limits_{(m,\beta) < (n,\alpha)} \mathscr{P}(m, \beta)_w & \text{otherwise} \end{cases} \tag{6.12}$$

for each $w \in \mathscr{X}$. It is clear that each $A_0^0(w) \subseteq el(\mathscr{M}_w)$ and $|A_0^0(w)| \leq \kappa$.

Since $\mathscr{M}$ is cartesian, we know that $\alpha^*\mathscr{M}_y = \mathscr{M}_z$. Since $\alpha : (\mathscr{R}_y, C, \psi_y) \longrightarrow (\mathscr{R}_z, C, \psi_z)$ is flat in $\mathscr{E}nt_C$, we use Lemma 6.6 with $A_0^0(y) \subseteq el(\mathscr{M}_y)$ and $A_0^0(z) \subseteq el(\alpha^*\mathscr{M}_y) = el(\mathscr{M}_z)$ to obtain $A_1^0(y) \hookrightarrow \mathscr{M}_y$ in $\mathbf{M}^C_{\mathscr{R}_y}(\psi_y)$ such that

$$|A_1^0(y)| \leq \kappa \qquad |\alpha^* A_1^0(y)| \leq \kappa \qquad A_0^0(y) \subseteq el(A_1^0(y)) \qquad A_0^0(z) \subseteq el(\alpha^* A_1^0(y)) \tag{6.13}$$

We now set $A_1^0(z) := \alpha^* A_1^0(y)$. Then, (6.13) can be rewritten as

$$|A_1^0(y)| \leq \kappa \qquad |A_1^0(z)| \leq \kappa \qquad A_0^0(y) \subseteq el(A_1^0(y)) \qquad A_0^0(z) \subseteq el(A_1^0(z)) \tag{6.14}$$

We observe here that since $\mathscr{X}$ is a poset, then $y = z$ implies $\alpha : y \longrightarrow z$ is the identity and hence $A_1^0(y) = A_1^0(z)$. For any $w \neq y, z$ in $\mathscr{X}$, we set $A_1^0(w) = A_0^0(w)$. Combining with (6.14), we have $A_0^0(w) \subseteq A_1^0(w)$ for every $w \in \mathscr{X}$ and each $|A_1^0(w)| \leq \kappa$.

Lemma 6.7 *Let $B \subseteq el_{\mathscr{X}}(\mathscr{M})$ with $|B| \leq \kappa$. Then, there is a submodule $\mathscr{Q} \hookrightarrow \mathscr{M}$ in $Mod^C - \mathscr{R}$ such that $B \subseteq el_{\mathscr{X}}(\mathscr{Q})$ and $|\mathscr{Q}| \leq \kappa$.*

Proof For any $m \in B \subseteq el_{\mathscr{X}}(\mathscr{M})$ we can choose, as in the proof of Theorem 4.9, a subobject $\mathscr{Q}_m \subseteq \mathscr{M}$ such that $m \in el_{\mathscr{X}}(\mathscr{Q}_m)$ and $|\mathscr{Q}_m| \leq \kappa$. Then, we set $\mathscr{Q} := \sum\limits_{m\in B} \mathscr{Q}_m$. In particular, $\mathscr{Q}$ is a quotient of $\bigoplus\limits_{m\in B} \mathscr{Q}_m$. Since $|B| \leq \kappa$, the result follows. □

Using Lemma 6.7, we now choose a submodule $\mathscr{Q}^0(n, \alpha) \hookrightarrow \mathscr{M}$ in $Mod^C - \mathscr{R}$ such that $\bigcup\limits_{w\in\mathscr{X}} A_1^0(w) \subseteq el_{\mathscr{X}}(\mathscr{Q}^0(n, \alpha))$ and $|\mathscr{Q}^0(n, \alpha)| \leq \kappa$. In particular, $A_1^0(w) \subseteq \mathscr{Q}^0(n, \alpha)_w$ for each $w \in \mathscr{X}$.

We now iterate this construction. Suppose we have constructed a submodule $\mathscr{Q}^l(n, \alpha) \hookrightarrow \mathscr{M}$ for every $l \leq m$ such that $\bigcup\limits_{w\in\mathscr{X}} A_1^l(w) \subseteq el_{\mathscr{X}}(\mathscr{Q}^l(n, \alpha))$ and $|\mathscr{Q}^l(n, \alpha)| \leq \kappa$. Then, we set $A_0^{m+1}(w) := \mathscr{Q}^m(n, \alpha)_w$ for each $w \in \mathscr{X}$. We then use Lemma 6. with $A_0^{m+1}(y) \subseteq el(\mathscr{M}_y)$ and $A_0^{m+1}(z) \subseteq el(\alpha^*\mathscr{M}_y) = el(\mathscr{M}_z)$ to obtain $A_1^{m+1}(y) \hookrightarrow \mathscr{M}_y$ in $\mathbf{M}^C_{\mathscr{R}_y}(\psi_y)$ such that

$$|A_1^{m+1}(y)| \leq \kappa \qquad |\alpha^* A_1^{m+1}(y)| \leq \kappa \qquad A_0^{m+1}(y) \subseteq el(A_1^{m+1}(y)) \qquad A_0^{m+1}(z) \subseteq el(\alpha^* A_1^{m+1}(y)) \tag{6.15}$$

We now set $A_1^{m+1}(z) := \alpha^* A_1^{m+1}(y)$. Then, (6.15) can be rewritten as

$$|A_1^{m+1}(y)| \leq \kappa \qquad |A_1^{m+1}(z)| \leq \kappa \qquad A_0^{m+1}(y) \subseteq el(A_1^{m+1}(y)) \qquad A_0^{m+1}(z) \subseteq el(A_1^{m+1}(z)) \tag{6.16}$$

For any $w \neq y, z$ in $\mathscr{X}$, we set $A_1^{m+1}(w) = A_0^{m+1}(w)$. Combining with (6.16), we have $A_0^{m+1}(w) \subseteq A_1^{m+1}(w)$ for every $w \in \mathscr{X}$ and each $|A_1^{m+1}(w)| \leq \kappa$.

Using Lemma 6.7, we now choose a submodule $\mathscr{Q}^{m+1}(n, \alpha) \hookrightarrow \mathscr{M}$ in $Mod^C - \mathscr{R}$ such that $\bigcup\limits_{w\in\mathscr{X}} A_1^{m+1}(w) \subseteq el_{\mathscr{X}}(\mathscr{Q}^{m+1}(n, \alpha))$ and $|\mathscr{Q}^{m+1}(n, \alpha)| \leq \kappa$. In particular, $A_1^{m+1}(w) \subseteq \mathscr{Q}^{m+1}(n, \alpha)_w$ for each $w \in \mathscr{X}$.

Finally, we set

$$\mathscr{P}(n, \alpha) := \varinjlim_{m\geq 0} \mathscr{Q}^m(n, \alpha) \tag{6.17}$$

in $Mod^C - \mathscr{R}$.

Lemma 6.8 *The family $\{\mathscr{P}(n, \alpha) | (n, \alpha) \in \mathbb{N} \times Mor(\mathscr{X})\}$ satisfies the following conditions.*

(1) $m \in el_{\mathscr{X}}(\mathscr{P}(1, \alpha_0))$, *where* α_0 *is the least element of* $Mor(\mathscr{X})$.
(2) $\mathscr{P}(n, \alpha) \subseteq \mathscr{P}(m, \beta)$, *whenever* $(n, \alpha) \leq (m, \beta)$ *in* $\mathbb{N} \times Mor(\mathscr{X})$
(3) *For each* $(n, \alpha : y \longrightarrow z) \in \mathbb{N} \times Mor(\mathscr{X})$, *the morphism* $\mathscr{P}(n, \alpha)^{\alpha} : \alpha^* \mathscr{P}(n, \alpha)_y \longrightarrow \mathscr{P}(n, \alpha)_z$ *is an isomorphism in* $\mathbf{M}^C_{\mathscr{R}_z}(\psi_z)$.
(4) $|\mathscr{P}(n, \alpha)| \leq \kappa$.

Proof The conditions (1) and (2) are immediate from the definition in (6.12). The condition (4) follows from (6.17) and the fact that each $|\mathscr{Q}^{m+1}(n, \alpha)| \leq \kappa$.

To prove (3), we notice that $\mathscr{P}(n, \alpha)_y$ may be expressed as the filtered union

$$A_1^0(y) \hookrightarrow \mathscr{Q}^0(n, \alpha)_y \hookrightarrow A_1^1(y) \hookrightarrow \mathscr{Q}^1(n, \alpha)_y \hookrightarrow \ldots \hookrightarrow A_1^{m+1}(y) \hookrightarrow \mathscr{Q}^{m+1}(n, \alpha)_y \hookrightarrow \ldots \tag{6.18}$$

of objects in $\mathbf{M}^C_{\mathscr{R}_y}(\psi_y)$. Since α^* is exact and a left adjoint, we can express $\alpha^* \mathscr{P}(n, \alpha)_y$ as the filtered union

$$\begin{aligned} &\alpha^* A_1^0(y) \hookrightarrow \alpha^* \mathscr{Q}^0(n, \alpha)_y \hookrightarrow \alpha^* A_1^1(y) \hookrightarrow \alpha^* \mathscr{Q}^1(n, \alpha)_y \hookrightarrow \ldots \hookrightarrow \alpha^* A_1^{m+1}(y) \\ &\hookrightarrow \alpha^* \mathscr{Q}^{m+1}(n, \alpha)_y \hookrightarrow \ldots \end{aligned} \tag{6.19}$$

in $\mathbf{M}^C_{\mathscr{R}_z}(\psi_z)$. Similarly, $\mathscr{P}(n, \alpha)_z$ may be expressed as the filtered union

$$A_1^0(z) \hookrightarrow \mathscr{Q}^0(n, \alpha)_z \hookrightarrow A_1^1(z) \hookrightarrow \mathscr{Q}^1(n, \alpha)_z \hookrightarrow \ldots \hookrightarrow A_1^{m+1}(z) \hookrightarrow \mathscr{Q}^{m+1}(n, \alpha)_z \hookrightarrow \ldots \tag{6.20}$$

in $\mathbf{M}^C_{\mathscr{R}_z}(\psi_z)$. By definition, we know that $A_1^m(z) = \alpha^* A_1^m(y)$ for each $m \geq 0$. From (6.19) and (6.20), it is clear that the filtered colimit of the isomorphisms $\alpha^* A_1^m(y) = A_1^m(z)$ induces an isomorphism $\mathscr{P}(n, \alpha)^{\alpha} : \alpha^* \mathscr{P}(n, \alpha)_y \longrightarrow \mathscr{P}(n, \alpha)_z$. □

Lemma 6.9 *Let* $\mathscr{M}$ *be a cartesian module over a flat representation* $\mathscr{R} : \mathscr{X} \longrightarrow \mathscr{E}nt_C$. *Choose* $m \in el_{\mathscr{X}}(\mathscr{M})$. *Let* $\kappa = max\{|\mathbb{N}|, |C|, |K|, |Mor(\mathscr{X})|, |Mor(\mathscr{R}_x)|, x \in \mathscr{X}\}$. *Then, there is a cartesian submodule* $\mathscr{P} \subseteq \mathscr{M}$ *with* $m \in el_{\mathscr{X}}(\mathscr{P})$ *such that* $|\mathscr{P}| \leq \kappa$.

Proof It is clear that $\mathbb{N} \times Mor(\mathscr{X})$ with the lexicographic ordering is filtered. We set

$$\mathscr{P} := \bigcup_{(n,\alpha) \in \mathbb{N} \times Mor(\mathscr{X})} \mathscr{P}(n, \alpha) \subseteq \mathscr{M} \tag{6.21}$$

in $Mod^C - \mathscr{R}$. It is immediate that $m \in el_{\mathscr{X}}(\mathscr{P})$. Since each $|\mathscr{P}(n, \alpha)| \leq \kappa$, it is clear that $|\mathscr{P}| \leq \kappa$.

We now consider a morphism $\beta : z \longrightarrow w$ in $\mathscr{X}$. Then, the family $\{(m, \beta)\}_{m \geq 1}$ is cofinal in $\mathbb{N} \times Mor(\mathscr{X})$ and hence it follows that

$$\mathscr{P} := \varinjlim_{m \geq 1} \mathscr{P}(m, \beta) \tag{6.22}$$

Since each $\mathscr{P}(m, \beta)^{\beta} : \beta^* \mathscr{P}(m, \beta)_z \longrightarrow \mathscr{P}(m, \beta)_w$ is an isomorphism, the filtered colimit $\mathscr{P}^{\beta} : \beta^* \mathscr{P}_z \longrightarrow \mathscr{P}_w$ is an isomorphism. □

Theorem 6.10 *Let* C *be a right semiperfect coalgebra over a field* K. *Let* $\mathscr{X}$ *be a poset and let* $\mathscr{R} : \mathscr{X} \longrightarrow \mathscr{E}nt_C$ *be an entwined* C*-representation of* $\mathscr{X}$. *Suppose that* $\mathscr{R}$ *is flat. Then,* $Cart^C - \mathscr{R}$ *is a Grothendieck category.*

Proof It is already clear that $Cart^C - \mathscr{R}$ satisfies the (AB5) condition. From Lemma 6.9, it is clear that any $\mathscr{M} \in Cart^C - \mathscr{R}$ can be expressed as a sum of a family $\{\mathscr{P}_m\}_{m \in el_{\mathscr{X}}(\mathscr{M})}$ of cartesian submodules such that each $|\mathscr{P}_m| \leq \kappa$. As such, isomorphism classes of cartesian modules $\mathscr{P}$ with $|\mathscr{P}| \leq \kappa$ form a family of generators for $Cart^C - \mathscr{R}$. □

7 Separability of the Forgetful Functor

Let $(\mathcal{R}, C, \psi)$ be an entwining structure. We consider the forgetful functor $\mathcal{F} : \mathbf{M}^C_{\mathcal{R}}(\psi) \longrightarrow \mathbf{M}_{\mathcal{R}}$. By [4, Lemma 2.4 & Lemma 3.1], we know that $\mathcal{F}$ has a right adjoint $\mathcal{G} : \mathbf{M}_{\mathcal{R}} \longrightarrow \mathbf{M}^C_{\mathcal{R}}(\psi)$ given by setting $\mathcal{G}(\mathcal{N}) := \mathcal{N} \otimes C$, i.e. $\mathcal{G}(\mathcal{N})(r) := \mathcal{N}(r) \otimes C$ for each $r \in \mathcal{R}$. The right $\mathcal{R}$-module structure on $\mathcal{G}(\mathcal{N})$ is given by $(n \otimes c) \cdot f := nf_\psi \otimes c^\psi$ for $f \in \mathcal{R}(r', r)$, $n \in \mathcal{N}(r)$ and $c \in C$.

We continue with $\mathscr{X}$ being a poset, C being a right semiperfect coalgebra and let $\mathscr{R} : \mathscr{X} \longrightarrow \mathscr{E}nt_C$ be an entwined C-representation. We denote by $\mathscr{L}in$ the category of small K-linear categories. Then, for each $x \in \mathscr{X}$, we may replace the entwining structure $(\mathscr{R}_x, C, \psi_x)$ by the K-linear category $\mathscr{R}_x$ to obtain a functor that we continue to denote by $\mathscr{R} : \mathscr{X} \longrightarrow \mathscr{L}in$. We consider modules over $\mathscr{R} : \mathscr{X} \longrightarrow \mathscr{L}in$ in the sense of Estrada and Virili [19, Definition 3.6] and denote their category by $Mod - \mathscr{R}$. Explicitly, an object $\mathscr{N}$ in $Mod - \mathscr{R}$ consists of a module $\mathscr{N}_x \in \mathbf{M}_{\mathscr{R}_x}$ for each $x \in \mathscr{X}$ as well as compatible morphisms $\mathscr{N}_\alpha : \mathscr{N}_x \longrightarrow \alpha_* \mathscr{N}_y$ (equivalently $\mathscr{N}^\alpha : \alpha^* \mathscr{N}_x \longrightarrow \mathscr{N}_y$) for each $\alpha : x \longrightarrow y$ in $\mathscr{X}$. The module $\mathscr{N}$ is said to be cartesian if each $\mathscr{N}^\alpha : \alpha^* \mathscr{N}_x \longrightarrow \mathscr{N}_y$ is an isomorphism. We denote by $Cart - \mathscr{R}$ the full subcategory of cartesian modules on $\mathscr{R}$.

For each $x \in \mathscr{X}$, we have a forgetful functor $\mathscr{F}_x : \mathbf{M}^C_{\mathscr{R}_x}(\psi_x) \longrightarrow \mathbf{M}_{\mathscr{R}_x}$ having right adjoint $\mathscr{G}_x : \mathbf{M}_{\mathscr{R}_x} \longrightarrow \mathbf{M}^C_{\mathscr{R}_x}(\psi_x)$. From the proofs of Propositions 2.3 and 2.4, it is clear that we have commutative diagrams

$$\begin{array}{ccc} \mathbf{M}^C_{\mathscr{R}_y}(\psi_y) & \xrightarrow{\alpha_*} & \mathbf{M}^C_{\mathscr{R}_x}(\psi_x) \\ \downarrow{\scriptstyle \mathscr{F}_y} & & \downarrow{\scriptstyle \mathscr{F}_x} \\ \mathbf{M}_{\mathscr{R}_y} & \xrightarrow{\alpha_*} & \mathbf{M}_{\mathscr{R}_x} \end{array} \qquad \begin{array}{ccc} \mathbf{M}^C_{\mathscr{R}_x}(\psi_x) & \xrightarrow{\alpha^*} & \mathbf{M}^C_{\mathscr{R}_y}(\psi_y) \\ \downarrow{\scriptstyle \mathscr{F}_x} & & \downarrow{\scriptstyle \mathscr{F}_y} \\ \mathbf{M}_{\mathscr{R}_x} & \xrightarrow{\alpha^*} & \mathbf{M}_{\mathscr{R}_y} \end{array} \qquad \begin{array}{ccc} \mathbf{M}_{\mathscr{R}_y} & \xrightarrow{\alpha_*} & \mathbf{M}_{\mathscr{R}_x} \\ \downarrow{\scriptstyle \mathscr{G}_y} & & \downarrow{\scriptstyle \mathscr{G}_x} \\ \mathbf{M}^C_{\mathscr{R}_y}(\psi_y) & \xrightarrow{\alpha_*} & \mathbf{M}^C_{\mathscr{R}_x}(\psi_x) \end{array} \tag{7.1}$$

for each $\alpha : x \longrightarrow y$ in $\mathscr{X}$.

Proposition 7.1 *Let $\mathscr{R} : \mathscr{X} \longrightarrow \mathscr{E}nt_C$ be an entwined C-representation. Then, the collection $\{\mathscr{F}_x : \mathbf{M}^C_{\mathscr{R}_x}(\psi_x) \longrightarrow \mathbf{M}_{\mathscr{R}_x}\}_{x \in \mathscr{X}}$ (resp. the collection $\{\mathscr{G}_x : \mathbf{M}_{\mathscr{R}_x} \longrightarrow \mathbf{M}^C_{\mathscr{R}_x}(\psi_x)\}_{x \in \mathscr{X}}$) together defines a functor $\mathscr{F} : Mod^C - \mathscr{R} \longrightarrow Mod - \mathscr{R}$ (resp. a functor $\mathscr{G} : Mod - \mathscr{R} \longrightarrow Mod^C - \mathscr{R}$).*

Proof We consider $\mathscr{M} \in Mod^C - \mathscr{R}$ and set $\mathscr{F}(\mathscr{M})_x := \mathscr{F}_x(\mathscr{M}_x) \in \mathbf{M}_{\mathscr{R}_x}$. For a morphism $\alpha : x \longrightarrow y$, we obtain from (7.1) a morphism $\mathscr{F}(\mathscr{M})_\alpha := \mathscr{F}_x(\mathscr{M}_\alpha) : \mathscr{F}_x(\mathscr{M}_x) \longrightarrow \mathscr{F}_x(\alpha_* \mathscr{M}_y) = \alpha_* \mathscr{F}_y(\mathscr{M}_y)$. This shows that $\mathscr{F}(\mathscr{M})$ is an object of $Mod - \mathscr{R}$. Similarly, it follows from (7.1) that for any $\mathscr{N} \in Mod - \mathscr{R}$, we have $\mathscr{G}(\mathscr{N}) \in Mod^C - \mathscr{R}$ obtained by setting $\mathscr{G}(\mathscr{N})_x := \mathscr{G}_x(\mathscr{N}_x) = \mathscr{N}_x \otimes C$. □

Proposition 7.2 *Let $\mathscr{R} : \mathscr{X} \longrightarrow \mathscr{E}nt_C$ be an entwined C-representation. Then, the functor $\mathscr{F} : Mod^C - \mathscr{R} \longrightarrow Mod - \mathscr{R}$ has a right adjoint, given by $\mathscr{G} : Mod - \mathscr{R} \longrightarrow Mod^C - \mathscr{R}$.*

Proof We consider $\mathscr{M} \in Mod^C - \mathscr{R}$ and $\mathscr{N} \in Mod - \mathscr{R}$ along with a morphism $\eta : \mathscr{F}(\mathscr{M}) \longrightarrow \mathscr{N}$ in $Mod - \mathscr{R}$. We will show how to construct a morphism $\zeta : \mathscr{M} \longrightarrow \mathscr{G}(\mathscr{N})$ in $Mod^C - \mathscr{R}$ corresponding to η.

For each $x \in \mathscr{X}$, we consider $\eta_x : \mathscr{F}(\mathscr{M})_x = \mathscr{F}_x(\mathscr{M}_x) \longrightarrow \mathscr{N}_x$ in $\mathbf{M}_{\mathscr{R}_x}$. By [4, Lemma 3.1], we already know that $(\mathscr{F}_x, \mathscr{G}_x)$ is a pair of adjoint functors, which gives us $\mathbf{M}_{\mathscr{R}_x}(\mathscr{F}_x(\mathscr{M}_x), \mathscr{N}_x) \cong \mathbf{M}^C_{\mathscr{R}_x}(\mathscr{M}_x, \mathscr{G}_x(\mathscr{N}_x))$. Accordingly, we define $\zeta_x : \mathscr{M}_x \longrightarrow \mathscr{G}_x(\mathscr{N}_x) = \mathscr{N}_x \otimes C$ by setting $\zeta_x(m') := \eta_x(r)(m'_0) \otimes m'_1$ for $m' \in \mathscr{M}_x(r)$, $r \in \mathscr{R}_x$. We now consider the diagrams

$$\begin{array}{ccc} \mathscr{F}_x(\mathscr{M}_x) & \xrightarrow{\eta_x} & \mathscr{N}_x \\ \downarrow{\scriptstyle \mathscr{F}_x(\mathscr{M}_\alpha)} & & \downarrow{\scriptstyle \mathscr{N}_\alpha} \\ \alpha_* \mathscr{F}_y(\mathscr{M}_y) & \xrightarrow{\alpha_*(\eta_y)} & \alpha_* \mathscr{N}_y \end{array} \quad \Rightarrow \quad \begin{array}{ccc} \mathscr{M}_x & \xrightarrow{\zeta_x} & \mathscr{G}_x(\mathscr{N}_x) \\ \downarrow{\scriptstyle \mathscr{M}_\alpha} & & \downarrow{\scriptstyle \mathscr{G}_x(\mathscr{N}_\alpha)} \\ \alpha_* \mathscr{M}_y & \xrightarrow{\alpha_*(\zeta_y)} & \alpha_* \mathscr{G}_y(\mathscr{N}_y) \end{array} \tag{7.2}$$

The left hand side diagram in (7.2) is commutative because $\eta : \mathscr{F}(\mathscr{M}) \longrightarrow \mathscr{N}$ is a morphism in $Mod - \mathscr{R}$. In order to prove that we have a morphism $\zeta : \mathscr{M} \longrightarrow \mathscr{G}(\mathscr{N})$ in $Mod^C - \mathscr{R}$, it suffices to show that this implies the commutativity of the right hand side diagram in (7.2).

We consider $m \in el(\mathscr{M}_x)$. Then, we have $\mathscr{G}_x(\mathscr{N}_\alpha)(\zeta_x(m)) = \mathscr{N}_\alpha(\eta_x(m_0)) \otimes m_1$. On the other hand, we have $\alpha_*(\zeta_y)(\mathscr{M}_\alpha(m)) = \eta_y((\mathscr{M}_\alpha(m))_0) \otimes (\mathscr{M}_\alpha(m))_1$. Since $\mathscr{M}_\alpha$ is C-colinear, we have $(\mathscr{M}_\alpha(m))_0 \otimes (\mathscr{M}_\alpha(m))_1 = \mathscr{M}_\alpha(m_0) \otimes m_1$. It follows that $\alpha_*(\zeta_y)(\mathscr{M}_\alpha(m)) = \eta_y(\mathscr{M}_\alpha(m_0)) \otimes m_1$. From the left hand side commutative diagram in (7.2), we get $\eta_y(\mathscr{M}_\alpha(m_0)) = \mathscr{N}_\alpha(\eta_x(m_0))$, which shows that the right hand diagram in (7.2) is commutative.

Similarly, we may show that a morphism $\zeta' : \mathscr{M} \longrightarrow \mathscr{G}(\mathscr{N})$ in $Mod^C - \mathscr{R}$ induces a morphism $\eta' : \mathscr{F}(\mathscr{M}) \longrightarrow \mathscr{N}$ in $Mod - \mathscr{R}$ and that these two associations are inverse to each other. This proves the result. □

We now recall that a functor $F : \mathcal{A} \longrightarrow \mathcal{B}$ is said to be separable if the natural transformation $\mathcal{A}(\_\_, \_\_) \longrightarrow \mathcal{B}(F(\_\_), F(\_\_))$ is a split monomorphism (see [26, 27]). If F has a right adjoint $G : \mathcal{B} \longrightarrow \mathcal{A}$, then F is separable if and only if there exists a natural transformation $\upsilon \in Nat(GF, 1_{\mathcal{A}})$ satisfying $\upsilon \circ \mu = 1_{\mathcal{A}}$, where μ is the unit of the adjunction (see [27, Theorem 1.2]).

We now consider the forgetful functor $\mathscr{F} : Mod^C - \mathscr{R} \longrightarrow Mod - \mathscr{R}$ as well as its right adjoint $\mathscr{G} : Mod - \mathscr{R} \longrightarrow Mod^C - \mathscr{R}$ constructed in Proposition 7.2. We will need an alternate description for the natural transformations $\mathscr{G}\mathscr{F} \longrightarrow 1_{Mod^C - \mathscr{R}}$.

Proposition 7.3 *A natural transformation $\upsilon \in Nat(\mathscr{G}\mathscr{F}, 1_{Mod^C - \mathscr{R}})$ corresponds to a collection of natural transformations $\{\upsilon_x \in Nat(\mathscr{G}_x\mathscr{F}_x, 1_{\mathbf{M}^C_{\mathscr{R}_x}(\psi_x)})\}_{x \in \mathscr{X}}$ such that for any $\alpha : x \longrightarrow y$ in $\mathscr{X}$ and object $\mathscr{M} \in Mod^C - \mathscr{R}$, we have a commutative diagram*

$$\begin{array}{ccc} \mathscr{G}_x\mathscr{F}_x(\mathscr{M}_x) & \xrightarrow{\upsilon_x(\mathscr{M}_x)} & \mathscr{M}_x \\ \downarrow{\scriptstyle \mathscr{G}_x\mathscr{F}_x(\mathscr{M}_\alpha)} & & \downarrow{\scriptstyle \mathscr{M}_\alpha} \\ \alpha_*\mathscr{G}_y\mathscr{F}_y(\mathscr{M}_y) & \xrightarrow{\alpha_*\upsilon_y(\mathscr{M}_y)} & \alpha_*\mathscr{M}_y \end{array} \tag{7.3}$$

in $\mathbf{M}^C_{\mathscr{R}_x}(\psi_x)$.

Proof We consider $\upsilon \in Nat(\mathscr{G}\mathscr{F}, 1_{Mod^C - \mathscr{R}})$. For $x \in \mathscr{X}$, we define the natural transformation $\upsilon_x \in Nat(\mathscr{G}_x\mathscr{F}_x, 1_{\mathbf{M}^C_{\mathscr{R}_x}(\psi_x)})$ by setting

$$\upsilon_x(\mathcal{M}) := \upsilon(ex^C_x(\mathcal{M}))_x : \mathscr{G}_x\mathscr{F}_x(\mathcal{M}) = \mathscr{G}_x\mathscr{F}_x((ex^C_x(\mathcal{M}))_x) \longrightarrow (ex^C_x(\mathcal{M}))_x = \mathcal{M} \tag{7.4}$$

for $\mathcal{M} \in \mathbf{M}^C_{\mathscr{R}_x}(\psi_x)$. We now consider $\mathscr{M} \in Mod^C - \mathscr{R}$. For $\alpha : x \longrightarrow y$ in $\mathscr{X}$, the morphism $\upsilon(\mathscr{M}) : \mathscr{G}\mathscr{F}(\mathscr{M}) \longrightarrow \mathscr{M}$ in $Mod^C - \mathscr{R}$ leads to a commutative diagram

$$\begin{array}{ccc} (\mathscr{G}\mathscr{F}(\mathscr{M}))_x = \mathscr{G}_x\mathscr{F}_x(\mathscr{M}_x) & \xrightarrow{\upsilon(\mathscr{M})_x} & \mathscr{M}_x \\ \downarrow{\scriptstyle \mathscr{G}_x\mathscr{F}_x(\mathscr{M}_\alpha)} & & \downarrow{\scriptstyle \mathscr{M}_\alpha} \\ \alpha_*(\mathscr{G}\mathscr{F}(\mathscr{M}))_y = \alpha_*\mathscr{G}_y\mathscr{F}_y(\mathscr{M}_y) & \xrightarrow{\alpha_*(\upsilon(\mathscr{M})_y)} & \alpha_*\mathscr{M}_y \end{array} \tag{7.5}$$

We now claim that $\upsilon(\mathscr{M})_x = (\upsilon(ex^C_x(\mathscr{M}_x)))_x = \upsilon_x(\mathscr{M}_x)$ for each $x \in \mathscr{X}$. For this, we consider the canonical morphism $\zeta : ex^C_x(\mathscr{M}_x) = ex^C_x(ev^C_x(\mathscr{M})) \longrightarrow \mathscr{M}$ in $Mod^C - \mathscr{R}$ corresponding to the adjoint pair (ex_x^C, ev_x^C) in Proposition 5.3. It is clear that $ev_x^C(\zeta) = id$. Then, we have commutative diagrams

$$\begin{array}{ccccccc} \mathscr{G}\mathscr{F}(ex_x^C(\mathscr{M}_x)) & \xrightarrow{\upsilon(ex_x^C(\mathscr{M}_x))} & ex_x^C(\mathscr{M}_x) & & \mathscr{G}_x\mathscr{F}_x(\mathscr{M}_x) & \xrightarrow{(\upsilon(ex_x^C(\mathscr{M}_x)))_x} & \mathscr{M}_x \\ \downarrow{\scriptstyle \mathscr{G}\mathscr{F}(\zeta)} & & \downarrow{\scriptstyle \zeta} & \Rightarrow & \downarrow{\scriptstyle id} & & \downarrow{\scriptstyle id} \\ \mathscr{G}\mathscr{F}(\mathscr{M}) & \xrightarrow{\upsilon(\mathscr{M})} & \mathscr{M} & & \mathscr{G}_x\mathscr{F}_x(\mathscr{M}_x) & \xrightarrow{\upsilon(\mathscr{M})_x} & \mathscr{M}_x \end{array} \tag{7.6}$$

This proves that $\upsilon(\mathscr{M})_x = (\upsilon(ex_x^C(\mathscr{M}_x)))_x = \upsilon_x(\mathscr{M}_x)$ for each $x \in \mathscr{X}$. The commutativity of the diagram (7.3) now follows from (7.5).

Conversely, given a collection of natural transformations $\{\upsilon_x \in Nat(\mathscr{G}_x\mathscr{F}_x, 1_{\mathbf{M}^C_{\mathscr{R}_x}(\psi_x)})\}_{x\in\mathscr{X}}$ satisfying (7.3) for each $\mathscr{M} \in Mod^C - \mathscr{R}$, we get $\upsilon(\mathscr{M}) : \mathscr{G}\mathscr{F}(\mathscr{M}) \longrightarrow \mathscr{M}$ in $Mod^C - \mathscr{R}$ by setting $\upsilon(\mathscr{M})_x = \upsilon_x(\mathscr{M}_x)$ for each $x \in \mathscr{X}$. From (7.3), it is clear that $\upsilon \in Nat(\mathscr{G}\mathscr{F}, 1_{Mod^C-\mathscr{R}})$. □

More explicitly, the diagram in (7.3) shows that for each $\alpha : x \longrightarrow y$ in $\mathscr{X}$ and $r \in \mathscr{R}_x$, we have a commutative diagram

$$\begin{array}{ccc} \mathscr{M}_x(r)\otimes C = (\mathscr{G}_x\mathscr{F}_x(\mathscr{M}_x))(r) & \xrightarrow{(\upsilon_x(\mathscr{M}_x))(r)} & \mathscr{M}_x(r) \\ \downarrow{\scriptstyle (\mathscr{G}_x\mathscr{F}_x(\mathscr{M}_\alpha))(r)} & & \downarrow{\scriptstyle \mathscr{M}_\alpha(r)} \\ \mathscr{M}_y(\alpha(r))\otimes C = (\mathscr{G}_y\mathscr{F}_y(\mathscr{M}_y))(\alpha(r)) = (\alpha_*\mathscr{G}_y\mathscr{F}_y(\mathscr{M}_y))(r) & \xrightarrow[=(\upsilon_y(\mathscr{M}_y))(\alpha(r))]{(\alpha_*\upsilon_y(\mathscr{M}_y))(r)} & (\alpha_*\mathscr{M}_y)(r) = \mathscr{M}_y(\alpha(r)) \end{array} \tag{7.7}$$

We note that all morphisms in (7.7) are C-colinear. We now give another interpretation of the space $Nat(\mathscr{G}\mathscr{F}, 1_{Mod^C-\mathscr{R}})$. For this, we consider a collection $\theta := \{\theta_x(r) : C \otimes C \longrightarrow \mathscr{R}_x(r,r)\}_{x\in\mathscr{X}, r\in\mathscr{R}_x}$ of K-linear maps satisfying the following conditions.

(1) Fix $x \in \mathscr{X}$ and $r \in \mathscr{R}_x$. Then, for $c, d \in C$, we have

$$\theta_x(r)(c \otimes d_1) \otimes d_2 = (\theta_x(r)(c_2 \otimes d))_{\psi_x} \otimes c_1{}^{\psi_x} \tag{7.8}$$

(2) Fix $x \in \mathscr{X}$ and $c, d \in C$. Then, for $f : s \longrightarrow r$ in $\mathscr{R}_x$, we have

$$(\theta_x(r)(c \otimes d)) \circ f = f_{\psi_x\psi_x} \circ (\theta_x(s)(c^{\psi_x} \otimes d^{\psi_x})) \tag{7.9}$$

(3) Fix $c, d \in C$. Then, for any $\alpha : x \longrightarrow y$ in $\mathscr{X}$ and $r \in \mathscr{R}_x$, we have

$$\alpha(\theta_x(r)(c \otimes d)) = \theta_y(\alpha(r))(c \otimes d) \tag{7.10}$$

The space of all such θ will be denoted by V_1.

Proposition 7.4 *Let $\theta \in V_1$. Then, θ induces a natural transformation $\upsilon \in Nat(\mathscr{G}\mathscr{F}, 1_{Mod^C-\mathscr{R}})$, such that for each $x \in \mathscr{X}$, $\upsilon_x \in Nat(\mathscr{G}_x\mathscr{F}_x, 1_{\mathbf{M}^C_{\mathscr{R}_x}(\psi_x)})$ is given by*

$$\upsilon_x(\mathcal{M}) : \mathcal{M} \otimes C \longrightarrow \mathcal{M} \qquad (m \otimes c) \mapsto \mathcal{M}(\theta_x(r)(m_1 \otimes c))(m_0) \tag{7.11}$$

for any $\mathcal{M} \in \mathbf{M}^C_{\mathscr{R}_x}(\psi_x)$, $r \in \mathscr{R}_x$, $m \in \mathcal{M}(r)$ and $c \in C$.

Proof From [4, Proposition 3.6], it follows that each υ_x as defined in (7.11) by the collection $\theta_x := \{\theta_x(r) : C \otimes C \longrightarrow \mathscr{R}_x(r,r)\}_{r\in\mathscr{R}_x}$ gives a natural transformation $\upsilon_x \in Nat(\mathscr{G}_x\mathscr{F}_x, 1_{\mathbf{M}^C_{\mathscr{R}_x}(\psi_x)})$. To prove the result, it therefore suffices to show the commutativity of the diagram (7.7) for any $\mathscr{M} \in Mod^C - \mathscr{R}$. Accordingly, for $\alpha : x \longrightarrow y$ in $\mathscr{X}$ and $r \in \mathscr{R}_x$, we have

$$((\mathscr{M}_\alpha(r)) \circ (\upsilon_x(\mathscr{M}_x))(r))(m \otimes c) = (\mathscr{M}_\alpha(r))(\mathscr{M}_x(\theta_x(r)(m_1 \otimes c))(m_0)) \tag{7.12}$$

for $m \otimes c \in \mathscr{M}_x(r) \otimes C$. On the other hand, we have

$$\begin{aligned}(((\upsilon_y(\mathscr{M}_y))(\alpha(r))) \circ ((\mathscr{G}_x\mathscr{F}_x(\mathscr{M}_\alpha))(r)))(m \otimes c) &= \mathscr{M}_y(\theta_y(\alpha(r))(\mathscr{M}_\alpha(m)_1 \otimes c))(\mathscr{M}_\alpha(r)(m))_0 \\ &= \mathscr{M}_y(\theta_y(\alpha(r))(m_1 \otimes c))(\mathscr{M}_\alpha(r)(m_0)) \\ &= \mathscr{M}_y(\alpha(\theta_x(r)(m_1 \otimes c)))(\mathscr{M}_\alpha(r)(m_0))\end{aligned} \tag{7.13}$$

The second equality in (7.13) follows from the C-colinearity of $\mathscr{M}_\alpha(r)$ and the third equality follows by applying condition (7.10). We now notice that for any $f \in \mathscr{R}_x(r,r)$, we have a commutative diagram

$$\begin{array}{ccc} \mathscr{M}_x(r) & \xrightarrow{\mathscr{M}_\alpha(r)} & \mathscr{M}_y(\alpha(r)) \\ {\scriptstyle \mathscr{M}_x(f)}\downarrow & & \downarrow{\scriptstyle \mathscr{M}_y(\alpha(f))} \\ \mathscr{M}_x(r) & \xrightarrow{\mathscr{M}_\alpha(r)} & \mathscr{M}_y(\alpha(r)) \end{array} \tag{7.14}$$

Applying (7.14) to $f = \theta_x(r)(m_1 \otimes c) \in \mathscr{R}_x(r,r)$, we obtain from (7.13) that

$$(((\upsilon_y(\mathscr{M}_y))(\alpha(r))) \circ ((\mathscr{G}_x\mathscr{F}_x(\mathscr{M}_\alpha))(r)))(m \otimes c) = (\mathscr{M}_\alpha(r))(\mathscr{M}_x(\theta_x(r)(m_1 \otimes c))(m_0)) \tag{7.15}$$

This proves the result. □

Fix $x \in \mathscr{X}$ and $r \in \mathscr{R}_x$. We now set

$$\mathscr{H}_y^{(x,r)} := \begin{cases} \mathscr{R}_y(\_\_, \alpha(r)) \otimes C & \text{if } \alpha : x \longrightarrow y \\ 0 & \text{if } x \not\leq y \end{cases} \tag{7.16}$$

for each $y \in \mathscr{X}$.

Lemma 7.5 *For each $x \in \mathscr{X}$ and $r \in \mathscr{R}_x$, the collection $\mathscr{H}^{(x,r)} := \{\mathscr{H}_y^{(x,r)}\}_{y \in \mathscr{X}}$ determines an object of $Mod^C - \mathscr{R}$.*

Proof For each $y \in \mathscr{X}$, it follows by [4, Lemma 2.4] that $\mathscr{H}_y^{(x,r)}$ is an object of $\mathbf{M}^C_{\mathscr{R}_y}(\psi_y)$. We consider $\beta : y \longrightarrow z$ in $\mathscr{X}$ and suppose we have $\alpha : x \longrightarrow y$, i.e., $x \leq y$. Then, for $r' \in \mathscr{R}_y$, we have an obvious morphism

$$\begin{aligned}\beta(\_\_) \otimes C : \mathscr{H}_y^{(x,r)}(r') = \mathscr{R}_y(r', \alpha(r)) \otimes C &\longrightarrow \beta_*(\mathscr{R}_z(\_\_, \beta\alpha(r)) \otimes C)(r') \\ &= \mathscr{R}_z(\beta(r'), \beta\alpha(r)) \otimes C\end{aligned} \tag{7.17}$$

which is C-colinear. To prove that $\mathscr{H}_y^{(x,r)} \longrightarrow \beta_*\mathscr{H}_z^{(x,r)}$ is a morphism in $\mathbf{M}^C_{\mathscr{R}_y}(\psi_y)$, it remains to show that for any $g : r'' \longrightarrow r'$ in $\mathscr{R}_y$, the following diagram commutes

$$\begin{array}{ccc} \mathscr{R}_y(r', \alpha(r)) \otimes C & \xrightarrow{\cdot g} & \mathscr{R}_y(r'', \alpha(r)) \otimes C \\ {\scriptstyle \beta(\_\_)\otimes C}\downarrow & & \downarrow{\scriptstyle \beta(\_\_)\otimes C} \\ \mathscr{R}_z(\beta(r'), \beta\alpha(r)) \otimes C & \xrightarrow{\cdot \beta(g)} & \mathscr{R}_z(\beta(r''), \beta\alpha(r)) \otimes C \end{array} \tag{7.18}$$

For $f \otimes c \in \mathscr{R}_y(r', \alpha(r)) \otimes C$, we have

$$\begin{aligned}(\beta(\_\_) \otimes C)((f \otimes c) \cdot g) &= (\beta(\_\_) \otimes C)(fg_{\psi_y} \otimes c^{\psi_y}) = \beta(f)\beta(g_{\psi_y}) \otimes c^{\psi_y} \\ &= \beta(f)\beta(g)_{\psi_z} \otimes c^{\psi_z} = (\beta(f) \otimes c) \cdot \beta(g)\end{aligned}$$

This shows that (7.18) is commutative. Finally, if $x \not\leq y$, then $0 = \mathscr{H}_y^{(x,r)} \longrightarrow \beta_*\mathscr{H}_z^{(x,r)}$ is obviously a morphism in $\mathbf{M}^C_{\mathscr{R}_y}(\psi_y)$. This proves the result. □

Proposition 7.6 *Let $\upsilon \in Nat(\mathscr{G}\mathscr{F}, 1_{Mod^C-\mathscr{R}})$. For each $x \in \mathscr{X}$ and $r \in \mathscr{R}_x$, define $\theta_x(r) : C \otimes C \longrightarrow \mathscr{R}_x(r,r)$ by setting*

$$\theta_x(r)(c \otimes d) := ((id \otimes \varepsilon_C) \circ (\upsilon_x(\mathscr{H}_x^{(x,r)})(r)))(id_r \otimes c \otimes d) \tag{7.19}$$

for $c, d \in C$. Then, the collection $\theta := \{\theta_x(r) : C \otimes C \longrightarrow \mathscr{R}_x(r,r)\}_{x\in\mathscr{X}, r\in\mathscr{R}_x}$ is an element of V_1.

Proof From the definition in (7.19), we have explicitly that

$$\theta_x(r)(c \otimes d) = ((id \otimes \varepsilon_C) \circ (\upsilon_x(\mathscr{R}_x(\_\_, r) \otimes C)(r)))(id_r \otimes c \otimes d) \tag{7.20}$$

Then, it follows from [4, Proposition 3.5] that $\theta_x(r)$ satisfies the conditions in (7.8) and (7.9). It remains to verify the condition (7.10). For this we take $\alpha : x \longrightarrow y$ in $\mathscr{X}$ and consider the commutative diagram

$$\begin{array}{ccccc}
\mathscr{R}_x(r',r)\otimes C\otimes C & \xrightarrow{\upsilon_x(\mathscr{H}_x^{(x,r)})(r')} & \mathscr{R}_x(r',r)\otimes C & \xrightarrow{id\otimes\varepsilon_C} & \mathscr{R}_x(r',r) \\
\downarrow{\scriptstyle \alpha(__)\otimes C\otimes C} & & \downarrow{\scriptstyle \alpha(__)\otimes C} & & \downarrow{\scriptstyle \alpha(__)} \\
\mathscr{R}_y(\alpha(r'),\alpha(r))\otimes C\otimes C & \xrightarrow{\upsilon_y(\mathscr{H}_y^{(x,r)})(\alpha(r'))} & \mathscr{R}_y(\alpha(r'),\alpha(r))\otimes C & \xrightarrow{id\otimes\varepsilon_C} & \mathscr{R}_y(\alpha(r'),\alpha(r))
\end{array} \tag{7.21}$$

for any $r, r' \in \mathscr{R}_x$. Since $\upsilon \in Nat(\mathscr{G}\mathscr{F}, 1_{Mod^C-\mathscr{R}})$, the commutativity of the left hand side square in (7.21) follows from (7.7). It is clear that the right hand square in (7.21) is commutative.

We notice that $\mathscr{H}_y^{(y,\alpha(r))} = \mathscr{H}_y^{(x,r)}$ in $\mathbf{M}^C_{\mathscr{R}_y}(\psi_y)$. Applying (7.21) with $r' = r \in \mathscr{R}_x$ and $id_r \otimes c \otimes d \in \mathscr{R}_x(r,r) \otimes C \otimes C$, it follows from (7.20) that $\alpha(\theta_x(r)(c \otimes d)) = \theta_y(\alpha(r))(c \otimes d)$. This proves (7.10). □

Proposition 7.7 *$Nat(\mathscr{G}\mathscr{F}, 1_{Mod^C-\mathscr{R}})$ is isomorphic to V_1.*

Proof From Propositions 7.4 and 7.6, we see that we have maps $\psi : V_1 \longrightarrow Nat(\mathscr{G}\mathscr{F}, 1_{Mod^C-\mathscr{R}})$ and $\phi : Nat(\mathscr{G}\mathscr{F}, 1_{Mod^C-\mathscr{R}}) \longrightarrow V_1$ in opposite directions.

We consider $\upsilon \in Nat(\mathscr{G}\mathscr{F}, 1_{Mod^C-\mathscr{R}})$. By Proposition 7.6, υ induces an element $\theta \in V_1$. Applying Proposition 7.4, θ induces an element in $Nat(\mathscr{G}\mathscr{F}, 1_{Mod^C-\mathscr{R}})$, which we denote by υ'. Then, υ and υ' are determined respectively by natural transformations $\{\upsilon_x \in Nat(\mathscr{G}_x\mathscr{F}_x, 1_{\mathbf{M}^C_{\mathscr{R}_x}(\psi_x)})\}_{x\in\mathscr{X}}$ and $\{\upsilon'_x \in Nat(\mathscr{G}_x\mathscr{F}_x, 1_{\mathbf{M}^C_{\mathscr{R}_x}(\psi_x)})\}_{x\in\mathscr{X}}$ satisfying compatibility conditions as in (7.3). From [4, Proposition 3.7], it follows that $\upsilon'_x = \upsilon_x$ for each $x \in \mathscr{X}$. Hence, $\upsilon' = \upsilon$ and $\psi \circ \phi = id$. Similarly, we can show that $\phi \circ \psi = id$. □

Theorem 7.8 *Let $\mathscr{X}$ be a partially ordered set. Let C be a right semiperfect K-coalgebra and let $\mathscr{R} : \mathscr{X} \longrightarrow \mathscr{E}nt_C$ be an entwined C-representation. Then, the functor $\mathscr{F} : Mod^C - \mathscr{R} \longrightarrow Mod - \mathscr{R}$ is separable if and only if there exists $\theta \in V_1$ such that*

$$\theta_x(r)(c_1 \otimes c_2) = \varepsilon_C(c) \cdot id_r \tag{7.22}$$

for every $x \in \mathscr{X}$, $r \in \mathscr{R}_x$ and $c \in C$.

Proof We suppose that $\mathscr{F} : Mod^C - \mathscr{R} \longrightarrow Mod - \mathscr{R}$ is separable. As mentioned before, this implies that there exists $\upsilon \in Nat(\mathscr{G}\mathscr{F}, 1_{Mod^C-\mathscr{R}})$ such that $\upsilon \circ \mu = 1_{Mod^C-\mathscr{R}}$, where μ is the unit of the adjunction $(\mathscr{F}, \mathscr{G})$. We set $\theta = \phi(\upsilon)$, where $\phi : Nat(\mathscr{G}\mathscr{F}, 1_{Mod^C-\mathscr{R}}) \longrightarrow V_1$ is the isomorphism described in the proof of Proposition 7.7. In particular, for every $x \in \mathscr{X}$, $r \in \mathscr{R}_x$, we have $\upsilon(\mathscr{H}^{(x,r)}) \circ \mu(\mathscr{H}^{(x,r)}) = id$. From (7.19), it now follows that for every $c \in C$, we have

$$\begin{aligned}
\theta_x(r)(c_1 \otimes c_2) &= ((id \otimes \varepsilon_C) \circ (\upsilon_x(\mathscr{H}_x^{(x,r)})(r)))(id_r \otimes c_1 \otimes c_2) \\
&= ((id \otimes \varepsilon_C) \circ (\upsilon_x(\mathscr{H}_x^{(x,r)})(r)) \circ \mu(\mathscr{H}^{(x,r)})_x(r))(id_r \otimes c) \\
&= (id \otimes \varepsilon_C)(id_r \otimes c) = \varepsilon_C(c) \cdot id_r
\end{aligned} \tag{7.23}$$

Conversely, suppose that there exists $\theta \in V_1$ satisfying the condition in (7.22). We set $\upsilon := \psi(\theta)$, where $\psi : V_1 \longrightarrow Nat(\mathscr{G}\mathscr{F}, 1_{Mod^C-\mathscr{R}})$ is the other isomorphism described in the proof of Proposition 7.7. We consider $\mathscr{M} \in Mod^C - \mathscr{R}$. By (7.11), we know that

$$\upsilon_x(\mathscr{M}_x) : \mathscr{M}_x \otimes C \longrightarrow \mathscr{M}_x \qquad (m \otimes c) \mapsto \mathscr{M}_x(\theta_x(r)(m_1 \otimes c))(m_0) \tag{7.24}$$

for any $x \in \mathscr{X}$, $r \in \mathscr{R}_x$, $m \in \mathscr{M}_x(r)$ and $c \in C$. We claim that $\upsilon \circ \mu = 1_{Mod^C-\mathscr{R}}$. For this, we see that

$$\begin{aligned} ((\upsilon(\mathscr{M}) \circ \mu(\mathscr{M}))_x(r))(m) &= (\upsilon_x(\mathscr{M}_x)(r))(m_0 \otimes m_1) \\ &= \mathscr{M}_x(\theta_x(r)(m_{01} \otimes m_1))(m_{00}) \\ &= \mathscr{M}_x(\theta_x(r)(m_{11} \otimes m_{12}))(m_0) \\ &= \varepsilon_C(m_1)m_0 = m \end{aligned} \tag{7.25}$$

This proves the result. □

We now turn to cartesian modules over entwined C-representations. For this, we assume additionally that $\mathscr{R} : \mathscr{X} \longrightarrow \mathscr{E}nt_C$ is flat. Then, it follows from Theorem 6.10 that $Cart^C - \mathscr{R}$ is a Grothendieck category. In particular, by taking $C = K$, we note that $Cart - \mathscr{R}$ is also a Grothendieck category.

Proposition 7.9 *Let $\mathscr{X}$ be a poset, C be a right semiperfect K-coalgebra and $\mathscr{R} : \mathscr{X} \longrightarrow \mathscr{E}nt_C$ be an entwined C-representation that is also flat. Then, the functor $\mathscr{F} : Mod^C - \mathscr{R} \longrightarrow Mod - \mathscr{R}$ restricts to a functor $\mathscr{F}^c : Cart^C - \mathscr{R} longrightarrow Cart - \mathscr{R}$. Additionally, $\mathscr{F}^c$ has a right adjoint $\mathscr{G}^c : Cart - \mathscr{R} \longrightarrow Cart^C - \mathscr{R}$.*

Proof We consider $\mathscr{M} \in Cart^C - \mathscr{R}$. We claim that $\mathscr{F}(\mathscr{M}) \in Mod - \mathscr{R}$ actually lies in the subcategory $Cart - \mathscr{R}$. Indeed, for $\alpha : x \longrightarrow y$ in $\mathscr{X}$, we have $\mathscr{F}(M)_\alpha : \mathscr{F}_x(\mathscr{M}_x) = \mathscr{M}_x \longrightarrow \alpha_*\mathscr{M}_y = \alpha_*\mathscr{F}_y(\mathscr{M}_y)$ in $\mathbf{M}_{\mathscr{R}_x}$. By adjunction, this corresponds to a morphism $\alpha^*\mathscr{M}_x \longrightarrow \mathscr{M}_y$ in $\mathbf{M}_{\mathscr{R}_y}$. But since $\mathscr{M} \in Cart^C - \mathscr{R}$, we already know that $\alpha^*\mathscr{M}_x \longrightarrow \mathscr{M}_y$ is an isomorphism. Hence, $\mathscr{F}^c\mathscr{M}) := \mathscr{F}(\mathscr{M}) \in Cart - \mathscr{R}$.

We also notice that $Cart^C - \mathscr{R}$ is closed under taking colimits in $Mod^C - \mathscr{R}$. Then $\mathscr{F}^c : Cart^C - \mathscr{R} \longrightarrow Cart - \mathscr{R}$ preserves colimits and we know from Theorem 6.10 that both $Cart^C - \mathscr{R}$ and $Cart - \mathscr{R}$ are Grothendieck categories. It now follows from [23, Proposition 8.3.27] that $\mathscr{F}^c$ has a right adjoint. □

Proposition 7.10 *Let $\mathscr{X}$ be a poset, C be a right semiperfect K-coalgebra and $\mathscr{R} : \mathscr{X} \longrightarrow \mathscr{E}nt_C$ be an entwined C-representation that is also flat. Suppose there exists $\theta \in V_1$ such that*

$$\theta_x(r)(c_1 \otimes c_2) = \varepsilon_C(c) \cdot id_r \tag{7.26}$$

for every $x \in \mathscr{X}$, $r \in \mathscr{R}_x$ and $c \in C$. Then, $\mathscr{F}^c : Cart^C - \mathscr{R} \longrightarrow Cart - \mathscr{R}$ is separable.

Proof From Theorem 7.8, it follows that $\mathscr{F} : Mod^C - \mathscr{R} \longrightarrow Mod - \mathscr{R}$ is separable. In other words, for any $\mathscr{M}, \mathscr{N} \in Mod^C - \mathscr{R}$, the canonical morphism $Mod^C - \mathscr{R}(\mathscr{M}, \mathscr{N}) \longrightarrow Mod - \mathscr{R}(\mathscr{F}(\mathscr{M}), \mathscr{F}(\mathscr{N}))$ is a split monomorphism. Since $Cart^C - \mathscr{R}$ and $Cart - \mathscr{R}$ are full subcategories of $Mod^C - \mathscr{R}$ and $Mod - \mathscr{R}$ respectively and $\mathscr{F}^c$ is a restriction of $\mathscr{F}$, the result follows. □

8 Separability of the Functor $\mathscr{G}$: $Mod - \mathscr{R} \longrightarrow Mod^C - \mathscr{R}$

We continue with $\mathscr{X}$ being a poset, C being a right semiperfect coalgebra and let $\mathscr{R} : \mathscr{X} \longrightarrow \mathscr{E}nt_C$ be an entwined C-representation. In this section, we will give conditions for the right adjoint $\mathscr{G} : Mod - \mathscr{R} \longrightarrow Mod^C - \mathscr{R}$ to be separable.

Putting $C = K$ in Proposition 5.3, we see that for each $x \in \mathscr{X}$, there is a functor $ex_x : \mathbf{M}_{\mathscr{R}_x} \longrightarrow Mod - \mathscr{R}$ having right adjoint $ev_x : Mod - \mathscr{R} \longrightarrow \mathbf{M}_{\mathscr{R}_x}$. In a manner similar to Proposition 7.3, we now can show that a natural transformation $\omega \in Nat(1_{Mod-\mathscr{R}}, \mathscr{F}\mathscr{G})$ consists of a collection of natural transformations $\{\omega_x \in Nat(1_{\mathbf{M}_{\mathscr{R}_x}}, \mathscr{F}_x\mathscr{G}_x)\}_{x\in\mathscr{X}}$ such that for any $\alpha : x \longrightarrow y$ in $\mathscr{X}$ and any $\mathscr{N} \in Mod-\mathscr{R}$, we have the following commutative diagram

$$\begin{array}{ccc} \mathscr{N}_x & \xrightarrow{\omega_x(\mathscr{N}_x)} & \mathscr{F}_x\mathscr{G}_x(\mathscr{N}_x) \\ \mathscr{N}_\alpha \downarrow & & \downarrow \mathscr{F}_x\mathscr{G}_x(\mathscr{N}_\alpha) \\ \alpha_*\mathscr{N}_y & \xrightarrow{\alpha_*\omega_y(\mathscr{N}_y)} & \alpha_*\mathscr{F}_y\mathscr{G}_y(\mathscr{N}_y) \end{array} \tag{8.1}$$

Here, $\omega_x \in Nat(1_{\mathbf{M}_{\mathscr{R}_x}}, \mathscr{F}_x\mathscr{G}_x)$ is determined by setting

$$\omega_x(\mathcal{N}) := \omega(ex_x(\mathcal{N}))_x : (ex_x(\mathcal{N}))_x = \mathcal{N} \longrightarrow \mathscr{F}_x\mathscr{G}_x(\mathcal{N}) = \mathscr{F}_x\mathscr{G}_x((ex_x(\mathcal{N}))_x) \tag{8.2}$$

for $\mathcal{N} \in \mathbf{M}_{\mathscr{R}_x}$. As in the proof of Proposition 7.3, we can also show that

$$\omega_x(\mathscr{N}_x) = \omega(ex_x(\mathscr{N}_x))_x = \omega(\mathscr{N})_x \tag{8.3}$$

for any $\mathscr{N} \in Mod-\mathscr{R}$ and $x \in \mathscr{X}$. More explicitly, for each $x \in \mathscr{X}$ and $r \in \mathscr{R}_x$, we have a commutative diagram

$$\begin{array}{ccc} \mathscr{N}_x(r) & \xrightarrow{(\omega_x(\mathscr{N}_x))(r)} & (\mathscr{F}_x\mathscr{G}_x(\mathscr{N}_x))(r) = \mathscr{N}_x(r)\otimes C \\ \mathscr{N}_\alpha(r) \downarrow & & \downarrow (\mathscr{F}_x\mathscr{G}_x(\mathscr{N}_\alpha))(r) \\ \mathscr{N}_y(\alpha(r)) = (\alpha_*\mathscr{N}_y)(r) & \xrightarrow[=\omega_y(\mathscr{N}_y)(\alpha(r))]{(\alpha_*\omega_y(\mathscr{N}_y))(r)} & (\alpha_*\mathscr{F}_y\mathscr{G}_y(\mathscr{N}_y))(r) = \mathscr{N}_y(\alpha(r))\otimes C \end{array} \tag{8.4}$$

We will now give another interpretation for the space $Nat(1_{Mod-\mathscr{R}}, \mathscr{F}\mathscr{G})$. For this, we consider a collection $\eta = \{\eta_x(s,r) : H^x_r(s) = \mathscr{R}_x(s,r) \longrightarrow H^x_r(s)\otimes C = \mathscr{R}_x(s,r)\otimes C : f \mapsto \hat{f}\otimes c_f\}_{x\in\mathscr{X}, r,s\in\mathscr{R}_x}$ of K-linear maps satisfying the following conditions:

(1) Fix $x \in \mathscr{X}$. Then, for $s' \xrightarrow{h} s \xrightarrow{f} r \xrightarrow{g} r'$ in $\mathscr{R}_x$, we have

$$\eta_x(s',r')(gfh) = \sum \widehat{gfh}\otimes c_{gfh} = g\hat{f}h_{\psi_x}\otimes c_f^{\psi_x} \in \mathscr{R}_x(s',r')\otimes C \tag{8.5}$$

(2) For $\alpha : x \longrightarrow y$ in $\mathscr{X}$ and $f \in \mathscr{R}_x(s,r)$ we have

$$\alpha(\hat{f})\otimes c_f = \widehat{\alpha(f)}\otimes c_{\alpha(f)} \in \mathscr{R}_y(\alpha(s),\alpha(r))\otimes C \tag{8.6}$$

The space of all such η will be denoted by W_1. We note that condition (1) is equivalent to saying that for each $x \in \mathscr{X}$, the element $\eta_x = \{\eta_x(s,r) : \mathscr{R}_x(s,r) \longrightarrow \mathscr{R}_x(s,r)\otimes C : f \mapsto \hat{f}\otimes c_f\}_{r,s\in\mathscr{R}_x} \in Nat(H^x, H^x\otimes C)$, i.e., η_x is a morphism in the category of $\mathscr{R}_x$-bimodules (functors $\mathscr{R}_x^{op}\otimes\mathscr{R}_x \longrightarrow Vect_K$). Here H^x is the canonical $\mathscr{R}_x$-bimodule that takes a pair of objects $(s,r) \in Ob(\mathscr{R}_x^{op}\otimes\mathscr{R}_x)$ to $\mathscr{R}_x(s,r)$. Further, $H^x\otimes C$ is the $\mathscr{R}_x$-bimodule defined by setting

$$(H^x\otimes C)(s,r) = \mathscr{R}_x(s,r)\otimes C \qquad (H^x\otimes C)(h,g)(f\otimes c) = gfh_{\psi_x}\otimes c^{\psi_x} \tag{8.7}$$

for $s' \xrightarrow{h} s \xrightarrow{f} r \xrightarrow{g} r'$ in $\mathscr{R}_x$ and $c \in C$.

Lemma 8.1 *There is a canonical morphism* $Nat(1_{Mod-\mathscr{R}}, \mathscr{F}\mathscr{G}) \longrightarrow W_1$.

Proof As mentioned above, any $\omega \in Nat(1_{Mod-\mathscr{R}}, \mathscr{F}\mathscr{G})$ corresponds to a collection of natural transformations $\{\omega_x \in Nat(1_{\mathbf{M}_{\mathscr{R}_x}}, \mathscr{F}_x\mathscr{G}_x)\}_{x\in\mathscr{X}}$ satisfying (8.1). From the proof of [4, Proposition

3.10], we know that each $\omega_x \in Nat(1_{\mathbf{M}_{\mathscr{R}_x}}, \mathscr{F}_x\mathscr{G}_x)$ corresponds to $\eta_x \in Nat(H^x, H^x \otimes C)$ determined by setting

$$\eta_x(s,r) : H_r^x(s) = \mathscr{R}_x(s,r) \longrightarrow H_r^x(s) \otimes C = \mathscr{R}_x(s,r) \otimes C \qquad \eta_x(s,r) := \omega_x(H_r^x)(s) \quad (8.8)$$

for $r, s \in \mathscr{R}_x$. Here, H_r^x is the right $\mathscr{R}_x$-module $H_r^x := \mathscr{R}_x(\_\_, r) : \mathscr{R}_x^{op} \longrightarrow Vect_K$. We now consider $\alpha : x \longrightarrow y$ in $\mathscr{X}$ and some $f \in \mathscr{R}_x(s,r)$. By applying Lemma 5.1 with $C = K$, we have $ex_x(H_r^x) \in Mod - \mathscr{R}$ which satisfies $(ex_x(H_r^x))_y = \alpha^* H_r^x = H_{\alpha(r)}^y$. Setting $\mathscr{N} = ex_x(H_r^x)$ in (8.4), we have

$$\begin{array}{ccc}
\mathscr{N}_x(s) = H_r^x(s) & \xrightarrow[=\eta_x(s,r)]{(\omega_x(H_r^x))(s)} & (\mathscr{F}_x\mathscr{G}_x(\mathscr{N}_x))(s) = H_r^x(s) \otimes C \\
\mathscr{N}_\alpha(s) \downarrow & & \downarrow (\mathscr{F}_x\mathscr{G}_x(\mathscr{N}_\alpha))(s) \\
\mathscr{N}_y(\alpha(s)) = H_{\alpha(r)}^y(\alpha(s)) & \xrightarrow[=\omega_y(H_{\alpha(r)}^y)(\alpha(s))]{\eta_y(\alpha(s),\alpha(r))} & \mathscr{N}_y(\alpha(s)) \otimes C = H_{\alpha(r)}^y(\alpha(s)) \otimes C
\end{array} \quad (8.9)$$

It follows that that the collection $\eta_x(s,r)$ satisfies condition (8.6). This proves the result. □

Proposition 8.2 *The spaces $Nat(1_{Mod-\mathscr{R}}, \mathscr{F}\mathscr{G})$ and W_1 are isomorphic.*

Proof We consider an element $\eta \in W_1$. As mentioned before, this gives a collection $\{\eta_x \in Nat(H^x, H^x \otimes C)\}_{x \in \mathscr{X}}$ satisfying the compatibility condition in (8.6). From the proof of [4, Proposition 3.10], it follows that each η_x corresponds to a natural transformation $\omega_x \in Nat(1_{\mathbf{M}_{\mathscr{R}_x}}, \mathscr{F}_x\mathscr{G}_x)$ which satisfies $\omega_x(H_r^x)(s) = \eta_x(s,r)$ for $r, s \in \mathscr{R}_x$. We claim that the collection $\{\omega_x\}_{x \in \mathscr{X}}$ satisfies the compatibility condition in (8.1) for each $\mathscr{N} \in Mod - \mathscr{R}$, thus determining an element $\omega \in Nat(1_{Mod-\mathscr{R}}, \mathscr{F}\mathscr{G})$.

We start with $\mathscr{N} = ex_x(H_r^x)$ for some $x \in \mathscr{X}$ and $r \in \mathscr{R}_x$. We consider a morphism $\alpha : y \longrightarrow z$ in $\mathscr{X}$. If $x \not\leq y$, then $\mathscr{N}_y = 0$ and the condition in (8.1) is trivially satisfied. Otherwise, let $\beta : x \longrightarrow y$ in $\mathscr{X}$ and set $s = \beta(r)$. In particular, $\mathscr{N}_y = \beta^* H_r^x = H_{\beta(r)}^y = H_s^y$ and $\mathscr{N}_z = H_{\alpha\beta(r)}^z = H_{\alpha(s)}^z$. Applying the condition (8.6), we see that the following diagram is commutative for any $s' \in \mathscr{R}_y$:

$$\begin{array}{ccc}
\mathscr{R}_y(s',s) = \mathscr{N}_y(s') & \xrightarrow[=\omega_y(\mathscr{N}_y)(s')]{\eta_y(s',s)} & \mathscr{N}_y(s') \otimes C = \mathscr{R}_y(s',s) \otimes C \\
\mathscr{N}_\alpha(s') \downarrow & & \downarrow (\mathscr{F}_y\mathscr{G}_y(\mathscr{N}_\alpha))(s') \\
\mathscr{R}_z(\alpha(s'),\alpha(s)) = \mathscr{N}_z(\alpha(s')) & \xrightarrow[=\omega_z(\mathscr{N}_z)(\alpha(s'))]{\eta_z(\alpha(s'),\alpha(s))} & \mathscr{N}_z(\alpha(s')) \otimes C = \mathscr{R}_z(\alpha(s'),\alpha(s)) \otimes C
\end{array} \quad (8.10)$$

In other words, the condition in (8.1) is satisfied for $\mathscr{N} = ex_x(H_r^x)$. From Theorem 5.5, we know that the collection

$$\{ex_x(H_r^x) | x \in \mathscr{X}, r \in \mathscr{R}_x\} \quad (8.11)$$

is a set of generators for $Mod - \mathscr{R}$. Accordingly, for any $\mathscr{N}' \in Mod - \mathscr{R}$, we can choose an epimorphism $\phi : \mathscr{N} \longrightarrow \mathscr{N}'$ where $\mathscr{N}$ is a direct sum of copies of objects in (8.11). Then, $\mathscr{N}$ satisfies (8.1) and we have commutative diagrams

$$\begin{array}{ccccc}
\mathscr{N}_y & \xrightarrow{\mathscr{N}_\alpha} & \alpha_*\mathscr{N}_z & \xrightarrow{\alpha_*\omega_z(\mathscr{N}_z)} & \alpha_*\mathscr{F}_z\mathscr{G}_z(\mathscr{N}_z) \\
\phi_y \downarrow & & \alpha_*\phi_z \downarrow & & \downarrow \alpha_*\mathscr{F}_z\mathscr{G}_z(\phi_z) \\
\mathscr{N}'_y & \xrightarrow{\mathscr{N}'_\alpha} & \alpha_*\mathscr{N}'_z & \xrightarrow{\alpha_*\omega_z(\mathscr{N}'_z)} & \alpha_*\mathscr{F}_z\mathscr{G}_z(\mathscr{N}'_z)
\end{array} \quad (8.12)$$

$$\begin{array}{ccccc} \mathscr{N}_y & \xrightarrow{\omega_y(\mathscr{N}_y)} & \mathscr{F}_y\mathscr{G}_y(\mathscr{N}_y) & \xrightarrow{\mathscr{F}_y\mathscr{G}_y(\mathscr{N}_\alpha)} & \alpha_*\mathscr{F}_z\mathscr{G}_z(\mathscr{N}_z) \\ \phi_y\downarrow & & \mathscr{F}_y\mathscr{G}_y(\phi_y)\downarrow & & \downarrow \alpha_*\mathscr{F}_z\mathscr{G}_z(\phi_z) \\ \mathscr{N}'_y & \xrightarrow{\omega_y(\mathscr{N}'_y)} & \mathscr{F}_y\mathscr{G}_y(\mathscr{N}'_y) & \xrightarrow{\mathscr{F}_y\mathscr{G}_y(\mathscr{N}'_\alpha)} & \alpha_*\mathscr{F}_z\mathscr{G}_z(\mathscr{N}'_z) \end{array} \qquad \begin{array}{ccc} \mathscr{N}_y & \xrightarrow{\omega_y(\mathscr{N}_y)} & \mathscr{F}_y\mathscr{G}_y(\mathscr{N}_y) \\ \mathscr{N}_\alpha\downarrow & & \downarrow \mathscr{F}_y\mathscr{G}_y(\mathscr{N}_\alpha) \\ \alpha_*\mathscr{N}_z & \xrightarrow{\alpha_*\omega_z(\mathscr{N}_z)} & \alpha_*\mathscr{F}_z\mathscr{G}_z(\mathscr{N}_z) \end{array} \tag{8.13}$$

for any $\alpha : y \longrightarrow z$ in $\mathscr{X}$. Since $\phi_y : \mathscr{N}_y \longrightarrow \mathscr{N}'_y$ is an epimorphism, it follows that $\mathscr{N}'$ also satisfies the condition in (8.1). This gives a morphism $W_1 \longrightarrow Nat(1_{Mod-\mathscr{R}}, \mathscr{F}\mathscr{G})$. It may be verified that this is inverse to the morphism $Nat(1_{Mod-\mathscr{R}}, \mathscr{F}\mathscr{G}) \longrightarrow W_1$ in Lemma 8.1, which proves the result. □

We will now give conditions for the functor $\mathscr{G} : Mod - \mathscr{R} \longrightarrow Mod^C - \mathscr{R}$ to be separable. Since $\mathscr{G}$ has a left adjoint, it follows (see [27, Theorem 1.2]) that $\mathscr{G}$ is separable if and only if there exists a natural transformation $\omega \in Nat(1_{Mod-\mathscr{R}}, \mathscr{F}\mathscr{G})$ such that $\nu \circ \omega = 1_{Mod-\mathscr{R}}$, where ν is the counit of the adjunction.

Theorem 8.3 *Let $\mathscr{X}$ be a partially ordered set, C be a right semiperfect K-coalgebra and let $\mathscr{R} : \mathscr{X} \longrightarrow \mathscr{E}nt_C$ be an entwined C-representation. Then, the functor $\mathscr{G} : Mod - \mathscr{R} \longrightarrow Mod^C - \mathscr{R}$ is separable if and only if there exists $\eta \in W_1$ such that*

$$id = (id \otimes \varepsilon_C) \circ \eta_x(s,r) : \mathscr{R}_x(s,r) \xrightarrow{\eta_x(s,r)} \mathscr{R}_x(s,r) \otimes C \xrightarrow{(id\otimes\varepsilon_C)} \mathscr{R}_x(s,r) \tag{8.14}$$

for each $x \in \mathscr{X}$ and $s, r \in \mathscr{R}_x$.

Proof First, we suppose that $\mathscr{G}$ is separable, i.e., there exists a natural transformation $\omega \in Nat(1_{Mod-\mathscr{R}}, \mathscr{F}\mathscr{G})$ such that $\nu \circ \omega = 1_{Mod-\mathscr{R}}$. Using Proposition 8.2, we consider $\eta \in W_1$ corresponding to ω.

By definition, the counit ν of the adjunction $(\mathscr{F}, \mathscr{G})$ is described as follows: for any $\mathscr{N} \in Mod - \mathscr{R}$, we have

$$\nu(\mathscr{N})_x(s) : \mathscr{N}_x(s) \otimes C \longrightarrow \mathscr{N}_x(s) \qquad n \otimes c \mapsto n\varepsilon_C(c) \tag{8.15}$$

for each $x \in \mathscr{X}$, $s \in \mathscr{R}_x$. We choose $x \in \mathscr{X}$, $r \in \mathscr{R}_x$ and set $\mathscr{N} = ex_x(H_r^x)$. Since $\nu \circ \omega = 1_{Mod-\mathscr{R}}$, it now follows from (8.8) that

$$id = \nu(ex_x(H_r^x))_x(s) \circ \omega(ex_x(H_r^x))_x(s) = (id \otimes \varepsilon_C) \circ \omega_x(H_r^x)(s) = (id \otimes \varepsilon_C) \circ \eta_x(s,r) \tag{8.16}$$

Conversely, suppose that we have $\eta \in W_1$ such that the condition in (8.14) is satisfied. Using the isomorphism in Proposition 8.2, we obtain the natural transformation $\omega \in Nat(1_{Mod-\mathscr{R}}, \mathscr{F}\mathscr{G})$ corresponding to η. Then, it is clear from (8.16) that $\nu(\mathscr{N}) \circ \omega(\mathscr{N}) = id$ for $\mathscr{N} = ex_x(H_r^x)$. Since $\{ex_x(H_r^)|x \in \mathscr{X}, r \in \mathscr{R}_x\}$ is a set of generators for $Mod - \mathscr{R}$, it follows that for any $\mathscr{N}' \in Mod - \mathscr{R}$, there is an epimorphism $\phi : \mathscr{N} \longrightarrow \mathscr{N}'$ such that $\nu(\mathscr{N}) \circ \omega(\mathscr{N}) = id$. We now consider the commutative diagram

$$\begin{array}{ccccc} \mathscr{N} & \xrightarrow{\omega(\mathscr{N})} & \mathscr{F}\mathscr{G}(\mathscr{N}) & \xrightarrow{\nu(\mathscr{N})} & \mathscr{N} \\ \phi\downarrow & & \mathscr{F}\mathscr{G}(\phi)\downarrow & & \downarrow\phi \\ \mathscr{N}' & \xrightarrow{\omega(\mathscr{N}')} & \mathscr{F}\mathscr{G}(\mathscr{N}') & \xrightarrow{\nu(\mathscr{N}')} & \mathscr{N}' \end{array} \tag{8.17}$$

Since the upper horizontal composition in (8.17) is the identity and ϕ is an epimorphism, it follows that $\nu(\mathscr{N}') \circ \omega(\mathscr{N}') = id$. This proves the result. □

9 $(\mathscr{F}, \mathscr{G})$ as a Frobenius Pair

In Sections 7 and 8, we have given conditions for the functor $\mathscr{F} : Mod^C - \mathscr{R} \longrightarrow Mod - \mathscr{R}$ and its right adjoint $\mathscr{G} : Mod - \mathscr{R} \longrightarrow Mod^C - \mathscr{R}$ to be separable. In this section, we will give necessary and sufficient conditions for $(\mathscr{F}, \mathscr{G})$ to be a Frobenius pair, i.e., $\mathscr{G}$ is both a right and a left adjoint of $\mathscr{F}$. First, we note that it follows from the characterization of Frobenius pairs (see for instance, [9, §1]) that $(\mathscr{F}, \mathscr{G})$ is a Frobenius pair if and only if there exist $\upsilon \in Nat(\mathscr{G}\mathscr{F}, 1_{Mod^C-\mathscr{R}})$ and $\omega \in Nat(1_{Mod-\mathscr{R}}, \mathscr{F}\mathscr{G})$ such that

$$\mathscr{F}(\upsilon(\mathscr{M})) \circ \omega(\mathscr{F}(\mathscr{M})) = id_{\mathscr{F}(\mathscr{M})} \qquad \upsilon(\mathscr{G}(\mathscr{N})) \circ \mathscr{G}(\omega(\mathscr{N})) = id_{\mathscr{G}(\mathscr{N})} \tag{9.1}$$

for any $\mathscr{M} \in Mod^C - \mathscr{R}$ and $\mathscr{N} \in Mod - \mathscr{R}$. Equivalently, for each $x \in \mathscr{X}$, we must have

$$\begin{aligned} (\mathscr{F}(\upsilon(\mathscr{M})))_x \circ \omega(\mathscr{F}(\mathscr{M}))_x &= \mathscr{F}_x(\upsilon_x(\mathscr{M}_x)) \circ \omega_x(\mathscr{F}_x(\mathscr{M}_x)) = id_{\mathscr{F}_x(\mathscr{M}_x)} \\ \upsilon(\mathscr{G}(\mathscr{N}))_x \circ \mathscr{G}(\omega(\mathscr{N}))_x &= \upsilon_x(\mathscr{G}_x(\mathscr{N}_x)) \circ \mathscr{G}_x(\omega_x(\mathscr{N}_x)) = id_{\mathscr{G}_x(\mathscr{N}_x)} \end{aligned} \tag{9.2}$$

for any $\mathscr{M} \in Mod^C - \mathscr{R}$ and $\mathscr{N} \in Mod - \mathscr{R}$.

Theorem 9.1 *Let $\mathscr{X}$ be a partially ordered set, C be a right semiperfect K-coalgebra and let $\mathscr{R} : \mathscr{X} \longrightarrow \mathscr{E}nt_C$ be an entwined C-representation. Let $\mathscr{F} : Mod^C - \mathscr{R} \longrightarrow Mod - \mathscr{R}$ be the forgetful functor and $\mathscr{G} : Mod - \mathscr{R} \longrightarrow Mod^C - \mathscr{R}$ its right adjoint. Then, $(\mathscr{F}, \mathscr{G})$ is a Frobenius pair if and only if there exist $\theta \in V_1$ and $\eta \in W_1$ such that*

$$\varepsilon_C(d)f = \sum \widehat{f} \circ \theta_x(r)(c_f \otimes d) \qquad \varepsilon_C(d)f = \sum \widehat{f_{\psi_x}} \circ \theta_x(r)(d^{\psi_x} \otimes c_f) \tag{9.3}$$

for every $x \in \mathscr{X}$, $r \in \mathscr{R}_x$, $f \in \mathscr{R}_x(r, s)$ and $d \in C$, where $\eta_x(r, s)(f) = \widehat{f} \otimes c_f$.

Proof We suppose there exist $\theta \in V_1$ and $\eta \in W_1$ satisfying (9.3) and consider $\mathscr{M} \in Mod^C - \mathscr{R}$, $\mathscr{N} \in Mod - \mathscr{R}$. Using the isomorphisms in Propositions 7.7 and 8.2, we obtain $\upsilon \in Nat(\mathscr{G}\mathscr{F}, 1_{Mod^C-\mathscr{R}})$ and $\omega \in Nat(1_{Mod-\mathscr{R}}, \mathscr{F}\mathscr{G})$ corresponding to θ and η respectively.

For fixed $x \in \mathscr{X}$, it follows that $\theta_x = \{\theta_x(r) : C \otimes C \longrightarrow \mathscr{R}_x(r, r)\}_{r \in \mathscr{R}_x}$ and the $\mathscr{R}_x$-bimodule morphism $\eta_x \in Nat(H^x, H^x \otimes C)$ satisfy the conditions in [4, Theorem 3.14]. Hence, we have

$$\mathscr{F}_x(\upsilon_x(\mathcal{M})) \circ \omega_x(\mathscr{F}_x(\mathcal{M})) = id_{\mathscr{F}_x(\mathcal{M})} \qquad \upsilon_x(\mathscr{G}_x(\mathcal{N})) \circ \mathscr{G}_x(\omega_x(\mathcal{N})) = id_{\mathscr{G}_x(\mathcal{N})} \tag{9.4}$$

for any $\mathcal{M} \in \mathbf{M}^C_{\mathscr{R}_x}(\psi_x)$ and $\mathcal{N} \in \mathbf{M}_{\mathscr{R}_x}$. In particular, (9.2) holds for $\mathscr{M}_x \in \mathbf{M}^C_{\mathscr{R}_x}(\psi_x)$ and $\mathscr{N}_x \in \mathbf{M}_{\mathscr{R}_x}$.

Conversely, suppose that $(\mathscr{F}, \mathscr{G})$ is a Frobenius pair. Then, there exist $\upsilon \in Nat(\mathscr{G}\mathscr{F}, 1_{Mod^C-\mathscr{R}})$ and $\omega \in Nat(1_{Mod-\mathscr{R}}, \mathscr{F}\mathscr{G})$ satisfying (9.2) for each $x \in \mathscr{X}$. Again using the isomorphisms in Propositions 7.7 and 8.2, we obtain corresponding $\theta \in V_1$ and $\eta \in W_1$.

We now consider $\mathcal{M} \in \mathbf{M}^C_{\mathscr{R}_x}(\psi_x)$ and $\mathcal{N} \in \mathbf{M}_{\mathscr{R}_x}$. Applying (9.2) with $\mathscr{M} = ex^C_x(\mathcal{M})$ and $\mathscr{N} = ex_x(\mathcal{N})$, we have

$$\mathscr{F}_x(\upsilon_x(\mathcal{M})) \circ \omega_x(\mathscr{F}_x(\mathcal{M})) = id_{\mathscr{F}_x(\mathcal{M})} \qquad \upsilon_x(\mathscr{G}_x(\mathcal{N})) \circ \mathscr{G}_x(\omega_x(\mathcal{N})) = id_{\mathscr{G}_x(\mathcal{N})} \tag{9.5}$$

It now follows from [4, Theorem 3.14] that $\theta_x = \{\theta_x(r) : C \otimes C \longrightarrow \mathscr{R}_x(r, r)\}_{r \in \mathscr{R}_x}$ and the $\mathscr{R}_x$-bimodule morphism $\eta_x \in Nat(H^x, H^x \otimes C)$ satisfy (9.3). This proves the result. □

Corollary 9.2 *Let $(\mathscr{F}, \mathscr{G})$ be a Frobenius pair. Then, for each $x \in \mathscr{X}$, $(\mathscr{F}_x, \mathscr{G}_x)$ is a Frobenius pair of adjoint functors.*

Proof This is immediate from (9.4). □

We consider $\alpha : x \longrightarrow y$ in $\mathscr{X}$. In (7.1), we observed directly that the functors $\{\mathscr{F}_x : \mathbf{M}^C_{\mathscr{R}_x}(\psi_x) \longrightarrow \mathbf{M}_{\mathscr{R}_x}\}_{x \in \mathscr{X}}$ commute with both α^* and α_*, while the functors $\{\mathscr{G}_x : \mathbf{M}_{\mathscr{R}_x} \longrightarrow \mathbf{M}^C_{\mathscr{R}_x}(\psi_x)\}_{x \in \mathscr{X}}$ commute only with α_*. We will now give a sufficient condition for the functors $\{\mathscr{G}_x : \mathbf{M}_{\mathscr{R}_x} \longrightarrow \mathbf{M}^C_{\mathscr{R}_x}(\psi_x)\}_{x \in \mathscr{X}}$ to commute with α^*.

Lemma 9.3 *Let $(\mathscr{F},\mathscr{G})$ be a Frobenius pair. Then, for any $\alpha : x \longrightarrow y$ in $\mathscr{X}$, we have a commutative diagram*

$$\begin{array}{ccc} \mathbf{M}_{\mathscr{R}_x} & \xrightarrow{\alpha^*} & \mathbf{M}_{\mathscr{R}_y} \\ \mathscr{G}_x \downarrow & & \downarrow \mathscr{G}_y \\ \mathbf{M}^C_{\mathscr{R}_x}(\psi_x) & \xrightarrow{\alpha^*} & \mathbf{M}^C_{\mathscr{R}_y}(\psi_y) \end{array} \tag{9.6}$$

Proof For $\mathcal{M} \in \mathbf{M}_{\mathscr{R}_x}$, we will show that $\mathscr{G}_y\alpha^*(\mathcal{M}) = \alpha^*\mathscr{G}_x(\mathcal{M}) \in \mathbf{M}^C_{\mathscr{R}_y}(\psi_y)$. From Corollary 9.2 we know that each $(\mathscr{F}_x, \mathscr{G}_x)$ is a Frobenius pair of adjoint functors. Using this fact and the commutative diagrams in (7.1), we now have that for any $\mathcal{N} \in \mathbf{M}^C_{\mathscr{R}_y}(\psi_y)$:

$$\begin{aligned} \mathbf{M}^C_{\mathscr{R}_y}(\psi_y)(\mathscr{G}_y\alpha^*(\mathcal{M}),\mathcal{N}) &= \mathbf{M}_{\mathscr{R}_x}(\mathcal{M},\alpha_*\mathscr{F}_y(\mathcal{N})) = \mathbf{M}_{\mathscr{R}_x}(\mathcal{M},\mathscr{F}_x\alpha_*(\mathcal{N})) \\ &= \mathbf{M}^C_{\mathscr{R}_y}(\psi_y)(\alpha^*\mathscr{G}_x(\mathcal{M}),\mathcal{N}) \end{aligned} \tag{9.7}$$

□

Proposition 9.4 *Let $(\mathscr{F},\mathscr{G})$ be a Frobenius pair. Suppose that $\mathscr{R} : \mathscr{X} \longrightarrow \mathscr{E}nt_C$ is flat. Then, $\mathscr{G} : Mod-\mathscr{R} \longrightarrow Mod^C-\mathscr{R}$ restricts to a functor $\mathscr{G}^c : Cart-\mathscr{R} \longrightarrow Cart^C-\mathscr{R}$.*

Proof For any $\mathscr{N} \in Cart-\mathscr{R}$, we claim that $\mathscr{G}(\mathscr{N}) \in Mod^C-\mathscr{R}$ actually lies in $Cart^C-\mathscr{R}$. By definition of $\mathscr{G}$, we have for any $\alpha : x \longrightarrow y$, a morphism $\mathscr{G}(\mathscr{N})_\alpha = \mathscr{G}_x(\mathscr{N}_\alpha) : \mathscr{G}_x(\mathscr{N}_x) \longrightarrow \mathscr{G}_x(\alpha_*(\mathscr{N}_y)) = \alpha_*(\mathscr{G}_y(\mathscr{N}_y))$ in $\mathbf{M}^C_{\mathscr{R}_x}(\psi_x)$ which corresponds to a morphism $\mathscr{G}(\mathscr{N})^\alpha : \alpha^*(\mathscr{G}_x(\mathscr{N}_x)) \longrightarrow \mathscr{G}_y(\mathscr{N}_y)$ in $\mathbf{M}^C_{\mathscr{R}_y}(\psi_y)$. Since $(\mathscr{F},\mathscr{G})$ is a Frobenius pair, it follows from Lemma 9.3 that $\mathscr{G}_y\alpha^*(\mathscr{N}_x) = \alpha^*\mathscr{G}_x(\mathscr{N}_x) \in \mathbf{M}^C_{\mathscr{R}_y}(\psi_y)$. Since $\mathcal{N}$ is cartesian, we know that $\alpha^*\mathscr{N}_x$ is isomorphic to $\mathscr{N}_y$ and hence $\mathscr{G}(\mathscr{N})^\alpha = \mathscr{G}_y(\mathscr{N}^\alpha)$ is an isomorphism. □

Corollary 9.5 *Let $(\mathscr{F},\mathscr{G})$ be a Frobenius pair. Suppose that $\mathscr{R} : \mathscr{X} \longrightarrow \mathscr{E}nt_C$ is flat. Then, $(\mathscr{F}^c,\mathscr{G}^c)$ is a Frobenius pair of adjoint functors between $Cart^C-\mathscr{R}$ and $Cart-\mathscr{R}$.*

Proof From Proposition 7.9, we know that $\mathscr{F} : Mod^C-\mathscr{R} \longrightarrow Mod-\mathscr{R}$ restricts to a functor $\mathscr{F}^c : Cart^C-\mathscr{R} \longrightarrow Cart-\mathscr{R}$. From Proposition 9.4, we know that $\mathscr{G} : Mod-\mathscr{R} \longrightarrow Mod^C-\mathscr{R}$ restricts to a functor $\mathscr{G}^c : Cart-\mathscr{R} \longrightarrow Cart^C-\mathscr{R}$ on the full subcategories of cartesian modules. Since $\mathscr{G}$ is both right and left adjoint to $\mathscr{F}$, it is clear that $\mathscr{G}^c$ is both right and left adjoint to $\mathscr{F}^c$. □

10 Constructing Entwined Representations

In this final section, we will give examples of how to construct entwined representations and describe modules over them. Let $(\mathcal{R}, C, \psi)$ be an entwining structure. Then, we consider the K-linear category $(C,\mathcal{R})_\psi$ defined as follows

$$Ob((C,\mathcal{R})_\psi) = Ob(\mathcal{R}) \qquad (C,\mathcal{R})_\psi(s,r) := Hom_K(C,\mathcal{R}(s,r)) \tag{10.1}$$

for $s, r \in \mathcal{R}$. The composition in $(C,\mathcal{R})_\psi$ is as follows: given $\phi : C \longrightarrow \mathcal{R}(s,r)$ and $\phi' : C \longrightarrow \mathcal{R}(t,s)$ respectively in $(C,\mathcal{R})_\psi(s,r)$ and $(C,\mathcal{R})_\psi(t,s)$, we set

$$\phi * \phi' : C \longrightarrow \mathcal{R}(t,r) \qquad c \mapsto \sum \phi(c_2)_\psi \circ \phi'(c_1^\psi) \tag{10.2}$$

Lemma 10.1 *Let $(\mathcal{R}, C, \psi)$ be an entwining structure. Then, there is a canonical functor $P_\psi : \mathbf{M}^C_{\mathcal{R}}(\psi) \longrightarrow \mathbf{M}_{(C,\mathcal{R})_\psi}$.*

Proof We consider $\mathcal{M} \in \mathbf{M}^C_{\mathcal{R}}(\psi)$. We will define $\mathcal{N} = P_\psi(\mathcal{M}) \in \mathbf{M}_{(C,\mathcal{R})_\psi}$ by setting $\mathcal{N}(r) := \mathcal{M}(r)$ for each $r \in (C, \mathcal{R})$. Given $\phi : C \longrightarrow \mathcal{R}(s, r)$ in $(C, \mathcal{R})_\psi(s, r)$, we define $m * \phi \in \mathcal{N}(s) = \mathcal{M}(s)$ by setting $m * \phi = \sum m_0\phi(m_1)$. Here, $\rho_{\mathcal{M}(r)}(m) = \sum m_0 \otimes m_1$ is the right C-comodule structure on $\mathcal{M}(r)$.

For $\phi' : C \longrightarrow \mathcal{R}(t, s)$ in $(C, \mathcal{R})_\psi(t, s)$, we now have

$$\begin{aligned} m * (\phi * \phi') &= \sum m_0(\phi * \phi')(m_1) = \sum m_0\phi(m_{12})_\psi \phi'(m_{11}^\psi) = \sum m_0\phi(m_2)_\psi \phi'(m_1^\psi) \\ (m * \phi) * \phi' &= \sum (m * \phi)_0\phi'((m * \phi)_1) = \sum (m_0\phi(m_1))_0\phi'((m_0\phi(m_1))_1) \\ &= \sum (m_{00}\phi(m_1)_\psi)\phi'(m_{01}^\psi) = \sum m_0\phi(m_2)_\psi \phi'(m_1^\psi) \end{aligned} \tag{10.3}$$

This proves the result. □

Lemma 10.2 *Let* $(\alpha, id) : (\mathcal{R}, C, \psi) \longrightarrow (\mathcal{S}, C, \psi')$ *be a morphism of entwining structures. Then,* $P_\psi \circ (\alpha, id)_* = \alpha_* \circ P_{\psi'} : \mathbf{M}^C_{\mathcal{S}}(\psi') \longrightarrow \mathbf{M}_{(C,\mathcal{R})}$.

Proof We begin with $\mathcal{N} \in \mathbf{M}^C_{\mathcal{S}}(\psi')$. From the construction in Lemma 10.1, it is clear that for any $r \in (C, \mathcal{R})_\psi$, we have $(P_\psi \circ (\alpha, id)_*)(\mathcal{N})(r) = (\alpha_* \circ P_{\psi'})(\mathcal{N})(r) = \mathcal{N}(\alpha(r))$. We set $\mathcal{N}_1 := (P_\psi \circ (\alpha, id)_*)(\mathcal{N})$ and $\mathcal{N}_2 := (\alpha_* \circ P_{\psi'})(\mathcal{N})$ and consider $n \in \mathcal{N}_1(r) = \mathcal{N}_2(r)$ as well as $\phi : C \longrightarrow \mathcal{R}(s, r)$ in $(C, \mathcal{R})_\psi(s, r)$. Then, in both $\mathcal{N}_1(s)$ and $\mathcal{N}_2(s)$, we have $n * \phi = \sum n_0\alpha(\phi(n_1))$. This proves the result. □

Now let $\mathscr{X}$ be a small category and $\mathscr{R} : \mathscr{X} \longrightarrow \mathscr{E}nt_C$ an entwined C-representation. By replacing each entwining structure $(\mathscr{R}_x, C, \psi_x)$ with the category $(C, \mathscr{R}_x)_{\psi_x}$, we obtain an induced representation $(C, \mathscr{R})_\psi : \mathscr{X} \longrightarrow \mathscr{L}in$ (we recall that $\mathscr{L}in$ is the category of small K-linear categories).

Proposition 10.3 *There is a canonical functor* $Mod^C - \mathscr{R} \longrightarrow Mod - (C, \mathscr{R})_\psi$.

Proof By definition, an object $\mathscr{M} \in Mod^C - \mathscr{R}$ consists of a collection $\{\mathscr{M}_x \in \mathbf{M}^C_{\mathscr{R}_x}(\psi_x)\}_{x\in\mathscr{X}}$ and for each $\alpha : x \longrightarrow y$ in $\mathscr{X}$, a morphism $\mathscr{M}_\alpha : \mathscr{M}_x \longrightarrow \alpha_*\mathscr{M}_y$ in $\mathbf{M}^C_{\mathscr{R}_x}(\psi_x)$. Applying the functors $P_{\psi_x} : \mathbf{M}^C_{\mathscr{R}_x}(\psi_x) \longrightarrow \mathbf{M}_{(C,\mathscr{R}_x)_{\psi_x}}$ for $x \in \mathscr{X}$ and using Lemma 10.2, the result is now clear. □

Now let C be finitely generated as a K-vector space and let C^* denote its K-linear dual. Then, the canonical map $C^* \otimes V \longrightarrow Hom_K(C, V)$ is an isomorphism for any vector space V. For an entwining structure $(\mathcal{R}, C, \psi)$, the category $(C, \mathcal{R})_\psi$ can now be rewritten as $(C^* \otimes \mathcal{R})_\psi$ where $(C^*\otimes\mathcal{R})_\psi(s, r) = C^*\otimes\mathcal{R}(s, r)$ for $s, r \in Ob((C^*\otimes\mathcal{R})_\psi) = Ob(\mathcal{R})$. Given $c^*\otimes f \in C^*\otimes\mathcal{R}(s, r)$ and $d^* \otimes g \in C^* \otimes \mathcal{R}(t, s)$, the composition in $(C^* \otimes \mathcal{R})_\psi$ is expressed as

$$(c^* \otimes f) \circ (d^* \otimes g) : C \longrightarrow \mathcal{R}(t, r) \qquad x \mapsto \sum c^*(x_2)d^*(x_1^\psi)(f_\psi \circ g) \tag{10.4}$$

for $x \in C$. It is important to note that when f and g are identity maps, the composition in (10.4) simplifies to

$$(c^* \otimes id_r) \circ (d^* \otimes id_r) : C \longrightarrow \mathcal{R}(t, r) \qquad x \mapsto \sum c^*(x_2)d^*(x_1)id_r \tag{10.5}$$

In other words, for the canonical morphism $C^* \longrightarrow C^* \otimes \mathcal{R}(r, r)$ given by $c^* \mapsto c^* \otimes id_r$ to be a morphism of algebras, we must use the opposite of the usual convolution product on C^*.

Similarly, given an entwined C-representation $\mathscr{R} : \mathscr{X} \longrightarrow \mathscr{E}nt_C$ with C finitely generated as a K-vector space, we can replace the induced representation $(C, \mathscr{R})_\psi : \mathscr{X} \longrightarrow \mathscr{L}in$ by $(C^* \otimes \mathscr{R})_\psi$. Then, $Mod - (C, \mathscr{R})_\psi$ may be replaced by $Mod - (C^* \otimes \mathscr{R})_\psi$.

Proposition 10.4 *Let* $\mathscr{X}$ *be a small category and* $\mathscr{R} : \mathscr{X} \longrightarrow \mathscr{E}nt_C$ *an entwined* C*-representation. Suppose that* C *is finitely generated as a* K*-vector space. Then, the categories* $Mod^C - \mathscr{R}$ *and* $Mod - (C^* \otimes \mathscr{R})_\psi$ *are equivalent.*

Proof By Proposition 10.3, we already know that any object in $Mod^C - \mathscr{R}$ may be equipped with a $(C^* \otimes \mathscr{R})_\psi$-module structure. For the converse, we consider some $\mathscr{M} \in Mod - (C^* \otimes \mathscr{R})_\psi$ and choose some $x \in \mathscr{X}$.

We make $\mathscr{M}_x$ into an $\mathscr{R}_x$-module as follows: for $f \in \mathscr{R}_x(s,r)$ and $m \in \mathscr{M}_x(r)$, we set $mf \in \mathscr{M}_x(s)$ to be $mf := m(\varepsilon_C \otimes f)$. By considering the canonical morphism $C^* \longrightarrow C^* \otimes \mathscr{R}_x(r,r)$, it follows that the right $(C^* \otimes \mathscr{R}_x)_{\psi_x}(r,r)$ module $\mathscr{M}_x(r)$ carries a right C^*-module structure. As observed in (10.5), here the product on C^* happens to be the opposite of the usual convolution product. Hence, the right C^*-module structure on $\mathscr{M}_x(r)$ leads to a left C^*-module structure on $\mathscr{M}_x(r)$ when C^* is equipped with the usual product. Since C is finite dimensional, it is well known (see, for instance, [18, §2.2]) that we have an induced right C-comodule structure on $\mathscr{M}_x(r)$. It may be verified by direct computation that $\mathscr{M}_x \in \mathbf{M}^C_{\mathscr{R}_x}(\psi_x)$. Finally, for a morphism $\alpha : x \longrightarrow y$ in $\mathscr{X}$, the map $\mathscr{M}_\alpha : \mathscr{M}_x \longrightarrow \alpha_* \mathscr{M}_y$ in $\mathbf{M}_{(C^* \otimes \mathscr{R}_x)_{\psi_x}}$ induces a morphism in $\mathbf{M}^C_{\mathscr{R}_x}(\psi_x)$. Hence, $\mathscr{M} \in Mod - (C^* \otimes \mathscr{R})_\psi$ may be treated as an object of $Mod^C - \mathscr{R}$. It may be directly verified that this structure is the inverse of the one defined by Propostion 10.3. □

Finally, we will give an example of constructing entwined representations starting from B-comodule categories, where B is a bialgebra. So let B be a bialgebra over K, having multiplication μ_B, unit map u_B as well as comultiplication Δ_B and counit map ε_B. Then, the notion of a "B-comodule category," which behaves like a B-comodule algebra with many objects, is implicit in the literature.

Definition 10.5 Let B be a K-bialgebra. We will say that a small K-linear category $\mathcal{R}$ is a right B-comodule category if it satisfies the following conditions:

(i) For any $r, s \in \mathcal{R}$, there is a coaction $\rho = \rho(r,s) : \mathcal{R}(r,s) \longrightarrow \mathcal{R}(r,s) \otimes B$, $f \mapsto \sum f_0 \otimes f_1$, making $\mathcal{R}(r,s)$ a right B-comodule. Further, $\rho(id_r) = id_r \otimes 1_B$ for each $r \in \mathcal{R}$.
(ii) For $f \in \mathcal{R}(r,s)$ and $g \in \mathcal{R}(s,t)$, we have

$$\rho(g \circ f) = (g \circ f)_0 \otimes (g \circ f)_1 = (g_0 \circ f_0) \otimes (g_1 f_1) \tag{10.6}$$

We have suppressed the summation signs in (10.6). We will always refer to a right B-comodule category as a co-B-category. We will only consider those K-linear functors between co-B-categories whose action on morphisms is B-colinear. Together, the co-B-categories form a new category, which we will denote by Cat^B.

Lemma 10.6 *Let B be a bialgebra over K. Let $\mathcal{R}$ be a co-B-category and let C be a right B-module coalgebra. The collection $\psi := \psi_{\mathcal{R}} = \{\psi_{rs} : C \otimes \mathcal{R}(r,s) \longrightarrow \mathcal{R}(r,s) \otimes C\}_{r,s \in \mathcal{R}}$ defined by setting*

$$\psi_{rs}(c \otimes f) = f_\psi \otimes c^\psi = f_0 \otimes cf_1 \qquad f \in \mathcal{R}(r,s),\ c \in C \tag{10.7}$$

makes $(\mathcal{R}, C, \psi)$ an entwining structure.

Proof We consider morphisms f, g in $\mathcal{R}$ so that gf is defined. Then, for $c \in C$, we see that

$$\begin{aligned} (gf)_\psi \otimes c^\psi &= (gf)_0 \otimes c(gf)_1 = (g_0 f_0) \otimes c(g_1 f_1) = g_\psi f_\psi \otimes c^{\psi\psi} \\ f_\psi \otimes \Delta_C(c^\psi) &= f_0 \otimes \Delta_C(cf_1) = f_0 \otimes c_1 f_1 \otimes c_2 f_2 = f_{00} \otimes c_1 f_{01} \otimes c_2 f_1 = f_{\psi\psi} \otimes c_1^\psi \otimes c_2^\psi \\ \varepsilon_C(c^\psi) f_\psi &= \varepsilon_C(c)\varepsilon_B(f_1) f_0 = \varepsilon_C(c) f \qquad \psi(c \otimes id_r) = id_r \otimes c1_B \end{aligned} \tag{10.8}$$

This proves the result. □

Proposition 10.7 *Let B be a K-bialgebra and let C be a right B-module coalgebra. If $\mathscr{X}$ is a small category, a functor $\mathscr{R}' : \mathscr{X} \longrightarrow Cat^B$ induces an entwined C-representation of $\mathscr{X}$*

$$\mathscr{R} : \mathscr{X} \longrightarrow \mathscr{E}nt_C \qquad x \mapsto (\mathscr{R}_x, C, \psi_x) := (\mathscr{R}'_x, C, \psi_{\mathscr{R}'_x}) \tag{10.9}$$

Proof It may be easily verified that the entwining structures constructed in Lemma 10.6 are functorial with respect to B-colinear functors between B-comodule categories. This proves the result. □

We now consider a representation $\mathscr{R}' : \mathscr{X} \longrightarrow Cat^B$ as in Proposition 10.7 and the corresponding entwined C-representation $\mathscr{R} : \mathscr{X} \longrightarrow \mathscr{E}nt_C$. By considering the underlying K-linear category of any co-B-category, we obtain an induced representation that we continue to denote by $\mathscr{R}' : \mathscr{X} \longrightarrow Cat^B \longrightarrow \mathscr{L}in$. We conclude by showing how entwined modules over $\mathscr{R}$ are related to modules over $\mathscr{R}'$ in the sense of Estrada and Virili [19].

Proposition 10.8 *Let B be a K-bialgebra and let C be a right B-module coalgebra. Let $\mathscr{X}$ be a small category, $\mathscr{R}' : \mathscr{X} \longrightarrow Cat^B$ a functor and let $\mathscr{R} : \mathscr{X} \longrightarrow \mathscr{E}nt_C$ be the corresponding entwined C-representation. Then, a module $\mathscr{M}$ over $\mathscr{R}$ consists of the following data:*

(1) A module $\mathscr{M}$ over the induced representation $\mathscr{R}' : \mathscr{X} \longrightarrow Cat^B \longrightarrow \mathscr{L}in$.

(2) For each $x \in \mathscr{X}$ and $r \in \mathscr{R}_x$ a right C-comodule structure $\rho_r^x : \mathscr{M}_x(r) \longrightarrow \mathscr{M}_x(r) \otimes C$ such that

$$\rho_s^x(mf) = \big(mf\big)_0 \otimes \big(mf\big)_1 = m_0 f_0 \otimes m_1 f_1$$

for every $f \in \mathscr{R}_x(s,r)$ and $m \in \mathscr{M}_x(r)$.

(3) For each morphism $\alpha : x \longrightarrow y$ in $\mathscr{X}$, the morphism $\mathscr{M}_\alpha(r) : \mathscr{M}_x(r) \longrightarrow (\alpha_*\mathscr{M}_y)(r)$ is C-colinear for each $r \in \mathscr{R}_x$.

Proof We consider a datum as described by the three conditions above. The conditions (1) and (2) ensure that each $\mathscr{M}_x \in \mathbf{M}^C_{\mathscr{R}_x}(\psi_x)$. For each $x \in \mathscr{X}$, there is a forgetful functor $\mathscr{F}_x : \mathbf{M}^C_{\mathscr{R}_x}(\psi_x) \longrightarrow \mathbf{M}_{\mathscr{R}_x}$. Let $\alpha : x \longrightarrow y$ be a morphism in $\mathscr{X}$. From (7.1), we know that $(\alpha, id)_* : \mathbf{M}^C_{\mathscr{R}_y}(\psi_y) \longrightarrow \mathbf{M}^C_{\mathscr{R}_x}(\psi_x)$ and $\alpha_* : \mathbf{M}_{\mathscr{R}_y} \longrightarrow \mathbf{M}_{\mathscr{R}_x}$ are well behaved with respect to these forgetful functors. For each $r \in \mathscr{R}_x$, if $\mathscr{M}_\alpha(r) : \mathscr{M}_x(r) \longrightarrow (\alpha_*\mathscr{M}_y)(r)$ is also C-colinear, it follows that $\mathscr{M}_\alpha$ is a morphism in $\mathbf{M}^C_{\mathscr{R}_x}(\psi_x)$. The result is now clear. □

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Declarations

Conflict of Interest The author has no relevant financial or non-financial interests to disclose.

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