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Complete Nevanlinna-Pick kernels and the characteristic function

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A R T I C L E   I N F O

A B S T R A C T

This note finds a new characterization of complete Nevanlinna-Pick kernels on the Euclidean unit ball.

The classical theory of Sz.-Nagy and Foias about the characteristic function is extended in this note to a commuting tuple $\boldsymbol{T}$ of bounded operators satisfying the natural positivity condition of $1/k$-contractivity for an irreducible unitarily invariant complete Nevanlinna-Pick kernel. The characteristic function is a multiplier from $H_k \otimes \mathcal{E}$ to $H_k \otimes \mathcal{F}$, *factoring* a certain positive operator, for suitable Hilbert spaces $\mathcal{E}$ and $\mathcal{F}$ depending on $\boldsymbol{T}$. There is a converse, which roughly says that if a kernel k *admits* a characteristic function, then it has to be an irreducible unitarily invariant complete Nevanlinna-Pick kernel. The characterization explains, among other things, why in the literature an analogue of the characteristic function for a Bergman contraction ($1/k$-contraction where k is the Bergman kernel), when viewed as a multiplier between two vector valued reproducing kernel Hilbert spaces, requires a different (vector valued) reproducing kernel Hilbert space as the domain.

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1. Introduction

This section sets up notations and tools that will be used throughout this note. Results start from Section 2.

1.1. Unitarily invariant kernels

A *unitarily invariant kernel* on the open Euclidean unit ball

$$\mathbb{B}_d = \{ \boldsymbol{z} = (z_1, \ldots z_d) \in \mathbb{C}^d : \|\boldsymbol{z}\| \overset{\text{def}}{=} (|z_1|^2 + \cdots + |z_d|^2)^{1/2} < 1 \}$$

is a reproducing kernel k of the form

$$k(\boldsymbol{z}, \boldsymbol{w}) = \sum_{n=0}^{\infty} a_n \langle \boldsymbol{z}, \boldsymbol{w} \rangle^n \quad (\boldsymbol{z}, \boldsymbol{w} \in \mathbb{B}_d) \tag{1.1}$$

for some sequence of strictly positive coefficients $\{a_n\}_{n \geq 0}$ with $a_0 = 1$. The corresponding reproducing kernel Hilbert space, to be denoted by H_k, is called a *unitarily invariant space*. The *generalized Bergman kernels*

$$k_m(\boldsymbol{z}, \boldsymbol{w}) = \left(\frac{1}{1 - \langle \boldsymbol{z}, \boldsymbol{w} \rangle} \right)^m ; \ m = 1, 2, \ldots, \tag{1.2}$$

are examples of unitarily invariant kernels. For $m = 1$, k_m is called the Drury-Arveson kernel. The Dirichlet kernel

$$k(z, w) = \sum_{n=0}^{\infty} \frac{1}{n+1} z^n \overline{w}^n \quad (z, w \in \mathbb{D})$$

is a unitarily invariant kernel and a motivating example in this note.

Since $a_0 = 1$, there exists a sequence of real numbers $\{b_n\}_{n=1}^{\infty}$ and an $\epsilon > 0$ such that the equality

$$\sum_{n=1}^{\infty} b_n \langle \boldsymbol{z}, \boldsymbol{w} \rangle^n = 1 - \frac{1}{\sum\limits_{n=0}^{\infty} a_n \langle \boldsymbol{z}, \boldsymbol{w} \rangle^n} \tag{1.3}$$

holds for all $\boldsymbol{z}, \boldsymbol{w} \in \mathbb{B}_d$ satisfying $\|\boldsymbol{z}\| < 1$ and $\|\boldsymbol{w}\| < \epsilon$. Moreover, if k is *non-vanishing* then this equality holds for all $\boldsymbol{z}, \boldsymbol{w} \in \mathbb{B}_d$. For any $\epsilon > 0$,

$$\text{span}\{ k_{\boldsymbol{w}} : \boldsymbol{w} \in \mathbb{B}_d, \|\boldsymbol{w}\| < \epsilon \}$$

of the set of *kernel functions* defined as

$$k_{\boldsymbol{w}}(\boldsymbol{z}) = k(\boldsymbol{z}, \boldsymbol{w}) \quad \text{for } \boldsymbol{z}, \boldsymbol{w} \in \mathbb{B}_d$$

is dense in H_k.

Let $\mathbb{Z}_+$ denote the set of all non-negative integers. Given $\alpha \in \mathbb{Z}_+^d$ and $\boldsymbol{z} \in \mathbb{C}^d$, we use the usual multi-index notations:

$$\alpha = (\alpha_1, \ldots, \alpha_d), \quad |\alpha| = \alpha_1 + \cdots + \alpha_d, \quad \alpha! = \alpha_1! \ldots \alpha_d!$$

and

$$\binom{|\alpha|}{\alpha} = \frac{|\alpha|!}{\alpha_1! \ldots \alpha_d!}, \qquad \boldsymbol{z}^\alpha = z_1^{\alpha_1} \ldots z_d^{\alpha_d}.$$

We set

$$a_\alpha = \begin{cases} a_{|\alpha|}\binom{|\alpha|}{\alpha}, & \alpha \in \mathbb{Z}_+^d \\ 0, & \alpha \in \mathbb{Z}^d \backslash \mathbb{Z}_+^d \end{cases} \quad \text{for } \alpha \in \mathbb{Z}^d, \text{ and } b_\alpha = b_{|\alpha|}\binom{|\alpha|}{\alpha} \text{ for } \alpha \in \mathbb{Z}_+^d \backslash \{0\}. \quad (1.4)$$

With the help of notations in (1.4), a unitarily invariant kernel k, defined in (1.1) can be written as

$$k(\boldsymbol{z}, \boldsymbol{w}) = \sum_{\alpha \in \mathbb{Z}_+^d} a_\alpha \boldsymbol{z}^\alpha \overline{\boldsymbol{w}^\alpha}.$$

It is clear that the monomials $\{\boldsymbol{z}^\alpha\}_{\alpha \in \mathbb{Z}_+^d}$ form an orthogonal basis for a unitarily invariant space H_k and that

$$\|\boldsymbol{z}^\alpha\|_{H_k}^2 = \frac{1}{a_\alpha}$$

for all $\alpha \in \mathbb{Z}_+^d$.

For a Hilbert space $\mathcal{E}$, let $\mathcal{O}(\mathbb{B}_d, \mathcal{E})$ be the class of all holomorphic $\mathcal{E}$-valued functions on $\mathbb{B}_d$. Then the vector valued Hilbert space $H_k(\mathcal{E})$ is defined as

$$H_k(\mathcal{E}) := \left\{ f \in \mathcal{O}(\mathbb{B}_d, \mathcal{E}) : f(\boldsymbol{z}) = \sum_{\alpha \in \mathbb{Z}_+^d} c_\alpha \boldsymbol{z}^\alpha, c_\alpha \in \mathcal{E} \right.$$

$$\left. \text{and } \|f\|^2 := \sum_{\alpha \in \mathbb{Z}_+^d} \|c_\alpha\|^2 \|\boldsymbol{z}^\alpha\|^2 < \infty \right\}.$$

As a Hilbert space, $H_k(\mathcal{E})$ is the same as $H_k \otimes \mathcal{E}$, the identification being

$$\sum_{\alpha \in \mathbb{Z}_+^d} c_\alpha \boldsymbol{z}^\alpha \rightarrow \sum_{\alpha \in \mathbb{Z}_+^d} (\boldsymbol{z}^\alpha \otimes c_\alpha).$$

1.2. 1/k-contractions

We shall denote a commuting d-tuple of bounded operators $(T_1, \ldots, T_d)$ by $\boldsymbol{T}$. For $\alpha \in \mathbb{Z}^d$, we set

$$\boldsymbol{T}^\alpha = \begin{cases} T_1^{\alpha_1} \ldots T_d^{\alpha_d}, & \alpha \in \mathbb{Z}_+^d \\ I, & \alpha \in \mathbb{Z}^d \backslash \mathbb{Z}_+^d \end{cases}.$$

The following definition, motivated by (1.3) in the context of a unitarily invariant kernel on $\mathbb{B}_d$ is from [9] and is of crucial use to us.

Definition 1.1. Let k be a unitarily invariant kernel on $\mathbb{B}_d$ and $\boldsymbol{T}$ be a commuting d-tuple of bounded operators such that the series

$$\sum_{\alpha \in \mathbb{Z}_+^d \backslash \{0\}} b_\alpha \boldsymbol{T}^\alpha (\boldsymbol{T}^\alpha)^*$$

converges in strong operator topology. The d-tuple $\boldsymbol{T}$ is called a $1/k$-contraction if

$$I - \sum_{\alpha \in \mathbb{Z}_+^d \backslash \{0\}} b_\alpha \boldsymbol{T}^\alpha (\boldsymbol{T}^\alpha)^* \geq 0.$$

In this case, we shall denote the unique positive square root of the positive operator $I - \sum_{\alpha \in \mathbb{Z}_+^d \backslash \{0\}} b_\alpha \boldsymbol{T}^\alpha (\boldsymbol{T}^\alpha)^*$ by $\Delta_{\boldsymbol{T}}$.

While we shall be content with the definition above, a vast generalization of this definition, applicable to other domains, appeared in [6]. A contraction T and a *commuting contractive tuple* (or a d-contraction in Arveson's terminology, see page 175 of [7] for example) $\boldsymbol{T} = (T_1, \ldots, T_d)$ are $1/k$-contractions when k is the Szegő kernel (on $\mathbb{D}$) and the Drury Arveson kernel (on $\mathbb{B}_d$) respectively.

Definition 1.2. A $1/k$-contraction $\boldsymbol{T} = (T_1, \ldots, T_d)$ is called pure if the series

$$\sum_{\alpha \in \mathbb{Z}_+^d} a_\alpha \boldsymbol{T}^\alpha \Delta_{\boldsymbol{T}}^2 (\boldsymbol{T}^\alpha)^*$$

converges strongly to I.

If k is the Drury-Arveson kernel and $\boldsymbol{T}$ is a d-contraction, then the series above converges strongly to I if and only if $\boldsymbol{T}$ is pure in the sense of Definition 3.1 of [8]. The compression of a pure $1/k$-contraction to a co-invariant subspace is a pure $1/k$-contraction. Theorem 2.2 will imply that any pure $1/k$-contraction is, in fact, the compression to a co-invariant subspace of a special $1/k$-contraction.

1.3. Admissible kernels

Let $\mathcal{E}$ and $\mathcal{F}$ be two Hilbert spaces. For a reproducing kernel Hilbert space H_k of holomorphic functions on $\mathbb{B}_d$, a *multiplier* from $H_k \otimes \mathcal{E}$ to $H_k \otimes \mathcal{F}$ is a $\mathcal{B}(\mathcal{E}, \mathcal{F})$-valued function φ on $\mathbb{B}_d$ with the property that if f is in $H_k \otimes \mathcal{E}$, then φf is in $H_k \otimes \mathcal{F}$. For such a φ, we shall let M_φ denote the operator of multiplication by φ. An application of the Closed Graph Theorem proves M_φ to be a bounded operator. The set of all such functions is denoted by $Mult(H_k \otimes \mathcal{E}, H_k \otimes \mathcal{F})$. Since the multiplication operators by coordinate functions will serve as the model operators, we need to restrict the class of kernels we work with.

Definition 1.3. A unitarily invariant kernel k is called admissible if the operators of multiplication by the co-ordinate functions M_{z_i} are bounded operators on H_k for $i = 1, \ldots, d$ and the d-tuple $\boldsymbol{M_z} = (M_{z_1}, \ldots, M_{z_d})$ is a $1/k$-contraction.

The M_{z_i} will sometimes be referred to as the *shift operators*. The name is self-explanatory. The generalized Bergman kernels k_m defined in (1.2) are examples of admissible kernels, see [15]. *In this paper, k will always denote an admissible kernel.*

1.4. Complete Nevanlinna-Pick kernels

Complete Nevanlinna-Pick kernels arose out of a question of Quiggin. In [19], he asked for a characterization of reproducing kernel Hilbert spaces on which Pick's theorem was true. The answer came a year later when McCullough showed in [13] that there is a characterization if one considers Nevanlinna-Pick interpolation problem for matrix-valued functions, viz., that the inverse of the kernel has only one positive square. Starting with [1] by Agler and McCarthy, the complete Nevanlinna-Pick kernels have been of constant interest over the last two decades - [3], [4], [10], [12] and [14] are representative publications. This list is by no means exhaustive and is growing. We shall see a new characterization of irreducible unitarily invariant complete Nevanlinna-Pick kernels on the Euclidean unit ball in Section 3.

Definition 1.4. A reproducing kernel s is said to have the $M_{m \times n}$ Nevanlinna-Pick property if, whenever $\boldsymbol{\lambda}_1, \ldots, \boldsymbol{\lambda}_N$ are points in $\mathbb{B}_d$ and $W_1, \ldots, W_N$ are m-by-n matrices such that

$$(I - W_i W_j^*)s(\boldsymbol{\lambda}_i, \boldsymbol{\lambda}_j) \geq 0,$$

then there exists a multiplier φ in the closed unit ball of

$$Mult(H_s \otimes \mathbb{C}^n, H_s \otimes \mathbb{C}^m)$$

such that $\phi(\boldsymbol{\lambda}_i) = W_i, i = 1, \ldots, N$. The kernel s is said to have the complete Nevanlinna-Pick property (CNPP) if it has the $M_{m \times n}$ Nevanlinna-Pick property for all positive integers m and n and the corresponding reproducing kernel Hilbert space H_s is called a complete Nevanlinna-Pick space.

Definition 1.5. A reproducing kernel Hilbert space H_s is irreducible if $s(\boldsymbol{z}, \boldsymbol{w}) \neq 0$ for all $\boldsymbol{z}, \boldsymbol{w} \in \mathbb{B}_d$, and $s_{\boldsymbol{w}}$ and $s_{\boldsymbol{v}}$ are linearly independent if $\boldsymbol{v} \neq \boldsymbol{w}$.

One of the main tools for us is the following well-known result. See Lemma 2.3 of [11] for a proof. For one variable, the result is in Lemma 7.33 of [2].

Lemma 1.6. Let H_s be a unitarily invariant space on $\mathbb{B}_d$ with reproducing kernel

$$s(\boldsymbol{z}, \boldsymbol{w}) = \sum_{n=0}^{\infty} a_n \langle \boldsymbol{z}, \boldsymbol{w} \rangle^n.$$

Then the following are equivalent:

(1) H_s is an irreducible complete Nevanlinna-Pick space.
(2) The sequence $\{b_n\}_{n=1}^{\infty}$ defined by (1.3) is a sequence of non-negative real numbers.

Definition 1.7. A reproducing kernel s is called a unitarily invariant complete Nevanlinna-Pick (CNP) kernel if

(1) it is of the form

$$s(\boldsymbol{z}, \boldsymbol{w}) = \sum_{n=0}^{\infty} a_n \langle \boldsymbol{z}, \boldsymbol{w} \rangle^n \quad (\boldsymbol{z}, \boldsymbol{w} \in \mathbb{B}_d)$$

for a sequence of strictly positive coefficients $\{a_n\}_{n \geq 0}$ with $a_0 = 1$, and
(2) H_s is an irreducible complete Nevanlinna-Pick space.

Unitarily invariant CNP kernels are admissible. See, [[9], Lemma 5.2]. *In this paper, s will always denote a unitarily invariant CNP kernel.*

In general, the generalized Bergman kernels k_m defined in (1.2) are not always unitarily invariant CNP. Some basic examples of unitarily invariant CNP kernels are the Drury-Arveson kernel and the Dirichlet kernel.

Definition 1.8. For any $t \geq 0$, we define a weighted Dirichlet space, to be denoted by $\mathcal{D}_t$, in the following way

$$\mathcal{D}_t := \left\{ f \in \mathcal{O}(\mathbb{D}, \mathbb{C}) : f(z) = \sum_{n=0}^{\infty} c_n z^n \text{ and } \|f\|_{\mathcal{D}_t}^2 := \sum_{n=0}^{\infty} (n+1)^t |c_n|^2 < \infty \right\}.$$

Note that $\mathcal{D}_1$ is the same as the Dirichlet space. The space $\mathcal{D}_t$ is a reproducing kernel Hilbert space and the corresponding reproducing kernel is unitarily invariant CNP (see, [[2], Corollary 7.41]).

There are complete Nevanlinna-Pick kernels on $\mathbb{B}_d$ which are not unitarily invariant. For example, Dirichlet-type spaces of functions analytic in the unit disc whose derivatives are square area integrable with superharmonic weights have complete Nevanlinna–Pick reproducing kernels, see [18], but in general, they are not always unitarily invariant.

2. Existence of the characteristic function

The characteristic function introduced by Sz.-Nagy and Foias has a long history of generating beautiful mathematics, see [16] and many references therein for the classical theory.

It was created in search of a complete unitary invariant for a contraction T and is defined as

$$\theta_T(z) = (-T + zD_{T^*}(I_{\mathcal{H}} - zT^*)^{-1}D_T)|_{\mathcal{D}_T},$$

on $\mathbb{D}$. The *defect operators* $D_T = (I_{\mathcal{H}} - T^*T)^{1/2}$ and $D_{T^*} = (I_{\mathcal{H}} - TT^*)^{1/2}$ satisfy $TD_T = D_{T^*}T$. Hence, the *defect space* $\mathcal{D}_T = \overline{\mathrm{Ran}}D_T$ is mapped by the characteristic function into the other defect space $\mathcal{D}_{T^*} = \overline{\mathrm{Ran}}D_{T^*}$. The $\mathcal{B}(\mathcal{D}_T, \mathcal{D}_{T^*})$ valued contractive holomorphic function θ_T is a complete unitary invariant for any completely non-unitary (c.n.u.) contraction T, i.e., a c.n.u. contraction T is unitarily equivalent to another c.n.u. contraction R if and only if the following diagram commutes for some unitary operators $\sigma_1 : \mathcal{D}_T \to \mathcal{D}_R$ and $\sigma_2 : \mathcal{D}_{T^*} \to \mathcal{D}_{R^*}$ and for all $z \in \mathbb{D}$:

$$
\begin{array}{ccc}
\mathcal{D}_T & \xrightarrow{\theta_T(z)} & \mathcal{D}_{T^*} \\
\sigma_1 \downarrow & & \downarrow \sigma_2 \\
\mathcal{D}_R & \xrightarrow[\theta_R(z)]{} & \mathcal{D}_{R^*}
\end{array}
$$

At this point, it is important to explain the interplay between a contraction T and the Szegő kernel $\mathbb{S}(z, w) = \frac{1}{1 - z\overline{w}}$ on $\mathbb{D}$. Clearly, T is a contraction if and only if $\frac{1}{\mathbb{S}}(T, T^*) = I - TT^*$ is a positive operator. It is a natural question whether a characteristic function can always be associated with a $1/k$-contraction for an admissible kernel k. Although, the answer is yes in case of the Drury-Arveson kernel, we shall see that the answer is no for a general admissible k. Hence, this section will culminate in finding a necessary and sufficient condition for a $1/k$-contraction to have a characteristic function.

2.1. Factoring a positive operator

The main result of this subsection is Proposition 2.4. Let E_0 be the *projection onto the one dimensional subspace of constant functions in H_k*.

Lemma 2.1. *For an admissible kernel k, we have $\Delta_{M_z} = E_0$. Moreover, the operator d-tuple $M_z = (M_{z_1}, \ldots, M_{z_d})$ is pure in the sense of Definition 1.2.*

Proof. The fact that Δ_{M_z} is E_0 can be seen by applying Δ_{M_z} on the linear combinations of kernel functions which form a dense set. The pureness of M_z follows by observing that with respect to the orthonormal basis

$$e(\alpha) = \sqrt{a_\alpha} z^\alpha \text{ for } \alpha \in \mathbb{Z}_+^d,$$

the operator $\sum_{|\alpha| \leq N} a_\alpha M_z^\alpha E_0 (M_z^\alpha)^*$ is the projection onto the finite dimensional space spanned by $\{e(\alpha) : \alpha \in \mathbb{Z}_+^d, |\alpha| \leq N\}$. Thus, $\sum_{|\alpha| \leq N} a_\alpha M_z^\alpha E_0 (M_z^\alpha)^*$ is an increasing sequence of projections converging to I in strong operator topology. $\square$

Our next tool is a construction by Arazy and Englis in [6], the roots of which go back to [5]. For a proof of the following theorem, see [[6], Theorem 1.3].

Theorem 2.2. *Let $T = (T_1, \ldots, T_d)$ be a pure $1/k$ contraction acting on a Hilbert space $\mathcal{H}$. Then the linear map $V_T : \mathcal{H} \to H_k \otimes \overline{\mathrm{Ran}}\Delta_T$ given by*

$$h \mapsto \sum_{\alpha \in \mathbb{Z}_+^d} a_\alpha z^\alpha \otimes \Delta_T (T^\alpha)^* h$$

is an isometry which satisfies

$$V_T^* (p(M_z) \otimes I_{\overline{\mathrm{Ran}}\Delta_T}) = p(T) V_T^*$$

for all polynomials p in d complex variables.

Let

$$M_z \otimes I_{\overline{\mathrm{Ran}}\Delta_T} \overset{\mathrm{def}}{=} (M_{z_1} \otimes I_{\overline{\mathrm{Ran}}\Delta_T}, \ldots, M_{z_d} \otimes I_{\overline{\mathrm{Ran}}\Delta_T}).$$

We shall suppress the suffix $\overline{\mathrm{Ran}}\Delta_T$ when it is obvious. An important consequence of this theorem about an admissible kernel is that every pure $1/k$-contraction is unitarily equivalent to the compression of $M_z \otimes I$ to a co-invariant subspace, viz., the range of the isometry V_T.

Definition 2.3. Let k be an admissible kernel on $\mathbb{B}_d$. Let $\boldsymbol{T} = (T_1, \dots, T_d)$ be a commuting $d-$ tuple of bounded operators acting on a Hilbert space $\mathcal{H}$. A positive operator X in $\mathcal{B}(\mathcal{H})$ with closed range is called $(k, \boldsymbol{T})$-factorable if there is a Hilbert space $\mathcal{E}$ and a bounded linear transformation $\Theta : H_k \otimes \mathcal{E} \to \mathcal{H}$ such that $X = \Theta\Theta^*$ and $\Theta(M_{z_i} \otimes I_{\mathcal{E}}) = T_i\Theta$ for all i.

See [7] for a discussion on factorable operators when k is the Drury-Arveson kernel and $\boldsymbol{T} = (T_1, \dots, T_d)$ is a d-contraction.

Proposition 2.4. *Assume the setup of Definition 2.3. Let $c_i = \|M_{z_i}\|^2$. Then, X is $(k, \boldsymbol{T})$-factorable if and only if*

(1) $c_i X - T_i X T_i^* \geq 0$, *for all i,*
(2) *the series* $P_{\boldsymbol{T}}(X) := \sum\limits_{\alpha \in \mathbb{Z}_+^d \setminus \{0\}} b_\alpha \boldsymbol{T}^\alpha X (\boldsymbol{T}^\alpha)^*$ *converges strongly such that $X - P_{\boldsymbol{T}}(X) \geq$ 0, and*
(3) *the series* $\sum\limits_{\alpha \in \mathbb{Z}_+^d} a_\alpha \boldsymbol{T}^\alpha (X - P_{\boldsymbol{T}}(X))(\boldsymbol{T}^\alpha)^*$ *converges strongly to X.*

Proof. Since X is a positive operator with closed range, we have

$$\mathrm{Ran}X = \mathrm{Ran}X^{1/2}$$

where $X^{1/2}$ is the unique positive square root of the operator X. If the conditions (1), (2) and (3) are satisfied, define for each i, a linear transformation $A_i : \mathrm{Ran}X \to \mathrm{Ran}X$ by

$$A_i X^{1/2} h = X^{1/2} T_i^* h \quad \text{for all } h \in \mathcal{H}.$$

Each A_i is a bounded operator as

$$\|A_i X^{1/2} h\|^2 = \langle T_i X T_i^* h, h \rangle \leq c_i \|X^{1/2} h\|^2.$$

Let $S_i = A_i^*$. The operator $S_i : \mathrm{Ran}X \to \mathrm{Ran}X$ is a bounded operator which satisfies

$$T_i X^{1/2} = X^{1/2} S_i.$$

In fact, the d-tuple $\boldsymbol{S} = (S_1, \dots, S_d)$ on $\mathrm{Ran}X$ is a $1/k$-contraction. To see that, we observe that for any $h \in \mathcal{H}$ we have

$$\langle X^{1/2}h, X^{1/2}h \rangle \geq \left\langle \sum_{\alpha \in \mathbb{Z}_+^d \setminus \{0\}} b_\alpha \boldsymbol{T}^\alpha X (\boldsymbol{T}^\alpha)^* h, h \right\rangle \quad \text{(by condition (2))}$$

$$= \sum_{\alpha \in \mathbb{Z}_+^d \setminus \{0\}} b_\alpha \langle \boldsymbol{T}^\alpha X (\boldsymbol{T}^\alpha)^* h, h \rangle$$

$$= \sum_{\alpha \in \mathbb{Z}_+^d \setminus \{0\}} b_\alpha \langle X^{1/2} \boldsymbol{S}^\alpha (\boldsymbol{S}^\alpha)^* X^{1/2} h, h \rangle$$

$$= \left\langle \sum_{\alpha \in \mathbb{Z}_+^d \setminus \{0\}} b_\alpha \boldsymbol{S}^\alpha (\boldsymbol{S}^*)^\alpha X^{1/2} h, X^{1/2} h \right\rangle.$$

More is true, viz., $\boldsymbol{S}$ is pure because

$$\left\langle \sum_{\alpha \in \mathbb{Z}_+^d} a_\alpha \boldsymbol{S}^\alpha \Delta_{\boldsymbol{S}}^2 (\boldsymbol{S}^\alpha)^* X^{1/2} h, X^{1/2} h \right\rangle = \sum_{\alpha \in \mathbb{Z}_+^d} a_\alpha \langle \boldsymbol{S}^\alpha \Delta_{\boldsymbol{S}}^2 (\boldsymbol{S}^\alpha)^* X^{1/2} h, X^{1/2} h \rangle$$

$$= \left\langle \sum_{\alpha \in \mathbb{Z}_+^d} a_\alpha \boldsymbol{T}^\alpha (X - P_{\boldsymbol{T}}(X)) (\boldsymbol{T}^\alpha)^* h, h \right\rangle$$

$$= \langle Xh, h \rangle \quad \text{(by condition (3))}.$$

Let $\mathcal{E} = \overline{\mathrm{Ran}} \Delta_{\boldsymbol{S}}$. By Theorem 2.2 we have an isometry

$$V_{\boldsymbol{S}} : \mathrm{Ran} X \to H_k \otimes \mathcal{E}$$

such that

$$V_{\boldsymbol{S}}^* (M_{z_i} \otimes I_{\mathcal{E}}) = S_i V_{\boldsymbol{S}}^*.$$

Defining $\Theta : H_k \otimes \mathcal{E} \to \mathcal{H}$ by $\Theta = X^{1/2} V_{\boldsymbol{S}}^*$, we have

$$\Theta \Theta^* = X^{1/2} V_{\boldsymbol{S}}^* V_{\boldsymbol{S}} X^{1/2} = X$$

and

$$\Theta(M_{z_i} \otimes I_{\mathcal{E}}) = X^{1/2} V_{\boldsymbol{S}}^* (M_{z_i} \otimes I_{\mathcal{E}}) = X^{1/2} S_i V_{\boldsymbol{S}}^* = T_i X^{1/2} V_{\boldsymbol{S}}^* = T_i \Theta.$$

Conversely, if there is a Hilbert space $\mathcal{E}$ and a bounded linear transformation

$$\Theta : H_k \otimes \mathcal{E} \to \mathcal{H}$$

such that $X = \Theta \Theta^*$ and $\Theta(M_{z_i} \otimes I_{\mathcal{E}}) = T_i \Theta$, then to prove (1), we note that

$$c_i X - T_i X T_i^* = c_i \Theta \Theta^* - T_i \Theta \Theta^* T_i^*$$

$$= c_i \Theta \Theta^* - \Theta(M_{z_i} \otimes I_{\mathcal{E}})(M_{z_i}^* \otimes I_{\mathcal{E}}) \Theta^*$$

$$= \Theta(c_i I - (M_{z_i} \otimes I_{\mathcal{E}})(M_{z_i}^* \otimes I_{\mathcal{E}})) \Theta^* \geq 0.$$

To prove (2) and (3), we fix an $N \in \mathbb{N}$. Then, we note that

$$X - \sum_{1 \le |\alpha| \le N} b_\alpha \boldsymbol{T}^\alpha X (\boldsymbol{T}^\alpha)^* = \Theta\Theta^* - \sum_{1 \le |\alpha| \le N} b_\alpha \boldsymbol{T}^\alpha \Theta\Theta^* (\boldsymbol{T}^\alpha)^*$$

$$= \Theta \left(I - \sum_{1 \le |\alpha| \le N} b_\alpha (\boldsymbol{M_z}^\alpha \otimes I_\mathcal{E})((\boldsymbol{M_z}^\alpha)^* \otimes I_\mathcal{E}) \right) \Theta^*$$

and

$$\sum_{|\alpha| \le N} a_\alpha \boldsymbol{T}^\alpha (X - P_{\boldsymbol{T}}(X))(\boldsymbol{T}^\alpha)^* = \sum_{|\alpha| \le N} a_\alpha \boldsymbol{T}^\alpha (\Theta\Theta^* - P_{\boldsymbol{T}}(\Theta\Theta^*))(\boldsymbol{T}^\alpha)^*$$

$$= \Theta \left(\sum_{|\alpha| \le N} a_\alpha (\boldsymbol{M_z}^\alpha \otimes I_\mathcal{E}) \Delta_{(\boldsymbol{M_z} \otimes I_\mathcal{E})}((\boldsymbol{M_z}^\alpha)^* \otimes I_\mathcal{E}) \right) \Theta^*.$$

Now, letting N go to ∞ completes the proof. □

2.2. An invariant subspace, and an associated tuple

The special case of an operator X which is $(k, \boldsymbol{T})$-factorable when the Hilbert space $\mathcal{H}$ is of the form $H_k \otimes \mathcal{E}$ for the same kernel k and the operators T_i are $M_{z_i} \otimes I_\mathcal{E}$ is of particular interest because it is straightforward then that the *factor* Θ is a multiplication operator.

Definition 2.5. A pure $1/k$-contraction $\boldsymbol{T} = (T_1, \dots, T_d)$ is said to admit a characteristic function if there exist a Hilbert space $\mathcal{E}$ and a $\mathcal{B}(\mathcal{E}, \overline{\mathrm{Ran}}\Delta_{\boldsymbol{T}})$-valued analytic function $\theta_{\boldsymbol{T}}$ on $\mathbb{B}_d$ such that $M_{\theta_{\boldsymbol{T}}}$ is a multiplication operator from $H_k \otimes \mathcal{E}$ to $H_k \otimes \overline{\mathrm{Ran}}\Delta_{\boldsymbol{T}}$ satisfying

$$I - V_{\boldsymbol{T}} V_{\boldsymbol{T}}^* = M_{\theta_{\boldsymbol{T}}} M_{\theta_{\boldsymbol{T}}}^*.$$

The following proposition follows from the fact that if an operator $\Theta : H_k \otimes \mathcal{E} \to H_k \otimes \mathcal{F}$ satisfies $\Theta(M_{z_i} \otimes I_\mathcal{E}) = (M_{z_i} \otimes I_\mathcal{F})\Theta$ for all $i = 1, \dots, d$, then there exists a $\mathcal{B}(\mathcal{E}, \mathcal{F})$-valued analytic function θ on $\mathbb{B}_d$ such that $\Theta = M_\theta$.

Proposition 2.6. *A pure $1/k$-contraction $\boldsymbol{T} = (T_1, \dots, T_d)$ admits a characteristic function if and only if the projection $I - V_{\boldsymbol{T}} V_{\boldsymbol{T}}^*$ is $(k, \boldsymbol{M_z} \otimes I)-$ factorable.*

By virtue of Theorem 2.2, we observe that the kernel of $V_{\boldsymbol{T}}^*$, i.e., the range of projection $I - V_{\boldsymbol{T}} V_{\boldsymbol{T}}^*$ is an invariant subspace of the tuple $\boldsymbol{M_z} \otimes I$.

Definition 2.7. Let $\boldsymbol{T} = (T_1, \dots, T_d)$ be a pure $1/k$-contraction. The associated d-tuple of commuting operators $\boldsymbol{B_T}$ is defined on $\mathrm{Ker}V_{\boldsymbol{T}}^*$ as

$$\boldsymbol{B_T} = ((M_{z_1} \otimes I)|_{\mathrm{Ker}V_{\boldsymbol{T}}^*}, \dots, (M_{z_d} \otimes I)|_{\mathrm{Ker}V_{\boldsymbol{T}}^*}).$$

We have reached the main theorem of this section which gives a neat equivalent condition for existence of a characteristic function. This is a crucial result because the proof of the characterization of unitarily invariant CNP kernels obtained in the next section requires this result.

Theorem 2.8. *A pure $1/k$-contraction $\boldsymbol{T} = (T_1, \ldots, T_d)$ acting on a Hilbert space $\mathcal{H}$ admits a characteristic function if and only if the associated tuple $\boldsymbol{B_T}$ is a $1/k$-contraction.*

Proof. It is trivial to note that $\boldsymbol{B_T}$ is a $1/k$-contraction if and only if the series

$$\sum_{\alpha \in \mathbb{Z}_+^d \setminus \{0\}} b_\alpha (\boldsymbol{M_z^\alpha} \otimes I)(I - V_{\boldsymbol{T}} V_{\boldsymbol{T}}^*)((\boldsymbol{M_z^\alpha})^* \otimes I)$$

converges strongly such that

$$(I - V_{\boldsymbol{T}} V_{\boldsymbol{T}}^*) - \sum_{\alpha \in \mathbb{Z}_+^d \setminus \{0\}} b_\alpha (\boldsymbol{M_z^\alpha} \otimes I)(I - V_{\boldsymbol{T}} V_{\boldsymbol{T}}^*)((\boldsymbol{M_z^\alpha})^* \otimes I) \geq 0.$$

We prove the easy side first. Let $\boldsymbol{T}$ admit a characteristic function. This means that there is a Hilbert space $\mathcal{E}$ and a $\mathcal{B}(\mathcal{E}, \overline{\mathrm{Ran}}\Delta_{\boldsymbol{T}})$-valued bounded analytic function $\theta_{\boldsymbol{T}}$ on $\mathbb{B}_d$ such that

$$I - V_{\boldsymbol{T}} V_{\boldsymbol{T}}^* = M_{\theta_{\boldsymbol{T}}} M_{\theta_{\boldsymbol{T}}}^*.$$

Thus, for any $N \geq 1$, we get

$$(I - V_{\boldsymbol{T}} V_{\boldsymbol{T}}^*) - \sum_{1 \leq |\alpha| \leq N} b_\alpha (\boldsymbol{M_z^\alpha} \otimes I)(I - V_{\boldsymbol{T}} V_{\boldsymbol{T}}^*)((\boldsymbol{M_z^\alpha})^* \otimes I)$$

$$= M_{\theta_{\boldsymbol{T}}} M_{\theta_{\boldsymbol{T}}}^* - \sum_{1 \leq |\alpha| \leq N} b_\alpha (\boldsymbol{M_z^\alpha} \otimes I) M_{\theta_{\boldsymbol{T}}} M_{\theta_{\boldsymbol{T}}}^* ((\boldsymbol{M_z^\alpha})^* \otimes I)$$

$$= M_{\theta_{\boldsymbol{T}}} \left(I - \sum_{1 \leq |\alpha| \leq N} b_\alpha (\boldsymbol{M_z^\alpha} \otimes I)((\boldsymbol{M_z^\alpha})^* \otimes I) \right) M_{\theta_{\boldsymbol{T}}}^*$$

and that completes the proof of this direction.

Conversely, let $c_i = \|M_{z_i}\|^2$. Since $V_{\boldsymbol{T}}$ is an isometry, $I - V_{\boldsymbol{T}} V_{\boldsymbol{T}}^*$ is the projection onto $\mathrm{Ker} V_{\boldsymbol{T}}^*$ which is invariant under $M_{z_i} \otimes I$ for each i. Let $P_{\mathrm{Ker} V_{\boldsymbol{T}}^*}$ be the projection of $H_k \otimes \overline{\mathrm{Ran}}\Delta_{\boldsymbol{T}}$ onto $\mathrm{Ker} V_{\boldsymbol{T}}^*$. Now for each i, define a linear operator $R_i : \mathrm{Ker} V_{\boldsymbol{T}}^* \to \mathrm{Ker} V_{\boldsymbol{T}}^*$ by $R_i = (M_{z_i} \otimes I)|_{\mathrm{Ker} V_{\boldsymbol{T}}^*}$. Note that $\|R_i\|^2 \leq c_i$. So we have

$$c_i (I - V_{\boldsymbol{T}} V_{\boldsymbol{T}}^*) - (M_{z_i} \otimes I)(I - V_{\boldsymbol{T}} V_{\boldsymbol{T}}^*)(M_{z_i}^* \otimes I)$$

$$= c_i P_{\mathrm{Ker} V_{\boldsymbol{T}}^*} - (M_{z_i} \otimes I) P_{\mathrm{Ker} V_{\boldsymbol{T}}^*} (M_{z_i}^* \otimes I)$$

$$= c_i P_{\mathrm{Ker} V_{\boldsymbol{T}}^*} - (M_{z_i} \otimes I) P_{\mathrm{Ker} V_{\boldsymbol{T}}^*} (M_{z_i}^* \otimes I) P_{\mathrm{Ker} V_{\boldsymbol{T}}^*}$$

$$\geq 0 \quad (\text{since } c_i I_{\operatorname{Ker} V_{\boldsymbol{T}}^*} - R_i R_i^* \geq 0).$$

Since the operator $I - V_{\boldsymbol{T}} V_{\boldsymbol{T}}^*$ acts on the Hilbert space $H_k \otimes \overline{\operatorname{Ran}} \Delta_{\boldsymbol{T}}$, we can consider the map $P_{\boldsymbol{M_z} \otimes I}$ as in Proposition 2.4 with $\boldsymbol{T}$ replaced by $\boldsymbol{M_z} \otimes I$. For the sake of notational brevity, we shall denote it by P for the rest of this proof. Then the given condition is

$$(I - V_{\boldsymbol{T}} V_{\boldsymbol{T}}^*) - P(I - V_{\boldsymbol{T}} V_{\boldsymbol{T}}^*) \geq 0.$$

Next we shall show that the series

$$\sum_{\alpha \in \mathbb{Z}_+^d} a_\alpha (\boldsymbol{M_z^\alpha} \otimes I)((I - V_{\boldsymbol{T}} V_{\boldsymbol{T}}^*) - P(I - V_{\boldsymbol{T}} V_{\boldsymbol{T}}^*))((\boldsymbol{M_z^\alpha})^* \otimes I)$$

converges strongly to $I - V_{\boldsymbol{T}} V_{\boldsymbol{T}}^*$. Note that if we define

$$S_N = \sum_{|\alpha| \leq N} a_\alpha (\boldsymbol{M_z^\alpha} \otimes I)((I - V_{\boldsymbol{T}} V_{\boldsymbol{T}}^*) - P(I - V_{\boldsymbol{T}} V_{\boldsymbol{T}}^*))((\boldsymbol{M_z^\alpha})^* \otimes I),$$

then S_N is an increasing sequence of positive operators. So if we can show that $\{S_N\}$ is bounded above then it will converge strongly.

Claim. $S_N \leq I - V_{\boldsymbol{T}} V_{\boldsymbol{T}}^*$ for every $N \in \mathbb{N}$.

Proof of the claim. Fix an $\epsilon > 0$ so that (1.3) is satisfied and consider the dense subspace

$$\left\{ \sum_{i=1}^l (k_{\boldsymbol{w}_i} \otimes \xi_i) : l \geq 1, \boldsymbol{w}_1, \ldots \boldsymbol{w}_l \in \mathbb{B}_d, \|\boldsymbol{w}_1\| < \epsilon, \ldots, \|\boldsymbol{w}_l\| < \epsilon, \xi_1, \ldots \xi_l \in \overline{\operatorname{Ran}} \Delta_{\boldsymbol{T}} \right\}$$

of $H_k \otimes \overline{\operatorname{Ran}} \Delta_{\boldsymbol{T}}$. It is enough to verify that

$$\left\langle S_N \sum_{i=1}^l (k_{\boldsymbol{w}_i} \otimes \xi_i), \sum_{i=1}^l (k_{\boldsymbol{w}_i} \otimes \xi_i) \right\rangle \leq \left\langle (I - V_{\boldsymbol{T}} V_{\boldsymbol{T}}^*) \sum_{i=1}^l (k_{\boldsymbol{w}_i} \otimes \xi_i), \sum_{i=1}^l (k_{\boldsymbol{w}_i} \otimes \xi_i) \right\rangle. \tag{2.1}$$

The left hand side after a simplification is

$$\sum_{|\alpha| \leq N} a_\alpha \sum_{i,j=1}^l \overline{\boldsymbol{w}_i}^\alpha \boldsymbol{w}_j^\alpha \left(1 - \sum_{\beta \in \mathbb{Z}_+^d \setminus \{0\}} b_\beta \overline{\boldsymbol{w}_i}^\beta \boldsymbol{w}_j^\beta \right) \langle (I - V_{\boldsymbol{T}} V_{\boldsymbol{T}}^*)(k_{\boldsymbol{w}_i} \otimes \xi_i), (k_{\boldsymbol{w}_j} \otimes \xi_j) \rangle.$$

This is no bigger than

$$\sum_{\alpha \in \mathbb{Z}_+^d} a_\alpha \sum_{i,j=1}^l \overline{\boldsymbol{w}_i}^\alpha \boldsymbol{w}_j^\alpha \left(1 - \sum_{\beta \in \mathbb{Z}_+^d \setminus \{0\}} b_\beta \overline{\boldsymbol{w}_i}^\beta \boldsymbol{w}_j^\beta \right) \langle (I - V_{\boldsymbol{T}} V_{\boldsymbol{T}}^*)(k_{\boldsymbol{w}_i} \otimes \xi_i), (k_{\boldsymbol{w}_j} \otimes \xi_j) \rangle$$

which is the same as

$$\sum_{i,j=1}^{l} \langle (I - V_{\boldsymbol{T}} V_{\boldsymbol{T}}^*)(k_{\boldsymbol{w}_i} \otimes \xi_i), (k_{\boldsymbol{w}_j} \otimes \xi_j) \rangle$$

and that is the right hand side of (2.1). This proves the claim.

So S_N converges strongly. From the calculations above it is also clear that S_N increases to $I - V_{\boldsymbol{T}} V_{\boldsymbol{T}}^*$.

By Proposition 2.4, the operator $I - V_{\boldsymbol{T}} V_{\boldsymbol{T}}^*$ is $(k, \boldsymbol{M}_z \otimes I)$-factorable. Finally, by Proposition 2.6, the pure $1/k$-contraction $\boldsymbol{T}$ admits a characteristic function. $\square$

Remark 2.9. The proof shows that the d-tuple $\boldsymbol{B}_{\boldsymbol{T}}$ is a pure $1/k$-contraction whenever it is a $1/k$-contraction, i.e., whenever $\boldsymbol{T}$ admits a characteristic function.

3. A characterization of unitarily invariant CNP kernels

This section contains the major result Theorem 3.4. The proof crucially uses Theorem 2.8.

Definition 3.1. An admissible kernel k is said to admit a characteristic function if every pure $1/k$-contraction admits a characteristic function.

3.1. Sufficiency of the complete Nevanlinna-Pick property

Note that (1.3) can also be written as

$$\sum_{\alpha \in \mathbb{Z}_+^d} a_\alpha \boldsymbol{z}^\alpha \overline{\boldsymbol{w}^\alpha} = 1 + \left(\sum_{\alpha \in \mathbb{Z}_+^d \setminus \{0\}} b_\alpha \boldsymbol{z}^\alpha \overline{\boldsymbol{w}^\alpha} \right) \left(\sum_{\alpha \in \mathbb{Z}_+^d} a_\alpha \boldsymbol{z}^\alpha \overline{\boldsymbol{w}^\alpha} \right).$$

This leads us to the following relation:

$$a_\alpha = \sum_{\beta \in \mathbb{Z}_+^d \setminus \{0\}} b_\beta a_{\alpha - \beta} \tag{3.1}$$

for all $\alpha \in \mathbb{Z}_+^d \setminus \{0\}$. In the series in (3.1), only finitely many summands are non-zero. Also, if the kernel is a unitarily invariant CNP kernel, then all the summands are positive.

Recall from [9] that if s is a unitarily invariant CNP kernel then $\boldsymbol{M}_{\boldsymbol{z}} = (M_{z_1}, \dots, M_{z_d})$ on H_s is a $1/s$-contraction and so is $\boldsymbol{M}_{\boldsymbol{z}} \otimes I_{\mathcal{E}}$ for any Hilbert space $\mathcal{E}$.

Theorem 3.2. *If s is a unitarily invariant CNP kernel, then s admits a characteristic function, i.e., every pure $1/s$-contraction admits a characteristic function.*

Proof. By Theorem 2.8, it is enough to show that the restriction of $\boldsymbol{M_z} \otimes I_{\mathcal{E}}$ to a closed invariant subspace $\mathcal{M}$ of $H_s \otimes \mathcal{E}$ is a $1/s$-contraction for any Hilbert space $\mathcal{E}$. Let $P_{\mathcal{M}}$ be the projection of $H_s \otimes \mathcal{E}$ onto the subspace $\mathcal{M}$. Let $T_i = (M_{z_i} \otimes I_{\mathcal{E}})|_{\mathcal{M}}$ for all $i = 1, \ldots, d$. For any $N \geq 1$,

$$
\begin{aligned}
\sum_{1 \leq |\alpha| \leq N} b_\alpha \boldsymbol{T}^\alpha (\boldsymbol{T}^\alpha)^* &= \sum_{1 \leq |\alpha| \leq N} b_\alpha P_{\mathcal{M}} (\boldsymbol{M_z^\alpha} \otimes I_{\mathcal{E}}) P_{\mathcal{M}} ((\boldsymbol{M_z^\alpha})^* \otimes I_{\mathcal{E}})|_{\mathcal{M}} \\
&\leq \sum_{1 \leq |\alpha| \leq N} b_\alpha P_{\mathcal{M}} (\boldsymbol{M_z^\alpha} \otimes I_{\mathcal{E}}) ((\boldsymbol{M_z^\alpha})^* \otimes I_{\mathcal{E}})|_{\mathcal{M}} \quad \text{(since } b_\alpha \geq 0) \\
&\leq \sum_{\alpha \in \mathbb{Z}_+^d \setminus \{0\}} b_\alpha P_{\mathcal{M}} (\boldsymbol{M_z^\alpha} \otimes I_{\mathcal{E}}) ((\boldsymbol{M_z^\alpha})^* \otimes I_{\mathcal{E}})|_{\mathcal{M}} \quad \text{(since } b_\alpha \geq 0) \\
&\leq I_{\mathcal{M}} \quad \text{(since } \boldsymbol{M_z} \otimes I_{\mathcal{E}} \text{ is a } 1/s- \text{ contraction).}
\end{aligned}
$$

Since this holds for all $N \geq 1$, the d-tuple $\boldsymbol{T} = (T_1, \ldots, T_d)$ is a $1/s$-contraction. This completes the proof. $\square$

Interestingly, the Bergman kernel does not admit a characteristic function in the sense of Definition 3.1 as is shown below.

Proposition 3.3. *The generalized Bergman kernels k_m defined in (1.2) do not admit characteristic functions for $m \geq 2$.*

Proof. For $\boldsymbol{z}, \boldsymbol{w} \in \mathbb{B}_d$, the expression for the generalized Bergman kernel k_m is $k_m(\boldsymbol{z}, \boldsymbol{w}) = \sum_{\alpha \in \mathbb{Z}_+^d} \sigma_m(\alpha) \boldsymbol{z}^\alpha \overline{\boldsymbol{w}^\alpha}$ where

$$
\sigma_m(\alpha) = \frac{(m + |\alpha| - 1)!}{\alpha!(m-1)!} \quad (\alpha \in \mathbb{Z}_+^d). \tag{3.2}
$$

We shall denote the Hilbert space corresponding to the kernel k_m by $\mathbb{H}_m$. Recall that $\{e_m(\alpha) = \sqrt{\sigma_m(\alpha)} \boldsymbol{z}^\alpha : \alpha \in \mathbb{Z}_+^d\}$ is an orthonormal basis for the Hilbert space $\mathbb{H}_m$.

For $N \geq 0$, define the subspace $\mathcal{H}_N = \operatorname{span}\{\boldsymbol{z}^\alpha : |\alpha| \leq N\}$ and define an operator tuple acting on $\mathcal{H}_N$ by $\boldsymbol{T}_N = P_{\mathcal{H}_N} \boldsymbol{M_z}|_{\mathcal{H}_N}$, where $P_{\mathcal{H}_N}$ is the projection of $\mathbb{H}_m$ onto $\mathcal{H}_N$. It is easy to check that the d-tuple $\boldsymbol{T}_N$ is a pure $1/k_m$-contraction, $\Delta_{\boldsymbol{T}_N} = E_0$ and

$$
\mathcal{M}_N \stackrel{\text{def}}{=} \operatorname{Ker} V_{\boldsymbol{T}_N}^* = \overline{\operatorname{span}}\{\boldsymbol{z}^\alpha : |\alpha| \geq N+1\}.
$$

Let $P_{\mathcal{M}_N}$ be the projection from $\mathbb{H}_m$ onto $\mathcal{M}_N$. Let $\boldsymbol{n}$ denote the multi-index $(n, 0, \ldots, 0)$.

We shall show that the pure $1/k_m$-contraction $\boldsymbol{T}_N$ does not admit a characteristic function for any $N \geq 0$. To that end, we consider the associated operator d-tuple of Definition 2.7. Note that

$$\left\langle \left(I_{\mathcal{M}_N} - \sum_{\alpha \in \mathbb{Z}_+^d \setminus \{0\}} b_\alpha \boldsymbol{M}_{\boldsymbol{z}}^\alpha P_{\mathcal{M}_N} (\boldsymbol{M}_{\boldsymbol{z}}^\alpha)^* \right) e_m(\boldsymbol{N}+\boldsymbol{2}), e_m(\boldsymbol{N}+\boldsymbol{2}) \right\rangle$$

$$= \left\langle (I - b_{\boldsymbol{1}} M_{z_1} P_{\mathcal{M}_N} M_{z_1}^*) e_m(\boldsymbol{N}+\boldsymbol{2}), e_m(\boldsymbol{N}+\boldsymbol{2}) \right\rangle$$

$$= 1 - \sigma_m(\boldsymbol{1}) \frac{\sigma_m(\boldsymbol{N}+\boldsymbol{1})}{\sigma_m(\boldsymbol{N}+\boldsymbol{2})} \quad (b_{\boldsymbol{1}} = \sigma_m(\boldsymbol{1}) \text{ by } (3.1)).$$

By putting values of the coefficients from (3.2) we get

$$\sigma_m(\boldsymbol{1}) \frac{\sigma_m(\boldsymbol{N}+\boldsymbol{1})}{\sigma_m(\boldsymbol{N}+\boldsymbol{2})} = \frac{m!}{(m-1)!} \frac{(m+|\boldsymbol{N}+\boldsymbol{1}|-1)!}{(\boldsymbol{N}+\boldsymbol{1})!(m-1)!} \frac{(\boldsymbol{N}+\boldsymbol{2})!(m-1)!}{(m+|\boldsymbol{N}+\boldsymbol{2}|-1)!}$$

$$= m \frac{(m+N)!}{(N+1)!} \frac{(N+2)!}{(m+N+1)!}$$

$$= m \frac{N+2}{m+N+1} > 1 \quad (\text{for } m \geq 2).$$

So the associated tuple in not a $1/k_m$-contraction. By Theorem 2.8, the d-tuple $\boldsymbol{T}_N$ does not admit a characteristic function for any $N \geq 0$. This completes the proof. □

3.2. Necessity of the complete Nevanlinna-Pick property

The above raises the natural question of characterizing all admissible kernels which admit a characteristic function. We shall answer this question completely. Note that the d-tuple $(0, \ldots, 0)$ of identically zero operators on any Hilbert space $\mathcal{H}$ is a pure $1/k$-contraction for any admissible kernel k on $\mathbb{B}_d$.

Theorem 3.4. *Let k be an admissible kernel on $\mathbb{B}_d$. Then the following are equivalent.*

(1) *The d-tuple $(0, \ldots, 0)$ of identically zero operators admits a characteristic function.*
(2) *The kernel k is unitarily invariant CNP.*
(3) *Any pure $1/k$-contraction admits a characteristic function.*

Proof. $(1) \Rightarrow (2)$: Let $\mathcal{H}$ be the space of all constant functions in H_k. Let $\boldsymbol{T} = (0, \ldots, 0)$ be the d-tuple of identically zero operators on $\mathcal{H}$. Then $\Delta_{\boldsymbol{T}} = I_{\mathcal{H}}$. Recall that E_0 is the projection of H_k onto the space of constant functions. If $V_{\boldsymbol{T}}$ is the isometry obtained in Theorem 2.2, then $V_{\boldsymbol{T}} V_{\boldsymbol{T}}^* = E_0$. Let $e(\alpha) = \sqrt{a_\alpha} z^\alpha$ and $\boldsymbol{n} = (n, 0, \ldots, 0)$. Since $\boldsymbol{T}$ admits a characteristic function, the associated operator tuple in Definition 2.7 is a $1/k$-contraction, see Theorem 2.8. This means that for all $n \geq 1$ we have

$$\left\langle \left(P_{\mathcal{H}^\perp} - \sum_{\alpha \in \mathbb{Z}_+^d \setminus \{0\}} b_\alpha \boldsymbol{M}_{\boldsymbol{z}}^\alpha P_{\mathcal{H}^\perp} (\boldsymbol{M}_{\boldsymbol{z}}^\alpha)^* \right) e(\boldsymbol{n}+\boldsymbol{1}), e(\boldsymbol{n}+\boldsymbol{1}) \right\rangle \geq 0.$$

In other words,

$$
\begin{aligned}
0 &\leq \left\langle \left(P_{\mathcal{H}^\perp} - \sum_{i=1}^n b_i \boldsymbol{M_z^i} P_{\mathcal{H}^\perp} (\boldsymbol{M_z^i})^* \right) e(\boldsymbol{n+1}), e(\boldsymbol{n+1}) \right\rangle \\
&= \left\langle \left(P_{\mathcal{H}^\perp} - \sum_{i=1}^n b_i M_{z_1}^i P_{\mathcal{H}^\perp} (M_{z_1}^i)^* \right) e(\boldsymbol{n+1}), e(\boldsymbol{n+1}) \right\rangle \\
&= 1 - \sum_{i=1}^n b_i \frac{a_{\boldsymbol{n+1-i}}}{a_{\boldsymbol{n+1}}} \\
&= \frac{1}{a_{\boldsymbol{n+1}}} \sum_{i=1}^{n+1} b_i a_{\boldsymbol{n+1-i}} - \frac{1}{a_{\boldsymbol{n+1}}} \sum_{i=1}^n b_i a_{\boldsymbol{n+1-i}} \quad \text{(by (3.1))} \\
&= \frac{b_{\boldsymbol{n+1}}}{a_{\boldsymbol{n+1}}} = \frac{b_{\boldsymbol{n+1}}}{a_{\boldsymbol{n+1}}} \quad \text{(by (1.4))}.
\end{aligned}
$$

This implies $b_{n+1} \geq 0$ for all $n \geq 1$. Since $b_1 = a_1$, we get $b_n \geq 0$ for all $n \geq 1$. Now by Lemma 1.6, k is a unitarily invariant CNP kernel. The implication $(2) \Rightarrow (3)$ has already been proved and for $(3) \Rightarrow (1)$, there is nothing to prove. $\square$

4. Explicit construction of the characteristic function

Recall that a unitarily invariant CNP kernel s is a reproducing kernel of the form

$$
s(\boldsymbol{z}, \boldsymbol{w}) = \sum_{n=0}^\infty a_n \langle \boldsymbol{z}, \boldsymbol{w} \rangle^n \tag{4.1}
$$

having the following properties:

(1) $a_0 = 1$ and $a_n > 0$ for all $n \geq 1$, and
(2) H_s is an irreducible complete Nevanlinna-Pick space.

By Lemma 1.6, for a unitarily invariant CNP kernel, (1.3) holds for all $\boldsymbol{z}, \boldsymbol{w} \in \mathbb{B}_d$ and the sequence $\{b_n\}_{n=1}^\infty$ defined by (1.3) is a sequence of non-negative real numbers.

4.1. $1/s$-contractions

We shall give an explicit construction of the characteristic function for any $1/s$-contraction (not necessarily pure). The next lemma plays an important role in extending the notion of characteristic function beyond pure $1/s$-contractions and also clarifies the motivation for definition of a pure $1/s$-contraction.

Lemma 4.1. *If $\boldsymbol{T} = (T_1, \ldots, T_d)$ is a $1/s$-contraction, then the series*

$$\sum_{\alpha \in \mathbb{Z}_+^d} a_\alpha \boldsymbol{T}^\alpha \Delta_{\boldsymbol{T}}^2 (\boldsymbol{T}^\alpha)^*$$

converges in the strong operator topology and the limiting operator is a positive contraction.

Proof. The positivity is clear if we can show the convergence. We shall show that the partial sums form an increasing sequence of positive operators bounded above by the identity operator. Hence the sequence will converge in strong operator topology with the limiting operator bounded above by the identity operator. To that end, consider $h \in \mathcal{H}$. Then for any $N \geq 1$,

$$
\begin{aligned}
\sum_{|\alpha| \leq N} a_\alpha \langle \boldsymbol{T}^\alpha \Delta_{\boldsymbol{T}}^2 (\boldsymbol{T}^\alpha)^* h, h \rangle &= \sum_{|\alpha| \leq N} a_\alpha \left\langle \boldsymbol{T}^\alpha \left(I - \sum_{\beta \in \mathbb{Z}_+^d \backslash \{0\}} b_\beta \boldsymbol{T}^\beta (\boldsymbol{T}^\beta)^* \right) (\boldsymbol{T}^\alpha)^* h, h \right\rangle \\
&\leq \sum_{|\alpha| \leq N} a_\alpha \| (\boldsymbol{T}^\alpha)^* h \|^2 - \sum_{1 \leq |\alpha| \leq N} \sum_{\beta \in \mathbb{Z}_+^d \backslash \{0\}} a_{\alpha-\beta} b_\beta \| (\boldsymbol{T}^\alpha)^* h \|^2 \\
&= \sum_{|\alpha| \leq N} a_\alpha \| (\boldsymbol{T}^\alpha)^* h \|^2 - \sum_{1 \leq |\alpha| \leq N} a_\alpha \| (\boldsymbol{T}^\alpha)^* h \|^2 \quad \text{(by (3.1))}.
\end{aligned}
$$

The last quantity is $\|h\|^2$ and that completes the proof. $\square$

As a direct consequence of this Lemma, we get the following Corollary.

Corollary 4.2. *Let $\boldsymbol{T} = (T_1, \ldots, T_d)$ be a $1/s$-contraction acting on a Hilbert space $\mathcal{H}$. Define an operator $V_{\boldsymbol{T}} : \mathcal{H} \to H_s \otimes \overline{\mathrm{Ran}} \Delta_{\boldsymbol{T}}$ by*

$$h \mapsto \sum_{\alpha \in \mathbb{Z}_+^d} a_\alpha \boldsymbol{z}^\alpha \otimes \Delta_{\boldsymbol{T}} (\boldsymbol{T}^\alpha)^* h.$$

Then $V_{\boldsymbol{T}}$ is a contraction which satisfies

$$V_{\boldsymbol{T}}^* (p(\boldsymbol{M_z}) \otimes I_{\overline{\mathrm{Ran}\Delta_{\boldsymbol{T}}}}) = p(\boldsymbol{T}) V_{\boldsymbol{T}}^*$$

for all polynomials p in d complex variables.

Note that for a pure $1/s$-contraction $\boldsymbol{T}$, the operator $V_{\boldsymbol{T}}$ defined here is the same as the one defined in Theorem 2.2. Here it is defined for a bigger class of operator tuples, albeit for a smaller class of kernels. By virtue of Corollary 4.2, we can generalize Definition 3.1 for unitarily invariant CNP kernels.

Definition 4.3. A $1/s$-contraction $\boldsymbol{T} = (T_1, \ldots, T_d)$ is said to admit a characteristic function if there exists a Hilbert space $\mathcal{E}$ and a $\mathcal{B}(\mathcal{E}, \overline{\mathrm{Ran}}\Delta_{\boldsymbol{T}})$-valued analytic function $\theta_{\boldsymbol{T}}$ on $\mathbb{B}_d$ such that $M_{\theta_{\boldsymbol{T}}}$ is a multiplication operator from $H_s \otimes \mathcal{E}$ to $H_s \otimes \overline{\mathrm{Ran}}\Delta_{\boldsymbol{T}}$ satisfying

$$I - V_{\boldsymbol{T}}V_{\boldsymbol{T}}^* = M_{\theta_{\boldsymbol{T}}}M_{\theta_{\boldsymbol{T}}}^*.$$

4.2. Taylor spectrum

Let $r\mathbb{B}_d$ be the open ball in $\mathbb{C}^d$ centered at origin of radius r. For $r = 1$, the following lemma is due to Hartz. The proof for any $r \geq 1$ is similar to the proof of [[11], Lemma 2.3].

Lemma 4.4. *Let s be a unitarily invariant CNP kernel defined in* (4.1)*. If the power series $\sum_{n=0}^{\infty} a_n t^n$ has radius of convergence $r \geq 1$, then $s(\boldsymbol{z}, \boldsymbol{w}) \neq 0$ for all $\boldsymbol{z}, \boldsymbol{w} \in \sqrt{r}\mathbb{B}_d$.*

Corollary 4.5. *Let s be a unitarily invariant CNP kernel defined in* (4.1)*. If the power series $\sum_{n=0}^{\infty} a_n t^n$ has radius of convergence $r \geq 1$, then the power series $\sum_{n=1}^{\infty} b_n t^n$ has radius of convergence greater than or equal to r.*

We denote by $\sigma(\boldsymbol{T}) \subset \mathbb{C}^d$ the Taylor spectrum of a commuting d-tuple of bounded operators $\boldsymbol{T} = (T_1, \ldots, T_d)$ on a Hilbert space. For $r = 1$, the following lemma can be found in [[9], Lemma 5.3]. For any $r \geq 1$, it can be proved using the same techniques.

Lemma 4.6. *Let s be a unitarily invariant CNP kernel defined in* (4.1) *and the power series $\sum_{n=0}^{\infty} a_n t^n$ has radius of convergence $r \geq 1$. If $\boldsymbol{T} = (T_1, \ldots, T_d)$ is a $1/s$-contraction, then $\sigma(\boldsymbol{T}) \subseteq \sqrt{r}\mathbb{B}_d$.*

4.3. Functional calculus

Note that for fixed $\boldsymbol{w} \in \mathbb{B}_d$, the series $\sum_{\alpha \in \mathbb{Z}_+^d} a_\alpha \overline{\boldsymbol{w}^\alpha} \boldsymbol{z}^\alpha$ defines an analytic function in a ball centered at origin of radius $\sqrt{r}/\|\boldsymbol{w}\| > \sqrt{r}$. By Corollary 4.5, the series $\sum_{\alpha \in \mathbb{Z}_+^d \setminus \{0\}} b_\alpha \overline{\boldsymbol{w}^\alpha} \boldsymbol{z}^\alpha$ also defines an analytic function in $(\sqrt{r}/\|\boldsymbol{w}\|)\mathbb{B}_d$. Now by [[20], Theorem III.9.9], we get that the two operator series $\sum_{\alpha \in \mathbb{Z}_+^d} a_\alpha \overline{\boldsymbol{w}^\alpha} \boldsymbol{T}^\alpha$ and $\sum_{\alpha \in \mathbb{Z}_+^d \setminus \{0\}} b_\alpha \overline{\boldsymbol{w}^\alpha} \boldsymbol{T}^\alpha$ converge in norm operator topology. By virtue of (1.3), we get

$$\left(\sum_{\alpha \in \mathbb{Z}_+^d} a_\alpha \overline{\boldsymbol{w}^\alpha} \boldsymbol{T}^\alpha \right) \left(I - \sum_{\alpha \in \mathbb{Z}_+^d \setminus \{0\}} b_\alpha \overline{\boldsymbol{w}^\alpha} \boldsymbol{T}^\alpha \right) = I.$$

For $\boldsymbol{w} \in \mathbb{B}_d$, we set the notation:

$$s_{\boldsymbol{w}}(\boldsymbol{T}) = \sum_{\alpha \in \mathbb{Z}_+^d} a_\alpha \overline{\boldsymbol{w}^\alpha} \boldsymbol{T}^\alpha.$$

Since the map $A \mapsto A^*$ is continuous in norm operator topology, we get

$$(s_{\boldsymbol{z}}(\boldsymbol{T}))^* = \sum_{\alpha \in \mathbb{Z}_+^d} a_\alpha \boldsymbol{z}^\alpha (\boldsymbol{T}^\alpha)^*$$

and

$$\left(\sum_{\alpha \in \mathbb{Z}_+^d} a_\alpha \boldsymbol{z}^\alpha (\boldsymbol{T}^\alpha)^* \right) \left(I - \sum_{\alpha \in \mathbb{Z}_+^d \setminus \{0\}} b_\alpha \boldsymbol{z}^\alpha (\boldsymbol{T}^\alpha)^* \right) = I$$

for all $\boldsymbol{z} \in \mathbb{B}_d$. This implies

$$\left(I - \sum_{\alpha \in \mathbb{Z}_+^d \setminus \{0\}} b_\alpha \boldsymbol{z}^\alpha (\boldsymbol{T}^\alpha)^* \right)^{-1} = \left(\sum_{\alpha \in \mathbb{Z}_+^d} a_\alpha \boldsymbol{z}^\alpha (\boldsymbol{T}^\alpha)^* \right) = (s_{\boldsymbol{z}}(\boldsymbol{T}))^* \qquad (4.2)$$

4.4. Construction of the characteristic function

Let $\boldsymbol{T} = (T_1, \dots, T_d)$ be a $1/s$-contraction acting on a Hilbert space $\mathcal{H}$. We denote by

$$\tilde{\mathcal{H}} \stackrel{\text{def}}{=} \oplus_{\alpha \in \mathbb{Z}_+^d \setminus \{0\}} \mathcal{H},$$

the infinite direct sum of the Hilbert space $\mathcal{H}$. For each multi-index $\alpha \in \mathbb{Z}_+^d \setminus \{0\}$, consider the polynomial $\psi_\alpha : \mathbb{B}_d \to \mathbb{C}$ given by

$$\psi_\alpha(\boldsymbol{z}) = (b_\alpha)^{1/2} \boldsymbol{z}^\alpha$$

and define the infinite operator tuple

$$\boldsymbol{Z} = (\psi_\alpha(z) I_{\mathcal{H}})_{\alpha \in \mathbb{Z}_+^d \setminus \{0\}}.$$

We shall also denote by $\boldsymbol{Z}$ the operator from $\tilde{\mathcal{H}}$ to $\mathcal{H}$ which maps $(h_\alpha)_{\alpha \in \mathbb{Z}_+^d \setminus \{0\}}$ to $\sum_{\alpha \in \mathbb{Z}_+^d \setminus \{0\}} (b_\alpha)^{1/2} \boldsymbol{z}^\alpha h_\alpha$. The operator $\boldsymbol{Z}$ is a strict contraction because

$$\|\boldsymbol{Z}\|^2 = \sum_{\alpha \in \mathbb{Z}_+^d \setminus \{0\}} b_\alpha |\boldsymbol{z}^\alpha|^2 = 1 - \frac{1}{s(\boldsymbol{z}, \boldsymbol{z})} < 1.$$

Define by $\tilde{\boldsymbol{T}}$ the infinite operator tuple

$$\tilde{\boldsymbol{T}} = (\psi_\alpha(\boldsymbol{T}))_{\alpha \in \mathbb{Z}_+^d \setminus \{0\}},$$

as well as the operator from $\tilde{\mathcal{H}}$ to $\mathcal{H}$ which maps $(h_\alpha)_{\alpha \in \mathbb{Z}_+^d \setminus \{0\}}$ to $\sum\limits_{\alpha \in \mathbb{Z}_+^d \setminus \{0\}} (b_\alpha)^{1/2} \boldsymbol{T}^\alpha h_\alpha$.

It is easy to check that $\tilde{\boldsymbol{T}}$ is a contraction if and only if $\boldsymbol{T}$ is a $1/s$-contraction. Also,

$$\tilde{\boldsymbol{T}} \tilde{\boldsymbol{T}}^* = \sum_{\alpha \in \mathbb{Z}_+^d \setminus \{0\}} b_\alpha \boldsymbol{T}^\alpha (\boldsymbol{T}^\alpha)^* = I_\mathcal{H} - \Delta_{\boldsymbol{T}}^2$$

and hence $\Delta_{\boldsymbol{T}}^2 = I_\mathcal{H} - \tilde{\boldsymbol{T}} \tilde{\boldsymbol{T}}^*$. Let $D_{\tilde{\boldsymbol{T}}}$ be the unique positive square root of the positive operator $I_{\tilde{H}} - \tilde{\boldsymbol{T}}^* \tilde{\boldsymbol{T}}$, and let $\mathcal{D}_{\tilde{\boldsymbol{T}}} = \overline{\mathrm{Ran}} D_{\tilde{\boldsymbol{T}}}$. By equation (I.3.4) of [17] we obtain the identity

$$\tilde{\boldsymbol{T}} D_{\tilde{\boldsymbol{T}}} = \Delta_{\boldsymbol{T}} \tilde{\boldsymbol{T}}. \tag{4.3}$$

Note that the operator $\boldsymbol{Z} \tilde{\boldsymbol{T}}^*$ is a strict contraction. So, $I_\mathcal{H} - \boldsymbol{Z} \tilde{\boldsymbol{T}}^*$ is invertible.

Definition 4.7. The characteristic function of a $1/s$-contraction $\boldsymbol{T} = (T_1, \ldots, T_d)$ is the analytic operator valued function $\theta_{\boldsymbol{T}} : \mathbb{B}_d \to \mathcal{B}(\mathcal{D}_{\tilde{\boldsymbol{T}}}, \overline{\mathrm{Ran}} \Delta_{\boldsymbol{T}})$ given by

$$\theta_{\boldsymbol{T}}(\boldsymbol{z}) = (-\tilde{\boldsymbol{T}} + \Delta_{\boldsymbol{T}} (I_\mathcal{H} - \boldsymbol{Z} \tilde{\boldsymbol{T}}^*)^{-1} \boldsymbol{Z} D_{\tilde{\boldsymbol{T}}})|_{\mathcal{D}_{\tilde{\boldsymbol{T}}}}. \tag{4.4}$$

The characteristic function $\theta_{\boldsymbol{T}}$ takes values in $\mathcal{B}(\mathcal{D}_{\tilde{\boldsymbol{T}}}, \overline{\mathrm{Ran}} \Delta_{\boldsymbol{T}})$ by virtue of (4.3). Since

$$I - \boldsymbol{Z} \tilde{\boldsymbol{T}}^* = I - \sum_{\alpha \in \mathbb{Z}_+^d \setminus \{0\}} b_\alpha \boldsymbol{z}^\alpha (\boldsymbol{T}^\alpha)^*,$$

we get by (4.2) that

$$(I - \boldsymbol{Z} \tilde{\boldsymbol{T}}^*)^{-1} = (s_{\boldsymbol{z}}(\boldsymbol{T}))^*.$$

Lemma 4.8. *The identity*

$$I - \theta_{\boldsymbol{T}}(\boldsymbol{z}) \theta_{\boldsymbol{T}}(\boldsymbol{w})^* = \frac{1}{s(\boldsymbol{z}, \boldsymbol{w})} \Delta_{\boldsymbol{T}} (s_{\boldsymbol{z}}(\boldsymbol{T}))^* s_{\boldsymbol{w}}(\boldsymbol{T}) \Delta_{\boldsymbol{T}}$$

holds for any $\boldsymbol{z}, \boldsymbol{w} \in \mathbb{B}_d$.

We omit the proof because it is a straightforward computation. In what follows, $V_{\boldsymbol{T}}$ is as in Theorem 2.2.

Lemma 4.9. *The identity*

$$V_{\boldsymbol{T}}^* (s_{\boldsymbol{w}} \otimes \xi) = s_{\boldsymbol{w}}(\boldsymbol{T}) \Delta_{\boldsymbol{T}} \xi$$

holds for any $\boldsymbol{w} \in \mathbb{B}_d$ and $\xi \in \overline{\mathrm{Ran}} \Delta_{\boldsymbol{T}}$.

Proof. Let h be any element of $\mathcal{H}$. Then

$$
\begin{aligned}
\langle V_{\boldsymbol{T}}^*(s_{\boldsymbol{w}} \otimes \xi), h \rangle &= \langle s_{\boldsymbol{w}} \otimes \xi, V_{\boldsymbol{T}} h \rangle \\
&= \langle s_{\boldsymbol{w}} \otimes \xi, \sum_{\alpha \in \mathbb{Z}_+^d} a_\alpha \boldsymbol{z}^\alpha \otimes \Delta_{\boldsymbol{T}}(\boldsymbol{T}^\alpha)^* h \rangle \\
&= \sum_{\alpha \in \mathbb{Z}_+^d} a_\alpha \overline{\boldsymbol{w}^\alpha} \langle \xi, \Delta_{\boldsymbol{T}}(\boldsymbol{T}^\alpha)^* h \rangle \\
&= \sum_{\alpha \in \mathbb{Z}_+^d} a_\alpha \overline{\boldsymbol{w}^\alpha} \langle \boldsymbol{T}^\alpha \Delta_{\boldsymbol{T}} \xi, h \rangle \\
&= \langle \sum_{\alpha \in \mathbb{Z}_+^d} a_\alpha \overline{\boldsymbol{w}^\alpha} \boldsymbol{T}^\alpha \Delta_{\boldsymbol{T}} \xi, h \rangle = \langle s_{\boldsymbol{w}}(\boldsymbol{T}) \Delta_{\boldsymbol{T}} \xi, h \rangle.
\end{aligned}
$$

Since h is arbitrary, we have the desired identity. □

Using Lemma 4.8 and Lemma 4.9, it is straightforward to see the following.

Corollary 4.10. *The identity*

$$
\langle V_{\boldsymbol{T}}^*(s_{\boldsymbol{w}} \otimes \xi), V_{\boldsymbol{T}}^*(s_{\boldsymbol{z}} \otimes \eta) \rangle = s(\boldsymbol{z}, \boldsymbol{w}) \langle (I - \theta_{\boldsymbol{T}}(\boldsymbol{z}) \theta_{\boldsymbol{T}}(\boldsymbol{w})^*) \xi, \eta \rangle
$$

holds for any $\boldsymbol{z}, \boldsymbol{w} \in \mathbb{B}_d$ *and* $\xi, \eta \in \overline{\mathrm{Ran}} \Delta_{\boldsymbol{T}}$.

The main result of this section is the next theorem which shows that for a $1/s$-contraction $\boldsymbol{T}$, the characteristic function $\theta_{\boldsymbol{T}}$ works as the multiplier required in Definition 4.3.

Theorem 4.11. *Given a $1/s$-contraction $\boldsymbol{T} = (T_1, \ldots, T_d)$, its characteristic function $\theta_{\boldsymbol{T}}$ (defined in (4.4)) is a multiplier from $H_s \otimes \mathcal{D}_{\tilde{\boldsymbol{T}}}$ to $H_s \otimes \overline{\mathrm{Ran}} \Delta_{\boldsymbol{T}}$ with $\|M_{\theta_{\boldsymbol{T}}}\| \leq 1$. Moreover the identity*

$$
V_{\boldsymbol{T}} V_{\boldsymbol{T}}^* + M_{\theta_{\boldsymbol{T}}} M_{\theta_{\boldsymbol{T}}}^* = I \tag{4.5}
$$

holds.

Proof. Define a linear map

$$
A : \mathrm{span}\{s_{\boldsymbol{w}} \otimes \xi : \boldsymbol{w} \in \mathbb{B}_d, \xi \in \overline{\mathrm{Ran}} \Delta_{\boldsymbol{T}}\} \to H_s \otimes \mathcal{D}_{\tilde{\boldsymbol{T}}}
$$

by

$$
A(s_{\boldsymbol{w}} \otimes \xi) = s_{\boldsymbol{w}} \otimes \theta_{\boldsymbol{T}}(\boldsymbol{w})^* \xi.
$$

For $\boldsymbol{z}, \boldsymbol{w} \in \mathbb{B}_d$ and $\xi, \eta \in \mathcal{E}$ we get

$$
\begin{aligned}
\langle A(s_{\boldsymbol{w}} \otimes \xi), A(s_{\boldsymbol{z}} \otimes \eta) \rangle &= \langle s_{\boldsymbol{w}} \otimes \theta_{\boldsymbol{T}}(\boldsymbol{w})^* \xi, s_{\boldsymbol{z}} \otimes \theta_{\boldsymbol{T}}(\boldsymbol{z})^* \eta \rangle \\
&= s(\boldsymbol{z}, \boldsymbol{w}) \langle \theta_{\boldsymbol{T}}(\boldsymbol{z}) \theta_{\boldsymbol{T}}(\boldsymbol{w})^* \xi, \eta \rangle \\
&= \langle (I - V_{\boldsymbol{T}} V_{\boldsymbol{T}}^*)(s_{\boldsymbol{w}} \otimes \xi), s_{\boldsymbol{z}} \otimes \eta \rangle \qquad (4.6)
\end{aligned}
$$

by Corollary 4.10. This implies that

$$
\|Ax\| \le \|x\|
$$

for all $x \in \operatorname{span}\{s_{\boldsymbol{w}} \otimes \xi : \boldsymbol{w} \in \mathbb{B}_d, \xi \in \overline{\operatorname{Ran}} \Delta_{\boldsymbol{T}}\}$. Thus, A extends to be a bounded linear operator from $H_s \otimes \overline{\operatorname{Ran}} \Delta_{\boldsymbol{T}}$ to $H_s \otimes \mathcal{D}_{\tilde{\boldsymbol{T}}}$. Now we shall prove that $A^* = M_{\theta_{\boldsymbol{T}}}$. For $f \in H_s \otimes \mathcal{D}_{\tilde{\boldsymbol{T}}}, \xi \in \overline{\operatorname{Ran}} \Delta_{\boldsymbol{T}}$ and $\boldsymbol{z} \in \mathbb{B}_d$, we have

$$
\langle (A^* f)(\boldsymbol{z}), \xi \rangle = \langle A^* f, s_{\boldsymbol{z}} \otimes \xi \rangle = \langle f, A(s_{\boldsymbol{z}} \otimes \xi) \rangle = \langle f, s_{\boldsymbol{z}} \otimes \theta_{\boldsymbol{T}}(\boldsymbol{z})^* \xi \rangle = \langle \theta_{\boldsymbol{T}}(\boldsymbol{z}) f(\boldsymbol{z}), \xi \rangle.
$$

So we get $(A^* f)(\boldsymbol{z}) = \theta_{\boldsymbol{T}}(\boldsymbol{z}) f(\boldsymbol{z})$. This implies that $A^* = M_{\theta_{\boldsymbol{T}}}$. The identity

$$
M_{\theta_{\boldsymbol{T}}} M_{\theta_{\boldsymbol{T}}}^* + V_{\boldsymbol{T}} V_{\boldsymbol{T}}^* = I_{H_k \otimes \mathcal{E}}
$$

follows from (4.6). $\square$

Corollary 4.12. *Given a $1/s$-contraction $\boldsymbol{T}$, its characteristic function $\theta_{\boldsymbol{T}}$ is a bounded analytic function on $\mathbb{B}_d$ with $\sup\limits_{\boldsymbol{z} \in \mathbb{B}_d} \|\theta_{\boldsymbol{T}}(z)\| \le 1$.*

Epilogue

In this note, we have proved the existence of characteristic function for a certain class, but did not show any utility. This will be the theme of a future paper. However, the principal utility of the characteristic function in the *pure* case is a straightforward consequence of what we have developed so far.

Definition. Given two $1/s$-contractions $\boldsymbol{T}$ and $\boldsymbol{R}$ on Hilbert spaces $\mathcal{H}$ and $\mathcal{K}$ respectively, the characteristic functions of $\boldsymbol{T}$ and $\boldsymbol{R}$ are said to coincide if there exist unitary operators $\tau : \mathcal{D}_{\tilde{\boldsymbol{T}}} \to \mathcal{D}_{\tilde{\boldsymbol{R}}}$ and $\tau_* : \overline{\operatorname{Ran}} \Delta_{\boldsymbol{T}} \to \overline{\operatorname{Ran}} \Delta_{\boldsymbol{R}}$ such that the following diagram commutes for all $\boldsymbol{z} \in \mathbb{B}_d$:

$$
\begin{array}{ccc}
\mathcal{D}_{\tilde{\boldsymbol{T}}} & \xrightarrow{\ \theta_T(z)\ } & \overline{\operatorname{Ran}} \Delta_{\boldsymbol{T}} \\
{\scriptstyle \tau} \downarrow & & \downarrow {\scriptstyle \tau_*} \\
\mathcal{D}_{\tilde{\boldsymbol{R}}} & \xrightarrow[\ \theta_R(z)\ ]{} & \overline{\operatorname{Ran}} \Delta_{\boldsymbol{R}}
\end{array}
$$

The characteristic functions of two unitarily equivalent $1/s$-contractions clearly coincide. It is somewhat of a surprise that the converse is true for at least pure $1/s$-contractions. This is achieved through the construction of a functional model. Using the isometry $V_{\boldsymbol{T}}$ as well as the identity (4.5), we get that every pure $1/s$-contraction $\boldsymbol{T} = (T_1, \ldots, T_d)$ acting on a Hilbert space $\mathcal{H}$ is unitarily equivalent to the commuting tuple $\mathbb{T} = (\mathbb{T}_1, \ldots, \mathbb{T}_d)$ on the functional space

$$\mathbb{H}_{\boldsymbol{T}} = (H_s \otimes \overline{\mathrm{Ran}}\Delta_{\boldsymbol{T}}) \ominus M_{\theta_{\boldsymbol{T}}}(H_s \otimes \mathcal{D}_{\tilde{\boldsymbol{T}}})$$

defined by $\mathbb{T}_i = P_{\mathbb{H}_{\boldsymbol{T}}}(M_{z_i} \otimes I_{\mathcal{D}_{\tilde{\boldsymbol{T}}^*}})|_{\mathbb{H}_{\boldsymbol{T}}}$ for $1 \leq i \leq d$. This functional model produces the following theorem whose proof is along the lines of the contents of [8].

Theorem. Two pure $1/s$-contractions are unitarily equivalent if and only if their characteristic functions coincide.

Acknowledgments

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