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Hypergeometric Functions of Type BC and Standard Multiplicities

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We study the Heckman–Opdam hypergeometric functions associated with a root system of type BC and a multiplicity function that is allowed to assume some nonpositive values (a standard multiplicity function). For such functions, we obtain positivity properties and sharp estimates that imply a characterization of the bounded hypergeometric functions. As an application, our results extend the known properties of Harish-Chandra's spherical functions on Riemannian symmetric spaces of the non-compact type G/K to spherical functions over homogeneous vector bundles on G/K, which are associated with certain small K-types.

1 Introduction

Let G be a connected non-compact real semisimple Lie group with finite center, K, a maximal compact subgroup of G, and $X = G/K$, the corresponding Riemannian symmetric space of the non-compact type. Harish-Chandra's theory of spherical functions on X has been extended into two different directions.

The 1st direction is Heckman–Opdam's theory of hypergeometric functions on root systems. It originated from the fact that, by restriction to a maximally flat subspace of X, the K-invariant functions on X become Weyl group-invariant functions

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on the corresponding Cartan subspace $\mathfrak{a}$. The analogues of Harish-Chandra's spherical functions are the hypergeometric functions associated with root systems, also known as the Heckman–Opdam's hypergeometric functions. Heckman and Opdam introduced them in the late 1980s and, in the middle 1990s, Cherednik contributed in simplifying their original definition; see [11, 21], as well as [1], for a recent overview of the theory. In the context of special functions associated with root systems, the symmetric space X is replaced by a triple $(\mathfrak{a}, \Sigma, m)$, where $\mathfrak{a}$ is a Euclidean real vector space (playing the role of the Cartan subspace of X), Σ is a root system in the real dual space $\mathfrak{a}^*$ of $\mathfrak{a}$, and $m : \Sigma \to \mathbb{C}$ is a multiplicity function, that is, a function on Σ that is invariant with respect to the Weyl group W of Σ. If $\mathfrak{a}$ is a Cartan subspace of the symmetric space X, Σ is the corresponding system of (restricted) roots and the multiplicity function m is given by the dimensions of the root spaces, then Heckman–Opdam's hypergeometric functions associated with $(\mathfrak{a}, \Sigma, m)$ are precisely the restrictions to $\mathfrak{a}$ of Harish-Chandra's spherical functions. Here and in the following, we adopt the convention of identifying the Cartan subspace $\mathfrak{a}$ with its diffeomorphic image $A = \exp(\mathfrak{a})$ inside G.

Notice that, even if the multiplicity function in a triple $(\mathfrak{a}, \Sigma, m)$ might be generally complex-valued, most of the properties of Heckman–Opdam's hypergeometric functions, and also the harmonic analysis associated with them, are known assuming that the multiplicity functions have values in $[0, +\infty)$. This is of course the most natural setting to include Harish-Chandra's spherical harmonic analysis on Riemannian symmetric spaces.

The 2nd direction extending Harish-Chandra's theory of spherical functions concerns the so-called τ-spherical functions, where τ is a unitary irreducible representation of K. These functions already appeared in the work of Godement [7] and Harish-Chandra [8, 9], and they agree with Harish-Chandra's spherical functions on X when τ is the trivial representation. They have been studied either in the context of the representation theory of G or in relation to the harmonic analysis on homogeneous vector bundles over X. Among the references related to the present paper, we mention [3, 6, 10, 11, 15, 18, 19, 23, 24, 29, 30]. Several new features appear when τ is nontrivial. For instance, the algebra $\mathbb{D}(G/K;\tau)$ of invariant differential operators acting on the sections of the homogeneous vector bundle might not be commutative. Let V_τ be the space of τ, and let $L^1(G//K;\tau)$ denote the convolution algebra of $\mathrm{End}(V_\tau)$-valued functions f on G satisfying $f(k_1 g k_2) = \tau(k_1^{-1}) f(g) \tau(k_2^{-1})$ for all $g \in G$ and $k_1, k_2 \in K$. It is a classical result that $\mathbb{D}(G/K;\tau)$ is commutative if and only if $L^1(G//K;\tau)$ is commutative. In this case, (G, K, τ) is said to be a Gelfand triple. A convenient criterion to check whether (G, K, τ)

is a Gelfand triple is due to Deitmar [5]. Namely, $\mathbb{D}(G/K;\tau)$ is commutative if and only if the restriction of τ to M, the centralizer in K of the Cartan subspace $\mathfrak{a}$, is multiplicity free. This happens for instance when $\tau|_M$ is irreducible, that is, τ is a small K-type. For example, a one-dimensional representation τ is necessarily a small K-type. For Gelfand triples, the theory of τ-spherical functions can be set up exactly as in the case of τ trivial. For instance, the τ-spherical functions can be equivalently defined either as joint eigenfunctions of $\mathbb{D}(G/K;\tau)$ or as characters of $L^1(G//K;\tau)$. Nevertheless, they are much more difficult to handle than in the trivial case and many of their properties are still not known.

The present paper is situated at a crossing of the two directions of extensions of Harish-Chandra's theory of spherical functions on X mentioned above. It finds its motivations in Shimeno's [29] paper, in Heckman's chapter [11, Chapter 5], and, more generally, in the recent paper by Oda and Shimeno [18] on τ-spherical functions corresponding to small K-types. When τ is a small K-type, a τ-spherical function is uniquely determined by its restriction to a Cartan subspace $\mathfrak{a}$ of G. By Schur's lemma, this restriction is of the form $\varphi \cdot \mathrm{id}$ where id is the identity on V_τ and φ is function on $\mathfrak{a}$, which is Weyl group invariant. The main theorem of [18] proves that, up to multiplication by an explicit nonvanishing smooth cosh-like factor, the function φ is a Heckman–Opdam's hypergeometric function. It corresponds to a triple $(\mathfrak{a}, \Sigma(\tau), m(\tau))$ in which $\Sigma(\tau)$ is possibly not the root system associated with the symmetric space X and $m(\tau)$ need not be positive. This motivates a systematic study of Heckman–Opdam's hypergeometric functions corresponding to multiplicity functions that may assume negative values.

In this paper, we take up this line of investigations in the case of the so-called standard multiplicities. More precisely, let Σ be a root system in $\mathfrak{a}^*$ of type BC_r, where r is the dimension of $\mathfrak{a}$, and let $m = (m_{\mathrm{s}}, m_{\mathrm{m}}, m_{\mathrm{l}})$ be a multiplicity function on Σ. Set

$$\mathcal{M}_1 = \{(m_{\mathrm{s}}, m_{\mathrm{m}}, m_{\mathrm{l}}) \in \mathcal{M} : m_{\mathrm{m}} > 0, m_{\mathrm{s}} > 0, m_{\mathrm{s}} + 2m_{\mathrm{l}} > 0\}.$$

In [11, Definition 5.5.1], the elements of $\mathcal{M}_1$ were called the *standard multiplicities*. Standard multiplicities have been introduced by Heckman and their definition is linked to the regularity of Harish-Chandra's c-function; see [11, Lemma 5.5.2] and (26) below.

All positive multiplicity functions are standard, but standard multiplicities also allow negative values for the long roots. Because of this, the arguments leading to the known properties of the hypergeometric functions corresponding to positive multiplicity functions do not apply to this case. By suitable modifications of their proofs, we show in this paper that various results, including positivity properties and estimates for hypergeometric functions, extend to standard multiplicities (see Proposition 3.5 and Theorems 3.10 and 3.11).

Let $\mathfrak{a}_{\mathbb{C}}^*$ denote the complex dual vector space of $\mathfrak{a}$, and consider the Heckman–Opdam's (symmetric and nonsymmetric) hypergeometric functions $F_\lambda(m,x)$ and $G_\lambda(m,x)$ with $\lambda \in \mathfrak{a}_{\mathbb{C}}^*$; see Subsection 2.2 for the definitions. Under the condition that $m \in \mathcal{M}_1$, we prove the following:

(1) F_λ and G_λ are real and strictly positive on $\mathfrak{a}$ for all $\lambda \in \mathfrak{a}^*$;
(2) $|F_\lambda| \leq F_{\mathrm{Re}\,\lambda}$ and $|G_\lambda| \leq G_{\mathrm{Re}\,\lambda}$ on $\mathfrak{a}$ for all $\lambda \in \mathfrak{a}_{\mathbb{C}}^*$;
(3) $\max\{|F_\lambda(x)|, |G_\lambda(x)|\} \leq \sqrt{|W|}e^{\max_{w\in W}(w\lambda)(x)}$ for all $\lambda \in \mathfrak{a}_{\mathbb{C}}^*$ and $x \in \mathfrak{a}$;
(4) $F_{\lambda+\mu}(x) \leq F_\mu(x)e^{\max_{w\in W}(w\lambda)(x)}$ and $G_{\lambda+\mu}(x) \leq G_\mu(x)e^{\max_{w\in W}(w\lambda)(x)}$ for all $\lambda \in \mathfrak{a}^*$, $\mu \in \overline{(\mathfrak{a}^*)^+}$, and $x \in \mathfrak{a}$;
(5) asymptotics on $\mathfrak{a}$ for F_λ when $\lambda \in \mathfrak{a}_{\mathbb{C}}^*$ is fixed but not necessarily regular;
(6) sharp estimates on $\mathfrak{a}$ for F_λ when $\lambda \in \mathfrak{a}^*$ is fixed but not necessarily regular.

The estimates (but not the asymptotics) pass by continuity to the boundary of $\mathcal{M}_1$ as well. The above properties extend to $\mathcal{M}_1$ the corresponding properties proved for nonnegative multiplicities by Opdam [21], Ho and Ólafsson [14], Schapira [26], Rösler *et al.* [25], and the authors and Pusti [16].

The set of real-valued multiplicities on which both functions F_λ and G_λ are naturally defined is

$$\mathcal{M}_0 = \{(m_{\mathrm{s}}, m_{\mathrm{m}}, m_{\mathrm{l}}) \in \mathcal{M} : m_{\mathrm{m}} \geq 0, m_{\mathrm{s}} + m_{\mathrm{l}} \geq 0\}.$$

Clearly, $\mathcal{M}_1 \subset \mathcal{M}_0$. In Section 3, we introduce and discuss other subsets of $\mathcal{M}$. For instance, the estimates (3) hold in fact for a larger set of multiplicities than $\mathcal{M}_1$; see Lemma 3.4.

Let $\rho(m) \in \mathfrak{a}^*$ be the half-sum of the positive roots in Σ, counted with their multiplicities; see (2). Moreover, let $C(\rho(m))$ denote the convex hull of the finite set $\{w\rho(m) : w \in W\}$. Suppose first that the triple $(\mathfrak{a}, \Sigma, m)$ is associated with a symmetric space X, and consider Harish-Chandra's parametrization of the spherical functions by the elements of $\mathfrak{a}_{\mathbb{C}}^*$. Let φ_λ denote the spherical function corresponding to $\lambda \in \mathfrak{a}_{\mathbb{C}}^*$. The spherical functions that are bounded can be identified with the elements of the spectrum of the (commutative) convolution algebra of L^1 K-invariant functions on X. They have been determined by the celebrated theorem of Helgason and Johnson [13]. It states that the spherical function φ_λ on X is bounded if and only if λ belongs to the tube domain in $\mathfrak{a}_{\mathbb{C}}^*$ over $C(\rho)$. It is a fundamental result in the L^1 spherical harmonic analysis on X. For instance, it implies that the spherical transform of a K-invariant L^1 function on X is holomorphic in the interior of the tube domain over $C(\rho)$. In particular, the spherical transform of an L^1 function cannot have compact support, unlike what happens for the classical Fourier transform.

The theorem of Helgason and Johnson has been recently extended to Heckman–Opdam's hypergeometric functions in [16] under the assumption that the multiplicity function m is positive. On the boundary of $\mathcal{M}_1$, there are nonzero multiplicities m for which $\rho(m) = 0$. However, $\rho(m) \neq 0$ for $m \in \mathcal{M}_1$. It is therefore natural to ask if Helgason–Johnson's characterization of boundedness by means of the tube domain over $C(\rho)$ holds for Heckman–Opdam's hypergeometric functions associated with arbitrary standard multiplicity functions $m \in \mathcal{M}_1$ and not only to the positive ones. The answer to this question is positive and is given by Theorem 3.11.

Let $\mathcal{M}_+$ denote the set of nonnegative multiplicity functions on a fixed root system of type BC. In Section 4, we consider the two-parameter deformations of $m = (m_s, m_m, m_l) \in \mathcal{M}_+$ of the form

$$m(\ell, \widetilde{\ell}) = (m_s + 2\ell, m_m + 2\widetilde{\ell}, m_l - 2\ell)\,,$$

where $\ell, \widetilde{\ell} \in \mathbb{R}$. The corresponding nonsymmetric and symmetric $(\ell, \widetilde{\ell})$-hypergeometric functions are respectively defined for $\lambda \in \mathfrak{a}_{\mathbb{C}}^*$ and $x \in \mathfrak{a}$ by

$$\begin{aligned} G_{\ell,\widetilde{\ell},\lambda}(m; x) &= u(x)^{-\ell} v(x)^{-\widetilde{\ell}} G_\lambda(m(\ell, \widetilde{\ell}); x), \\ F_{\ell,\widetilde{\ell},\lambda}(m; x) &= u(x)^{-\ell} v(x)^{-\widetilde{\ell}} F_\lambda(m(\ell, \widetilde{\ell}); x), \end{aligned}$$

where $u(x)$ and $v(x)$ are suitable products of hyperbolic cosine functions depending on the roots; see (35) and (36). Given $m = (m_s, m_m, m_l) \in \mathcal{M}_+$, then $m(\ell, \widetilde{\ell}) \in \mathcal{M}_1$ if and only if $-\frac{m_s}{2} < \ell < \frac{m_s}{2} + m_l$ and $\widetilde{\ell} > -\frac{m_m}{2}$. For these values, the results proved in Section 3 for $G_\lambda(m(\ell, \widetilde{\ell}))$ and $F_\lambda(m(\ell, \widetilde{\ell}))$ extend to corresponding results for the $(\ell, \widetilde{\ell})$-hypergeometric functions, but some care is needed to take the factors $u^{-\ell}$ and $v^{-\widetilde{\ell}}$ into account. In particular, Theorem 4.7 characterizes the $F_{\ell,\widetilde{\ell},\lambda}$s that are bounded. This theorem as well as other properties extends to all ℓs for which $\left|\ell - (m_l - 1)/2\right| < (m_s + m_l + 1)/2$, thanks to the symmetry relation $F_{\ell,\widetilde{\ell},\lambda} = F_{-\ell+m_l-1,\widetilde{\ell},\lambda}$ proved in (44).

The final section of this paper, Section 5, is devoted to geometric examples based on [11, 29] and [18]. Namely, we prove that the τ-spherical functions on G/K associated with a small K-type τ and for which the root system of G/K is of type BC

can be described as symmetric $(\ell, \widetilde{\ell})$-hypergeometric functions on the root system $\Sigma(\tau)$ of [18] and specific choices of a multiplicity function $m \in \mathcal{M}_+$ and of the deformation parameters $(\ell, \widetilde{\ell})$. Our general results from Section 4 apply therefore to these cases.

2 Notation and Preliminaries

In this section, we collect the basic notation and some preliminary results. We refer to [11] and [16] for a more extended exposition.

If $F(m)$ is a function on a space X that depends on a parameter m, we will denote its value at $x \in X$ by $F(m; x)$ rather than $F(m)(x)$. Given two nonnegative functions f and g on a same domain D, we write $f \asymp g$ if there exist positive constants C_1 and C_2 so that $C_1 \leq \frac{f(x)}{g(x)} \leq C_2$ for all $x \in D$.

2.1 Root systems

Let $\mathfrak{a}$ be an r-dimensional real Euclidean vector space with inner product $\langle \cdot, \cdot \rangle$, and let $\mathfrak{a}^*$ be the dual space of $\mathfrak{a}$. For $\lambda \in \mathfrak{a}^*$, let $x_\lambda \in \mathfrak{a}$ be determined by the condition that $\lambda(x) = \langle x, x_\lambda \rangle$ for all $x \in \mathfrak{a}$. The assignment $\langle \lambda, \mu \rangle := \langle x_\lambda, x_\mu \rangle$ defines an inner product in $\mathfrak{a}^*$. Let $\mathfrak{a}_\mathbb{C}$ and $\mathfrak{a}_\mathbb{C}^*$ respectively denote the complexifications of $\mathfrak{a}$ and $\mathfrak{a}^*$. The $\mathbb{C}$-bilinear extension to $\mathfrak{a}_\mathbb{C}$ and $\mathfrak{a}_\mathbb{C}^*$ of the inner products on $\mathfrak{a}^*$ and $\mathfrak{a}$ will also be indicated by $\langle \cdot, \cdot \rangle$. We denote by $|\cdot| = \langle \cdot, \cdot \rangle^{1/2}$ the associated norm. We shall often employ the notation

$$\lambda_\alpha := \frac{\langle \lambda, \alpha \rangle}{\langle \alpha, \alpha \rangle} \tag{1}$$

for elements $\lambda, \alpha \in \mathfrak{a}_\mathbb{C}^*$ with $|\alpha| \neq 0$.

Let Σ be a root system in $\mathfrak{a}^*$ with set of positive roots of the form $\Sigma^+ = \Sigma_{\mathrm{s}}^+ \sqcup \Sigma_{\mathrm{m}}^+ \sqcup \Sigma_{\mathrm{l}}^+$, where

$$\Sigma_s^+ = \Big\{\frac{\beta_j}{2} : 1 \leq j \leq r\Big\}, \quad \Sigma_m^+ = \Big\{\frac{\beta_j \pm \beta_i}{2} : 1 \leq i < j \leq r\Big\}, \quad \Sigma_l^+ = \{\beta_j : 1 \leq j \leq r\}.$$

The positive Weyl chamber $\mathfrak{a}^+$ consists of the elements $x \in \mathfrak{a}$ for which $\alpha(x) > 0$ for all $\overline{\mathfrak{a}^+} = \{x \in \mathfrak{a} : \alpha(x) \geq 0 \text{ for all } \alpha \in \Sigma^+\}$. Dually, the positive Weyl chamber $(\mathfrak{a}^*)^+$ consists of the elements $\lambda \in \mathfrak{a}^*$ for which $\langle \lambda, \alpha \rangle > 0$ for all $\alpha \in \Sigma^+$. Its closure is denoted $\overline{(\mathfrak{a}^*)^+}$.

We assume that the elements of Σ_{l}^+ form an orthogonal basis of $\mathfrak{a}^*$ and that they all have the same norm p. We denote by W the Weyl group of Σ. It acts on the roots by permutations and sign changes. For $a \in \{\mathrm{s}, \mathrm{m}, \mathrm{l}\}$, set $\Sigma_a = \Sigma_a^+ \sqcup (-\Sigma_a^+)$. A multiplicity function on Σ is a W-invariant function on Σ. It is therefore given by a triple $m =$

$(m_{\mathrm{s}}, m_{\mathrm{m}}, m_{\mathrm{l}})$ of complex numbers so that m_a is the (constant) value of m on Σ_a for $a \in \{\mathrm{s}, \mathrm{m}, \mathrm{l}\}$.

It will be convenient to refer to a root system Σ as above as of type BC_r even if some values of m are zero. This means that root systems of type C_r will be considered as being of type BC_r with $m_{\mathrm{s}} = 0$ and $m_{\mathrm{m}} \neq 0$. Likewise, the direct products of rank-one root systems $(BC_1)^r$ and $(A_1)^r$ will be considered of type BC_r with $m_{\mathrm{m}} = 0$ and with $m_{\mathrm{m}} = m_{\mathrm{s}} = 0$, respectively.

The dimension r of $\mathfrak{a}$ is called the *(real) rank* of the triple $(\mathfrak{a}, \Sigma, m)$. Finally, we set

$$\rho(m) = \frac{1}{2} \sum_{\alpha \in \Sigma^+} m_\alpha \alpha = \frac{1}{2} \sum_{j=1}^{r} \left(\frac{m_{\mathrm{s}}}{2} + m_{\mathrm{l}} + (j-1) m_{\mathrm{m}} \right) \beta_j \in \mathfrak{a}^* . \tag{2}$$

Example 2.1 (Geometric multiplicities). For special values of the multiplicities $m = (m_{\mathrm{s}}, m_{\mathrm{m}}, m_{\mathrm{l}})$, triples $(\mathfrak{a}, \Sigma, m)$ as above appear as restricted root systems of a large family of irreducible symmetric spaces G/K of the non-compact type. Among these spaces, a remarkable class consists of the non-compact Hermitian symmetric spaces; see [12] or [29]. For these spaces $m_{\mathrm{l}} = 1$. Their root systems and multiplicity functions are listed in Table 1. The literature on Hermitian symmetric spaces refers to the situations where $\Sigma = C_r$ and $\Sigma = BC_r$ as to *Cases I* and *II*, respectively.

In the following, the triples $m = (m_{\mathrm{l}}, m_{\mathrm{m}}, m_{\mathrm{s}})$ appearing as root multiplicities of Riemannian symmetric spaces of the non-compact type and root system of type BC will be called *geometric multiplicities*. Other examples of geometric multiplicities will be considered in Section 5.

Notice that in this paper, we adopt the notation commonly used in the theory of symmetric spaces. It differs from the notation in the work of Heckman and Opdam in the following ways. The root system R and the multiplicity function k used by Heckman and Opdam are related to our Σ and m by the relations $R = \{2\alpha : \alpha \in \Sigma\}$ and $k_{2\alpha} = m_\alpha/2$ for $\alpha \in \Sigma$.

2.2 Cherednik operators and hypergeometric functions

In this subsection, we review some basic notions on the hypergeometric functions associated with root systems. This theory has been developed by Heckman, Opdam, and Cherednik. We refer the reader to [11], [21], and [22] for more details and further references.

Table 1 Root multiplicities of irreducible non-compact Hermitian symmetric spaces

G	K	Σ	$(m_{\mathrm{s}}, m_{\mathrm{m}}, m_{\mathrm{l}})$
$\mathrm{SU}(p,q)$	$\mathrm{S}(\mathrm{U}(p)\times\mathrm{U}(q))$	$p=q: C_p$	$p=q: (0,2,1)$
		$p<q: BC_p$	$p<q: (2(q-p),2,1)$
$\mathrm{SO}_0(p,2)$	$\mathrm{SO}(p)\times\mathrm{SO}(2)$	C_2	$(0,p-2,1)$
$\mathrm{SO}^*(2n)$	$\mathrm{U}(n)$	$n$even: C_n	$n$even: $(0,4,1)$
		n odd: BC_n	n odd: $(4,4,1)$
$\mathrm{Sp}(n,\mathbb{R})$	$\mathrm{U}(n)$	C_n	$(0,1,1)$
$\mathfrak{e}_{6(-14)}$	$\mathrm{Spin}(10)\times\mathrm{U}(1)$	BC_2	$(8,6,1)$
$\mathfrak{e}_{7(-25)}$	$\mathrm{ad}(\mathfrak{e}_6)\times\mathrm{U}(1)$	C_3	$(0,8,1)$

Let $\mathrm{S}(\mathfrak{a}_{\mathbb{C}})$ denote the symmetric algebra over $\mathfrak{a}_{\mathbb{C}}$ considered as the space of polynomial functions on $\mathfrak{a}_{\mathbb{C}}^*$, and let $\mathrm{S}(\mathfrak{a}_{\mathbb{C}})^W$ be the subalgebra of W-invariant elements. Every $p \in \mathrm{S}(\mathfrak{a}_{\mathbb{C}})$ defines a constant-coefficient differential operators $p(\partial)$ on $A_{\mathbb{C}}$ and on $\mathfrak{a}_{\mathbb{C}}$ such that $\xi(\partial) = \partial_\xi$ is the directional derivative in the direction of ξ for all $\xi \in \mathfrak{a}$. The algebra of the differential operators $p(\partial)$ with $p \in \mathrm{S}(\mathfrak{a}_{\mathbb{C}})$ will also be indicated by $\mathrm{S}(\mathfrak{a}_{\mathbb{C}})$.

Set $\mathfrak{a}_{\mathrm{reg}} = \{x \in \mathfrak{a} : \alpha(x) \neq 0 \text{ for all } \alpha \in \Sigma\}$. Let $\mathcal{R}$ denote the algebra of functions on $\mathfrak{a}_{\mathrm{reg}}$ generated by 1 and

$$g_\alpha = (1-e^{-2\alpha})^{-1} \qquad (\alpha \in \Sigma^+)\,. \tag{3}$$

Notice that for $\xi \in \mathfrak{a}$ and $\alpha \in \Sigma^+$, we have

$$\begin{aligned} \partial_\xi g_\alpha &= 2\alpha(\xi)\big(g_\alpha^2 - g_\alpha\big) \\ g_{-\alpha} &= 1 - g_\alpha\,. \end{aligned} \tag{4}$$

Hence, $\mathcal{R}$ is stable under the actions of $S(\mathfrak{a}_{\mathbb{C}})$ and of the Weyl group. Let $\mathbb{D}_{\mathcal{R}} = \mathcal{R} \otimes S(\mathfrak{a}_{\mathbb{C}})$ be the algebra of differential operators on $\mathfrak{a}_{\mathrm{reg}}$ with coefficients in $\mathcal{R}$. The Weyl group W acts on $\mathbb{D}_{\mathcal{R}}$ according to

$$w\big(\varphi \otimes p(\partial)\big) := w\varphi \otimes (wp)(\partial).$$

We indicate by $\mathbb{D}_{\mathcal{R}}^W$ the subalgebra of W-invariant elements of $\mathbb{D}_{\mathcal{R}}$. The space $\mathbb{D}_{\mathcal{R}} \otimes \mathbb{C}[W]$ can be naturally endowed with the structure of an associative algebra.

The differential component of a differential-reflection operator $P \in \mathbb{D}_{\mathcal{R}} \otimes \mathbb{C}[W]$ is the differential operator $\beta(P) \in \mathbb{D}_{\mathcal{R}}$ such that

$$Pf = \beta(P)f \tag{5}$$

for all $f \in \mathbb{C}[A_{\mathbb{C}}]^W$. If $P = \sum_{w \in W} P_w \otimes w$ with $P_w \in \mathbb{D}_{\mathcal{R}}$, then $\beta(P) = \sum_{w \in W} P_w$.

For $\xi \in \mathfrak{a}$ the Cherednik operator (or Dunkl–Cherednik operator) $T_\xi(m) \in \mathbb{D}_{\mathcal{R}} \otimes \mathbb{C}[W]$ is defined by

$$T_\xi(m) := \partial_\xi - \rho(m)(\xi) + \sum_{\alpha \in \Sigma^+} m_\alpha \alpha(\xi)(1 - e^{-2\alpha})^{-1} \otimes (1 - r_\alpha). \tag{6}$$

The Cherednik operators commute with each other (cf. [21, Section 2]). Therefore, the map $\xi \to T_\xi(m)$ on $\mathfrak{a}$ extends uniquely to an algebra homomorphism $p \to T_p(m)$ of $\mathrm{S}(\mathfrak{a}_{\mathbb{C}})$ into $\mathbb{D}_{\mathcal{R}} \otimes \mathbb{C}[W]$. If $p \in \mathrm{S}(\mathfrak{a}_{\mathbb{C}})^W$, then $D_p(m) := \beta\big(T_p(m)\big)$ turns out to be in $\mathbb{D}_{\mathcal{R}}^W$.

Let $p_L \in \mathrm{S}(\mathfrak{a}_{\mathbb{C}})^W$ be defined by $p_L(\lambda) := \langle \lambda, \lambda \rangle$ for $\lambda \in \mathfrak{a}_{\mathbb{C}}^*$. Then, $T_{p_L}(m) = \sum_{j=1}^r T_{\xi_j}(m)^2$, where $\{\xi_j\}_{j=1}^r$ is any orthonormal basis of $\mathfrak{a}$, is called the Heckman–Opdam Laplacian. Explicitly,

$$T_{p_L}(m) = L(m) + \langle \rho(m), \rho(m) \rangle - \sum_{\alpha \in \Sigma^+} m_\alpha \frac{\langle \alpha, \alpha \rangle}{\sinh^2 \alpha} \otimes (1 - r_\alpha), \tag{7}$$

where

$$L(m) := L_{\mathfrak{a}} + \sum_{\alpha \in \Sigma^+} m_\alpha \coth \alpha \, \partial_{x_\alpha}; \tag{8}$$

$L_{\mathfrak{a}}$ is the Laplace operator on $\mathfrak{a}$ and ∂_{x_α} is the directional derivative in the direction of the element $x_\alpha \in \mathfrak{a}$ corresponding to α in the identification of $\mathfrak{a}$ and $\mathfrak{a}^*$ under $\langle \cdot, \cdot \rangle$, as at the beginning of Subsection 2.1. See [26, (1)] and [27, Section 2.6] for the computation of (7). In (7) and (8), we have set

$$\sinh \alpha = \frac{e^\alpha - e^{-\alpha}}{2} = \frac{e^\alpha}{2}(1 - e^{-2\alpha}), \tag{9}$$

$$\coth \alpha = \frac{1 + e^{-2\alpha}}{1 - e^{-2\alpha}} = \frac{2}{1 - e^{-2\alpha}} - 1. \tag{10}$$

Moreover,

$$D_{p_L}(m) = L(m) + \langle \rho(m), \rho(m) \rangle.$$

Set $\mathbb{D}(m) := \{D_p(m) : p \in \mathrm{S}(\mathfrak{a}_{\mathbb{C}})^W\}$. The map $\gamma(m) : \mathbb{D}(m) \to \mathrm{S}(\mathfrak{a}_{\mathbb{C}})^W$ defined by

$$\gamma(m)\big(D_p(m)\big)(\lambda) := p(\lambda) \tag{11}$$

is called the Harish-Chandra homomorphism. It defines an algebra isomorphism of $\mathbb{D}(m)$ onto $\mathrm{S}(\mathfrak{a}_{\mathbb{C}})^W$ (see [11, Theorem 1.3.12 and Remark 1.3.14]). From Chevalley's theorem, it therefore follows that $\mathbb{D}(m)$ is generated by $r(= \dim \mathfrak{a})$ elements. The next lemma will play a decisive role for us.

Lemma 2.2. For every multiplicity function m, the algebra $\mathbb{D}(m)$ is the centralizer of $L(m)$ in $\mathbb{D}_{\mathcal{R}}^W$.

Proof. This is [11, Theorem 1.3.12]. See also [11, Remark 1.3.14] for its extension to complex-valued multiplicities. ■

Let $\lambda \in \mathfrak{a}_{\mathbb{C}}^*$ be fixed. The Heckman–Opdam hypergeometric function with spectral parameter λ is the unique W-invariant analytic solution $F_\lambda(m)$ of the system of differential equations

$$D_p(m)f = p(\lambda)f \qquad (p \in \mathrm{S}(\mathfrak{a}_{\mathbb{C}})^W), \tag{12}$$

which satisfies $f(0) = 1$. The nonsymmetric hypergeometric function with spectral parameter λ is the unique analytic solution $G_\lambda(m)$ of the system of differential equations

$$T_\xi(m)g = \lambda(\xi)g \qquad (\xi \in \mathfrak{a}), \tag{13}$$

which satisfies $g(0) = 1$. These functions are linked by the relation

$$F_\lambda(m; x) = \frac{1}{|W|} \sum_{w \in W} G_\lambda(m; w^{-1}x) \qquad (x \in \mathfrak{a}), \tag{14}$$

where $|W|$ denotes the cardinality of W. The existence of the hypergeometric functions requires suitable assumptions on m. We refer the reader to Appendix A for additional information.

For geometric multiplicities, $\mathbb{D}(m)$ coincides with the algebra of radial components of the G-invariant differential operators on G/K. Moreover, $F_\lambda(m)$ agrees with the restriction to $A \equiv \mathfrak{a}$ of Harish-Chandra's spherical function on G/K with spectral parameter λ. A geometric interpretation of the functions $G_\lambda(m)$ has been recently given in [17].

3 Hypergeometric Functions Associated with Standard Multiplicities

In this section, we look at the hypergeometric functions associated with standard multiplicities. Basic estimates and a characterization of the bounded hypergeometric functions (for these sets of multiplicity functions) are established.

Let $\mathcal{M}$ denote the set of real-valued multiplicity functions $m = (m_{\mathrm{s}}, m_{\mathrm{m}}, m_{\mathrm{l}})$ on a root system Σ of type BC_r. We will consider the following subsets of $\mathcal{M}$:

$$\mathcal{M}_+ = \{m \in \mathcal{M} : \ m_\alpha \geq 0 \text{ for every } \alpha \in \Sigma\} \tag{15}$$

$$\mathcal{M}_0 = \{(m_{\mathrm{s}}, m_{\mathrm{m}}, m_{\mathrm{l}}) \in \mathcal{M} : m_{\mathrm{m}} \geq 0, m_{\mathrm{s}} + m_{\mathrm{l}} \geq 0\}, \tag{16}$$

$$\mathcal{M}_1 = \{(m_{\mathrm{s}}, m_{\mathrm{m}}, m_{\mathrm{l}}) \in \mathcal{M} : m_{\mathrm{m}} > 0, m_{\mathrm{s}} > 0, m_{\mathrm{s}} + 2m_{\mathrm{l}} > 0\}, \tag{17}$$

$$\mathcal{M}_2 = \{(m_{\mathrm{s}}, m_{\mathrm{m}}, m_{\mathrm{l}}) \in \mathcal{M} : m_{\mathrm{m}} \geq 0, m_{\mathrm{l}} \geq 0, m_{\mathrm{s}} + m_{\mathrm{l}} \geq 0\}, \tag{18}$$

$$\mathcal{M}_3 = \{(m_{\mathrm{s}}, m_{\mathrm{m}}, m_{\mathrm{l}}) \in \mathcal{M} : m_{\mathrm{m}} \geq 0, m_{\mathrm{l}} \leq 0, m_{\mathrm{s}} + 2m_{\mathrm{l}} \geq 0\}. \tag{19}$$

So $\mathcal{M}_+$ consists of the nonnegative multiplicity functions and $\mathcal{M}_1 = (\mathcal{M}_+ \cup \mathcal{M}_3)^0$, the interior of $\mathcal{M}_+ \cup \mathcal{M}_3$. For real-valued multiplicities, $\mathcal{M}_0$ is the natural set for which both hypergeometric functions $G_\lambda(m)$ and $F_\lambda(m)$ are defined for all $\lambda \in \mathfrak{a}_{\mathbb{C}}^*$; see Appendix A. Recall that the elements of $\mathcal{M}_1$ are called standard multiplicity functions (see [11, Definition 5.5.1]). These sets of multiplicities are represented in Figure 1.

3.1 Basic estimates

Under the assumption that $m \in \mathcal{M}_+$, the positivity properties of $F_\lambda(m)$ and $G_\lambda(m)$ for $\lambda \in \mathfrak{a}^*$ as well as their basic estimates have been proved by Schapira [26, Section 3.1], by refining some ideas from [21, Section 6]. Under the same assumption, Schapira's estimates for $F_\lambda(m)$ and $G_\lambda(m)$ have been sharpened by Rösler *et al.* [25, Section 3]. The following proposition collects their results.

Proposition 3.1. Let $m \in \mathcal{M}_+$. Then, the following properties hold.

(a) For all $\lambda \in \mathfrak{a}^*$, the functions $G_\lambda(m)$ and $F_\lambda(m)$ are real and strictly positive.

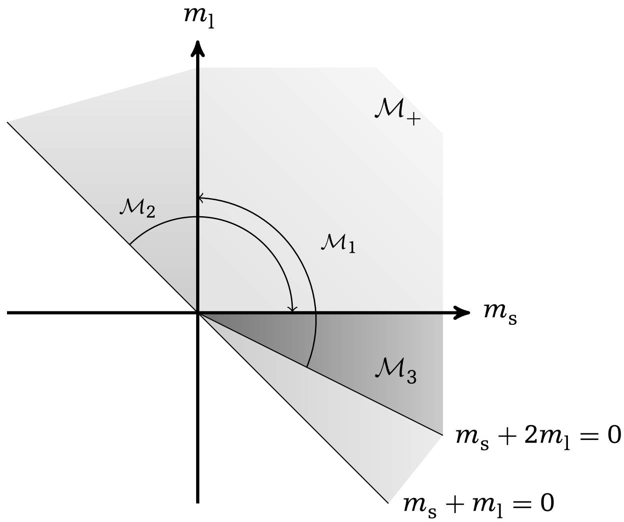

Fig. 1. Sets of *BC* multiplicities.

(b) For all $\lambda \in \mathfrak{a}^*_{\mathbb{C}}$,

$$|G_\lambda(m)| \le G_{\mathrm{Re}\,\lambda}(m), |F_\lambda(m)| \quad \le F_{\mathrm{Re}\,\lambda}(m). \tag{20}$$

(c) For all $\lambda \in \mathfrak{a}^*$ and all $x \in \mathfrak{a}$,

$$G_\lambda(m;x) \le G_0(m;x)e^{\max_w (w\lambda)(x)} F_\lambda(m;x) \quad \le F_0(m;x)e^{\max_w (w\lambda)(x)}. \tag{21}$$

More generally, for all $\lambda \in \mathfrak{a}^*$, $\mu \in \overline{(\mathfrak{a}^*)^+}$, and all $x \in \mathfrak{a}$,

$$G_{\lambda+\mu}(m;x) \le G_\mu(m;x)e^{\max_w (w\lambda)(x)} F_{\lambda+\mu}(m;x) \quad \le F_\mu(m;x)e^{\max_w (w\lambda)(x)}. \tag{22}$$

(In the above estimates, $\max_w$ denotes the maximum over all $w \in W$.)

Proposition 3.1 lists the sharpest estimates known so far for Heckman–Opdam's hypergeometric functions. The following lemma, even if quite elementary, is the key result allowing us to modify their proofs and to extend these estimates to multiplicities $m \in \mathcal{M}_3$. It will play a similar role also for extending the asymptotics and the boundedness characterization.

Lemma 3.2. Let $m = (m_s, m_m, m_l) \in \mathcal{M}_0$ and $\beta \in \Sigma_l^+$. Then, the following inequalities hold for all $x \in \mathfrak{a}$:

(a) $\frac{m_s}{2} + m_l \frac{1}{1+e^{-\beta(x)}} \geq 0$ if $m \in \mathcal{M}_+ \cup \mathcal{M}_3$ and

(b) $\frac{m_s}{2} + m_l \frac{1+e^{-2\beta(x)}}{(1+e^{-\beta(x)})^2} \geq 0$ if $m \in \mathcal{M}_2 \cup \mathcal{M}_3$.

Proof. If $m \in \mathcal{M}_+ \cup \mathcal{M}_3$, then $m_s \geq 0$ and $m_l \geq -\frac{1}{2}m_s$. Hence, for every $0 \leq C \leq 1$, we have $\frac{m_s}{2} + Cm_l \geq (1-C)\frac{m_s}{2} \geq 0$. We therefore obtain both (a) and (b) for these ms by observing that for every $t = -\beta(x) \in \mathbb{R}$, we have

$$0 \leq \frac{1}{1+e^t} \leq 1 \qquad \text{and} \qquad 0 \leq \frac{1+e^{2t}}{(1+e^t)^2} \leq 1.$$

Suppose now that $m \in \mathcal{M}_2$. Since $m_l \geq 0$, $m_s + m_l \geq 0$, and $\frac{1+|z|^2}{|1+z|^2} \geq \frac{1}{2}$ for all $z \in \mathbb{C}$, we immediately obtain that for all $t = -\beta(x) \in \mathbb{R}$, we have

$$\frac{m_s}{2} + m_l \frac{1+e^{2t}}{(1+e^t)^2} \geq \frac{1}{2}(m_s + m_l) \geq 0.$$

This proves the lemma. ■

Remark 3.3. Since

$$\lim_{t\to+\infty} \left(\frac{m_s}{2} + m_l \frac{1}{1+e^t}\right) = \frac{m_s}{2} \quad \text{and} \quad \left(\frac{1}{2}m_s + m_l \frac{1+e^{2t}}{(1+e^t)^2}\right)\Big|_{t=0} = \frac{1}{2}m_s + m_l,$$

it is clear that the inequality (a) of Lemma 3.2 cannot extend to $m \in \mathcal{M}_0 \setminus (\mathcal{M}_+ \cup \mathcal{M}_3)$ and (b) cannot extend to $m \in \mathcal{M}_0 \setminus (\mathcal{M}_2 \cup \mathcal{M}_3)$.

To use the methods from [25, Section 3], we will also need an extension of the real case of Opdam's [21, Proposition 6.1 (1)] estimates. We will do this for $m \in \mathcal{M}_2 \cup \mathcal{M}_3$. Notice that the complex variable version of Opdam's [21, Proposition 6.1 (2)] estimates has been recently extended by Ho and Ólafsson [14, Appendix A] to $m \in \mathcal{M}_2$ and to a domain in $\mathfrak{a}_{\mathbb{C}}$, which is much larger than the original one considered by Opdam. For $m \in \mathcal{M}_2$, the inequality (b) of Lemma 3.2 was noticed in the proof of [14, Proposition 5.A and p. 25].

Lemma 3.4. Suppose that $m \in \mathcal{M}_2 \cup \mathcal{M}_3$. Then, for all $\lambda \in \mathfrak{a}_{\mathbb{C}}^*$ and $x \in \mathfrak{a}$,

$$|G_\lambda(m;x)| \le \sqrt{|W|} e^{\max_w \mathrm{Re}(w\lambda)(x)}, |F_\lambda(m;x)| \quad \le \sqrt{|W|} e^{\max_w \mathrm{Re}(w\lambda)(x)}. \tag{23}$$

Proof. Regrouping the terms corresponding to β and $\beta/2$ and setting $z = x \in \mathfrak{a}$, we can rewrite the 1st term on the right-hand side of the 1st displayed equation in [21, p. 101] as

$$\sum_{w\in W, \alpha\in\Sigma_{\mathrm{m}}^+} m_\alpha \frac{\alpha(\xi)(1-e^{-4\alpha(x)})}{(1-e^{-2\alpha(x)})^2} |\varphi_w - \varphi_{r_\alpha w}|^2 e^{-2\mu(x)}$$
$$+ \sum_{w\in W, \beta\in\Sigma_{\mathrm{l}}^+} \left(m_{\beta/2} \frac{\frac{\beta}{2}(\xi)(1-e^{-2\beta(x)})}{(1-e^{-\beta(x)})^2} + m_\beta \frac{\beta(\xi)(1-e^{-4\beta(x)})}{|1-e^{-2\beta(x)})^2} \right) |\varphi_w - \varphi_{r_\beta w}|^2 e^{-2\mu(x)}$$

in which $\alpha(\xi)(1-e^{-4\alpha(x)})$ is positive for all $\alpha \in \Sigma^+$ when $\xi \in \mathfrak{a}$ is chosen in the same Weyl chamber of x. (In [21, p. 101], there is an additional multiplicative constant $-\frac{1}{2}$. Our argument takes into account this sign change.) The coefficient of $|\varphi_w - \varphi_{r_\beta w}|^2 e^{-2\mu(x)}$ for $\beta \in \Sigma_{\mathrm{m}}^+$ is then clearly positive. For $\beta \in \Sigma_{\mathrm{l}}^+$, this coefficient is equal to

$$\frac{\beta(\xi)(1-e^{-2\beta(x)})}{(1-e^{-\beta(x)})^2} \left(\frac{m_{\mathrm{s}}}{2} + m_{\mathrm{l}} \frac{1+e^{-2\beta(x)}}{(1+e^{-\beta(x)})^2} \right),$$

which is positive by Lemma 3.2(b). Therefore, the same proof as in [21, Proposition 6.1] allows us to obtain the required inequalities. ■

The following proposition extends the positivity properties and basic estimates from [26, Section 3.1] and [25, Section 3] to the multiplicity functions in $\mathcal{M}_3$.

Proposition 3.5. Proposition 3.1 holds for $m \in \mathcal{M}_3$.

Proof. Proofs of (a) and (b) and (21) follow the same steps as the proofs of [26, Lemma 3.1 and Proposition 3.1]. So we shall just indicate what has to be modified in the proofs when considering $m \in \mathcal{M}_3$ instead of $m \in \mathcal{M}_+$.

Suppose $G_\lambda(m)$ is not positive, and let $x \in \mathfrak{a}$ be a zero of $G_\lambda(m)$ of minimal norm. As in [26, Lemma 3.1], one has to distinguish whether x is regular or singular. If x is regular, take ξ in the same chamber of x. Evaluation at x of the equation

$$T_\xi(m) G_\lambda(m) = \lambda(\xi) G_\lambda(m) \tag{24}$$

yields

$$\partial_\xi G_\lambda(m;x) = \sum_{\alpha\in\Sigma^+} m_\alpha \frac{\alpha(\xi)}{1-e^{-2\alpha(x)}}\big[G_\lambda(m;r_\alpha x) - G_\lambda(m;x)\big] + \big(\rho(m)+\lambda\big)(\xi)G_\lambda(m;x)$$

in which $G_\lambda(m;x)$ vanishes. In the sum over Σ^+, the coefficient of $G_\lambda(m;r_\alpha x) - G_\lambda(m;x)$ is always nonnegative for $\alpha \in \Sigma^+_{\mathrm{m}}$. Moreover, grouping together those corresponding to $\beta/2$ and $\beta \in \Sigma^+_{\mathrm{l}}$, we obtain as coefficient of $G_\lambda(m;r_\beta x) - G_\lambda(m;x)$

$$m_{\beta/2}\frac{\frac{\beta}{2}(\xi)}{1-e^{-\beta(x)}} + m_\beta \frac{\beta(\xi)}{1-e^{-2\beta(x)}} = \frac{\beta(\xi)}{1-e^{-\beta(x)}}\Big[\frac{m_{\mathrm{s}}}{2} + m_{\mathrm{l}}\frac{1}{1+e^{-\beta(x)}}\Big], \tag{25}$$

which is positive by Lemma 3.2(a).

If x is singular, let $I = \{\alpha \in \Sigma^+ : \alpha(x) = 0\}$, and let ξ be in the same face of x so that $\alpha(\xi) = 0$ for all $\alpha \in I$. In this case, (24) evaluated at x gives

$$\begin{aligned}\partial_\xi G_\lambda(m;x) = -\sum_{\alpha\in I} m_\alpha \frac{\alpha(\xi)}{\langle\alpha,\alpha\rangle}\partial_{x_\alpha} G_\lambda(m;x) \\ + \sum_{\alpha\in\Sigma^+_{\mathrm{m}}\setminus I} m_{\mathrm{m}} \frac{\alpha(\xi)}{1-e^{-2\alpha(x)}}\big[G_\lambda(m;r_\alpha x) - G_\lambda(m;x)\big] \\ + \sum_{\beta\in\Sigma^+_{\mathrm{l}}\setminus I}\Big[m_{\beta/2}\frac{\frac{\beta}{2}(\xi)}{1-e^{-\beta(x)}} + m_\beta\frac{\beta(\xi)}{1-e^{-2\beta(x)}}\Big]\big[G_\lambda(m;r_\beta x) - G_\lambda(m;x)\big] \\ + \big(\rho(m)+\lambda\big)(\xi)G_\lambda(m;x),\end{aligned}$$

in which the 1st sum and $G_\lambda(m;x)$ vanish and the sum over $\Sigma^+_{\mathrm{l}} \setminus I$ has coefficient as in (25).

In both cases, one can argue as in [26, Lemma 3.1] by replacing the multiplicities $k_\alpha \geq 0$ in that proof with $\frac{m_{\mathrm{s}}}{2} + m_{\mathrm{l}}\frac{1}{1+e^{-\beta(x)}}$ and using Lemma 3.2(a).

Having that $G_{\mathrm{Re}\,\lambda}(m)$ is real and positive, to prove (b) for $G_\lambda(m)$, one can make the same grouping and substitution of multiplicities, as done above for $\partial_\xi G_\lambda(m;x)$, inside the formula for $\partial_\xi|Q_\lambda|^2(x)$ appearing in the proof of [26, Proposition 3.1 (a)]. A 3rd application of the same grouping in the formula of $\partial_\xi R_\lambda(x)$ in the proof of [26, Proposition 3.1 (b)] yields (21).

Finally, (22) has been proven for multiplicities $m_\alpha \geq 0$ in [25, Theorem 3.3] using the original versions of (a)–(c) and (21) together with a clever application of the Phragmén–Lindelöf principle that does not involve the root multiplicities. With Opdam's estimate for $m \in \mathcal{M}_3$, as in Lemma 3.4, the original proof from [25] extends to the case of $m \in \mathcal{M}_3$ and yields (22). ■

3.2 Asymptotics

We now investigate the asymptotic behavior of the hypergeometric functions $F_\lambda(m)$ for $m \in \mathcal{M}_1$. For this, we follow [16]. Before we state our results, it is useful to recall the methods adopted in [16]. Let m be an arbitrary nonnegative multiplicity function. One of the main results in [16] is Theorem 2.11, where a series expansion away from the walls was obtained for $F_\lambda(m)$ for all $\lambda \in \mathfrak{a}_{\mathbb{C}}^*$ (even when λ is nongeneric). Recall from [11, Section 4.2] (or from [16, Section 1.2] in symmetric space notation) that, for generic λ, the function $F_\lambda(m)$ is given on $\mathfrak{a}^+$ by $\sum_{w\in W} c(m; w\lambda)\Phi_{w\lambda}(m)$, where $c(m;\lambda)$ is Harish-Chandra's c-function (see (A1) in the appendix) and $\Phi_\lambda(m;x)$ admits the series expansion

$$\Phi_\lambda(m;x) = e^{(\lambda-\rho(m))(x)} \sum_{\mu\in 2\Lambda} \Gamma_\mu(m;\lambda)e^{-\mu(x)} \qquad (x \in \mathfrak{a}^+).$$

The coefficients $\Gamma_\mu(m,\lambda)$ are determined from the recursion relations

$$\langle \mu, \mu - 2\lambda\rangle \Gamma_\mu(m,\lambda) = 2 \sum_{\alpha\in\Sigma^+} m_\alpha \sum_{n\in\mathbb{N}, \mu-2n\alpha\in\Lambda} \Gamma_{\mu-2n\alpha}(m,\lambda)\langle \mu + \rho(m) - 2n\alpha - \lambda, \alpha\rangle,$$

with the initial condition that $\Gamma_0(m,\lambda) = 1$. The above defines $\Gamma_\mu(m;\lambda)$ as meromorphic functions on $\mathfrak{a}_{\mathbb{C}}^*$.

For a nongeneric point $\lambda = \lambda_0$, a series expansion for $F_\lambda(m)$ was obtained in [16] following the steps given below.

Step I. List the possible singularities of the c-function and the coefficients $\Gamma_\mu(m;\lambda)$ at $\lambda = \lambda_0$.

Step II. Identify a polynomial p so that

$$\lambda \to p(\lambda)\Big(\sum_{w\in W} c(m; w\lambda) e^{(\lambda-\rho(m))(x)} \sum_{\mu\in 2\Lambda} \Gamma_\mu(m;\lambda)e^{-\mu(x)}\Big)$$

is holomorphic in a neighborhood of λ_0.

Step III. Write $F_{\lambda_0}(m) = a\ \partial(\pi)(pF_\lambda(m))|_{\lambda=\lambda_0}$ where $\partial(\pi)$ is the differential operator corresponding to the highest degree homogenous term in p and a, is a nonzero constant. This gives the series expansion of $F_{\lambda_0}(m)$.

A careful examination shows that the same proofs go through even with the assumption that the multiplicity function belongs to $\mathcal{M}_1$. Indeed, the possible singularities of the c-function and the $\Gamma_\mu(m)$ are contained in the same set of hyperplanes as listed in [16, Lemma 2.3]. Hence, the same polynomials and differential operators can be used in Steps II and III above. The crucial detail to be checked is the computation of the constant $b_0(m;\lambda_0)$ appearing in [16, Lemma 2.6]. The explicit expression in terms of Harish-Chandra's c-function shows that, for root systems of type $BC_r \setminus C_r$, the function $b_0(m;\lambda_0)$ is nonzero if and only if

$$\prod_{\alpha\in\Sigma_s^+}\Gamma\Big(\frac{(\lambda_0)_\alpha}{2}+\frac{m_s}{4}+\frac{1}{2}\Big)\Gamma\Big(\frac{(\lambda_0)_\alpha}{2}+\frac{m_s}{4}+\frac{m_l}{2}\Big)\prod_{\alpha\in\Sigma_m^+}\Gamma\Big(\frac{(\lambda_0)_\alpha}{2}+\frac{m_m}{4}+\frac{1}{2}\Big)\Gamma\Big(\frac{(\lambda_0)_\alpha}{2}+\frac{m_m}{4}\Big) \tag{26}$$

(where the 2nd factor in the product does not appear if the rank r is one) is nonsingular. It is clear that the expression in (26) is nonsingular for $\lambda_0 \in \mathfrak{a}_\mathbb{C}^*$ with $\mathrm{Re}\,\lambda_0 \in \overline{(\mathfrak{a}^*)^+}$ if $m \in \mathcal{M}_1$. Notice that the nonvanishing of $b_0(m;\lambda_0)$ identifies the main term in the expansion of $F_{\lambda_0}(m;x)$ as $\frac{b_0(m;\lambda_0)}{\pi_0(\rho_0(m))}\pi_0(x)e^{(\lambda_0-\rho(m))(x)}$, where $\rho_0(m)$ is defined as in [16, (58)]. Notice also that $c(m;\lambda_0)$, and hence $b_0(m;\lambda_0)$, can vanish for $m \in \mathcal{M}_0 \setminus \mathcal{M}_1$.

It follows from the above that [16, Theorem 2.11] and, as a consequence, [16, Theorem 3.1] continue to hold true for a multiplicity function $m \in \mathcal{M}_1$. We state it below and refer to [16] for any unexplained notation.

Theorem 3.6. Suppose $m \in \mathcal{M}_1$. Let $\lambda_0 \in \mathfrak{a}_\mathbb{C}^*$ with $\mathrm{Re}\,\lambda_0 \in \overline{(\mathfrak{a}^*)^+}$, and let $x_0 \in \mathfrak{a}^+$ be fixed. Then, there are constants $C_1 > 0, C_2 > 0$ and $b > 0$ (depending on m, λ_0 and x_0) so that for all $x \in x_0 + \mathfrak{a}^+$:

$$\Big|\frac{F_{\lambda_0}(m;x)e^{-(\mathrm{Re}\,\lambda_0-\rho(m))(x)}}{\pi_0(x)} - \Big(\frac{b_0(m;\lambda_0)}{\pi_0(\rho_0(m))}e^{i\,\mathrm{Im}\,\lambda_0(x)} + \sum_{w\in W_{\mathrm{Re}\,\lambda_0}\setminus W_{\lambda_0}}\frac{b_w(m;\lambda_0)\pi_{w,\lambda_0}(x)}{c_0\pi_0(x)}e^{iw\,\mathrm{Im}\,\lambda_0(x)}\Big)\Big|$$
$$\leq C_1(1+\beta(x))^{-1} + C_2(1+\beta(x))^{|\Sigma_{\lambda_0}^+|}e^{-b\beta(x)}, \tag{27}$$

where $\beta(x)$ is the minimum of $\alpha(x)$ over the simple roots $\alpha \in \Sigma^+$ and the term $C_1(1+\beta(x))^{-1}$ on the right-hand side of (27) does not occur if $\langle\alpha,\lambda_0\rangle \neq 0$ for all $\alpha \in \Sigma$.

Notice that, for fixed $x_0 \in \mathfrak{a}^+$, we have $\beta(x) \asymp |x|$ as $x \to \infty$ in $x_0 + \overline{\mathfrak{a}^+}$, where $|x|$ is the Euclidean norm on $\mathfrak{a}$.

Likewise, one can extend to $m \in \mathcal{M}_1$ the following corollary, which restates Theorem 3.6 in the special case where $\lambda_0 \in \overline{(\mathfrak{a}^*)^+}$.

Corollary 3.7. Suppose $m \in \mathcal{M}_1$. Let $\lambda_0 \in \overline{(\mathfrak{a}^*)^+}$, and let $x_0 \in \mathfrak{a}^+$ be fixed. Then, there are constants $C_1 > 0$, $C_2 > 0$, and $b > 0$ (depending on m, λ_0 and x_0) so that for all $x \in x_0 + \overline{\mathfrak{a}^+}$,

$$\left|F_{\lambda_0}(m;x) - \frac{b_0(m;\lambda_0)}{\pi_0(m;\rho_0)}\pi_0(x)e^{(\lambda_0-\rho(m))(x)}\right| \leq$$
$$\leq \left[C_1(1+\beta(x))^{-1} + C_2(1+\beta(x))^{|\Sigma^+_{\lambda_0}|}e^{-b\beta(x)}\right]\pi_0(x)e^{(\lambda_0-\rho(m))(x)}. \quad (28)$$

The term $C_1(1+\beta(x))^{-1}$ on the right-hand side of (28) does not occur if $\langle\alpha,\lambda_0\rangle \neq 0$ for all $\alpha \in \Sigma$.

It might be useful to observe that for $m \in \mathcal{M}_0$, the only obstruction to (27) is that $b_0(m;\lambda_0) \neq 0$. For arbitrary values of $\lambda_0 \in \mathfrak{a}^*_{\mathbb{C}}$ with $\operatorname{Re}\lambda_0 \in \overline{(\mathfrak{a}^*)^+}$, by (26), this happens if and only if

$$m_{\mathrm{s}} > -2\,, \qquad m_{\mathrm{s}} + 2m_{\mathrm{l}} > 0 \quad \text{and, if } r > 1 \quad m_{\mathrm{m}} > 0.$$

Corollary 3.8. Let $m \in \mathcal{M}_0$. Then, Theorem 3.6 holds for every $\lambda_0 \in \mathfrak{a}^*_{\mathbb{C}}$ with $\operatorname{Re}\lambda_0 \in \overline{(\mathfrak{a}^*)^+}$ such that (26) is nonsingular.

3.3 Sharp estimates

In this subsection, we assume that $m \in \mathcal{M}_3$. Let $\mathcal{M}_3^0$ denote the interior of $\mathcal{M}_3$. Since $\mathcal{M}_3^0 \subset \mathcal{M}_1$, the results of both Subsections 3.1 and 3.2 are available on $\mathcal{M}_3^0$.

Let $\lambda \in \mathfrak{a}^*$. Using the nonsymmetric hypergeometric functions $G_\lambda(m)$, their relation to the hypergeometric function and the system of differential-reflection equations they satisfy, Schapira [26] proved the following local Harnack principle for the hypergeometric function $F_\lambda(m)$: for all $x \in \overline{\mathfrak{a}^+}$,

$$\nabla F_\lambda(m;x) = -\frac{1}{|W|}\sum_{w\in W} w^{-1}(\rho(m)-\lambda)G_\lambda(m;wx), \quad (29)$$

the gradient being taken with respect to the space variable $x \in \mathfrak{a}$. It holds for every multiplicity function m for which both $G_\lambda(m)$ and $F_\lambda(m)$ are defined. See [26, Lemma

3.4]. Since $\partial_\xi F = \langle \nabla F, \xi \rangle$ and since $G_\lambda(m)$ and $F_\lambda(m)$ are real and nonnegative for $m \in \mathcal{M}_3$, one obtains as in *loc. cit.* that for all $\xi \in \mathfrak{a}$,

$$\partial_\xi \left(e^{K_\xi \frac{\langle \xi, \cdot \rangle}{|\xi|^2}} F_\lambda(m; \cdot) \right) \geq 0,$$

where $K_\xi = \max_{w \in W} (\rho(m) - \lambda)(w\xi)$. See Appendix B for a proof. This in turn yields the following subadditivity property, which is implicit in [26] for $m \in \mathcal{M}_+$ and in fact holds also for $m \in \mathcal{M}_3^0$ and, by continuity, on $\mathcal{M}_3$.

Lemma 3.9. Suppose $m \in \mathcal{M}_3$. Let $\lambda \in \mathfrak{a}^*$. Then, for all $x, x_1 \in \mathfrak{a}$, we have

$$F_\lambda(m; x + x_1) e^{\min_{w \in W} (\rho(m) - \lambda)(wx_1)} \leq F_\lambda(m; x) \leq F_\lambda(m; x + x_1) e^{\max_{w \in W} (\rho(m) - \lambda)(wx_1)}. \tag{30}$$

In particular, if $\lambda \in \overline{(\mathfrak{a}^*)^+}$ and $x_1 \in \mathfrak{a}^+$, then

$$F_\lambda(m; x + x_1) e^{-(\rho(m) - \lambda)(x_1)} \leq F_\lambda(m; x) \leq F_\lambda(m; x + x_1) e^{(\rho(m) - \lambda)(x_1)} \tag{31}$$

for all $x \in \mathfrak{a}$.

Together with Corollary 3.7, the above lemma yields the following global estimates of $F_\lambda(m; x)$.

Theorem 3.10. Let $m \in \mathcal{M}_3^0 \subset \mathcal{M}_1$ and $\lambda_0 \in \overline{(\mathfrak{a}^*)^+}$. Then, for all $x \in \overline{\mathfrak{a}^+}$, we have

$$F_{\lambda_0}(m; x) \asymp \Big[\prod_{\alpha \in \Sigma_{\lambda_0}^0} (1 + \alpha(x)) \Big] e^{(\lambda_0 - \rho(m))(x)}, \tag{32}$$

where $\Sigma_{\lambda_0}^0 = \{\alpha \in \Sigma_{\mathrm{s}}^+ \cup \Sigma_{\mathrm{m}}^+ : \langle \alpha, \lambda_0 \rangle = 0\}$.

Proof. Same as in [16, Theorem 3.4]. ■

We end this section with the following characterization of the bounded hypergeometric functions corresponding to multiplicity functions $m \in \mathcal{M}_1$.

Theorem 3.11. Let $m \in \mathcal{M}_1$. Then, $F_\lambda(m)$ is bounded if and only if $\lambda \in C(\rho(m)) + i\mathfrak{a}^*$, where $C(\rho(m))$ is the convex hull of the set $\{w\rho(m) : \ w \in W\}$.

Proof. The same arguments as in [16, Theorem 4.2] work using the results in this section. ∎

4 Applications and Developments

In this section, we consider two-parameter deformations of the multiplicities in $\mathcal{M}_+$ and study a class of hypergeometric functions associated with them. As we shall see in Section 5, for specific values of $m \in \mathcal{M}_+$ and of the deformation parameters $(\ell, \widetilde{\ell})$, these hypergeometric functions turn out to agree with the τ-spherical functions on the homogeneous vector bundles over G/K, when τ is a small K-type and G/K has root system of type BC. The general properties proved in this section will provide for most of the BC cases in [18] symmetry properties, estimates, aymptotics, and a characterization of the τ-spherical functions that are bounded.

4.1 A two-parameter deformation of a multiplicity function

Let $(\mathfrak{a}, \Sigma, m)$ be a triple as in Subsection 2.1, with $m = (m_{\rm s}, m_{\rm m}, m_{\rm l})$. For any two parameters $\ell, \widetilde{\ell}$ we define a deformation $m(\ell, \widetilde{\ell})$ of m as follows:

$$m_\alpha(\ell, \widetilde{\ell}) = \begin{cases} m_{\rm s} + 2\ell & \text{if } \alpha \in \Sigma_{\rm s} \\ m_{\rm m} + 2\widetilde{\ell} & \text{if } \alpha \in \Sigma_{\rm m} \\ m_{\rm l} - 2\ell & \text{if } \alpha \in \Sigma_{\rm l}. \end{cases} \tag{33}$$

We shall suppose in the following that $m \in \mathcal{M}_0$, $\ell, \widetilde{\ell} \in \mathbb{R}$ and $\widetilde{\ell} \geq -m_{\rm m}/2$. When this does not cause any confusion, we shall shorten the notations $m(\ell, 0)$ and $m(0, \widetilde{\ell})$ and write $m(\ell)$ and $m(\widetilde{\ell})$, respectively. Since $m_{\rm s}(\ell, \widetilde{\ell}) + m_{\rm l}(\ell, \widetilde{\ell}) = m_{\rm s} + m_{\rm l}$, the above assumptions ensure that $m(\ell, \widetilde{\ell}) \in \mathcal{M}_0$. The two deformations, in ℓ and $\widetilde{\ell}$, are independent. So, $m(\ell, \widetilde{\ell}) = m(\ell)(\widetilde{\ell}) = m(\widetilde{\ell})(\ell)$. Since we are assuming that $\widetilde{\ell} \geq -m_{\rm m}/2$, the additional parameter $\widetilde{\ell}$ does not increase the range of possible multiplicities of the middle roots. Its relevance will appear in the definitions the of $(\ell, \widetilde{\ell})$-hypergeometric functions in (48) and (49).

The following theorem shows that every element of $\mathcal{M}_+ \cup \mathcal{M}_3$ is of the form $m(\ell)$ for some $m \in \mathcal{M}_+$. For a fixed $m = (m_{\rm s}, m_{\rm m}, m_{\rm l})$, we shall use the notation

$$\ell_{\min}(m) = -\frac{m_{\rm s}}{2} \qquad \text{and} \qquad \ell_{\max}(m) = \frac{m_{\rm s}}{2} + m_{\rm l}. \tag{34}$$

We simply write $\ell_{\min}$ and $\ell_{\max}$ when this does not cause any ambiguity.

Lemma 4.1. Let $m^0 = (m^0_{\mathrm{s}}, m^0_{\mathrm{m}}, m^0_{\mathrm{l}}) \in \mathcal{M}_0$. Then, $m^0 \in \mathcal{M}_+ \cup \mathcal{M}_3$ if and only if there are $m = (m_{\mathrm{s}}, m_{\mathrm{m}}, m_{\mathrm{l}}) \in \mathcal{M}_+$ and $\ell \in \mathbb{R}$ so that $m^0 = m(\ell)$ and $\ell \in [\ell_{\min}(m), \ell_{\max}(m)]$. Moreover, $m^0 \in \mathcal{M}_1 = (\mathcal{M}_+ \cup \mathcal{M}_3)^0$ if and only if $m = (m_{\mathrm{s}}, m_{\mathrm{m}}, m_{\mathrm{l}}) \in \mathcal{M}_+$ is as above and $\ell \in]\ell_{\min}(m), \ell_{\max}(m)[$.

Proof. If $m^0 \in \mathcal{M}_+ \cup \mathcal{M}_3$, then $m^0 = m(\ell)$ for

$$m = (m_{\mathrm{s}}, m_{\mathrm{m}}, m_{\mathrm{l}}) = (m^0_{\mathrm{s}} + m^0_{\mathrm{l}}, m^0_{\mathrm{m}}, 0) \in \mathcal{M}_+ \quad \text{and} \quad \ell = -\frac{m^0_{\mathrm{l}}}{2}.$$

Observe that ℓ satisfies $-\frac{m_{\mathrm{s}}}{2} = -\frac{m^0_{\mathrm{s}}+m^0_{\mathrm{l}}}{2} \leq \ell \leq \frac{m^0_{\mathrm{s}}+m^0_{\mathrm{l}}}{2} = \frac{m_{\mathrm{s}}}{2} + m_{\mathrm{l}}$ because $m^0_{\mathrm{s}} \geq 0$ and $m^0_{\mathrm{s}} + 2m^0_{\mathrm{l}} \geq 0$. The inequalities for ℓ are strict if $m^0 \in \mathcal{M}_1$ since in this case $m^0_{\mathrm{s}} > 0$.

Conversely, suppose that $m^0 = m(\ell)$ for $m \in \mathcal{M}_+$ and ℓ as in the statement. Then, $-m_{\mathrm{s}} \leq 2\ell \leq m_{\mathrm{s}} + 2m_{\mathrm{l}}$ and $m_{\mathrm{s}} + m_{\mathrm{l}} = m^0_{\mathrm{s}} + m^0_{\mathrm{l}}$. Hence,

$$m^0_{\mathrm{s}} = m_{\mathrm{s}} + 2\ell \leq 2(m_{\mathrm{s}} + m_{\mathrm{l}}) = 2(m^0_{\mathrm{s}} + m^0_{\mathrm{l}}), \quad \text{that is,} \quad m^0_{\mathrm{s}} + 2m^0_{\mathrm{l}} \geq 0.$$

Moreover, $m^0_{\mathrm{s}} = m_{\mathrm{s}} + 2\ell \geq m_{\mathrm{s}} - m_{\mathrm{s}} = 0$. Thus, $m^0 \in \mathcal{M}_+ \cup \mathcal{M}_3$. All the inequalities are strict if $\ell \in]\ell_{\min}(m), \ell_{\max}(m)[$. Hence, in this case, $m^0 \in \mathcal{M}_1$. ■

Lemma 4.1 is pictured in Figure 2 below. Recall that in this paper, we consider the root system C_r as a root system BC_r with $m_{\mathrm{s}} = 0$. The diagonal segments belong to lines $m_{\mathrm{s}} + m_{\mathrm{l}} =$ constant. If $m^0 \in \mathcal{M}_+ \cup \mathcal{M}_3$ belongs to such a line, the corresponding element $m \in \mathcal{M}_+$ in the 1st part of the proof, is the intersection of this line with the m_{s}-axis. The specific segments drawn are those passing through geometric multiplicities $(m_{\mathrm{s}}, m_{\mathrm{m}}, m_{\mathrm{l}} = 1)$: the segments contain the values of $(m_{\mathrm{s}}(\ell), m_{\mathrm{l}}(\ell)) = (m_{\mathrm{s}} + 2\ell, 1 - 2\ell)$ with $\ell \in [\ell_{\min}, \ell_{\max}]$.

4.2 $(\ell, \widetilde{\ell})$-Cherednik operators

We keep the notation of Subsection 4.1. Let u and v be the W-invariant functions on $\mathfrak{a}$ defined by

$$u(x) = \prod_{j=1}^{r} \cosh\Big(\frac{\beta_j(x)}{2}\Big), \tag{35}$$

$$v(x) = \prod_{1 \leq i < j \leq r} \cosh\Big(\frac{\beta_j(x) - \beta_i(x)}{2}\Big) \cosh\Big(\frac{\beta_j(x) + \beta_i(x)}{2}\Big). \tag{36}$$

G	m_{s}	$\ell_{\min}$	$\ell_{\max}$
with $\Sigma = C_r$	0	0	1
$\mathrm{SO}^*(2(2n+1))$	4	-2	3
$\mathfrak{e}_{6(-14)}$	8	-4	5
$\mathrm{SU}(p,q)$ $q > p$	$2(q-p)$	$p-q$	$q-p+1$

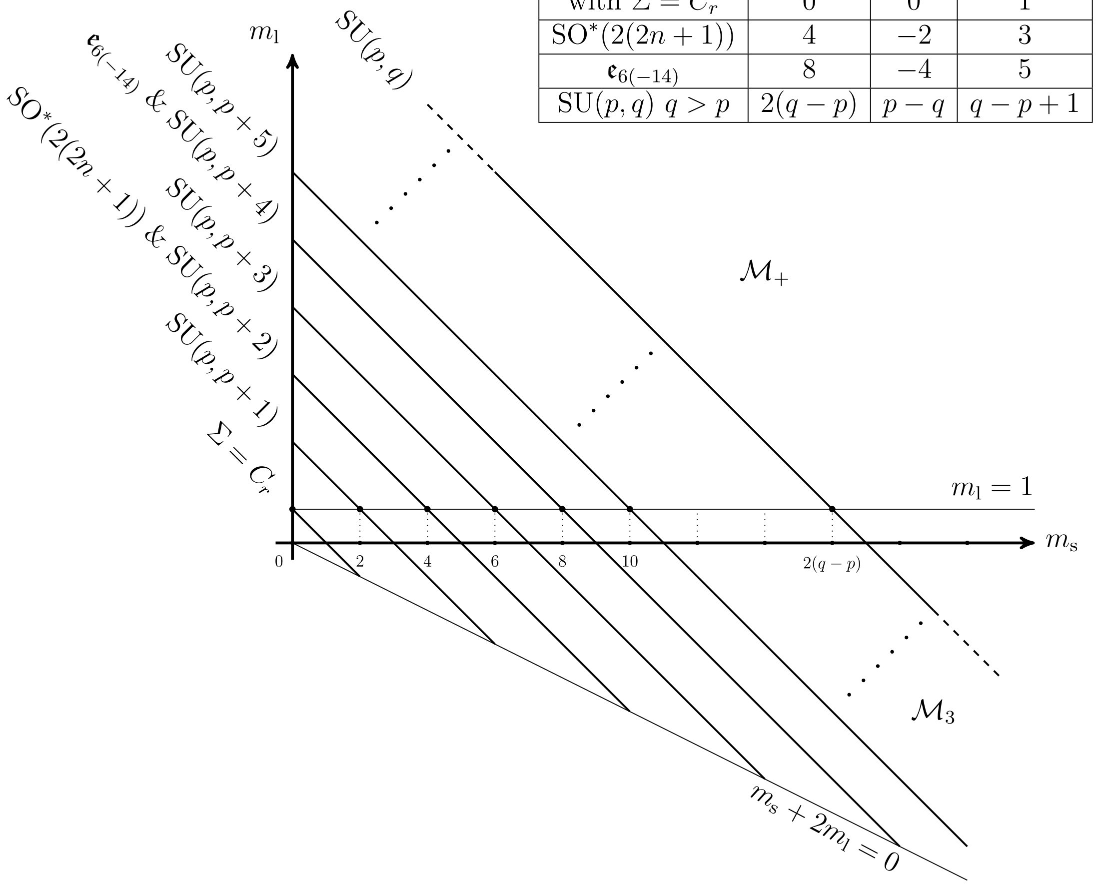

Fig. 2. $(m_{\mathrm{s}}(\ell), m_{\mathrm{l}}(\ell))$ for geometric $(m_{\mathrm{s}}, m_{\mathrm{l}} = 1)$ and $\ell \in [\ell_{\min}, \ell_{\max}]$.

For $\xi \in \mathfrak{a}$ and $\ell, \widetilde{\ell} \in \mathbb{R}$, we define the Cherednik operator $T_{\ell,\widetilde{\ell},\xi}$ by

$$T_{\ell,\widetilde{\ell},\xi}(m) = u^{-\ell} v^{-\widetilde{\ell}} \circ T_\xi(m(\ell,\widetilde{\ell})) \circ u^\ell v^{\widetilde{\ell}}. \tag{37}$$

A simple computation (using that u and v commute with $1 - r_\alpha$ for $\alpha \in \Sigma$) shows that

$$T_{\ell,\widetilde{\ell},\xi}(m) = T_\xi(m(\ell,\widetilde{\ell})) + \ell u^{-1}\partial_\xi(u) + \widetilde{\ell} v^{-1}\partial_\xi(v),$$

where

$$u^{-1}\partial_\xi(u) = \frac{1}{2}\sum_{j=1}^{r}\beta_j(\xi)\tanh\left(\frac{\beta_j}{2}\right) \tag{38}$$

and

$$v^{-1}\partial_\xi(v) = \frac{1}{2}\sum_{1\le i<j\le r}\left[(\beta_j(\xi)-\beta_i(\xi))\tanh\left(\frac{\beta_j-\beta_i}{2}\right)+(\beta_j(\xi)+\beta_i(\xi))\tanh\left(\frac{\beta_j+\beta_i}{2}\right)\right]. \tag{39}$$

Let $\mathcal{R}_0$ be the algebra of functions on $\mathfrak{a}_{\mathrm{reg}}$ generated by 1 and $(1 \pm e^{-\alpha})^{-1}$ with $\alpha \in \Sigma^+$; see [10, pp. 63 and 64]. So $\mathcal{R}$ is a subalgebra of $\mathcal{R}_0$. In general, $T_{\ell,\widetilde{\ell},\xi} \in \mathbb{D}_{\mathcal{R}_0} \otimes \mathbb{C}[W]$, where $\mathbb{D}_{\mathcal{R}_0} = \mathcal{R}_0 \otimes S(\mathfrak{a}_{\mathbb{C}})$. However, if $\widetilde{\ell} = 0$, then $T_{\ell,\xi} = T_{\ell,0,\xi} \in \mathbb{D}_{\mathcal{R}} \otimes \mathbb{C}[W]$. By construction, $\{T_{\ell,\widetilde{\ell},\xi}(m) : \xi \in \mathfrak{a}\}$ is a commutative family of differential-reflection operators. So, the map $\xi \to T_{\ell,\widetilde{\ell},\xi}(m)$ extends uniquely to an algebra homomorphism $p \to T_{\ell,\widetilde{\ell},p}(m)$ from $S(\mathfrak{a}_{\mathbb{C}})$ to $\mathbb{D}_{\mathcal{R}_0} \otimes \mathbb{C}[W]$ such that $T_{\ell,\widetilde{\ell},p}(m) = u^{-\ell}v^{-\widetilde{\ell}} \circ T_p(m(\ell,\widetilde{\ell})) \circ v^{\widetilde{\ell}}u^{\ell}$. In particular, one can define the $(\ell,\widetilde{\ell})$-Heckman–Opdam Laplacian

$$T_{\ell,\widetilde{\ell},p_L}(m) = \sum_{j=1}^{r} T_{\ell,\widetilde{\ell},\xi_j}(m)^2 = u^{-\ell}v^{-\widetilde{\ell}} \circ T_{p_L}(m(\ell)) \circ u^{\ell}v^{\widetilde{\ell}}, \tag{40}$$

where $\{\xi_j\}_{j=1}^r$ is any orthonormal basis of $\mathfrak{a}$ and p_L is defined by $p_L(\lambda) := \langle\lambda,\lambda\rangle$ for $\lambda \in \mathfrak{a}^*_{\mathbb{C}}$.

Next, we compute the differential part of the $(\ell,\widetilde{\ell})$-Heckman–Opdam Laplacian in a closed form, allowing us to deduce some symmetry properties that will be useful in later sections. For an arbitrary root system Σ on $\mathfrak{a}^*$ and a multiplicity function m, let us set

$$f_\Sigma(m) = \sum_{\alpha\in\Sigma^+}\frac{m_\alpha(2-m_\alpha-2m_{2\alpha})\langle\alpha,\alpha\rangle}{(e^{\alpha}-e^{-\alpha})^2}. \tag{41}$$

We start by recalling the following lemma.

Lemma 4.2. Let Σ be an arbitrary root system on $\mathfrak{a}^*$ and m a multiplicity function. If $\delta_\Sigma(m)^{\frac{1}{2}} = \prod_{\alpha\in\Sigma^+}(e^{\alpha}-e^{-\alpha})^{\frac{m_\alpha}{2}}$, then

$$\delta_\Sigma(m)^{\frac{1}{2}} \circ \left(L_\Sigma(m) + \langle\rho_\Sigma(m),\rho_\Sigma(m)\rangle\right) \circ \delta_\Sigma(m)^{-\frac{1}{2}} = L_{\mathfrak{a}} + f_\Sigma(m),$$

where $L_\Sigma(m)$ is the Heckman–Opdam Laplacian associated with (Σ, m) and $\rho_\Sigma(m)$ is the half sum of positive roots.

Proof. See [11, Theorem 2.1.1]. ■

From now onwards, Σ will denote a root system of type BC, as in Section 2. Recall the operator β from (5). The same definition extends β to an operator from $\mathbb{D}_{\mathcal{R}_0} \otimes \mathbb{C}[W]$ to $\mathbb{D}_{\mathcal{R}_0}$. If $p \in S(\mathfrak{a}_{\mathbb{C}})$, then $\beta\big(T_{\ell,\widetilde{\ell},p}(m)\big) = u^{-\ell} v^{-\widetilde{\ell}} \circ \beta\big(T_p(m(\ell,\widetilde{\ell}))\big) \circ u^{\ell} v^{\widetilde{\ell}} \in \mathbb{D}_{\mathcal{R}_0}$. Set $D_{\ell,\widetilde{\ell},p}(m) = \beta\big(T_{\ell,\widetilde{\ell},p}(m)\big)$ for $p \in S(\mathfrak{a}_{\mathbb{C}})^W$. Furthermore, set

$$\mathbb{D}_{\ell,\widetilde{\ell}}(m) = \{D_{\ell,\widetilde{\ell},p}(m) : p \in S(\mathfrak{a}_{\mathbb{C}})^W\}. \tag{42}$$

Lemma 4.3. Consider the root system $\widetilde{\Sigma} = 2\Sigma$ and the multiplicity function $\widetilde{m} = (m_{\rm s} + m_{\rm l}, m_{\rm m} + 2\widetilde{\ell}, 0)$. Then, we have the identity

$$u^{-\frac{m_{\rm s}}{2}} v^{-\frac{m_{\rm m}}{2}} \circ D_{\ell,\widetilde{\ell},p_L}(m) \circ u^{\frac{m_{\rm s}}{2}} v^{\frac{m_{\rm m}}{2}} =$$

$$L_{\widetilde{\Sigma}}(\widetilde{m}) + \langle \rho_{\widetilde{\Sigma}}(\widetilde{m}), \rho_{\widetilde{\Sigma}}(\widetilde{m})\rangle + f_{\Sigma}(m(\ell,\widetilde{\ell})) - f_{\widetilde{\Sigma}}(\widetilde{m}).$$

Proof. We start with

$$\delta_{\Sigma}(m(\ell,\widetilde{\ell}))^{\frac{1}{2}} \circ \big(L_{\Sigma}(m(\ell,\widetilde{\ell})) + \langle \rho(\Sigma, m(\ell,\widetilde{\ell})), \rho(\Sigma, m(\ell,\widetilde{\ell}))\rangle\big) \circ \delta_{\Sigma}(m(\ell,\widetilde{\ell}))^{-\frac{1}{2}}. \tag{43}$$

The above, by Lemma 4.2, equals

$$L_{\mathfrak{a}} + \sum_{\alpha>0} \frac{m_\alpha(\ell,\widetilde{\ell})\left(2 - m_\alpha(\ell,\widetilde{\ell}) - 2m_{2\alpha}(\ell,\widetilde{\ell})\right)\langle\alpha,\alpha\rangle}{(e^{\alpha} - e^{-\alpha})^2} = L_{\mathfrak{a}} + f_{\Sigma}(m(\ell,\widetilde{\ell})).$$

Replacing $L_{\mathfrak{a}}$ with

$$\delta_{\widetilde{\Sigma}}(\widetilde{m})^{\frac{1}{2}} \circ \big(L_{\widetilde{\Sigma}}(\widetilde{m}) + \langle \rho_{\widetilde{\Sigma}}(\widetilde{m}), \rho_{\widetilde{\Sigma}}(\widetilde{m})\rangle\big) \circ \delta_{\widetilde{\Sigma}}(\widetilde{m})^{-\frac{1}{2}} - f_{\widetilde{\Sigma}}(\widetilde{m}),$$

we get the result as

$$\delta_{\Sigma}(m(\ell,\widetilde{\ell}))^{\frac{1}{2}} \delta_{\widetilde{\Sigma}}(\widetilde{m})^{-\frac{1}{2}} = 2^M u^{-(\frac{m_s}{2}+\ell)} v^{-(\frac{m_m}{2}+\widetilde{\ell})},$$

where M is a constant depending on m and r. ■

Corollary 4.4. In the above notation, for all $l \in \mathbb{R}$ and $\widetilde{\ell} \geq -\frac{m_{\rm m}}{2}$,

$$D_{\ell,\widetilde{\ell},p_L}(m) = D_{-\ell+m_{\rm l}-1,\widetilde{\ell},p_L}\,. \tag{44}$$

Proof. A simple computation shows that

$$f_\Sigma(m(\ell,\widetilde{\ell})) = f_\Sigma(m(\widetilde{\ell})) + \frac{p^2}{4}\ell(\ell+1-m_{\rm l})\sum_{j=1}^{r}\frac{1}{\cosh^2\frac{\beta_j}{2}}\,.$$

The stated equality follows then from Lemma 4.3. ■

Proposition 4.5. In the above notation, $\mathbb{D}_{\ell,\widetilde{\ell}}(m) = u^{-\ell}v^{-\widetilde{\ell}} \circ \mathbb{D}(m(\ell,\widetilde{\ell})) \circ u^{\ell}v^{\widetilde{\ell}}$ is a commutative subalgebra of $\mathbb{D}_{\mathcal{R}_0}^W$ of rank r. It is the centralizer of $D_{\ell,\widetilde{\ell},p_L}$ in $\mathbb{D}_{\mathcal{R}_0}^W$. As a consequence, $\mathbb{D}_{\ell,\widetilde{\ell}}(m) = \mathbb{D}_{-\ell+m_{\rm l}-1,\widetilde{\ell}}(m)$. In particular, $\mathbb{D}_{\ell,\widetilde{\ell}}(m) = \mathbb{D}_{-\ell,\widetilde{\ell}}(m)$ if $m_{\rm l} = 1$.

If $\widetilde{\ell} = 0$, then $\mathcal{R}_0$ can be replaced by $\mathcal{R}$. Furthermore, if $m = (m_{\rm s}, m_{\rm m}, m_{\rm l} = 1)$ is geometric and $\ell \in \mathbb{Z}$, $\widetilde{\ell} = 0$, then $\mathbb{D}_{\ell,0}(m) = \delta_\ell(\mathbb{D}(G/K;\tau_\ell))$.

Proof. Because of (38), (39) and since u and v are W-invariant, one can easily check that the conjugation by $u^\ell v^{\widetilde{\ell}}$ maps $\mathbb{D}_{\mathcal{R}_0}^W$ into itself and $\mathbb{D}_{\mathcal{R}}^W$ into itself if $\widetilde{\ell} = 0$. Now, the proof follows from the more general result in [18, Lemma 5.2], showing that the centralizers of $D_{p_L}(m(\ell,\widetilde{\ell}))$ in $\mathbb{D}_{\mathcal{R}_0}^W$ and $\mathbb{D}_{\mathcal{R}}^W$ agree, together with (44). Here, $\delta_\ell(\mathbb{D}(G/K,\tau_\ell)$ denotes the τ_ℓ-radial components of the commutative algebra $\mathbb{D}(G/K;\tau_\ell)$ (see Subsection 5.1). ■

4.3 $(\ell,\widetilde{\ell})$-Hypergeometric functions

Let $\lambda \in \mathfrak{a}_{\mathbb{C}}^*$ be fixed. The *$(\ell,\widetilde{\ell})$-Heckman–Opdam hypergeometric function with spectral parameter* λ is the unique W-invariant analytic solution $F_{\ell,\widetilde{\ell},\lambda}(m)$ of the system of differential equations

$$D_{\ell,\widetilde{\ell},p}(m)f = p(\lambda)f \qquad (p \in \mathrm{S}(\mathfrak{a}_{\mathbb{C}})^W), \tag{45}$$

which satisfies $f(0) = 1$.

The *nonsymmetric $(\ell,\widetilde{\ell})$-hypergeometric function with spectral parameter* λ is the unique analytic solution $G_{\ell,\widetilde{\ell},\lambda}(m)$ of the system of differential equations

$$T_{\ell,\widetilde{\ell},\xi}(m)g = \lambda(\xi)g \qquad (\xi \in \mathfrak{a}), \tag{46}$$

which satisfies $g(0) = 1$.

The symmetric and nonsymmetric $(\ell,\widetilde{\ell})$-Heckman–Opdam hypergeometric functions are therefore (suitably normalized) joint eigenfunctions of the commutative algebras $\mathbb{D}_{\ell,\widetilde{\ell}}(m)$ and $\{T_{\ell,\widetilde{\ell},p} : p \in S(\mathfrak{a}_{\mathbb{C}})\}$, respectively. Notice that the equality for $\mathbb{D}_{\ell,\widetilde{\ell}}(m)$ in Proposition 4.5 yields

$$F_{\ell,\widetilde{\ell},\lambda}(m) = F_{-\ell+m_l-1,\widetilde{\ell},\lambda}(m) \qquad (\lambda \in \mathfrak{a}_{\mathbb{C}}^*). \tag{47}$$

On the other hand, an analogous symmetry relation is generally not true for $G_{\ell,\widetilde{\ell},\lambda}(m)$ as can be seen from rank one examples. We omit the details.

By definition, $F_{\ell,\widetilde{\ell},\lambda}(m;x)$ is W-invariant in $x \in \mathfrak{a}$ and in $\lambda \in \mathfrak{a}_{\mathbb{C}}^*$. Furthermore, since $u^\ell v^{\widetilde{\ell}} \circ D_{\ell,\widetilde{\ell},p}(m) \circ u^{-\ell}v^{-\widetilde{\ell}} = D_p(m(\ell,\widetilde{\ell}))$ and $u^\ell v^{\widetilde{\ell}} \circ T_{\ell,\widetilde{\ell},\xi}(m) \circ u^{-\ell}v^{-\widetilde{\ell}} = T_\xi(m(\ell,\widetilde{\ell}))$, one obtains for all $\lambda \in \mathfrak{a}_{\mathbb{C}}^*$,

$$F_{\ell,\widetilde{\ell},\lambda}(m) = u^{-\ell}v^{-\widetilde{\ell}}F_\lambda(m(\ell,\widetilde{\ell})), \tag{48}$$

$$G_{\ell,\widetilde{\ell},\lambda}(m) = u^{-\ell}v^{-\widetilde{\ell}}G_\lambda(m(\ell,\widetilde{\ell})). \tag{49}$$

As in the case $\ell = \widetilde{\ell} = 0$,

$$F_{\ell,\widetilde{\ell},\lambda}(m;x) = \frac{1}{|W|}\sum_{w\in W} G_{\ell,\widetilde{\ell},\lambda}(m;w^{-1}x) \qquad (x \in \mathfrak{a}). \tag{50}$$

As a consequence of (48) and of the corresponding properties of the Heckman–Opdam hypergeometric functions (see, e.g., [11, Theorem 4.4.2] and [21, Theorem 3.15]), the $F_{\ell,\widetilde{\ell},\lambda}(m;x)$ and $G_{\ell,\widetilde{\ell},\lambda}(m;x)$ are entire functions of $\lambda \in \mathfrak{a}_{\mathbb{C}}^*$, analytic functions in $x \in \mathfrak{a}$ and meromorphic functions of $m = (m_s, m_m, m_l) \in \mathbb{C}^3$. Let $\mathcal{M}_{\mathbb{C},0} = \{m = (m_s, m_m, m_l) \in \mathbb{C}^3 : \mathrm{Re}(m_s + m_l) \geq 0\}$. Observe that $m \in \mathcal{M}_{\mathbb{C},0}$ if and only if $m(\ell) \in \mathcal{M}_{\mathbb{C},0}$. One can show (see Appendix A below) that for fixed $(\lambda, x) \in \mathfrak{a}_{\mathbb{C}}^* \times \mathfrak{a}$, the functions $F_\lambda(m;x)$ and $G_\lambda(m;x)$ are holomorphic in a neighborhood of $\mathcal{M}_{\mathbb{C},0}$. It follows that $F_{\ell,\widetilde{\ell},\lambda}(m;x)$ and $G_{\ell,\widetilde{\ell},\lambda}(m;x)$ are holomorphic near each $m = (m_s, m_m, m_l) \in \mathcal{M}_0$.

4.4 Estimates and asymptotics for the $(\ell,\widetilde{\ell})$-hypergeometric functions

Let $m = (m_s, m_m, m_l) \in \mathcal{M}_+$. Using (48) and (49), we can obtain from the results of Section 3 some estimates and asymptotic properties for the $(\ell,\widetilde{\ell})$-hypergeometric functions. The factors $u^{-\ell}$ and $v^{-\widetilde{\ell}}$ require some additional care for the asymptotics.

By Lemma 4.1, we have $m(\ell,\widetilde{\ell}) \in \mathcal{M}_+ \cup \mathcal{M}_3$ for $m \in \mathcal{M}_+$ and $\ell \in [\ell_{\min}(m), \ell_{\max}(m)], \widetilde{\ell} \geq -\frac{m_m}{2}$. This implies that Proposition 3.5, Theorem 3.6, and

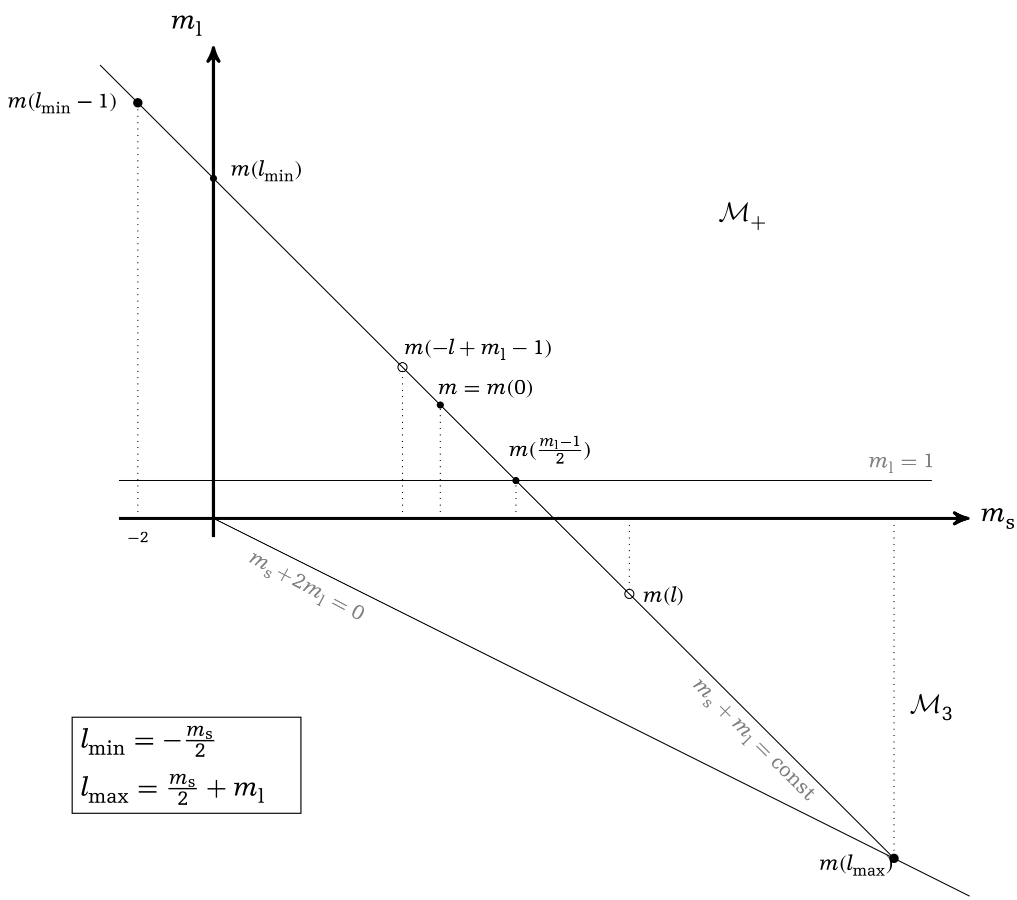

Fig. 3. Symmetries of $m(\ell)$ around $m(\frac{m_l-1}{2})$ if $m_l \geq 1$.

Corollary 3.7 hold true for $F_\lambda(m(\ell,\widetilde{\ell}))$ for all $m \in \mathcal{M}_+$ and $\ell \in \,]\ell_{\min}(m), \ell_{\max}(m)[$. In turn, (48) yields analogous statements for the $(\ell,\widetilde{\ell})$-hypergeometric functions. Recall that $\ell_{\min} = -\frac{m_s}{2}$ and $\ell_{\max} = \frac{m_s}{2} + m_l$. (We are omitting from the notation the dependence on m of $\ell_{\min}$ and $\ell_{\max}$). So $\ell_{\min} \leq 0 < \ell_{\max}$.

Recalling also that $F_{\ell,\widetilde{\ell},\lambda} = F_{-\ell+l-1,\widetilde{\ell},\lambda}$, we see that we can extend the inequalities for $F_{\ell,\widetilde{\ell},\lambda}$ to $\ell \in]-\frac{m_s}{2}-1, \frac{m_s}{2}+m_l[$ (or to $\ell \in [-\frac{m_s}{2}-1, \frac{m_s}{2}+m_l]$ where the extension by continuity in the multiplicity parameter is possible; see Theorem A.1 in the appendix). For a fixed multiplicity $m = (m_s, m_m, m_l)$, the symmetries of $F_{\ell,\widetilde{\ell},\lambda}$ in the ℓ-parameter can be pictured as symmetries of the deformations $m(\ell)$ around the point $m(\frac{m_l-1}{2})$, as in Figure 3.

Similarly, the inequalities of Lemma 3.4 extend to any $\ell \in \mathbb{R}$ because either the condition $\ell \le \ell_{\max}$ or $-\ell + m_l - 1 \le \ell_{\max}$ is always satisfied. In the 1st case, $m(\ell, \widetilde{\ell}) \in \mathcal{M}_2 \cup \mathcal{M}_3$; in the other, $m(-\ell + m_l + 1, \widetilde{\ell}) \in \mathcal{M}_2 \cup \mathcal{M}_3$.

We leave to the reader the simple task of modifying the statements of Theorem 3.6 and Corollary 3.7 in this case, and we collect the estimates obtained for $G_{\ell,\widetilde{\ell},\lambda}(m)$ and $F_{\ell,\widetilde{\ell},\lambda}(m)$ in the corollary below.

Corollary 4.6. Let $m = (m_s, m_m, m_l) \in \mathcal{M}_+$ be a nonnegative multiplicity function on a root system of type BC_r, and let $\ell, \widetilde{\ell} \in \mathbb{R}$, $\widetilde{\ell} \ge -\frac{m_m}{2}$. Then, the following properties hold.

(a) For all $\ell \le \ell_{\max}(m)$, $\lambda \in \mathfrak{a}_{\mathbb{C}}^*$ and $x \in \mathfrak{a}$,

$$|G_{\ell,\widetilde{\ell},\lambda}(m;x)| \le \sqrt{|W|}\, u^{-\ell}(x) v^{-\widetilde{\ell}}(x) e^{\max_w (w\lambda)(x)}, \tag{51}$$

and for all $\ell \in \mathbb{R}$, $\lambda \in \mathfrak{a}_{\mathbb{C}}^*$ and $x \in \mathfrak{a}$,

$$|F_{\ell,\widetilde{\ell},\lambda}(m;x)| = |F_{-\ell+m_l-1,\lambda}(m;x)| \le \sqrt{|W|}\, u^{-\ell}(x) v^{-\widetilde{\ell}}(x) e^{\max_w (w\lambda)(x)}. \tag{52}$$

(b) Suppose $\ell \in [\ell_{\min} - 1, \ell_{\max}]$. For all $\lambda \in \mathfrak{a}^*$, the function $F_{\ell,\widetilde{\ell},\lambda}(m)$ is real and strictly positive. For all $\lambda \in \mathfrak{a}_{\mathbb{C}}^*$,

$$|F_{\ell,\widetilde{\ell},\lambda}(m)| \le F_{\ell,\widetilde{\ell},\mathrm{Re}\,\lambda}(m). \tag{53}$$

Moreover, for all $\lambda \in \mathfrak{a}^*$ and all $x \in \mathfrak{a}$,

$$F_{\ell,\widetilde{\ell},\lambda}(m;x) \le F_{\ell,\widetilde{\ell},0}(m;x) e^{\max_w (w\lambda)(x)}. \tag{54}$$

More generally, for all $\lambda \in \mathfrak{a}^*$, $\mu \in \overline{(\mathfrak{a}^*)^+}$, and all $x \in \mathfrak{a}$,

$$F_{\ell,\widetilde{\ell},\lambda+\mu}(m;x) \le F_{\ell,\widetilde{\ell},\mu}(m;x) e^{\max_w (w\lambda)(x)}. \tag{55}$$

The same properties hold for $G_{\ell,\widetilde{\ell},\lambda}(m)$ provided $\ell \in [\ell_{\min}, \ell_{\max}]$.

(c) Suppose $\ell \in [\ell_{\min} - 1, \ell_{\max}]$, $\widetilde{\ell} \ge 0$, $\lambda \in \overline{(\mathfrak{a}^*)^+}$, and $x_1 \in \mathfrak{a}^+$. Further, assume that $m_l \ge 1$. Then, for all $x \in \mathfrak{a}$, we have

$$F_{\ell,\widetilde{\ell},\lambda}(m;x+x_1) e^{-(\lambda+\rho(m(2\widetilde{\ell})))(x_1)} \le F_{\ell,\widetilde{\ell},\lambda}(m;x) \le F_{\ell,\widetilde{\ell},\lambda}(m;x+x_1) e^{(\lambda+\rho(m(2\widetilde{\ell})))(x_1)}. \tag{56}$$

(d) Suppose $\ell \in]\ell_{\min}, \ell_{\max}[$, $\widetilde{\ell} > 0$ if $m_{\rm m} = 0$, $\widetilde{\ell} \geq 0$ if $m_{\rm m} > 0$, and $\lambda \in \overline{(\mathfrak{a}^*)^+}$. Then, for all $x \in \overline{\mathfrak{a}^+}$, we have

$$F_{\ell,\widetilde{\ell},\lambda}(m;x) \quad \asymp \quad \Big[\prod_{\alpha\in\Sigma_\lambda^0}(1+\alpha(x))\Big]e^{(\lambda-\rho(m(2\widetilde{\ell})))(x)}, \tag{57}$$

where $\Sigma_\lambda^0 = \{\alpha \in \Sigma_{\rm s}^+ \cup \Sigma_{\rm m}^+ : \langle\alpha,\lambda\rangle = 0\}$. The asymptotics (57) extend to $\ell \in]\ell_{\min}-1, \ell_{\max}[$ if $m_{\rm s}+m_{\rm l} \geq 1$.

Proof. We prove (c) and (d). For (c), first, assume that $\ell \in [\ell_{\min}, \ell_{\max}]$. Since $F_{\ell,\widetilde{\ell},\lambda} = F_{-\ell+m_{\rm l}-1,\widetilde{\ell},\lambda}$, we may assume that $\ell \geq \frac{m_{\rm l}-1}{2} \geq 0$. To keep track of the exponents appearing in our formulas, it will be convenient to introduce the following notation:

$$\sum_{1\leq i<j\leq r}(\beta_j \pm \beta_i) = \sum_{1\leq i<j\leq r}(\beta_j + \beta_i) + \sum_{1\leq i<j\leq r}(\beta_j - \beta_i) = \sum_{j=1}^{r} 2(j-1)\beta_j. \tag{58}$$

For $a \geq 0$ and $b \geq 0$,

$$\cosh a \cosh b \leq \cosh(a+b) \leq \cosh a\, e^b.$$

Hence, if $\ell \geq 0$, $\widetilde{\ell} \geq 0$, and x, x_1 are as above,

$$\begin{aligned} u^\ell(x)u^\ell(x_1) &\leq u^\ell(x+x_1) \leq u^\ell(x)e^{\frac{\ell}{2}\sum_{j=1}^r \beta_j(x_1)}, \\ v^{\widetilde{\ell}}(x)v^{\widetilde{\ell}}(x_1) &\leq v^{\widetilde{\ell}}(x+x_1) \leq v^{\widetilde{\ell}}(x)e^{\frac{\widetilde{\ell}}{2}\sum_{1\leq i<j\leq r}(\beta_j\pm\beta_i)(x_1)}. \end{aligned} \tag{59}$$

Notice that

$$\rho(m(2\widetilde{\ell})) = \rho(m(\ell,\widetilde{\ell})) + \frac{\ell}{2}\sum_{j=1}^{r}\beta_j + \frac{\widetilde{\ell}}{2}\sum_{1\leq i<j\leq r}(\beta_j \pm \beta_i). \tag{60}$$

By (31), which applies since $m(\ell,\widetilde{\ell}) \in \mathcal{M}_3$,

$$F_\lambda(m(\ell,\widetilde{\ell}); x+x_1)e^{-(\lambda+\rho(m(\ell,\widetilde{\ell})))(x_1)} \leq F_\lambda(m(\ell,\widetilde{\ell}); x) \leq F_\lambda(m(\ell,\widetilde{\ell}); x+x_1)e^{(\lambda+\rho(m(\ell,\widetilde{\ell})))(x_1)}.$$

Multiplying each term of the inequality by $u(x)^{-\ell}v(x)^{-\widetilde{\ell}}$ and using the definition of $F_{\ell,\widetilde{\ell},\lambda}$, we obtain

$$\frac{u^{\ell}(x+x_1)v^{\widetilde{\ell}}(x+x_1)}{u^{\ell}(x)v^{\widetilde{\ell}}(x)}F_{\ell,\widetilde{\ell},\lambda}(m;x+x_1)e^{-(\lambda+\rho(m(\ell,\widetilde{\ell})))(x_1)}$$
$$\leq F_{\ell,\widetilde{\ell},\lambda}(m;x) \leq \frac{u^{\ell}(x+x_1)v^{\widetilde{\ell}}(x+x_1)}{u^{\ell}(x)v^{\widetilde{\ell}}(x)}F_{\ell,\widetilde{\ell},\lambda}(m;x+x_1)e^{(\lambda+\rho(m(\ell,\widetilde{\ell})))(x_1)}.$$

The right-hand side inequality in (c) follows from the 2nd inequalities in (59) and (60). Again, using (60) (and (58)), the expression on the left-hand side inequality of (d) becomes

$$\frac{u^{\ell}(x+x_1)v^{\widetilde{\ell}}(x+x_1)}{u^{\ell}(x_1)v^{\widetilde{\ell}}(x_1)}e^{\frac{\ell}{2}\sum_{j=1}^{r}\beta_j(x_1)}e^{\frac{\widetilde{\ell}}{2}\sum_{1\leq i<j\leq r}(\beta_j\pm\beta_i)(x_1)}F_{\ell,\widetilde{\ell},\lambda}(m;x+x_1)e^{-(\lambda+\rho(m(2\widetilde{\ell})))(x_1)}.$$

This implies (c) as

$$\frac{u^{\ell}(x+x_1)v^{\widetilde{\ell}}(x+x_1)}{u^{\ell}(x_1)v^{\widetilde{\ell}}(x_1)}e^{\frac{\ell}{2}\sum_{j=1}^{r}\beta_j(x_1)}e^{\frac{\widetilde{\ell}}{2}\sum_{1\leq i<j\leq r}(\beta_j\pm\beta_i)(x_1)} \geq 1.$$

For $\ell \in [\ell_{\min}-1, \ell_{\min}[$, notice that $\ell_{\min} \leq -\ell+m_1-1 \leq \ell_{\max}$ as $m_1 \geq 1$, and use the equality $F_{\ell,\widetilde{\ell},\lambda} = F_{-\ell+m_1-1,\widetilde{\ell},\lambda}$. Next, we prove (d). For $a \geq 0$, we have $\cosh a \asymp e^{a}$. Hence,

$$u^{-\ell}(x) \asymp e^{-\frac{\ell}{2}\sum_{j=1}^{r}\beta_j(x)} v^{-\widetilde{\ell}}(x) \quad \asymp e^{\frac{-\widetilde{\ell}}{2}\sum_{1\leq i<j\leq r}(\beta_j\pm\beta_i)(x)}. \tag{61}$$

Since $m(\ell,\widetilde{\ell}) \in \mathcal{M}_3^0$, we have by Theorem 3.10,

$$F_{\lambda}(m(\ell,\widetilde{\ell});x) \asymp \Big[\prod_{\alpha\in\Sigma_{\lambda}^{0}}(1+\alpha(x))\Big]e^{(\lambda-\rho(m(\ell,\widetilde{\ell})))(x)}.$$

Then, (57) follows by multiplying both sides of this asymptotics by $u^{-\ell}(x)v^{-\widetilde{\ell}}(x)$, together with (61) and (60).

If $\ell \in]\ell_{\min}-1, \ell_{\min}[$, then $-\ell+m_1-1 \in]-\ell_{\min}+m_1-1, \ell_{\max}[$. The lower bound satisfies $-\ell_{\min}+m_1-1 \geq \ell_{\min}$ if and only if $m_s+m_1 \geq 1$. In this case, the above asymptotics hold for $-l+m_1-1$ as well. They lead to (57) using $F_{\ell,\widetilde{\ell},\lambda}(m) = F_{-\ell+m_1-1,\widetilde{\ell},\lambda}(m)$. ■

4.5 Bounded $(\ell,\widetilde{\ell})$-hypergeometric functions

In this section, we address the problem of characterizing the $(\ell,\widetilde{\ell})$-hypergeometric functions that are bounded. Recall that we are considering a multiplicity function $m = (m_{\rm s}, m_{\rm m}, m_{\rm l}) \in \mathcal{M}_+$ and its deformation $m(\ell,\widetilde{\ell})$ as in (33). To apply the estimates from Subsection 4.4, we will assume that $\ell \in [\ell_{\min}, \ell_{\max}]$. The asymptotics from Theorem 3.6 hold under the stronger assumption that $\ell \in]\ell_{\min}, \ell_{\max}[$.

Recall also that

$$\rho(m(\ell,\widetilde{\ell})) = \frac{1}{2}\sum_{\alpha\in\Sigma^+} m_\alpha(\ell,\widetilde{\ell})\alpha = \rho(m) - \frac{\ell}{2}\sum_{j=1}^{r}\beta_j + \frac{\widetilde{\ell}}{2}\sum_{1\le i<j\le r}(\beta_j \pm \beta_i),$$

where $\sum_{1\le i<j\le r}(\beta_j \pm \beta_i)$ is given by (58), and so,

$$\rho(m(2\widetilde{\ell})) = \rho(m(\ell,\widetilde{\ell})) + \frac{\ell}{2}\sum_{j=1}^{r}\beta_j + \frac{\widetilde{\ell}}{2}\sum_{1\le i<j\le r}(\beta_j \pm \beta_i).$$

Notice that, if we suppose $\ell \in [0, \ell_{\max}], \widetilde{\ell} \ge -\frac{m_{\rm s}}{2}$, then $m(\ell,\widetilde{\ell}) \in \mathcal{M}_+ \cup \mathcal{M}_3$ by Lemma 4.1. Hence, $\rho(m(\ell,\widetilde{\ell})) \in \overline{(\mathfrak{a}^*)^+}$. Notice also that $\rho(m(2\widetilde{\ell})) \ge \rho(m)$ if $\widetilde{\ell} \ge 0$.

Theorem 4.7. Assume that $m_{\rm l} \ge 1$ and $\widetilde{\ell} \ge 0$ if $m_{\rm m} > 0$, $\widetilde{\ell} > 0$ if $m_{\rm m} = 0$. Fix $\ell \in \mathbb{R}$ with $\ell \in]\ell_{\min} - 1, \ell_{\max}[$. Then, the $(\ell,\widetilde{\ell})$-hypergeometric function $F_{\ell,\widetilde{\ell},\lambda}(m)$ is bounded if and only if $\lambda \in C(\rho(m(2\widetilde{\ell}))) + i\mathfrak{a}^*$, where $C(\rho(m(2\widetilde{\ell})))$ is the convex hull of the set $\{w\rho(m(2\widetilde{\ell})) :\ w \in W\}$.

Proof. First, note that $m(\ell,\widetilde{\ell}) \in \mathcal{M}_1$. Since $F_{\ell,\widetilde{\ell},\lambda} = F_{-\ell+m_{\rm l}-1,\widetilde{\ell},\lambda}$, we may assume that $0 \le \frac{m_{\rm l}-1}{2} \le \ell < \ell_{\max}$. We show that $F_{\ell,\widetilde{\ell},\lambda}$ is bounded if $\lambda \in C(\rho(m(2\widetilde{\ell}))) + i\mathfrak{a}^*$. Let $\lambda \in C(\rho(m(2\widetilde{\ell}))) + i\mathfrak{a}^*$, and consider the holomorphic function $\lambda \to F_{\ell,\widetilde{\ell},\lambda}(m;x)$ (for a fixed x). Since

$$|F_{\ell,\widetilde{\ell},\lambda}(m;x)| \le F_{\ell,\widetilde{\ell},\mathrm{Re}\,\lambda}(m;x) = u^{-\ell}(x)v^{-\widetilde{\ell}}(x)F_{\mathrm{Re}\,\lambda}(m(\ell,\widetilde{\ell});x),$$

we can argue as in the proof of [16, Theorem 4.2] to obtain that the maximum of $|F_{\ell,\widetilde{\ell},\lambda}(m;x)|$ is attained at $\{w\rho(m(2\widetilde{\ell})) :\ w \in W\}$. That is,

$$|F_{\ell,\lambda}(m;x)| \le u^{-\ell}(x)v^{-\widetilde{\ell}}(x)F_{\rho(m(2\widetilde{\ell}))}(m(\ell,\widetilde{\ell});x).$$

As noticed before, $\rho(m(\ell,\widetilde{\ell})) \in \overline{(\mathfrak{a}^*)^+}$. Applying (22), we obtain

$$F_{\rho(m(2\widetilde{\ell}))}(m(\ell,\widetilde{\ell});x) = F_{\rho(m(\ell,\widetilde{\ell}))+\frac{\ell}{2}\sum_{j=1}^r \beta_j+\frac{\widetilde{\ell}}{2}\sum_{1\le i<j\le r}(\beta_j\pm\beta_i)}(m(\ell,\widetilde{\ell});x)$$
$$\le F_{\rho(m(\ell,\widetilde{\ell}))}(m(\ell,\widetilde{\ell});x)e^{\max_w w\left[\frac{\ell}{2}\sum_{j=1}^r \beta_j(x)+\frac{\widetilde{\ell}}{2}\sum_{1\le i<j\le r}(\beta_j\pm\beta_i)(x)\right]}.$$

Now, $F_{\rho(m(\ell,\widetilde{\ell}))}(m(\ell,\widetilde{\ell});x) = 1$; see, for example, [16, Lemma 4.1] (the proof extends to every multiplicity function for which the symmetric and nonsymmetric hypergeometric functions are defined). Choose x_0 in the W-orbit of x so that

$$\max_w w\left[\frac{\ell}{2}\sum_{j=1}^r \beta_j(x)+\frac{\widetilde{\ell}}{2}\sum_{1\le i<j\le r}(\beta_j\pm\beta_i)(x)\right] = \frac{\ell}{2}\sum_{j=1}^r \beta_j(x_0)+\frac{\widetilde{\ell}}{2}\sum_{1\le i<j\le r}(\beta_j\pm\beta_i)(x_0).$$

Since u and v are W-invariant, from the above, we get

$$|F_{\ell,\widetilde{\ell},\lambda}(m;x)| \le u^{-\ell}(x_0)v^{-\widetilde{\ell}}(x_0)e^{\frac{\ell}{2}\sum_{j=1}^r \beta_j(x_0)}e^{\frac{\widetilde{\ell}}{2}\sum_{1\le i<j\le r}(\beta_j\pm\beta_i)(x_0)} \le 2^M,$$

where M is a constant depending on $\ell, \widetilde{\ell}$ and r.

To prove the other way, we proceed as in [16] (see pages 251 and 252). Let λ_0 be such that $\mathrm{Re}\,\lambda_0 \in \overline{(\mathfrak{a}^*)^+} \setminus C(\rho(m(2\widetilde{\ell})))$. Let $x_1 \in \mathfrak{a}^+$ be such that $(\mathrm{Re}\,\lambda_0 - \rho(m(2\widetilde{\ell})))(x_1) > 0$. If $F_{\ell,\widetilde{\ell},\lambda_0}(m)$ is bounded, we have

$$\lim_{t\to\infty} F_{\ell,\widetilde{\ell},\lambda_0}(m;tx_1)e^{-t(\mathrm{Re}\,\lambda_0-\rho(m(2\widetilde{\ell})))(x_1)}t^{-d} = 0.$$

Since, $F_{\ell,\widetilde{\ell}\lambda_0}(m;x) = u^{-\ell}(x)v^{-\widetilde{\ell}}(x)F_{\lambda_0}(m(\ell,\widetilde{\ell});x)$ and $\cosh a$ behaves like e^a for large positive a, we have

$$F_{\lambda_0}(m(\ell,\widetilde{\ell});tx_1)e^{-t(\mathrm{Re}\,\lambda_0-\rho(m(\ell,\widetilde{\ell})))(x_1)}t^{-d} \to 0 \qquad t\to\infty.$$

So, by Theorem 3.6 (which is applicable as $m(\ell,\widetilde{\ell}) \in \mathcal{M}_1$), the proof can be completed as in [16, Theorem 4.2]. ■

Remark 4.8. The proof of Theorem 4.7 shows that the $(\ell,\widetilde{\ell})$-hypergeometric function $F_{\ell,\widetilde{\ell},\lambda}(m)$ is bounded for $\lambda \in C(\rho(m(2\widetilde{\ell}))) + i\mathfrak{a}^*$, provided $\ell \in [\ell_{\min}-1, \ell_{\max}]$ and $\widetilde{\ell} \ge 0$. For general root systems of type BC_r, we cannot prove that when $\ell = \ell_{\max}$ the function $F_{\ell,\widetilde{\ell},\lambda}(m)$ is not bounded for $\lambda \notin C(\rho(m(2\widetilde{\ell}))) + i\mathfrak{a}^*$. This is because the function b_0 appearing in Theorem 3.6 might vanish. However, the above theorem continues to hold for $\ell = \ell_{\max}$ provided $r = 1$. Indeed, for $\ell = \ell_{\max}$, we have $m_s(\ell) + 2m_l(\ell) = 0$ and, by (26), the function $b_0(m(\ell_{\max}); \lambda_0)$ vanishes for $\lambda_0 \in \mathfrak{a}^*_{\mathbb{C}}$ with $\operatorname{Re}\lambda_0 \in \overline{(\mathfrak{a}^*)^+}$, if and only if there is $j \in \{1, \ldots, r\}$ such that $(\lambda_0)_{\beta_j} = 0$. Outside these λ_0, the asymptotics from Theorem 3.6 still hold. In the rank-one case, there is only one λ_0 for which the asymptotics do not hold, namely $\lambda_0 = 0$, which is not outside $C(\rho(m)) + i\mathfrak{a}^*$. So the above result holds under the assumption $\ell \in [\ell_{\min}, \ell_{\max}]$, in the rank-one case.

Remark 4.9. It is natural to expect that $F_{\ell,\widetilde{\ell},\lambda}$ is bounded by one if $\lambda \in C(\rho(m(2\widetilde{\ell}))) + i\mathfrak{a}^*$; though we are not able to prove this. However, this holds true for the geometric case as can be seen from Corollary 5.2 and from the integral formula (66) together with the case-by-case computation that $\rho_{G/K} = \rho(m(2\widetilde{\ell}))$; see Subsection 5.2.

5 Some Geometric Cases

5.1 Spherical functions on line bundles over Hermitian symmetric spaces

In this subsection, we shall assume that $\widetilde{\ell} = 0$ and consequently suppress the index $\widetilde{\ell}$ from the notation.

Let $m = (m_s, m_m, m_l = 1)$ be a geometric multiplicity function corresponding to a non-compact Hermitian symmetric space G/K, as in Example 2.1. Let τ_ℓ be a fixed one-dimensional unitary representations of K. So $\ell \in \mathbb{Z}$ (or $\ell \in \mathbb{R}$ by passing to universal covering spaces). Let E_ℓ denote the homogeneous line bundle on G/K associated with τ_ℓ. The smooth τ_ℓ-spherical sections of E_ℓ can be identified with the space $C^\infty(G//K; \tau_\ell)$ of functions on G satisfying $f(k_1 g k_2) = \tau_{-\ell}(k_1 k_2) f(g)$ for all $g \in G$ and $k_1, k_2 \in K$. Recall our convention of identifying the Cartan subspace $\mathfrak{a}$ with its diffeomorphic image $A = \exp(\mathfrak{a}) \subset G$. So every element $\varphi \in C^\infty(G//K; \tau_\ell)$ is uniquely determined by its Weyl group-invariant restriction $\varphi|_{\mathfrak{a}}$ to $A = \mathfrak{a}$. Recall also the notation $\mathbb{D}(G/K; \tau_\ell)$ for the (commutative) algebra of the G-invariant differential operators on E_ℓ. Then, the τ_ℓ-spherical functions on G/K are the (suitably normalized) joint eigenfunctions of $\mathbb{D}(G/K; \tau_\ell)$ belonging to $C^\infty(G//K; \tau_\ell)$. For a fixed ℓ, they are parametrized by $\lambda \in \mathfrak{a}^*_{\mathbb{C}}$ (modulo the Weyl group). We denote by $\varphi_{\ell,\lambda}$ the τ_ℓ-spherical function of spectral parameter λ.

Many authors have studied the τ_ℓ-spherical functions by relating them to hypergeometric functions on root systems. See, for example, [11, Sections 5.2–5.5], [28, Section 6], [29, Sections 3 and 5], and [14]. Indeed, $\mathbb{D}_\ell(m) = \delta_\ell(\mathbb{D}_\ell(G/K))$, where δ_ℓ denotes the τ_ℓ-radial component in the sense of Harish-Chandra [10]; see [18, Section 3.6]. More specifically,

$$D_{\ell,p_L} = \delta_\ell(\Omega) - \tau_{-\ell}(\Omega_{\mathfrak{m}}), \tag{62}$$

where Ω and $-\Omega_{\mathfrak{m}}$ are the Casimir operator on $\mathfrak{g}_{\mathbb{C}}$ and $\mathfrak{m}_{\mathbb{C}}$, respectively. See [29, Lemma 2.4] and [18, Section 3.5]. Hence,

$$\varphi_{\ell,\lambda}|_{\mathfrak{a}} = F_{\ell,\lambda}(m) = u^{-\ell} F_\lambda(m(\ell)). \tag{63}$$

Since $m_1 = 1$, the classical symmetry $\varphi_{\ell,\lambda} = \varphi_{-\ell,\lambda}$ is a special case of the symmetry $F_{\ell,\lambda} = F_{-\ell+m_1-1,\lambda}$ from (47).

In the usual parametrization, the trivial representation of K corresponds to $\ell = 0$. In this case, $C^\infty(G//K;\tau_0)$ is the space of the smooth K-invariant functions on G/K, $\mathbb{D}_0(G/K)$ coincides with the algebra $\mathbb{D}(G/K)$ of G-invariant differential operators on G/K, and the τ_0-spherical functions $\varphi_{0,\lambda}$ are precisely Harish-Chandra's spherical functions φ_λ on G/K.

The following proposition summarizes the basic properties of the τ_ℓ-spherical functions. As in the K-biinvariant case, they can be explicitly given by an integral formula, which extends to arbitrary real ℓs the classical integral formula by Harish-Chandra's for the spherical function $\varphi_\lambda = \varphi_{0,\lambda}$ on G/K.

Proposition 5.1. For $\ell \in \mathbb{R}$ and $\lambda \in \mathfrak{a}^*_{\mathbb{C}}$, set

$$\varphi_{\ell,\lambda}(g) = \int_{K/Z(G)} e^{(\lambda-\rho)(H(gk))} \tau_\ell(k\kappa(gk)^{-1})\, \mathrm{d}k, \qquad g \in G. \tag{64}$$

Then, the set of functions $\varphi_{\ell,\lambda}$, $\lambda \in \mathfrak{a}^*_{\mathbb{C}}$ exhausts the class of (elementary) τ_ℓ-spherical functions on G. Two such functions $\varphi_{\ell,\lambda}$ and $\varphi_{\ell,\mu}$ are equal if and only if $\mu = w\lambda$ for some $w \in W$. Moreover, for a fixed $g \in G$, $\varphi_{\ell,\lambda}(g)$ is holomorphic in $(\lambda,\ell) \in \mathfrak{a}^*_{\mathbb{C}} \times \mathbb{C}$. Furthermore, $\varphi_{\ell,\lambda} = \varphi_{-\ell,\lambda}$.

Proof. See [28, Proposition 6.1]. ■

The integral formula together with the properties of Harish-Chandra's spherical functions φ_λ ($\ell = 0$ case) automatically implies several of the properties of the τ_ℓ-spherical functions we are studying in this paper. Nevertheless, others, such as their positivity or the full characterization of the bounded spherical functions, do not seem to follow from (64). We collect all properties in the following corollary. Recall that $\ell_{\max} = \frac{m_s}{2} + 1$ because $m_l = 1$.

Corollary 5.2. Let $\ell \in \mathbb{R}$, then we have the following:

(1) $\varphi_{\ell,\lambda}$ is real valued for $\lambda \in \mathfrak{a}^*$;

(2) $\varphi_{\ell,\lambda}|_{\mathfrak{a}}$ is positive for $\lambda \in \mathfrak{a}^*$ and $|\ell| \le \ell_{\max}$;

(3) $|\varphi_{\ell,\lambda}| \le \varphi_{\mathrm{Re}\,\lambda}$ for $\lambda \in \mathfrak{a}^*_{\mathbb{C}}$;

(4) $|\varphi_{\ell,\lambda}(m)| \le 1$ for $\lambda \in C(\rho(m)) + i\mathfrak{a}^*$, where $C(\rho(m))$ is the convex hull of the set $\{w\rho(m) : w \in W\}$;

(5) let $|\ell| < \ell_{\max}$. The τ_ℓ-spherical function $\varphi_{\ell,\lambda}(m)$ on G/K is bounded if and only if $\lambda \in C(\rho(m)) + i\mathfrak{a}^*$, where $C(\rho(m))$ is the convex hull of the set $\{w\rho(m) : w \in W\}$. Moreover, $|\varphi_{\ell,\lambda}(m; x)| \le 1$ for all $\lambda \in C(\rho(m)) + i\mathfrak{a}^*$ and $x \in \mathfrak{a}$.

Proof. The 1st property follows from the equality $\varphi_{\ell,\lambda} = \varphi_{-\ell,\lambda}$ because $\tau_\ell + \tau_{-\ell}$ is real valued. For part (2), by the symmetry in ℓ, we can suppose that $\ell \ge 0$. So $m(\ell) \in \mathcal{M}_1$. The positivity of $\varphi_{\ell,\lambda}|_{\mathfrak{a}}$ follows then from Propositions 3.1(a) and 3.5. Part (3) is a consequence of the integral formula, and the fact that τ_ℓ is a unitary character. Part (4) follows from (3) and the theorem by Helgason and Johnson characterizing the parameters λ for which Harish-Chandra's spherical functions φ_λ are bounded; see [13]. Finally, for (5), we can suppose that $\ell \ge 0$ so that $m(\ell) \in \mathcal{M}_1$. Then, (5) is an immediate consequence of (3) together with the 2nd part of Theorem 4.7. ■

5.2 τ-Spherical functions for other small K-types τ

We start by briefly recalling the main result from [18], which expresses spherical functions associated with small K-types as hypergeometric functions multiplied with an explicit function involving hyperbolic sines and cosines. We refer to [18] for the proofs of the properties mentioned here and for further information.

Let G be a non-compact connected real semisimple Lie group with finite center. Let $G = KAN$ be an Iwasawa decomposition, and let M be the centralizer of A in K. Recall that a K-type (τ, V_τ) (i.e., an irreducible representation of K) is called *small* if it is irreducible as an M-module. We denote by φ^τ_λ the τ-spherical function on G with spectral parameter $\lambda \in \mathfrak{a}^*$. Moreover, with a slight abuse of notation, we denote by $\varphi^\tau_\lambda|_{\mathfrak{a}}$ the W-invariant scalar function such that the restriction of φ^τ_λ to $A \equiv \mathfrak{a}$ is $\varphi^\tau_\lambda|_{\mathfrak{a}}\,\mathrm{id}$, where id denotes the identity operator on V_τ.

Theorem 5.3. (See [18, Theorem 1.6].) Suppose (τ, V) is a small K-type of a non-compact real simple Lie group G with finite center. If G is a simply connected split Lie group $\widetilde{G}_2$ of type G_2, we further suppose that τ is not the small type π_2 specified in [18, Theorem 2.2]. Then, there exists a root system $\Sigma(\tau)$ in $\mathfrak{a}^*$ and a multiplicity function $m(\tau)$ on $\Sigma(\tau)$ such that

$$\varphi_\lambda^\tau|_{\mathfrak{a}} = \widetilde{\delta}_{G/K}^{-\frac{1}{2}} \sim \widetilde{\delta}_{\Sigma(\tau)}(m(\tau))^{\frac{1}{2}} \sim F_\lambda(\Sigma(\tau), m(\tau))$$

for all $\lambda \in \mathfrak{a}^*_{\mathbb{C}}$.

In the above, $F_\lambda(\Sigma(\tau), m(\tau))$ stands for the Heckman–Opdam hypergeometric function associated with the triple $(\mathfrak{a}, \Sigma(\tau), m(\tau))$. Here, we recall that we use the symmetric space notation, and hence, our $\Sigma(\tau)$ and $m(\tau)$ are related to Σ^τ and $\mathbf{k}^\tau$ of [18] by $\Sigma^\tau = 2\Sigma(\tau)$ and $\mathbf{k}^\tau_{2\alpha} = \frac{m_\alpha(\tau)}{2}$. For any root system (Σ, m), the function $\widetilde{\delta}_\Sigma(m)$ is defined as

$$\widetilde{\delta}_\Sigma(m) = \prod_{\alpha \in \Sigma^+} \left| \frac{\sinh \alpha}{|\alpha|} \right|^{m_\alpha} \tag{65}$$

and $\widetilde{\delta}_{G/K}$ corresponds to the root system for G/K.

The classification of the small K-types τ and the concrete choices (which are one or two) of pairs $(\Sigma(\tau), m(\tau))$ occurring in Theorem 5.3 can be found in [18, Section 2]. Clearly, the trivial K-type is always small. If G/K is Hermitian, a K-type τ is small if and only if it is a one-dimensional unitary character τ_ℓ, as in Subsection 5.1. All other small K-types have dimension bigger than one.

As in the scalar and the Hermitian cases, the τ-spherical functions corresponding to a small K-type can be represented by the integral formula

$$\varphi_\lambda^\tau(g) = \int_K e^{(\lambda - \rho_{G/K})(H(gk))} \tau(k_K(gk))^{-1}\, \mathrm{d}k, \tag{66}$$

see [18, (3.7)], where $\rho_{G/K}$ is computed from the root system for G/K. Formula (66) is a special instance of [31, 9.1.5 and p. 300] and [3, (3.9)].

Based on the case-by-case analysis of [18, Section 2], we now show that every τ-spherical function for a symmetric space G/K with restricted root system of type BC and for which $\dim \tau \geq 2$ is a $(\ell, \widetilde{\ell})$-hypergeometric function for suitable choices of a multiplicity function $m = (m_{\mathrm{s}}, m_{\mathrm{m}}, m_{\mathrm{l}}) \in \mathcal{M}_+$, defined on a root subsystem of $\Sigma(\tau)$, and of values ℓ and $\widetilde{\ell}$. It follows, in particular, that m is in general different from both the original root multiplicity function of G/K and $m(\tau)$.

This identification, together with (66), allows us to prove an analogue of Corollary 5.2, in which one has to replace $\varphi_{\ell,\lambda}$ with $\varphi^{\tau}_{\lambda}|_{\mathfrak{a}}$ and the condition $|\ell| \leq \ell_{\max}$ by the conditions $\ell_{\min}(m) - 1 \leq \ell_{\max}(m)$ and $\widetilde{\ell} \geq -\frac{m_{\mathrm{m}}}{2}$. Notice that the condition on ℓ is equivalent to the more symmetric condition $\left|\ell - (m_{\mathrm{l}} - 1)/2\right| \leq (m_{\mathrm{s}} + m_{\mathrm{l}} + 1)/2$. The precise statement depends on the specific values of $\ell_{\min}(m) = -m_{\mathrm{s}}/2$ and $\ell_{\max}(m) = m_{\mathrm{s}}/2 + m_{\mathrm{l}}$ determined below. We leave to the reader the task of writing down the precise statement in each case. In fact, the symmetry (47) shows the multiplicity functions $m(\ell, \widetilde{\ell})$ and $m(-\ell + m_{\mathrm{l}} - 1, \widetilde{\ell})$ are different if $\ell \neq \frac{m_{\mathrm{l}}-1}{2}$ but yield to the same τ-spherical function.

As remarked in [18, Section 2], the classification of K-types can be given at the level of Lie algebras of non-compact type since a small K-type of G always lifts to that of a finite cover of G. We keep the convention that a root multiplicity 0 is equivalent to the fact that the corresponding root does not belong to the fixed root system.

5.2.1 *The case of* $\mathfrak{g} = \mathfrak{sp}(p,1)$, $p \geq 2$

Let $G = \mathrm{Sp}(p,1)$ (which is simply connected) and $K = \mathrm{Sp}(p) \times \mathrm{Sp}(1)$. The small K-types are precisely the representations of the form $\tau_n = 1 \otimes \tau'_n$, the tensor product of the trivial representation of $\mathrm{Sp}(p)$ and the n-dimensional irreducible representation of $\mathrm{Sp}(1) = \mathrm{SU}(2)$ with $n \in \mathbb{N}$. The original root system is given by $\Sigma = \{\pm\alpha, \pm 2\alpha\}$ (of type BC_1), with multiplicities $m_{\pm\alpha} = 4(p-1)$ and $m_{\pm 2\alpha} = 3$. For the small type τ_n, we have $\Sigma(\tau_n) = \Sigma$ and $m(\tau_n)_{\pm} = (4p - 2 \pm 2n, 1 \mp 2n)$; see [18, Section 2.3]. The associated spherical function is given by

$$\varphi^{\tau_n}_{\lambda}|_{\mathfrak{a}} = (\cosh \alpha)^{-1 \mp n}\, F_{\lambda}(\Sigma, m(\tau_n)_{\pm}). \tag{67}$$

Notice that $m(\tau_n)_+ = m(\ell_n)$ for $m = (4(p-1), 3) \in \mathcal{M}_+$ and $\ell_n = n + 1$. Moreover, $\ell_{\min} = -2(p-1)$ and $\ell_{\max} = 2p + 1$. So $m(\tau_n)_+ \in \mathcal{M}_+ \cup \mathcal{M}_3$ provided $n \leq 2p$ and $m(\tau_n)_+ \in \mathcal{M}_1$ if $n < 2p$. Notice that $\rho(m) = \rho_{G/K} = (2p+1)\alpha$.

Since $m(\tau_n)_- = m(-\ell + m_{\mathrm{l}} - 1)$, where $-\ell + m_{\mathrm{l}} - 1 = -n + 1$, the fact that the two multiplicity functions lead to the same τ_n-spherical function is a special instance of the symmetry (47).

5.2.2 *The case of* $\mathfrak{g} = \mathfrak{so}(2r,1)$, $r \geq 2$

Let $G = \mathrm{Spin}(2r,1)$ be the double cover of $\mathrm{SO}(2r,1)$ and $K = \mathrm{Spin}(2r)$. Then, G is simply connected. The nontrivial small K-types are the irreducible representations $\tau^{\pm}_s$ with highest weight $(s/2,\dots,s/2,\pm s/2)$ in standard notation, where $s\in\mathbb{N}$. The case $s=1$ corresponds to the positive and negative spin representations and the corresponding τ-spherical analysis was studied in [4].

The root system of G/K, say $\{\pm\alpha\}$, is of type A_1 and $m_\alpha=2r-1$. According to [18, Section 2.4], $\Sigma(\tau_s^\pm)=\{\pm\alpha/2,\pm\alpha\}$, $m(\tau_s^\pm)=(-2s,2r+2s-1)$ and

$$\varphi_\lambda^{\tau_s^\pm}|_{\mathfrak{a}}=\cosh^s\Big(\frac{\alpha}{2}\Big)\,F_\lambda(\Sigma(\tau_s^\pm),m(\tau_s^\pm)).$$

Then, $m(\tau_s^\pm)=m(\ell_s)$ for $m=(0,2r-1)\in\mathcal{M}_+$ on $\Sigma(\tau_s^\pm)$ and $\ell_s=-s$. Notice that $\ell_{s,\min}=0$ and $\ell_{s,\max}=2r-1$. The symmetry (47) gives (with $\widetilde{\ell}=0$) $F_{-s,\lambda}(m)=F_{s+2r-2,\lambda}(m)$. For $s=1$, the symmetric of $\ell_s=-1$ is $2r-1$ and $m(2r-1)\in\mathcal{M}_3$ (but $\notin\mathcal{M}_1$). Observe also that $\rho(m)=\rho_{G/K}=(r-1/2)\alpha$.

5.2.3 *The case of* $\mathfrak{g}=\mathfrak{so}(p,q)$, $p>q\geq 3$

Let G be the double cover of $\mathrm{Spin}(p,q)$ (which is simply connected) and $K=\mathrm{Spin}(p)\times\mathrm{Spin}(q)$. The root systems of G/K is $\{\pm\beta_j;1\leq j\leq q\}\cup\{\pm\beta_j\pm\beta_i;1\leq i<j\leq q\}$, with root multiplicities $m_{\beta_j}=p-q$ and $m_{\beta_j\pm\beta_i}=1$. According to [18, Section 2.5], the small K-types are of two forms:

(i) $\tau=1\otimes\sigma$, where σ is the spin representation of $\mathrm{Spin}(q)$ if q is odd, and either of the two spin representations of $\mathrm{Spin}(q)$ if q is even;

(ii) $\tau=\sigma\otimes 1$, where σ is either of the two spin representations of $\mathrm{Spin}(p)$ if p is even and q is odd.

In Case (i), $\Sigma(\tau)=\{\pm\beta_j;1\leq j\leq q\}\cup\{\frac{\pm\beta_j\pm\beta i}{2};1\leq i<j\leq q\}$ with $m(\tau)=(0,1,p-q)$. In Case (ii), $\Sigma(\tau)=\{\pm\frac{\beta_j}{2};1\leq j\leq q\}\cup\{\frac{\pm\beta_j\pm\beta i}{2};1\leq i<j\leq q\}\cup\{\pm\beta_j;1\leq j\leq q\}$ with $m(\tau)=(2(p-q),1,-(p-q))$. Furthermore, $\varphi_\lambda^\tau|_{\mathfrak{a}}$ equals

$$\prod_{1\leq i\leq j\leq q}\Big(\cosh\Big(\frac{\beta_j+\beta_i}{2}\Big)\cosh\Big(\frac{\beta_j-\beta_i}{2}\Big)\Big)^{-1/2}F_\lambda(\Sigma(\tau),m(\tau)),\qquad\text{in case (i)},$$

and

$$\prod_{j=1}^q\Big(\cosh\Big(\frac{\beta_j}{2}\Big)\Big)^{-(p-q)}\prod_{1\leq i\leq j\leq q}\Big(\cosh\Big(\frac{\beta_j+\beta_i}{2}\Big)\cosh\Big(\frac{\beta_j-\beta_i}{2}\Big)\Big)^{-1/2}F_\lambda(\Sigma(\tau),m(\tau)),\qquad\text{in case (ii)}.$$

The two cases come from the multiplicity on $\Sigma(\tau)$ equal to $m=(0,0,p-q)\in\mathcal{M}_+$ and

(ii) $(\ell, \widetilde{\ell}) = (0, 1/2)$ in case (i), so that $m(\ell, \widetilde{\ell}) = (0, 1, p-q)$;
$(\ell, \widetilde{\ell}) = (p-q, 1/2)$ in case (ii), so that $m(\ell, \widetilde{\ell}) = (2(p-q), 1, -(p-q))$.

Notice that $\ell_{\min}(m) = 0$ and $\ell_{\max}(m) = m_1 = p - q \geq 1$. Hence, in both cases, $m(\ell, \widetilde{\ell}) \in \mathcal{M}_+ \cup \mathcal{M}_3$ and $m(\ell, \widetilde{\ell}) \notin \mathcal{M}_1$. The symmetry (47) with $\ell + m_1 - 1 = -\ell + p - q - 1$ yields the equalities

$$F_{0,1/2,\lambda}(m) = F_{p-q-1,1/2,\lambda}(m), \qquad \text{in Case (i)},$$

$$F_{p-q,1/2,\lambda}(m) = F_{-1,1/2,\lambda}(m), \qquad \text{in Case (ii)}.$$

Since $m_1 = p - q \geq 1$, we see for instance that Theorem 4.7 applies to case (i). Notice also that $\rho_{G/K} = \rho(m(2\widetilde{\ell})) = \sum_{j=1}^{q} \left(\frac{p-q}{2} + (j-1)\right) e_j$.

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A The Heckman–Opdam Hypergeometric Functions as Functions of Their Parameters

Let $\mathfrak{a}$ be an r-dimensional Euclidian real vector space, with an inner product $\langle \cdot, \cdot \rangle$, and let Σ be a root system Σ in $\mathfrak{a}^*$ of Weyl group W. Let $\mathcal{M}_{\mathbb{C}}$ denote the set of complex-valued multiplicity functions $m = \{m_\alpha\}$ on Σ. (Hence, $\mathcal{M}_{\mathbb{C}} \equiv \mathbb{C}^d$ where d is the number of Weyl group orbits in Σ.)

In this appendix, we collect the regularity properties of the (symmetric and non-symmetric) hypergeometric functions $F_\lambda(m; x)$ and $G_\lambda(m; x)$ as functions of $(m, x, \lambda) \in \mathcal{M}_{\mathbb{C}} \times \mathfrak{a}_{\mathbb{C}} \times \mathfrak{a}^*_{\mathbb{C}}$. Most of the results are known but scattered in the literature.

Recall the Gindikin–Karpelevich formula for Harish-Chandra's c-function:

$$c(m; \lambda) = \frac{\widetilde{c}(m; \lambda)}{\widetilde{c}(m; \rho(m))} \qquad (\lambda \in \mathfrak{a}^*_{\mathbb{C}}), \tag{A.1}$$

where $\widetilde{c}(m; \lambda)$ is given in terms of the positive indivisible roots $\alpha \in \Sigma_{\mathrm{i}}^+$ by

$$\widetilde{c}(m; \lambda) = \prod_{\alpha \in \Sigma_{\mathrm{i}}^+} \frac{2^{-\lambda_\alpha}\, \Gamma(\lambda_\alpha)}{\Gamma\left(\frac{\lambda_\alpha}{2} + \frac{m_\alpha}{4} + \frac{1}{2}\right) \Gamma\left(\frac{\lambda_\alpha}{2} + \frac{m_\alpha}{4} + \frac{m_{2\alpha}}{2}\right)} \tag{A.2}$$

and Γ is the classical gamma function. Set (see [20, p. 196])

$$\mathcal{M}_{F,\mathrm{reg}} = \{m \in \mathcal{M}_{\mathbb{C}} : \ \frac{1}{\widetilde{c}(m;\rho(m))} \text{ is not singular}\}.$$

For an irreducible representation $\delta \in \widehat{W}$ and $m \in \mathcal{M}_{\mathbb{C}}$, let $\varepsilon_\delta(m) = \sum_{\alpha\in\Sigma^+} m_\alpha(1 - \chi_\delta(r_\alpha)/\chi_\delta(\mathrm{id}))$, where r_α is the reflection across $\ker(\alpha)$ and χ_δ is the character of δ. Let d_δ be the lowest embedding degree of δ in $\mathbb{C}[\mathfrak{a}_{\mathbb{C}}]$, and set (see [21, Definition 3.13])

$$\mathcal{M}_{G,\mathrm{reg}} = \{m \in \mathcal{M} : \ \mathrm{Re}(\varepsilon_\delta(m)) + d_\delta > 0 \text{ for all } \delta \in \widehat{W} \setminus \{\mathrm{triv}\}\}.$$

Finally, let

$$\Omega = \{x \in \mathfrak{a} : \ |\alpha(x)| < \pi \text{ for all } \alpha \in \Sigma^+\}.$$

The regularity properties of the (symmetric and nonsymmetric) functions are summarized in the following theorem.

Theorem A.1. The hypergeometric function $F_\lambda(m;x)$ is holomorphic in $(m,x,\lambda) \in \mathcal{M}_{F,\mathrm{reg}} \times (\mathfrak{a} + i\Omega) \times \mathfrak{a}^*_{\mathbb{C}}$ and satisfies

$$F_\lambda(m;wx) = F_\lambda(m;x) = F_{w\lambda}(m;x) \qquad (m \in \mathcal{M}_{F,\mathrm{reg}},\ w \in W,\ x \in \mathfrak{a} + i\Omega, \lambda \in \mathfrak{a}^*_{\mathbb{C}}).$$

The (nonsymmetric) hypergeometric function $G_\lambda(m;x)$ is a holomorphic function of $(m,x,\lambda) \in \mathcal{M}_{G,\mathrm{reg}} \times (\mathfrak{a} + i\Omega) \times \mathfrak{a}^*_{\mathbb{C}}$.

Proof. For $\mathfrak{a} + i\Omega$ replaced by a W-invariant tubular domain V of $\mathfrak{a}$ in $\mathfrak{a}_{\mathbb{C}}$, these results are due to Opdam; see [20, Theorem 2.8] and [21, Theorem 3.15]. See also [11, Theorem 4.4.2]. The remark that the maximal tubular domain V is $\mathfrak{a} + i\Omega$ was made by Jacques Faraut; see [2, Remark 3.17]. ■

Proposition A.2. Set

$$\mathcal{M}_{\mathbb{C},+} = \{m \in \mathcal{M}_{\mathbb{C}} : \ \mathrm{Re}(m_\alpha) \geq 0 \text{ for every } \alpha \in \Sigma^+\} \tag{A.3}$$

$$\mathcal{M}_{\mathbb{C},0} = \{m \in \mathcal{M}_{\mathbb{C}} : \mathrm{Re}(m_\alpha + m_{2\alpha}) \geq 0 \text{ for every } \alpha \in \Sigma_{\mathrm{i}}^+\}. \tag{A.4}$$

Then,

$$\mathcal{M}_{\mathbb{C},+} \subset \mathcal{M}_{\mathbb{C},0} \subset \mathcal{M}_{F,\mathrm{reg}} \cap \mathcal{M}_{G,\mathrm{reg}}.$$

Moreover, $\mathcal{M}_{\mathbb{C},0}$ is stable under deformations by $\ell \in \mathbb{C}$ as in (33): $m \in \mathcal{M}_{\mathbb{C},0}$ if and only if $m(\ell) \in \mathcal{M}_{\mathbb{C},0}$.

Proof. The inclusion $\mathcal{M}_{\mathbb{C},0} \subset \mathcal{M}_{F,\mathrm{reg}}$ was observed in [11, Remark 4.4.3] and follows from the properties of the gamma function. Notice that $\mathrm{Re}(\rho(m)_\alpha) \geq \mathrm{Re}(\frac{m_\alpha}{2} + m_{2\alpha})$ for $\alpha \in \Sigma_{\mathrm{i}}^+$, with simplifications in the factor of $c(m;\rho(m))$ corresponding to α in case of equality with $\frac{m_\alpha}{2} + m_{2\alpha}$ real. For the inclusion $\mathcal{M}_{\mathbb{C},0} \subset \mathcal{M}_{G,\mathrm{reg}}$, observe that $\varepsilon_\delta(m) = \sum_{\alpha\in\Sigma_{\mathrm{i}}^+}(m_\alpha + m_{2\alpha})(1 - \chi_\delta(r_\alpha)/\chi_\delta(\mathrm{id}))$, with $\chi_\delta(r_\alpha)/\chi_\delta(\mathrm{id})) \in [-1,1]$. ■

B Some Computations

In this appendix, we prove two inequalities stated in Subsection 3.3. We begin showing that for every $\lambda \in \mathfrak{a}^*$ and $\xi \in \mathfrak{a}$,

$$\partial_\xi\Big(e^{K_\xi \frac{\langle \xi,\cdot\rangle}{|\xi|^2}} F_\lambda(m;\cdot)\Big) \geq 0. \tag{B.1}$$

Firstly, as in the proof of [27,Lemma 3.4], for every $x \in \overline{\mathfrak{a}^+}$, $\xi \in \mathfrak{a}$, and $\lambda \in \mathfrak{a}^*$, we have

$$\partial_\xi F_\lambda(x) = \frac{1}{|W|}\sum_{w\in W}(\lambda - \rho(m))(w\xi)G_\lambda(wx)$$

(where ξ is fixed and acts on the x-variable). Here, $G_\lambda(wx) \geq 0$ for $m \in \mathcal{M}_3$ (as in [27] for $m \in \mathcal{M}_+$). Set $K_\xi = \max_{w\in W}(\rho(m) - \lambda)(w\xi)$. So $K_\xi = -\min_{w\in W}(\lambda - \rho(m))(w\xi)$ and

$$\partial_\xi F_\lambda(x) \geq -K_\xi \frac{1}{|W|}\sum_{w\in W} G_\lambda(wx) = -K_\xi F_\lambda(x).$$

Since $x \mapsto F_\lambda(x)$ and $\xi \mapsto K_\xi$ are W-invariant, the last inequality extends to all $x \in \mathfrak{a}$. Hence, for every $x, \xi \in \mathfrak{a}$ and every $\lambda \in \mathfrak{a}^*$, we have

$$\partial_\xi\big(e^{K_\xi \frac{\langle \xi,\cdot\rangle}{|\xi|^2}} F_\lambda(m;\cdot)\big) = e^{K_\xi \frac{\langle \xi,\cdot\rangle}{|\xi|^2}}\Big(\partial_\xi F_\lambda + K_\xi F_\lambda\Big) \geq 0.$$

The previous argument is essentially the one leading to (8) in [27, Lemma 3.3].

We pass from (B.1) to the inequality (30) in Lemma 3.9 by using the following arguments, which are repeatedly used in [27]. This is why we say that Lemma 3.9 is implicit in [27].

Let $\lambda \in \mathfrak{a}^*$ and $x \in \mathfrak{a}$ be fixed. Set

$$f(t) = e^{K_\xi \frac{\langle \xi, x+t\xi\rangle}{|\xi|^2}} F_\lambda(m; x + t\xi).$$

By [27, Lemma 3.2], and since

$$f'_d(t) = \frac{d}{dt}\Big|_{t\searrow 0} f(t) = \partial_\xi\Big(e^{K_\xi \frac{\langle \xi,\cdot\rangle}{|\xi|^2}} F_\lambda(m;\cdot)\Big)(x) \geq 0\,,$$

we conclude that $f(t)$ is increasing on $[0, +\infty[$, and hence,

$$e^{K_\xi \frac{\langle \xi,x\rangle}{|\xi|^2}} e^{K_\xi t} F_\lambda(m; x + t\xi) = f(t) \geq f(0) = e^{K_\xi \frac{\langle \xi,x\rangle}{|\xi|^2}} F_\lambda(m;x),$$

that is,

$$e^{K_\xi t} F_\lambda(m; x + t\xi) \geq F_\lambda(m; x) \tag{B.2}$$

for all $t \geq 0$. Choose first $\xi = x_1 \in \mathfrak{a}$ and $t = 1$. Then, (B.2) gives

$$e^{\max_{w \in W}(\rho(m)-\lambda)(wx_1)} F_\lambda(m; x + x_1) \leq F_\lambda(m; x). \tag{B.3}$$

Replace x by $x + x_1$, and choose $\xi = -x_1$ and $t = 1$. Then, (B.2) gives

$$e^{-\min_{w \in W}(\rho(m)-\lambda)(wx_1)} F_\lambda(m; x) \geq F_\lambda(m; x + x_1), \tag{B.4}$$

as required.

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