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IMPROVED FRACTIONAL HARDY INEQUALITIES FOR DUNKL GRADIENT

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Abstract. We prove an improved fractional Hardy inequality in the Dunkl setting for the weighted space $L^p(\mathbb{R}^N, d\mu_k(x))$. Also we prove a similar inequality for half-space.

1. Introduction and main theorems

A classical Hardy inequality is of the form

$$\int_{\mathbb{R}^N} |\nabla u|^p dx \geqslant \left(\frac{|N-p|}{p}\right)^p \int_{\mathbb{R}^N} \frac{|u(x)|^p}{|x|^p} dx,$$

for $u \in C_0^\infty(\mathbb{R}^N)$ or $u \in C_0^\infty(\mathbb{R}^N \setminus \{0\})$ respectively with respect to $1 \leqslant p < N$ or $p > N$. It is known that the constant $\left(\frac{|N-p|}{p}\right)^p$ is sharp and never attained in the corresponding spaces $\dot{W}_p^1(\mathbb{R}^N)$ or $\dot{W}_p^1(\mathbb{R}^N \setminus \{0\})$ respectively. In a remarkable paper [7], Frank and Seiringer have proven the sharp Hardy inequality with a remainder term. Their result is as follows: for $p \geqslant 2$ and $0 < s < 1$ and for some positive constants $C_{N,s,p}$ and c_p

$$\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{|u(x)-u(y)|^p}{|x-y|^{N+ps}} dxdy - C_{N,s,p}\int_{\mathbb{R}^N} \frac{|u(x)|^p}{|x|^{ps}} dx$$
$$\geqslant c_p \int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{|v(x)-v(y)|^p}{|x-y|^{N+ps}} \frac{dx}{|x|^{(N-ps)/2}} \frac{dy}{|y|^{(N-ps)/2}}, \tag{1}$$

where $v := |x|^{(N-ps)/p} u$. The result is true for all $u \in C_0^\infty(\mathbb{R}^N)$ if $ps < N$ and for all $u \in C_0^\infty(\mathbb{R}^N \setminus \{0\})$ if $ps > N$. The same authors proved a similar fractional Hardy inequality on half-space in [8], which states that: for $p \geqslant 2$, $0 < s < 1$ and $ps \neq 1$

$$\int_{\mathbb{R}^N_+}\int_{\mathbb{R}^N_+} \frac{|u(x)-u(y)|^p}{|x-y|^{N+ps}} dxdy - D_{N,p,s}\int_{\mathbb{R}^N_+} \frac{|u(x)|^p}{x_N^{ps}} dx$$
$$\geqslant c_p \int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{|v(x)-v(y)|^p}{|x-y|^{N+ps}} \frac{dx}{x_N^{(1-ps)/2}} \frac{dy}{y_N^{(1-ps)/2}}, \tag{2}$$

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where $D_{N,p,s}$ and c_p are positive constants and $v := x_N^{(1-ps)/p}u$. More generalized version of (1) and (2) in the Dunkl setting are studied in [4]. Combining the results due to Abdellaoui et al. in [1, 2, 3] we can get an improved fractional Hardy inequality for $1 < p < \infty$ which is stated below.

Let $0 < s < 1$, $ps < N$, $1 < q < p < \infty$ and $\Omega \subset \mathbb{R}^N$ be a bounded domain. Then we have

$$\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\frac{|u(x)-u(y)|^p}{|x-y|^{N+ps}}dxdy - C_{N,p,s}\int_{\mathbb{R}^N}\frac{|u(x)|^p}{|x|^{ps}}dx \geqslant C\int_{\Omega}\int_{\Omega}\frac{|u(x)-u(y)|^p}{|x-y|^{N+qs}}dxdy \tag{3}$$

for all functions $u \in C_0^\infty(\Omega)$. The constant $C_{N,p,s}$ is the sharp constant in the fractional Hardy inequality obtained by Frank et al. in [7] and the constant C is positive and depends on N, q, s and the domain Ω. Unlike in [7] the result is true for all $1 < p < \infty$ and the remainder term here is a p-norm of a fractional gradient.

In the proof of fractional Hardy inequalities mentioned in (1), (2) and (3), various properties of the kernel of the form $|x-y|^{-(N+\delta)}$ with $\delta > -N$ play an important role. When it comes to the Dunkl case we use a generalized kernel Φ_δ, $\delta > -d_k$ which is defined in (13). The kernel Φ_δ is defined through the Dunkl translation operator defined in Section 3. The kernel Φ_δ, $\delta > -d_k$ was introduced by Gorbachev et al. in [9] to study Riesz potential and maximal function for Dunkl transform. Authors of the article [4] proved certain generalized optimal fractional Hardy inequalities for $\mathbb{R}^N$, half-space and for the cone. Our main aims of this paper is to prove a generalized version of (3) in the Dunkl setting. Our first main result is recorded in the following theorem.

THEOREM 1. *Let $\Omega \subset \mathbb{R}^N$ be a bounded G-invariant domain. Let $1 < q < p < \infty$, $ps \leqslant d_k$ and $0 < s < 1$. Then for all $u \in C_0^\infty(\Omega)$*

$$\iint_{\mathbb{R}^N\times\mathbb{R}^N}|u(x)-u(y)|^p\Phi_{ps}(x,y)d\mu_k(x)d\mu_k(y) - \Lambda_{d_k,s,p}\int_{\mathbb{R}^N}\frac{|u(x)|^p}{|x|^{ps}}d\mu_k(x), \geqslant C\iint_{\Omega\times\Omega}|u(x)-u(y)|^p\Phi_{qs}(x,y)d\mu_k(x)d\mu_k(y), \tag{4}$$

where

$$\Lambda_{d_k,s,p} = 2\int_0^1 r^{ps-1}|1-r^{(d_k-ps)/p}|^p\Phi(r)dr, \tag{5}$$

with

$$\Phi(r) = \begin{cases} \frac{\Gamma(\frac{d_k}{2})}{\sqrt{\pi}\Gamma(\frac{d_k-1}{2})}\int_0^\pi \frac{\sin^{d_k-2}\theta}{(1-2r\cos\theta+r^2)^{\frac{d_k+ps}{2}}}d\theta & \text{for } N \geqslant 2 \\ \left(\tau_r^k(|.|^{-d_k-ps}) + \tau_{-r}^k(|.|^{-d_k-ps})\right)(1) & \text{for } N = 1 \end{cases}$$

and C is a positive constant depending on Ω, d_k, q and s.

We have adopted the ideas introduced in [1, 2, 3] and in [7] to prove Theorem 1. A slight modification of the techniques of the proof of Theorem 1 will lead to the following improved fractional Hardy inequality for half-space.

THEOREM 2. *Let $\Omega \subset \mathbb{R}^N_+$ be a bounded G-invariant domain. Let $1 < q < p < \infty$ with $ps < 1$ and $0 < s < 1$. Then for all $u \in C_0^\infty(\Omega)$*

$$\iint\limits_{\mathbb{R}^N_+ \times \mathbb{R}^N_+} |u(x) - u(y)|^p \Phi_{ps}(x,y) d\mu_k(x) d\mu_k(y) - \Lambda^0_{d_k,s,p} \int_{\mathbb{R}^N_+} \frac{|u(x)|^p}{x_N^{ps}} d\mu_k(x)$$
$$\geqslant C \iint\limits_{\Omega \times \Omega} |u(x) - u(y)|^p \Phi_{qs}(x,y) d\mu_k(x) d\mu_k(y), \tag{6}$$

where $\Lambda^0_{d_k,s,p}$ is given as

$$\Lambda^0_{d_k,s,p} := c_{k_1}^{-1} 2^{-\lambda_{k_1}} \frac{\Gamma((1+ps)/2)}{\Gamma((d_k+ps)/2)} \int_0^1 |1 - r^{\frac{ps-1}{p}}|^p \frac{dr}{(1-r)^{1+ps}}. \tag{7}$$

and $C = C(\Omega, d_k, q, s)$ is a positive constant.

By choosing the multiplicity function $k \equiv 0$ in Theorem 2 we obtain the following corollary. As far as we know, this inequality is not known in the Euclidean setting.

COROLLARY 1. *Let $0 < s < 1$ and $ps < 1$. Also let Ω be a bounded domain of $\mathbb{R}^N$. Then for all $1 < q < p < \infty$ and for all functions $u \in C_0^\infty(\Omega)$ the following inequality holds:*

$$\int_{\mathbb{R}^N_+} \int_{\mathbb{R}^N_+} \frac{|u(x) - u(y)|^p}{|x-y|^{N+ps}} dx dy - D_{N,p,s} \int_{\mathbb{R}^N_+} \frac{|u(x)|^p}{x_N^{ps}} dx$$
$$\geqslant C \int_\Omega \int_\Omega \frac{|u(x) - u(y)|^p}{|x-y|^{N+qs}} dx dy. \tag{8}$$

The constant $D_{N,p,s}$ is sharp and is given by

$$D_{N,p,s} = c_{N-1} \frac{\Gamma(\frac{1+ps}{2})}{\Gamma(\frac{N+ps}{2})} \int_0^1 |1 - r^{\frac{ps-1}{p}}|^p \frac{dr}{(1-r)^{1+ps}}, \tag{9}$$

with $c_{N-1} = 2^{\frac{N-3}{2}} \int_{\mathbb{R}^{N-1}} e^{-|x'|^2/2} dx'$. The constant C is positive and depends on N, q, s and the domain Ω.

The paper is organized as follows. In Section 2 we give a brief introduction to the Dunkl theory. In Section 3 we prove Picone's inequality and some lemmas on weighted Sobolev spaces. Section 4 is devoted to the proof of improved fractional Hardy inequality on $L^p(\mathbb{R}^N, d\mu_k(x))$. We use a slight modification of this idea to prove a similar Hardy inequality on the half-space in Section 5.

2. Basics of the Dunkl theory

In this section we give some basics on Dunkl theory which we will be using in the coming sections. The preliminaries of the Dunkl analysis can be found in [5, 11, 12, 13]. Let $\langle\cdot,\cdot\rangle$ denote the standard inner product on $\mathbb{R}^N$ and $|\cdot| := \sqrt{\langle\cdot,\cdot\rangle}$. For a non-zero element α in $\mathbb{R}^N$ the reflection in the hyperplane $\langle\alpha\rangle^\perp$ is defined as

$$\sigma_\alpha(x) = x - 2\frac{\langle\alpha,x\rangle}{|\alpha|^2}\alpha.$$

DEFINITION 1. Let $R \subset \mathbb{R}^N\backslash\{0\}$ be a finite set. Then R is called a root system, if

(1) $R\cap\mathbb{R}\alpha = \{\pm\alpha\}$ for all $\alpha\in R$
(2) $\sigma_\alpha(R) = R$ for all $\alpha\in R$.

A root system can be written as the disjoint union $R_+\cup(-R_+)$ and this R_+ and $(-R_+)$ is separated by a hyper plane passing through the origin. Here R_+ is the set of positive roots of the root system R. The subgroup $G = G(R) \subseteq O(N,\mathbb{R})$ which is generated by reflections $\{\sigma_\alpha : \alpha\in R\}$ is called reflection group (or Coxeter-group) associated with R. For the convenience of the calculations we assume that R is normalized, that is $\langle\alpha,\alpha\rangle = 2$ for all $\alpha\in R$. A G-invariant function k defined on R, that is $k(g\alpha) = k(\alpha)$ for all $g\in G$, is called a multiplicity function. An example of a root system on $\mathbb{R}^N$ is $A_N = \{\pm e_i\}$, where $\{e_i : 1\leqslant i\leqslant N\}$ is the standard basis for $\mathbb{R}^N$. In this case σ_{e_i} send e_i to $-e_i$ leaving other vectors e_j fixed and the corresponding Coxeter group is $\mathbb{Z}_2^N$. Given any function k' defined on R we can always define a G-invariant function k by $k(x) = \sum\limits_{\alpha\in R} k'(\sigma_\alpha x)$.

For $j\in\{1,2,\ldots,N\}$ the differential-difference operators T_j (the Dunkl operators) defined by

$$T_jf(x) = \partial_jf(x) + E_jf(x), \quad f\in C^1(\mathbb{R}^N),$$

where $E_jf(x) = \sum\limits_{\alpha\in R_+} k(\alpha)\alpha_j\frac{f(x)-f(\sigma_\alpha x)}{\langle\alpha,x\rangle}$ and $\alpha = (\alpha_1,\alpha_2,\ldots,\alpha_N)$. The Dunkl operators T_j's are a generalization of the partial differential operators in the classical analysis. As in the classical case we can define the Dunkl gradient $\nabla_k = (T_1,T_2,\ldots,T_N)$ and the Dunkl Laplacian Δ_k as $\Delta_k = \sum_{j=1}^N T_j^2$.

One of the important properties of the Dunkl operators is that they commute, that is $T_iT_j = T_jT_i$. Also for every $f,g\in C^1(\mathbb{R}^N)$ and for every $1\leqslant j\leqslant N$, one can see that $T_j(fg) = T_j(f)g + fT_j(g)$ when at least one of the functions is G-invariant.

Fix a reflection group G and a multiplicity function k. We can define the G-invariant homogeneous weight function $h_k^2(x) = \prod\limits_{\alpha\in R_+} |\langle x,\alpha\rangle|^{2k(\alpha)}$ of degree $2\gamma_k$, where $\gamma_k := \sum\limits_{\alpha\in R_+} k(\alpha)$.

Throughout the paper we denote the weighted measure $h_k^2(x)dx$ as $d\mu_k(x)$. Further we use the notations $d_k := N + 2\gamma_k$ and $\lambda_k := \frac{d_k-2}{2}$.

If for $g \in \mathscr{S}(\mathbb{R}^N)$, the space of Schwartz class functions, and a bounded $f \in C^1(\mathbb{R}^N)$, then

$$\int_{\mathbb{R}^N} T_j f(x) g(x) d\mu_k(x) = -\int_{\mathbb{R}^N} f(x) T_j g(x) d\mu_k(x).$$

For a fixed $y \in \mathbb{R}^N$, it is known that there exists a unique real analytic solution $f(x) = E_k(x,y)$ for the system $T_i f = y_i f$, $1 \leqslant i \leqslant N$, satisfying $f(0) = 1$. The kernel $E_k(x,y)$ is called the Dunkl kernel and it is clearly a generalization of the exponential functions $e^{\langle x, \rangle y}$. Dunkl kernel enjoys many properties similar to classical exponential function. We refer [11] and the references there in for further reading on Dunkl kernel.

Dunkl transform is defined as a generalization of Fourier transform. For $u \in L^1(\mathbb{R}^N, d\mu_k(x))$, its Dunkl transform is defined by

$$\mathscr{F}_k u(\xi) = c_k^{-1} \int_{\mathbb{R}^N} u(x) E_k(-i\xi, x) d\mu_k(x),$$

where $c_k^{-1} := \int_{\mathbb{R}^N} e^{-\|x\|^2/2} d\mu_k(x)$. Dunkl translation operator is defined through the Dunkl transform. The Dunkl translation $\tau_y^k f$ is defined by $\mathscr{F}_k(\tau_y^k f)(\xi) = E_k(iy, \xi)\mathscr{F}_k f(\xi)$ and it makes sense for all $f \in L^2(\mathbb{R}^N, d\mu_k(x))$ as $E_k(iy, \xi)$ is a bounded function. Dunkl translation has the property $\tau_y^k f(x) = \tau_{-x}^k f(-y)$.

3. Fractional Sobolev spaces and some auxiliary lemmas

We begin the section by stating three algebraic lemmas which we will use later to prove the main theorems.

LEMMA 1. [7, Frank, Seiringer] *Let $p \geqslant 1$. Then for all $0 \leqslant t \leqslant 1$ and $a \in \mathbb{C}$ one has*

$$|a-t|^p \geqslant (1-t)^{p-1}(|a|^p - 1). \tag{10}$$

For $p > 1$ this inequality is strict unless $a = 1$ or $t = 0$. Moreover, if $p \geqslant 2$ then for all $0 \leqslant t \leqslant 1$ and all $a \in \mathbb{C}$ one has

$$|a-t|^p \geqslant (1-t)^{p-1}\big(|a|^p - t\big) + c_p t^{p/2} |a-1|^p, \tag{11}$$

with $0 < c_p \leqslant 1$ and c_p is given by

$$c_p := \min_{0<\tau<1/2} \big((1-\tau)^p - \tau^p + p\tau^{p-1}\big). \tag{12}$$

For $p = 2$, (11) is an equality with $c_2 = 1$. For $p > 2$, (11) is a strict equality unless $a = 1$ or $t = 0$.

LEMMA 2. [10, P. Lindqvist] *For any* $1 < p < 2$ *there exist a positive constant* c *depending on* p *such thar for all* $a, b \in \mathbb{R}^N$ *we have:*

$$|a|^p - |b|^p - p|b|^{p-2}\langle b, a-b\rangle \geqslant c\frac{|a-b|^2}{(|a|+|b|)^{2-p}}$$

and for $p \geqslant 2$

$$|a|^p - |b|^p - p|b|^{p-2}\langle b, a-b\rangle \geqslant \frac{|a-b|^2}{2^{p-1}-1}.$$

LEMMA 3. [2, B. Abdellaoui, F. Mahmoudi] *Let* $1 \leqslant p \leqslant 2$ *and* $0 \leqslant t \leqslant 1$ *and* $a \in \mathbb{R}$. *Then for some positive constant* c *depending only on* p *we have the following inequality:*

$$|a-t|^p - (1-t)^{p-1}(|a|^p - t) \geqslant c\frac{|a-1|^2 t}{(|a-t|+|1-t|)^{2-p}}.$$

3.1. Weighted Sobolev spaces

Define the kernel Φ_δ with $\delta > -d_k$

$$\Phi_\delta(x,y) := \frac{1}{\Gamma((d_k+\delta)/2)}\int_0^\infty s^{\frac{d_k+\delta}{2}-1}\tau_y^k\big(e^{-s|.|^2}\big)(x)ds, \qquad (x,y) \neq (0,0). \tag{13}$$

The kernel $\Phi_\delta(\cdot,\cdot)$ was first considered by Gorbachev et al. [9] in connection with the study of Riesz potential and maximal function for Dunkl transform. If the multiplicity function is identically zero, that is $k \equiv 0$, then $d_k = N$ and τ_y^k reduces to the Euclidean translation operator and hence from the integral formula

$$\frac{1}{|y|^{d-\alpha}} = \frac{1}{\Gamma((d-\alpha)/2)}\int_0^\infty s^{(d-\alpha)/2-1}e^{-s|y|^2}ds$$

the kernel $\Phi_\delta(x,y)$ becomes the kernel $|x-y|^{-N-\delta}$. From this understanding we define fractional Sobolev space in the Dunkl setting by using $\Phi_\delta(x,y)$.

Let Ω be an open subset of $\mathbb{R}^N$ containing origin. Let $s \in (0,1)$ and $1 < p < \infty$. Then we define the fractional Sobolev space $W_k^{s,p}(\Omega)$ with the kernel Φ_{ps} as

$$W_k^{s,p}(\Omega) := \Big\{u \in L^p(\Omega, d\mu_k(x)) : \iint_{\Omega\times\Omega} |u(x)-u(y)|^p\Phi_{ps}(x,y)d\mu_k(x)d\mu_k(y) < \infty\Big\},$$

and the norm is given by

$$\|u\|_{W_k^{s,p}(\Omega)} = \Big(\int_\Omega |u|^p d\mu_k(x)\Big)^{\frac{1}{p}} + \Big(\iint_{\Omega\times\Omega} |u(x)-u(y)|^p\Phi_{ps}(x,y)d\mu_k(x)d\mu_k(y)\Big)^{\frac{1}{p}}.$$

Let $C_0^\infty(\Omega)$ be the set of compactly supported smooth functions on Ω. We define the Sobolev space $W_{k,0}^{s,p}(\Omega)$ as the completion of $C_0^\infty(\Omega)$ with the norm $\|.\|_{W_k^{s,p}(\Omega)}$.

PROPOSITION 1. *Let $\Omega \subset \mathbb{R}^N$ be open and G-invariant. Let $u \in W_k^{s,p}(\Omega)$ and let $A \subset \Omega$ such that A is compact and u is supported in A. Define an extension $\tilde{u}$ on $\mathbb{R}^N$ as $\tilde{u}(x) = u(x)$ when $x \in \Omega$ and $\tilde{u}(x) = 0$ when $x \in \mathbb{R}^N \setminus \Omega$. Then $\tilde{u}$ belongs to $W_k^{s,p}(\mathbb{R}^N)$ and*

$$\|\tilde{u}\|_{W_k^{s,p}(\mathbb{R}^N)} \leqslant C(\Omega, A, d_k, p, s)\|u\|_{W_k^{s,p}(\Omega)}.$$

Proof. By the definition of $\tilde{u}$ it is clear that $\tilde{u} \in L^p(\mathbb{R}^N, d\mu_k(x))$. Since Φ_{ps} is symmetric on x and y, we can write

$$\begin{aligned}
&\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} |\tilde{u}(x)-\tilde{u}(y)|^p \Phi_{ps}(x,y)d\mu_k(x)d\mu_k(y) \\
&= \int_{\Omega}\int_{\Omega} |u(x)-u(y)|^p \Phi_{ps}(x,y)d\mu_k(x)d\mu_k(y) \\
&\quad + 2\int_{\Omega}\left(\int_{\mathbb{R}^N\setminus\Omega} |u(x)|^p \Phi_{ps}(x,y)d\mu_k(y)\right)d\mu_k(x). \qquad (14)
\end{aligned}$$

Since $u \in W_k^{s,p}(\Omega)$

$$\int_{\Omega}\int_{\Omega} |u(x)-u(y)|^p \Phi_{ps}(x,y)d\mu_k(x)d\mu_k(y) < \infty.$$

Also u is supported in A and hence for any $y \in \mathbb{R}^N \setminus \Omega$

$$|u(x)|^p \Phi_{ps}(x,y) = |u(x)|^p \chi_A(x)\Phi_{ps}(x,y).$$

Now by [9, Lemma 2.3]

$$\Phi_{ps}(x,y) = \int_{\mathbb{R}^N}\left(|x|^2+|y|^2-2\langle y,\eta\rangle\right)^{-\frac{d_k+ps}{2}} d\mu_k^x(\eta),$$

where μ_k^x is a probability Borel measure whose support is contained in $\mathrm{Co}(G)$, the convex hull of G-orbit of x in $\mathbb{R}^N$ (see also [11]). It is easy to see that for any $\eta \in \mathrm{Co}(G)$

$$\left(|x|^2+|y|^2-2\langle y,\eta\rangle\right)^{\frac{1}{2}} \geqslant \min_{\sigma\in G}|\sigma y - x|.$$

Using this and the fact that μ_k^x is a probability measure we get

$$\Phi_{ps}(x,y) \leqslant \left(\min_{\sigma\in G}|\sigma y - x|\right)^{-(d_k+ps)}.$$

Since Ω is G-invariant we find that $y \in \mathbb{R}^N \setminus \Omega$ implies $\sigma y \in \mathbb{R}^N \setminus \Omega$ for any $\sigma \in G$. Using the fact that A is compact and and Ω is bounded we have $dist(\sigma y, \partial A) \geqslant dist(\partial\Omega, \partial A) > 0$ for all $\sigma \in G$ and $y \in \mathbb{R}^N \setminus \Omega$.

But $\min_{\sigma\in G}|\sigma y - x| \geqslant \min_{\sigma\in G}(dist(\sigma y, \partial A))$ and hence we can write

$$\begin{aligned}
&\int_{\Omega}\left(\int_{\mathbb{R}^N\setminus\Omega} |u(x)|^p \Phi_{ps}(x,y)d\mu_k(y)\right)d\mu_k(x) \\
&\leqslant \|u\|^p_{L^p(\Omega, d\mu_k(x))}\int_{\mathbb{R}^N\setminus\Omega}\frac{d\mu_k(y)}{dist(\partial\Omega,\partial A)^{d_k+ps}}.
\end{aligned}$$

Since $dist(\partial\Omega, \partial A) > 0$ and $d_k + ps > d_k$ the integral

$$\int_{\mathbb{R}^N\setminus\Omega} \frac{1}{dist(\partial\Omega, \partial A)^{d_k+ps}} d\mu_k(y)$$

is finite. Finiteness of the above integral together with (14) we find that

$$\|\tilde{u}\|_{W_k^{s,p}(\mathbb{R}^N)} \leqslant C(d_k, p, s, A, \Omega)\|u\|_{W_k^{s,p}(\Omega)}.$$

For $1 < p < \infty$ and $0 < \beta < \frac{d_k - ps}{2}$ we define the kernel K_p^β as

$$K_p^\beta(x,y) = \frac{\Phi_{ps}(x,y)}{|x|^\beta |y|^\beta}.$$

We also define the weighted fractional Sobolev space $W_k^{s,p,\beta}(\Omega)$, with $0 \in \Omega$, as

$$W_k^{s,p,\beta}(\Omega) := \left\{ u \in L^p(\Omega, \frac{d\mu_k(x)}{|x|^{2\beta}}) : \iint\limits_{\Omega\times\Omega} |u(x) - u(y)|^p K_p^\beta(x,y) d\mu_k(x) d\mu_k(y) < \infty \right\}$$

endowed with the norm

$$\|u\|_{W_k^{s,p,\beta}(\Omega)} := \left(\int_\Omega |u(x)|^p \frac{d\mu_k(x)}{|x|^{2\beta}} \right)^{\frac{1}{p}} + \left(\iint\limits_{\Omega\times\Omega} |u(x) - u(y)|^p K_p^\beta(x,y) d\mu_k(x) d\mu_k(y) \right)^{\frac{1}{p}}.$$

For $1 < q < p$ and $0 < \beta < \frac{d_k - qs}{2}$ we define the space $W_k^{s,p,q,\beta}(\Omega)$ as follows:

$$W_k^{s,p,q,\beta}(\Omega) := \left\{ u \in L^p(\Omega, \frac{d\mu_k(x)}{|x|^{2\beta}}) : \iint\limits_{\Omega\times\Omega} |u(x) - u(y)|^p K_q^\beta(x,y) d\mu_k(x) d\mu_k(y) < \infty \right\},$$

where the norm is given by

$$\|u\|_{W_k^{s,p,q,\beta}(\Omega)} := \left(\int_\Omega |u(x)|^p \frac{d\mu_k(x)}{|x|^{2\beta}} \right)^{\frac{1}{p}} + \left(\iint\limits_{\Omega\times\Omega} |u(x) - u(y)|^p K_q^\beta(x,y) d\mu_k(x) d\mu_k(y) \right)^{\frac{1}{p}}. \tag{15}$$

Let us denote $W_{k,0}^{s,p,q,\beta}(\Omega)$ as the completion $C_0^\infty(\Omega)$ with respect to the norm of $W_k^{s,p,q,\beta}(\Omega)$.

Let $\Omega \subset \mathbb{R}^N$ be a bounded G-invariant domain with $0 \in \Omega$. Using similar arguments used in Proposition 1 we can say that, if $u \in C_0^\infty(\Omega)$, with a compact support $A \subset \Omega$, then there exists a function $\tilde{u}$, which is an extension of u, belongs to $W_{k,0}^{s,p,q,\beta}(\mathbb{R}^N)$ such that

$$\|\tilde{u}\|_{W_k^{s,p,q,\beta}(\mathbb{R}^N)} \leqslant C\|u\|_{W_k^{s,p,q,\beta}(\Omega)}, \tag{16}$$

where $C = C(\Omega, A, d_k, p, q, s)$ is a positive constant.

REMARK 1. If Ω is a bounded G-invariant domain of $\mathbb{R}^N$, we can attach $W^{s,p}_{k,0}(\Omega)$ with an equivalent norm $|||.|||_{W^{s,p}_{k,0}}$

$$|||u|||_{W^{s,p}_{k,0}(\Omega)} := \left(\iint_{\Omega\times\Omega} |u(x)-u(y)|^p \Phi_p(x,y) d\mu_k(x) d\mu_k(y) \right)^{\frac{1}{p}}. \tag{17}$$

To prove the equivalence of the norms $\|.\|_{W^{s,p}_k(\Omega)}$ and $|||.|||_{W^{s,p}_{k,0}(\Omega)}$, we need to prove a Poincaré type inequality, $\|u\|^p_{L^p(\Omega, d\mu_k(x))} \leqslant C|||u|||_{W^{s,p}_{k,0}(\Omega)}$ for $u \in C^\infty_0(\Omega)$ and some positive constant C. We denote the extension $\tilde{u}$ of u to $\mathbb{R}^N$ as u itself.

Let $B_r \subset \mathbb{R}^N \setminus \Omega$ and let $\mathscr{O}(B_r)$ be the G-orbit of B_r. Since Ω is G-invariant and bounded we have $\mathscr{O}(B_r) \subset \mathbb{R}^N \setminus \Omega$. For $x \in \Omega$ and $y \in \mathscr{O}(B_r)$, write $|u(x)|^p = |u(x) - u(y)|^p \Phi_{ps}(x,y) \Phi^{-1}_{ps}(x,y)$. Since Ω is G-invariant and $\Phi_{ps}(x,y) \geqslant (\max_{\sigma\in G} |\sigma y - x|)^{-(d_k+ps)}$, we can write

$$\begin{aligned}
\mu_k(\mathscr{O}(B_r))|u(x)|^p &\leqslant \sup_{x\in\Omega, y\in\mathscr{O}(B_r)} \Phi^{-1}_{ps}(x,y) \int_{\mathscr{O}(B_r)} u(x)-u(y)|^p \Phi_{ps}(x,y) d\mu_k(y) \\
&\leqslant \sup_{x\in\Omega,\ y\in\mathscr{O}(B_r)} \left(\max_{\sigma\in G} |\sigma y - x| \right)^{d_k+ps} \int_{\mathscr{O}(B_r)} u(x)-u(y)|^p \Phi_{ps}(x,y) d\mu_k(y) \\
&\leqslant \sup_{x\in\Omega,\ y\in\mathscr{O}(B_r)} |x-y|^{d_k+ps} \int_{\mathscr{O}(B_r)} u(x)-u(y)|^p \Phi_{ps}(x,y) d\mu_k(y) \\
&\leqslant diam(\Omega \cup \mathscr{O}(B_r)) \int_{\mathscr{O}(B_r)} u(x)-u(y)|^p \Phi_{ps}(x,y) d\mu_k(y).
\end{aligned}$$

Now by integrating both sides over Ω with respect to x, we get the desired Poincaré type inequality.

3.2. Picone's inequality

Picone's inequality (Lemma 2.3) for the Sobolev space on $\mathbb{R}^N$ was proved in [2]. We need to prove Picone's Inequality for the Sobolev space $W^{s,p,q,\beta}_k(\Omega)$ defined through the fynction Φ_{ps}. Now for $w \in W^{s,p,q,\beta}_{k,0}(\mathbb{R}^N)$, we define

$$L(w)(x) = P.V. \int_{\mathbb{R}^N} |w(x)-w(y)|^{p-2}(w(x)-w(y)) K^\beta_q(x,y) d\mu_k(x) d\mu_k(y)$$

and for $v, w \in W^{s,p,q,\beta}_{k,0}(\mathbb{R}^N)$, we have

$$\langle L(w), v\rangle = \int_{\mathbb{R}^N}\int_{\mathbb{R}^N} |w(x)-w(y)|^{p-2}(w(x)-w(y))(v(x)-v(y)) K^\beta_q(x,y) d\mu_k(x) d\mu_k(y).$$

THEOREM 3. *Let $Q = \mathbb{R}^N \times \mathbb{R}^N \setminus (\mathscr{C}\Omega \times \mathscr{C}\Omega)$ and w be a positive function in $W_{k,0}^{s,p,q,\beta}(\Omega)$ with $L(w)(x) \geqslant 0$ for all x in Ω. Then for all $u \in C_0^\infty(\Omega)$ the following Picone's inequality holds:*

$$\frac{1}{2}\iint_Q |u(x)-u(y)|^p \Phi_{qs}(x,y)\frac{d\mu_k(x)d\mu_k(y)}{|x|^\beta|y|^\beta} \geqslant \left\langle L(w), \frac{|u|^p}{w^{p-1}} \right\rangle.$$

Proof. Let $v(x) = \frac{|u(x)|^p}{|w(x)|^{p-1}}$,

$$\begin{aligned}\langle L(w), v\rangle &= \int_\Omega v(x)\int_{\mathbb{R}^N} |w(x)-w(y)|^{p-2}(w(x)-w(y))K_q^\beta(x,y)d\mu_k(x)d\mu_k(y)\\ &= \int_\Omega \frac{|u(x)|^p}{|w(x)|^{p-1}}\int_{\mathbb{R}^N} |w(x)-w(y)|^{p-2}(w(x)-w(y))K_q^\beta(x,y)d\mu_k(x)d\mu_k(y).\end{aligned}$$

Since $K_q^\beta(x,y) = K_q^\beta(y,x)$, we can write

$$\begin{aligned}\langle L(w), v\rangle = \iint_Q &\left(\frac{|u(x)|^p}{|w(x)|^{p-1}} - \frac{|u(y)|^p}{|w(y)|^{p-1}}\right)\\ &\times |w(x)-w(y)|^{p-2}(w(x)-w(y))K_q^\beta(x,y)d\mu_k(x)d\mu_k(y).\end{aligned}$$

Define the function $g = u/w$ and obtain

$$\begin{aligned}\langle L(w), v\rangle &= \frac{1}{2}\big(|g(x)|^p w(x) - |g(y)|^p w(y)\big)\\ &\quad\times |w(x)-w(y)|^{p-2}(w(x)-w(y))K_q^\beta(x,y)d\mu_k(x)d\mu_k(y)\\ &= \frac{1}{2}\iint_Q \big[|u(x)-u(y)|^p - \phi(x,y)\big]K_q^\beta(x,y)d\mu_k(x)d\mu_k(y),\end{aligned}$$

where

$$\phi(x,y) = |u(x)-u(y)|^p - (|g(x)|^p w(x) - |g(y)|^p w(y))|w(x)-w(y)|^{p-2}(w(x)-w(y)).$$

It is enough to prove $\phi \geqslant 0$ to get the desired inequality

$$\langle L(w), v\rangle \leqslant \frac{1}{2}\iint_Q |u(x)-u(y)|^p K_q^\beta(x,y)d\mu_k(x)d\mu_k(y).$$

Since ϕ is symmetric we can assume that $w(x) \geqslant w(y)$. Putting $t = w(y)/w(x)$, $a = u(x)/u(y)$ and applying the inequality (10) we see that $\phi \geqslant 0$.

LEMMA 4. *Let $0 < \beta < \frac{d_k - qs}{2}$, $1 < q < p < \infty$, $0 < s < 1$ and let $0 < \alpha < \frac{d_k - qs - 2\beta}{p-1}$. For $w(x) = |x|^{-\alpha}$ we have the following equality for a.e. non zero x in $\mathbb{R}^N$*

$$L(w) = \Lambda(\alpha)\frac{w^{p-1}}{|x|^{qs+2\beta}},$$

where $\Lambda(\alpha)$ is a positive constant.

Proof. For w given in the statement, we have

$$L(w)(x) = P.V. \int_{\mathbb{R}^N} |w(x) - w(y)|^{p-2}(w(x) - w(y))K_q^\beta(x,y)d\mu_k(y).$$

Let $r = |x|$ and $\rho = |y|$. Also let $x = rx'$ and $y = \rho y'$ with $x', y' \in \mathbb{S}^{N-1}$. With this setting, we can write

$$L(w)(x) = \int_0^\infty \int_{\mathbb{S}^{N-1}} \frac{|r^{-\alpha} - \rho^{-\alpha}|^{p-2}(r^{-\alpha} - \rho^{-\alpha})\Phi_{qs}(rx' - \rho y')}{r^\beta \rho^\beta} \rho^{2\lambda_k+1} d\sigma_k(y')d\rho.$$

Let $t = \rho/r$. Using [9, Lemma 2.3] we have the following properties for Φ_δ

$$\Phi_\delta(rx', \rho y') = r^{-d_k-\delta}\Phi_\delta(x', ty') \tag{18}$$

and

$$P(t) := \int_{\mathbb{S}^{N-1}} \Phi_{qs}(x', ty')d\sigma_k(y') = \frac{\Gamma(\frac{d_k}{2})}{\sqrt{\pi}\Gamma(\frac{d_k-1}{2})} \int_0^\pi \frac{\sin^{d_k-2}\theta}{(1 - 2t\cos\theta + t^2)^{\frac{d_k+qs}{2}}} d\theta.$$

With these properties, we can write

$$L(w)(x) = \frac{r^{-\alpha(p-1)}}{r^{2\beta+qs}} \int_0^\infty |1 - t^{-\alpha}|^{p-2}(1 - t^{-\alpha})t^{2\lambda_k+1-\beta}P(t)dt = \Lambda(\alpha)\frac{w^{p-1}(x)}{|x|^{2\beta+qs}},$$

where $\Lambda(\alpha) = \int_0^\infty \varphi(t)dt$ with $\varphi(t) = |1 - t^{-\alpha}|^{p-2}(1 - t^{-\alpha})t^{2\lambda_k+1-\beta}P(t)$. Now we need to check the convergence of the integral $\int_0^\infty \varphi(t)dt$. With the change of variable $t \to \frac{1}{t}$ and using the fact that $P(\frac{1}{t}) = t^{d_k+qs}P(t)$ we can write

$$\int_0^1 \varphi(t)dt = -\int_1^\infty (t^\alpha - 1)^{p-1}t^{\beta+ps-1}P(t)dt$$

and with this, $\Lambda(\alpha)$ becomes

$$\Lambda(\alpha) = \int_1^\infty (t^\alpha - 1)^{p-1}P(t)\left(t^{d_k-1-\beta-\alpha(p-1)} - t^{\beta+qs-1}\right)dt. \tag{19}$$

Observe that $P(t)$ is similar to $\frac{1}{t^{d_k+qs}}$ as t tends to ∞ and $P(t)$ is dominated by a constant multiple of $\frac{1}{|t-1|^{1+qs}}$ as t tends to 1. Together with this understanding and using the assumption on α and β, as $t \to \infty$ we have

$$(t^\alpha - 1)^{p-1}P(t)\left(t^{d_k-1-\beta-\alpha(p-1)} - t^{\beta+qs-1}\right) \simeq \frac{1}{t^{1+\beta+qs}} \tag{20}$$

and as $t \to 1$ we have

$$(t^\alpha - 1)^{p-1}P(t)\left(t^{d_k-1-\beta-\alpha(p-1)} - t^{\beta+qs-1}\right) \simeq (t-1)^{p-1-qs}. \tag{21}$$

One can easily see that the similar function written on the right-hand side of (20) and (21) are integrable on the intervals $(2,\infty)$ and $(1,2)$ respectively. This gives $\Lambda(\alpha)$ is finite. Now since $0<\alpha(p-1)<d_k-qs-2\beta$,

$$\left(t^{d_k-1-\beta-\alpha(p-1)}-t^{\beta+qs-1}\right)>0$$

and hence from the expression of $\Lambda(\alpha)$ in (19) we conclude $\Lambda(\alpha)>0$.

We have just proved above that under the given assumptions

$$L(w)=\Lambda(\alpha)\frac{w^{p-1}}{|x|^{qs+2\beta}}.$$

Now Picone's inequality proved in Theorem 3 for this w gives that

$$\begin{aligned}2\Lambda(\alpha)\int_{\mathbb{R}^N}\frac{|u(x)|^p}{|x|^{qs+2\beta}}d\mu_k(x)&=\left\langle L(w),\frac{|u|^p}{w^{p-1}}\right\rangle\\&\leqslant\iint\limits_{\mathbb{R}^N\times\mathbb{R}^N}|u(x)-u(y)|^pK_q^{\beta}(x,y)d\mu_k(x)d\mu_k(y).\end{aligned}\tag{22}$$

REMARK 2. Now choose Ω to be a bounded G-invariant domain containing origin and let $u\in C_0^\infty(\Omega)$. Then as we described earlier we have an extension function $\tilde{u}$ of $u\in W_k^{s,p,q,\beta}(\Omega)$. Using (22) for $\tilde{u}$ together with the equations (15) and (16) we find

$$\begin{aligned}2\Lambda(\alpha)\int_{\mathbb{R}^N}\frac{|\tilde{u}(x)|^p}{|x|^{qs+2\beta}}d\mu_k(x)&\leqslant\iint\limits_{\mathbb{R}^N\times\mathbb{R}^N}|\tilde{u}(x)-\tilde{u}(y)|^pK_q^{\beta}(x,y)d\mu_k(x)d\mu_k(y)\\&\leqslant\|\tilde{u}\|_{W_k^{s,p,q,\beta}(\mathbb{R}^N)}\leqslant C\|u\|_{W_k^{s,p,q,\beta}(\Omega)}.\end{aligned}$$

4. Hardy inequality on $\mathbb{R}^N$

In this section we give the proof of Theorem 1. The following lemma is needed for proving Theorem 1.

LEMMA 5. *Fix* $\alpha=\frac{d_k-ps}{p}$, $\beta=\frac{d_k-ps}{2}$ *and let* $w(x)=|x|^{-\alpha}$. *Let* $u\in C_0^\infty(\mathbb{R}^N)$ *and define* $v(x)=u(x)/w(x)$. *Then for all* $1<q<p<\infty$ *and for a given positive constant* C *the following inequality holds:*

$$\begin{aligned}&\iint\limits_{\mathbb{R}^N\times\mathbb{R}^N}|v(x)-v(y)|^pK_q^{\beta}(x,y)d\mu_k(x)d\mu_k(y)\\&\qquad\geqslant C\iint\limits_{\mathbb{R}^N\times\mathbb{R}^N}|u(x)-u(y)|^p\Phi_{qs}(x,y)d\mu_k(x)d\mu_k(y).\end{aligned}$$

Proof. Let

$$\begin{aligned} f_1(x,y) &:= |v(x)-v(y)|^p K_q^\beta(x,y) \\ &= \frac{|w(y)u(x)-w(x)u(y)|^p}{\big(w(x)w(y)\big)^{\frac{p}{2}}}\Phi_{qs}(x,y) \\ &= \left|\big((u(y)-u(x)\big)-\frac{u(y)}{w(y)}\big(w(x)-w(y)\big)\right|^p \left(\frac{w(y)}{w(x)}\right)^{\frac{p}{2}}\Phi_{qs}(x,y). \end{aligned}$$

Observing the symmetry of $f_1(x,y)$ we define $f_2(x,y)$ in the following way

$$f_2(x,y) := \left|\big((u(x)-u(y)\big)-\frac{u(x)}{w(x)}\big(w(y)-w(x)\big)\right|^p \left(\frac{w(x)}{w(y)}\right)^{\frac{p}{2}}\Phi_{qs}(x,y).$$

Now the integral

$$H_k(v) := \iint_{\mathbb{R}^N\times\mathbb{R}^N} |v(x)-v(y)|^p K_q^\beta(x,y)d\mu_k(x)d\mu_k(y)$$

can be written as

$$H_k(v) = \frac{1}{2}\iint_{\mathbb{R}^N\times\mathbb{R}^N}(f_1(x,y)+f_2(x,y))d\mu_k(x)d\mu_k(y).$$

Also let

$$Q(x,y) = \frac{\big(w(x)w(y)\big)^{\frac{p}{2}}}{w(x)^p+w(y)^p} \text{ and } D(x,y) = \left(\frac{w(x)}{w(y)}\right)^{\frac{p}{2}} + \left(\frac{w(y)}{w(x)}\right)^{\frac{p}{2}}.$$

It is clear that $Q(x,y) \leqslant C$ and $Q(x,y)D(x,y)=1$ for all x and y. So for $p \geqslant 2$ we can apply the Lemma 2 to obtain the following inequality

$$\begin{aligned} f_1(x,y) \geqslant CQ(x,y)\left(\frac{w(y)}{w(x)}\right)^{\frac{p}{2}} \Big[& |u(x)-u(y)|^p\Phi_{qs}(x,y) \\ & - p|u(x)-u(y)|^{p-2}\Phi_{qs}(x,y)\langle u(x)-u(y), \frac{u(y)}{w(y)}(w(x)-w(y))\rangle \\ & + c(p)|\frac{u(y)}{w(y)}(w(x)-w(y))|^p\Phi_{qs}(x,y)\Big] \end{aligned} \tag{23}$$

and for $1<p<2$, again by using Lemma 2, we can write

$$\begin{aligned} f_1(x,y) \geqslant CQ(x,y)\left(\frac{w(y)}{w(x)}\right)^{\frac{p}{2}} \Big[& |u(x)-u(y)|^p\Phi_{qs}(x,y) \\ & + p|u(x)-u(y)|^{p-2}\Phi_{qs}(x,y)\langle u(x)-u(y), \frac{u(y)}{w(y)}(w(x)-w(y))\rangle\Big]. \end{aligned} \tag{24}$$

Now combining equations (23) and (24), we can write for $1 < p < \infty$,

$$f_1(x,y) \geqslant \left[CQ(x,y) \left(\frac{w(y)}{w(x)} \right)^{\frac{p}{2}} |u(x)-u(y)|^p \Phi_{qs}(x,y) \right] \\ - pCQ(x,y) \left(\frac{w(y)}{w(x)} \right)^{\frac{p}{2}} |u(x)-u(y)|^{p-1} \Phi_{qs}(x,y) \left| \frac{u(y)}{w(y)} \right| |w(x)-w(y)| \Big].$$

Similarly, we can calculate

$$f_2(x,y) \geqslant \left[CQ(x,y) \left(\frac{w(x)}{w(y)} \right)^{\frac{p}{2}} |u(x)-u(y)|^p \Phi_{qs}(x,y) \right] \\ - pCQ(x,y) \left(\frac{w(x)}{w(y)} \right)^{\frac{p}{2}} |u(x)-u(y)|^{p-1} \Phi_{qs}(x,y) \left| \frac{u(x)}{w(x)} \right| |w(x)-w(y)| \Big].$$

Now by using the estimates of f_1 and f_2 we obtain

$$H_k(v) \geqslant C \iint_{\mathbb{R}^N \times \mathbb{R}^N} |u(x)-u(y)|^p \Phi_{qs}(x,y) d\mu_k(x) d\mu_k(y) \\ - C_1 \iint_{\mathbb{R}^N \times \mathbb{R}^N} \big(h_1(x,y) + h_2(x,y) \big) d\mu_k(x) d\mu_k(y), \tag{25}$$

where

$$h_1(x,y) = Q(x,y) \left(\frac{w(y)}{w(x)} \right)^{\frac{p}{2}} |u(x)-u(y)|^{p-1} \Phi_{qs}(x,y) \left| \frac{u(y)}{w(y)} \right| |w(x)-w(y)|$$

and

$$h_2(x,y) = Q(x,y) \left(\frac{w(x)}{w(y)} \right)^{\frac{p}{2}} |u(x)-u(y)|^{p-1} \Phi_{qs}(x,y) \left| \frac{u(x)}{w(x)} \right| |w(x)-w(y)|.$$

Since $h_1(x,y) = h_2(y,x)$ we have

$$\iint_{\mathbb{R}^N \times \mathbb{R}^N} h_1(x,y) d\mu_k(x) d\mu_k(y) = \iint_{\mathbb{R}^N \times \mathbb{R}^N} h_2(x,y) d\mu_k(x) d\mu_k(y). \tag{26}$$

Therefore, it is sufficient to estimate one of the integral. Now by Young's inequality we can write

$$\iint_{\mathbb{R}^N \times \mathbb{R}^N} h_2(x,y) d\mu_k(x) d\mu_k(y) \leqslant \varepsilon \iint_{\mathbb{R}^N \times \mathbb{R}^N} |u(x)-u(y)|^p \Phi_{qs}(x,y) d\mu_k(x) d\mu_k(y) \\ + C(\varepsilon) \iint_{\mathbb{R}^N \times \mathbb{R}^N} G(x,y) d\mu_k(x) d\mu_k(y), \tag{27}$$

where

$$G(x,y) = Q(x,y)^p \left(\frac{w(x)}{w(y)}\right)^{\frac{p}{2}} \left|\frac{u(x)}{w(x)}\right|^p |w(x)-w(y)|^p \Phi_{qs}(x,y).$$

The proof will be completed if we can establish

$$\iint_{\mathbb{R}^N\times\mathbb{R}^N} G(x,y)d\mu_k(x)d\mu_k(y) \leqslant C \iint_{\mathbb{R}^N\times\mathbb{R}^N} |v(x)-v(y)|^p K_q^\beta(x,y)d\mu_k(x)d\mu_k(y). \tag{28}$$

Let us calculate

$$\iint_{\mathbb{R}^N\times\mathbb{R}^N} G(x,y)d\mu_k(x)d\mu_k(y)$$

$$= \iint_{\mathbb{R}^N\times\mathbb{R}^N} \frac{u(x)^p w(x)^{p(p-1)}|w(x)-w(y)|^p}{\left(w(x)^p+w(y)^p\right)^p}\Phi_{qs}(x,y)d\mu_k(x)d\mu_k(y)$$

$$= \int_{\mathbb{R}^N} u(x)^p \int_{\mathbb{R}^N} \frac{||x|^\alpha-|y|^\alpha|^p|y|^{\alpha p(p-1)}}{(|x|^{\alpha p}+|y|^{\alpha p})^p}\Phi_{qs}(x,y)d\mu_k(y)d\mu_k(x).$$

Let $|x|=r$ and $|y|=\rho$ with $x=rx'$ and $y=\rho y'$. Also write $t=\rho/r$ and $d\sigma_k(y') = h_k^2(y')d\sigma(y')$ with $d\sigma(y')$ as the Eulidean surface measure on the sphere $\mathbb{S}^{N-1}$. Then we have

$$\iint_{\mathbb{R}^N\times\mathbb{R}^N} G(x,y)d\mu_k(x)d\mu_k(y)$$

$$= \int_{\mathbb{R}^N} u(x)^p \int_0^\infty \frac{|r^\alpha-\rho^\alpha|^p\rho^{\alpha p(p-1)+2\lambda_k+1}}{(r^{p\alpha}+\rho^{p\alpha})^p}\int_{\mathbb{S}^{N-1}}\Phi_{qs}(rx',\rho y')d\sigma_k(y')d\rho d\mu_k(x)$$

$$= \int_{\mathbb{R}^N} \frac{u(x)^p}{|x|^{qs}} \int_0^\infty \frac{|1-t^\alpha|^p t^{\alpha p(p-1)+2\lambda_k+1}}{(1+t^{\alpha p})^p}\int_{\mathbb{S}^{N-1}}\Phi_{qs}(x',ty')d\sigma_k(y')dtdx$$

$$= I\int_{\mathbb{R}^N}\frac{u(x)^p}{|x|^{qs}}d\mu_k(x),$$

with

$$I = \int_0^\infty \frac{|1-t^\alpha|^p t^{\alpha p(p-1)+2\lambda_k+1}}{(1+t^{\alpha p})^p}P(t)dt.$$

Here we set

$$P(t) = \int_{\mathbb{S}^{N-1}}\Phi_{qs}(x',ty')d\sigma_k(y')$$

and used the property of the kernel $\Phi_{qs}(rx',\rho y') = r^{-d_k-qs}\Phi_{qs}(x',ty')$ (see [9, Lemma 2.3] for a proof). By proceeding with the similar steps used in Lemma 4 we get I is finite. Since we chose $w(x)=|x|^{-\frac{d_k-ps}{p}}$ and $u=vw$ we have

$$\iint_{\mathbb{R}^N\times\mathbb{R}^N} G(x,y)d\mu_k(x)d\mu_k(y) = I\int_{\mathbb{R}^N}\frac{|v(x)|^p}{|x|^{qs+(d_k-ps)}}d\mu_k(x).$$

Set $\beta_0 = \frac{d_k - ps}{2} < \frac{d_k - qs}{2}$ and apply (22) for v, to get

$$\iint\limits_{\mathbb{R}^N \times \mathbb{R}^N} G(x,y)d\mu_k(x)d\mu_k(y) \leqslant C \iint\limits_{\mathbb{R}^N \times \mathbb{R}^N} |v(x)-v(y)|^p K_q^{\beta}(x,y)d\mu_k(x)d\mu_k(y). \tag{29}$$

Thus we proved our claim in (28). Now by considering the inequalities (25), (26), (27) and (29) we get the desired inequality

$$\iint\limits_{\mathbb{R}^N \times \mathbb{R}^N} |u(x)-u(y)|^p \Phi_{qs}(x,y)d\mu_k(x)d\mu_k(y) \leqslant C \iint\limits_{\mathbb{R}^N \times \mathbb{R}^N} |v(x)-v(y)|^p K_q^{\beta}(x,y)d\mu_k(x)d\mu_k(y).$$

Let Ω be a bounded G-invariant domain on $\mathbb{R}^N$ containing origin. Also let $u \in C_0^\infty(\Omega)$ and $\tilde{u}$ be its extension to $\mathbb{R}^N$ as explained earlier (see Proposition 1). As $u = vw$ we let the extension of v as $\tilde{v}$ and $\tilde{u} = \tilde{v}w$. Now using (16) and Lemma 5 together, we get

$$\begin{aligned}
\iint\limits_{\Omega \times \Omega} |v(x)-v(y)|^p K_q^{\beta}(x,y)d\mu_k(x)d\mu_k(y) &\geqslant C \iint\limits_{\mathbb{R}^N \times \mathbb{R}^N} |\tilde{v}(x)-\tilde{v}(y)|^p K_q^{\beta}(x,y)d\mu_k(x)d\mu_k(y) \\
&\geqslant C \iint\limits_{\mathbb{R}^N \times \mathbb{R}^N} |\tilde{u}(x)-\tilde{u}(y)|^p \Phi_{qs}(x,y)d\mu_k(x)d\mu_k(y) \\
&\geqslant C \iint\limits_{\Omega \times \Omega} |u(x)-u(y)|^p \Phi_{qs}(x,y)d\mu_k(x)d\mu_k(y).
\end{aligned} \tag{30}$$

4.1. Proof of Theorem 1

The main idea of the proof is to show that

$$\iint\limits_{\mathbb{R}^N \times \mathbb{R}^N} |u(x)-u(y)|^p \Phi_{ps}(x,y)d\mu_k(x)d\mu_k(y) - \Lambda_{d_k,s,p} \int_{\mathbb{R}^N} \frac{|u(x)|^p}{|x|^{ps}} d\mu_k(x) \geqslant C \iint\limits_{\Omega \times \Omega} |v(x)-v(y)|^p K_q^{\beta}(x,y)d\mu_k(x)d\mu_k(y), \tag{31}$$

for some positive constant C. Then by using Lemma 5 we reach the desired inequality. In order to prove (31) we need to consider two different cases $p \geqslant 2$ and $1 < p < 2$.

Case 1: $p \geqslant 2$

From [4], we have

$$\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}|u(x)-u(y)|^p\Phi_{ps}(x,y)d\mu_k(x)d\mu_k(y)-C_{d_k,s,p}\int_{\mathbb{R}^N}\frac{|u(x)|^p}{|x|^{ps}}d\mu_k(x)$$
$$\geqslant c_p\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}|v(x)-v(y)|^p\Phi_{ps}(x,y)\frac{d\mu_k(x)}{|x|^{(d_k-ps)/2}}\frac{d\mu_k(y)}{|y|^{(d_k-ps)/2}}.$$

But for $\Omega\subset\mathbb{R}^N$ bounded, we have $\Phi_{ps}(x,y)\geqslant C(\Omega)\Phi_{qs}(x,y)$ on Ω. Using this we can write

$$\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}|v(x)-v(y)|^p\Phi_{ps}(x,y)\frac{d\mu_k(x)}{|x|^{(d_k-ps)/2}}\frac{d\mu_k(y)}{|y|^{(d_k-ps)/2}}$$
$$\geqslant C(\Omega)\iint\limits_{\Omega\times\Omega}|v(x)-v(y)|^pK_q^\beta(x,y)d\mu_k(x)d\mu_k(y)$$

and it gives the claim given in (31) for $p\geqslant 2$.

Case 2: $1<p<2$

We define f_1 and f_2 same as described in the proof of Lemma 5. We split the domain $\Omega\times\Omega$ in accordance with the values of $w(x)$ and $w(y)$ as

$$D_1=\{(x,y)\in\Omega\times\Omega: w(y)\leqslant w(x)\}\text{ and }D_2=\{(x,y)\in\Omega\times\Omega: w(x)<w(y)\}. \quad (32)$$

Now

$$\begin{aligned}C(\Omega)H_\Omega(v):=&\iint\limits_{\Omega\times\Omega}|v(x)-v(y)|^pK_q^\beta(x,y)d\mu_k(x)d\mu_k(y)\\ =&\iint\limits_{\Omega\times\Omega}f_1(x,y)d\mu_k(x)d\mu_k(y)\\ =&\iint\limits_{D_1}f_1(x,y)d\mu_k(x)d\mu_k(y)+\iint\limits_{D_2}f_2(x,y)d\mu_k(x)d\mu_k(y)\\ =&I_1+I_2.\end{aligned}$$

We will first estimate the integral in I_1. We can write

$$\begin{aligned}J_1(x,y):=&\left|\big((u(y)-u(x)\big)-\frac{u(y)}{w(y)}\big(w(x)-w(y)\big)\right|^p\left(\frac{w(y)}{w(x)}\right)^{\frac{p}{2}}\\ =&\frac{\left|\big((u(y)-u(x)\big)-\frac{u(y)}{w(y)}\big(w(x)-w(y)\big)\right|^p}{\left|u(x)-u(y)\right|+\left|\frac{u(y)}{w(x)}\big(w(x)-w(y)\big)\right|^{(2-p)\frac{p}{2}}}\left(\frac{w(y)}{w(x)}\right)^{\frac{p}{2}}\\ &\times\left|u(x)-u(y)\right|+\left|\frac{u(y)}{w(x)}\big(w(x)-w(y)\big)\right|^{(2-p)\frac{p}{2}}.\end{aligned}$$

Now applying Hölder's inequality, we obtain

$$I_1 \leqslant I_{1,1} \times I_{1,2}. \tag{33}$$

Here we denote

$$I_{1,1} = \left(\iint_{D_1} \frac{\left|\left((u(y)-u(x)\right) - \frac{u(y)}{w(y)}\left(w(x)-w(y)\right)\right|^2}{\left|u(x)-u(y)\right| + \left|\frac{u(y)}{w(x)}\left(w(x)-w(y)\right)\right|^{(2-p)}} \frac{w(y)}{w(x)} \Phi_{qs}(x,y)d\mu_k(x)d\mu_k(y) \right)^{\frac{p}{2}}$$

and

$$I_{1,2} = \left(\iint_{D_1} \left|\left((u(y)-u(x)\right) - \frac{u(y)}{w(y)}\left(w(x)-w(y)\right)\right|^p \Phi_{qs}(x,y)d\mu_k(x)d\mu_k(y) \right)^{\frac{2-p}{p}}.$$

From (30), we get

$$\begin{aligned} I_{1,2}^{\frac{2}{2-p}} &\leqslant C_1 \iint_{\Omega\times\Omega} |u(x)-u(y)|^p \Phi_{qs}(x,y)d\mu_k(x)d\mu_k(y) \\ &\quad + \iint_{\Omega\times\Omega} \left|\frac{u(y)}{w(y)}(w(x)-w(y))\right|^p \Phi_{qs}(x,y)d\mu_k(x)d\mu_k(y) \\ &\leqslant \iint_{\Omega\times\Omega} |v(x)-v(y)|^p K_q^{\beta}(x,y)d\mu_k(x)d\mu_k(y) = C(\Omega)H_{\Omega}(v). \end{aligned} \tag{34}$$

Thus we arrive at

$$I_{1,2} \leqslant C(\Omega)H_{(\Omega)}^{\frac{2-p}{2}}(v). \tag{35}$$

An application of Lemma 3 with $t = \frac{w(y)}{w(x)}$, $a = \frac{v(x)}{v(y)}$ we find for $(x,y) \in D_1$

$$\begin{aligned} &\frac{\left|\left((u(y)-u(x)\right) - \frac{u(y)}{w(y)}\left(w(x)-w(y)\right)\right|^2}{\left|u(x)-u(y)\right| + \left|\frac{u(y)}{w(x)}\left(w(x)-w(y)\right)\right|^{(2-p)}} \frac{w(y)}{w(x)} \\ &= \frac{w(x)^p |v(y)|^p |a-1|^2 t}{(|a-t| + |1-t|^{2-p})} \\ &\leqslant w(x)^p |v(y)|^p \left(|a-t|^p - (1-t)^{p-1}(|a|^p - t)\right) \\ &= w(x)^p |v(y)|^p \left(\left|\frac{v(x)}{v(y)} - \frac{w(y)}{w(x)}\right|^p - \left(1 - \frac{w(y)}{w(x)}\right)^{p-1} \left(\left|\frac{v(x)}{v(y)}\right|^p - \frac{w(y)}{w(x)} \right) \right) \\ &= |u(x)u(y)|^p - \Big((w(x)-w(y))^{p-2}(w(x)-w(y)) \Big) \left(\frac{|u(x)|^p}{w(x)^{p-1}} - \frac{|u(y)|^p}{w(y)^{p-1}} \right). \end{aligned} \tag{36}$$

Further using (34) and (36) the first integral $I_{1,1}$ in (33) becomes

$$\begin{aligned}
&C(\Omega)I_{1,1}^{2/p}\\
&\leqslant \iint\limits_{\mathbb{R}^N\times\mathbb{R}^N} |u(x)-u(y)|^p\Phi_{qs}(x,y)d\mu_k(x)d\mu_k(y)\\
&\quad -\iint\limits_{\mathbb{R}^N\times\mathbb{R}^N}\left(\frac{|u(x)|^p}{w(x)^{p-1}}-\frac{|u(y)|^p}{w(y)^{p-1}}\right)|w(x)-w(y)|^{p-2}(w(x)-w(y))\Phi_{qs}(x,y)d\mu_k(x)d\mu_k(y)\\
&=\iint\limits_{\mathbb{R}^N\times\mathbb{R}^N} |u(x)-u(y)|^p\Phi_{ps}(x,y)d\mu_k(x)d\mu_k(y)-\Lambda_{d_k,s,p}\int_{\mathbb{R}^N}\frac{|u(x)|^p}{|x|^{ps}}d\mu_k(x). \qquad (37)
\end{aligned}$$

This gives that

$$\begin{aligned}
I_1 &= \iint\limits_{D_1} f_1(x,y)d\mu_k(x)d\mu_k(y)\\
&\leqslant C(\Omega)H_\Omega^{\frac{2-p}{2}}(v)\\
&\quad\times\left(\iint\limits_{\mathbb{R}^N\times\mathbb{R}^N}|u(x)-u(y)|^p\Phi_{ps}(x,y)d\mu_k(x)d\mu_k(y)-\Lambda_{d_k,s,p}\int_{\mathbb{R}^N}\frac{|u(x)|^p}{|x|^{ps}}d\mu_k(x)\right)^{\frac{p}{2}}.
\end{aligned} \tag{38}$$

The same arguments allow us to write

$$\begin{aligned}
I_2 &= \iint\limits_{D_2} f_2(x,y)d\mu_k(x)d\mu_k(y)\\
&\leqslant C(\Omega)H_\Omega^{\frac{2-p}{2}}(v)\\
&\quad\times\left(\iint\limits_{\mathbb{R}^N\times\mathbb{R}^N}|u(x)-u(y)|^p\Phi_{ps}(x,y)d\mu_k(x)d\mu_k(y)-\Lambda_{d_k,s,p}\int_{\mathbb{R}^N}\frac{|u(x)|^p}{|x|^{ps}}d\mu_k(x)\right)^{\frac{p}{2}}.
\end{aligned} \tag{39}$$

Now put (38) and (39) together with the fact $C(\Omega)H_\Omega(v)=I_1+I_2$ to get

$$\begin{aligned}
H_\Omega(v) &\leqslant C(\Omega)H_\Omega^{\frac{2-p}{2}}(v)\\
&\quad\times\left(\iint\limits_{\mathbb{R}^N\times\mathbb{R}^N}|u(x)-u(y)|^p\Phi_{ps}(x,y)d\mu_k(x)d\mu_k(y)-\Lambda_{d_k,s,p}\int_{\mathbb{R}^N}\frac{|u(x)|^p}{|x|^{ps}}d\mu_k(x)\right)^{\frac{p}{2}}
\end{aligned}$$

and hence

$$\begin{aligned}
H_\Omega(v)\leqslant C(\Omega)&\iint\limits_{\mathbb{R}^N\times\mathbb{R}^N}|u(x)-u(y)|^p\Phi_{ps}(x,y)d\mu_k(x)d\mu_k(y)\\
&-\Lambda_{d_k,s,p}\int_{\mathbb{R}^N}\frac{|u(x)|^p}{|x|^{ps}}d\mu_k(x).
\end{aligned}$$

Now the case 1 and case 2 together provide the claim

$$\iint_{\mathbb{R}^N\times\mathbb{R}^N} |u(x)-u(y)|^p\Phi_{ps}(x,y)d\mu_k(x)d\mu_k(y)-\Lambda_{d_k,s,p}\int_{\mathbb{R}^N}\frac{|u(x)|^p}{|x|^{ps}}d\mu_k(x)$$
$$\geqslant C(\Omega)\iint_{\Omega\times\Omega}|v(x)-v(y)|^pK_q^{\beta}(x,y)d\mu_k(x)d\mu_k(y). \tag{40}$$

for all $1\leqslant q<p<\infty$. Hence the desired inequality (1.4) stated in the Theorem 1 follows from (40) together with Lemma 5.

5. Fractional Hardy inequality on half-space

Let R be a root system on $\mathbb{R}^{N-1}$ and k be a multiplicity function from R to $(0,\infty)$. Define the root system R_1 on $\mathbb{R}^N_+$ as $R_1:=R\times\{0\}$. We use the same notation G for the corresponding Coxeter group. Also extend the multiplicity function k to k_1 by defining $k_1(x,0)=k(x)$ where $x\in R$. With the root system R_1 and the multiplicity function k_1 on $\mathbb{R}^N_+$ we can write the kernel Φ_{qs} on $\mathbb{R}^N_+$ with $1<q<\infty$ and $0<s<1$ as

$$\Phi_{qs}(x,y)=\frac{1}{\Gamma((d_{k_1}+qs)/2)}\int_0^{\infty}s^{\frac{d_k+qs}{2}-1}e^{-s|x_N-y_N|^2}\tau^k_{y'}(e^{-s|.|^2})(x')ds.$$

For an element $x\in\mathbb{R}^N_+$ we write $x=(x',x_N)$ where $x'\in\mathbb{R}^{N-1}$ and $x_N>0$. Using the properties of Dunkl translation and gamma function we can perform the following calculations

$$\begin{aligned}
&\int_{\mathbb{R}^{N-1}}\Phi_{qs}(x,y)d\mu_k(y')\\
&=\frac{1}{\Gamma((d_{k_1}+qs)/2)}\int_{\mathbb{R}^{N-1}}\int_0^{\infty}s^{\frac{d_{k_1}+qs}{2}-1}e^{-s|x_N-y_N|^2}\tau^k_{y'}(e^{-s|.|^2})(x')ds\,d\mu_k(y')\\
&=\frac{1}{\Gamma((d_{k_1}+qs)/2)}\int_{\mathbb{R}^{N-1}}\int_0^{\infty}s^{\frac{d_{k_1}+qs}{2}-1}e^{-s(|x_N-y_N|^2+|x'-y'|^2)}ds\,d\mu_k(y')\\
&=\int_{\mathbb{R}^{N-1}}\frac{d\mu_k(y')}{(|x_n-y_N|^2+|x'-y'|^2)^{\frac{d_{k_1}+qs}{2}}}\\
&=\|\mathbb{S}^{N-2}\|_k\int_0^{\infty}\frac{1}{(|x_N-y_N|^2+r^2)^{\frac{d_{k_1}+qs}{2}}}r^{d_k-2}dr\\
&=\|\mathbb{S}^{N-2}\|_k\frac{1}{|x_N-y_N|^{1+qs}}\int_0^{\infty}\frac{t^{d_k-2}}{(1+t^2)^{\frac{d_{k_1}+qs}{2}}}dt\\
&=\|\mathbb{S}^{N-2}\|_k\frac{1}{|x_N-y_N|^{1+qs}}\frac{\Gamma((d_{k_1}-1)/2)\Gamma((1+qs)/2)}{\Gamma((d_{k_1}+qs)/2)}.
\end{aligned} \tag{41}$$

The constant $\|\mathbb{S}^{N-2}\|_k$ in (41) is given as

$$\|\mathbb{S}^{N-2}\|_k=\int_{\mathbb{S}^{N-2}}d\mu_k(x)=\frac{c_k^{-1}}{2^{\lambda_k}\Gamma(\frac{d_k}{2})}.$$

Let $\Omega \subset \mathbb{R}^N_+$ be an open G-invariant subset and let $w_0 \in W^{s,p,q,\beta}_{k,0}(\Omega)$. Define

$$L_0(w_0)(x) := P.V. \int_{\mathbb{R}^N_+} |w_0(x) - w_0(y)|^{p-2}(w_0(x) - w_0(y))K^{\beta}_{q,0}(x,y)d\mu_k(x)d\mu_k(y),$$

where

$$K^{\beta}_{q,0}(x,y) = \frac{\Phi_{qs}(x,y)}{x_N^{\beta} y_N^{\beta}}.$$

Also let $\Omega \subset \mathbb{R}^N_+$ be bounded and we denote $Q_0 = \mathbb{R}^N_+ \times \mathbb{R}^N_+ \setminus (\mathscr{C}\Omega \times \mathscr{C}\Omega)$. Then by the same arguments as in the proof of Theorem 3 we can conclude a Picone's inequality for half-space, that is

$$\frac{1}{2} \iint_{Q_0} |u(x) - u(y)|^p \Phi_{qs}(x,y) \frac{d\mu_k(x)d\mu_k(y)}{x_N^{\beta} y_N^{\beta}} \geqslant \left\langle L_0(w_0), \frac{|u|^p}{w_0^{p-1}} \right\rangle, \tag{42}$$

for all functions $u \in C_0^{\infty}(\Omega)$ and for all positive function $w \in W^{s,p,q,\beta}_{k,0}(\Omega)$.

Let $0 < \beta < \frac{1-qs}{2}$, $0 < \alpha < \frac{1-qs-2\beta}{p-1}$ and $w_0(x) = x_N^{-\alpha}$. Then for almost every non zero $x \in \mathbb{R}^N$ we have

$$L_0(w_0) = \Lambda_0(\alpha) \frac{w_0^{p-1}}{x_N^{qs+2\beta}} \tag{43}$$

for a positive constant $\Lambda_0(\alpha)$. The proof of this can be done with similar steps of the proof of the Lemma 4. Denoting $r = x_N$, $\rho = y_N$ and using the calculations in (41), we get

$$L_0(w_0)(x) = \|\mathbb{S}^{N-2}\|_k \frac{\Gamma((d_{k_1}-1)/2)\Gamma((1+qs)/2)}{\Gamma((d_{k_1}+qs)/2)} \times \int_0^{\infty} \frac{|r^{-\alpha} - \rho^{-\alpha}|^{p-2}(r^{-\alpha} - \rho^{-\alpha})\Phi_{qs}(rx', \rho y')}{r^{\beta}\rho^{\beta}|r-\rho|^{1+qs}} d\rho.$$

Set $t = r/\rho$,

$$\begin{aligned} L_0(w_0)(x) &= \|\mathbb{S}^{N-2}\|_k \frac{\Gamma((d_{k_1}-1)/2)\Gamma((1+qs)/2)}{\Gamma((d_{k_1}+qs)/2)} \frac{r^{-\alpha(p-1)}}{r^{2\beta}} \int_0^{\infty} \frac{|1-t^{-\alpha}|^{p-2}(1-t^{-\alpha})}{t^{\beta}|1-t|^{1+qs}} dt \\ &= \Lambda_0(\alpha) \frac{w^{p-1}(x)}{x_N^{2\beta+qs}}, \end{aligned}$$

where the constant

$$\Lambda_0(\alpha) = \|\mathbb{S}^{N-2}\|_k \frac{\Gamma((d_{k_1}-1)/2)\Gamma((1+qs)/2)}{\Gamma((d_{k_1}+qs)/2)} \int_0^{\infty} \frac{|1-t^{-\alpha}|^{p-2}(1-t^{-\alpha})}{t^{\beta}|1-t|^{1+qs}} dt.$$

It remains to show that $\Lambda_0(\alpha)$ is positive. Splitting the integral in to two domains; $(0,1)$ and $(1,\infty)$ and use the change of variable $t \to 1/t$ on $(0,1)$ we can write $\Lambda_0(\alpha)$ as

$$\Lambda_0(\alpha) = \int_1^{\infty} \frac{(t^{\alpha}-1)^{p-1}}{|1-t|^{1+qs}} \left(t^{-\beta-\alpha(p-1)} - t^{\beta+qs-1}\right) dt.$$

A repetition of same arguments in the proof of Lemma 4 will show that $\Lambda_0(\alpha)$ is positive.

Use the identity (43) and the Picone's inequality for half-space given in (42) together to see that

$$2\Lambda_0(\alpha)\int_{\mathbb{R}^N_+}\frac{|u(x)|^p}{x_N^{qs+2\beta}}d\mu_k(x) = \left\langle L_0(w_0), \frac{|u|^p}{w_0^{p-1}}\right\rangle \leqslant \iint\limits_{\mathbb{R}^N_+\times\mathbb{R}^N_+} |u(x)-u(y)|^p K^{\beta}_{q,0}(x,y)d\mu_k(x)d\mu_k(y).$$

LEMMA 6. *Fix* $\alpha=\beta=\frac{1-ps}{p}$ *and let* $w_0(x)=x_N^{-\alpha}$. *Let* $u\in C_0^\infty(\mathbb{R}^N)$ *and define* $v(x):=u(x)/w(x)$. *Then for all* $1<q<p<\infty$ *and for a given positive constant* C *the following inequality holds*

$$\iint\limits_{\mathbb{R}^N_+\times\mathbb{R}^N_+} |v(x)-v(y)|^p K^{\beta}_{q,0}(x,y)d\mu_k(x)d\mu_k(y) \geqslant C\iint\limits_{\mathbb{R}^N_+\times\mathbb{R}^N_+} |u(x)-u(y)|^p \Phi_{qs}(x,y)d\mu_k(x)d\mu_k(y).$$

Proof. We will prove the lemma by following the proof of Lemma 5. Replacing K and w by K_0 and w_0 we can define the functions f_1 and f_2 as:

$$\begin{aligned} f_1(x,y) &:= |v(x)-v(y)|^p K^{\beta}_{q,0}(x,y)\\ &= \frac{|w_0(y)u(x)-w_0(x)u(y)|^p}{\big(w_0(x)w_0(y)\big)^{\frac{p}{2}}}\Phi_{qs}(x,y)\\ &= \left|\big(u(y)-u(x)\big)-\frac{u(y)}{w_0(y)}\big(w_0(x)-w_0(y)\big)\right|^p\left(\frac{w_0(y)}{w_0(x)}\right)^{\frac{p}{2}}\Phi_{qs}(x,y);\end{aligned}$$

$$f_2(x,y) := \left|\big(u(x)-u(y)\big)-\frac{u(x)}{w_0(x)}\big(w_0(y)-w_0(x)\big)\right|^p\left(\frac{w_0(x)}{w_0(y)}\right)^{\frac{p}{2}}\Phi_{qs}(x,y).$$

Proceeding with similar steps of the proof of Lemma 5 we arrive at

$$\begin{aligned}\iint\limits_{\mathbb{R}^N_+\times\mathbb{R}^N_+} G(x,y)d\mu_k(x)d\mu_k(y) &= \iint\limits_{\mathbb{R}^N_+\times\mathbb{R}^N_+}\frac{u(x)^p w_0(x)^{p(p-1)}|w_0(x)-w_0(y)|^p}{\big(w_0(x)^p+w_0(y)^p\big)^p}\Phi_{qs}(x,y)d\mu_k(x)d\mu_k(y)\\ &= \int_{\mathbb{R}^N_+}u(x)^p\int_{\mathbb{R}^N_+}\frac{|x_N^\alpha-y_N^\alpha|^p y_N^{\alpha p(p-1)}}{(x_N^{\alpha p}+y_N^{\alpha p})^p}\Phi_{qs}(x,y)d\mu_k(y)d\mu_k(x), \qquad (44)\end{aligned}$$

where

$$G(x,y) = Q(x,y)^p \left(\frac{w_0(x)}{w_0(y)}\right)^{\frac{p}{2}} \left|\frac{u(x)}{w_0(x)}\right|^p |w_0(x) - w_0(y)|^p \Phi_{qs}(x,y).$$

By the definition of the root system we can write

$$\Phi_{qs}(x,y) = \frac{1}{\Gamma(\frac{d_k+qs}{2})} \int_0^\infty s^{\frac{d_k+qs}{2}-1} e^{-s|x_N-y_N|^2} \tau_{y'}^{k_1}(e^{-s|.|^2})(x')ds.$$

Using this and the properties of Dunkl translation(see [13, Proposition 2.4]), the integral become

$$\begin{aligned}
&\int_{\mathbb{R}_+^N} \frac{|x_N^\alpha - y_N^\alpha|^p y_N^{\alpha p(p-1)}}{(x_N^{\alpha p} + y_N^{\alpha p})^p} \Phi_{qs}(x,y) d\mu_k(y) \\
&= \frac{1}{\Gamma(\frac{d_k+qs}{2})} \int_0^\infty \frac{|x_N^\alpha - y_N^\alpha|^p y_N^{\alpha p(p-1)}}{(x_N^{\alpha p} + y_N^{\alpha p})^p} \\
&\quad \times \int_{\mathbb{R}^{N-1}} \int_0^\infty s^{\frac{d_k+qs}{2}-1} e^{-s|x_N-y_N|^2} \tau_{y'}^{k_1}(e^{-s|.|^2})(x') ds d\mu_{k_1}(y') dy_N \\
&= \frac{1}{\Gamma(\frac{d_k+qs}{2})} \int_0^\infty \frac{|x_N^\alpha - y_N^\alpha|^p y_N^{\alpha p(p-1)}}{(x_N^{\alpha p} + y_N^{\alpha p})^p} \\
&\quad \times \int_{\mathbb{R}^{N-1}} \int_0^\infty s^{\frac{d_k+qs}{2}-1} e^{-s|x_N-y_N|^2+|y'|^2} ds d\mu_{k_1}(y') dy_N.
\end{aligned} \tag{45}$$

Using the polar coordinates and integrating, we have

$$\begin{aligned}
&\int_{\mathbb{R}^{N-1}} \frac{1}{(|x_N-y_N|^2+|y'|^2)^{\frac{d_k+qs}{2}}} d\mu_k(y') \\
&= \|\mathbb{S}^{N-2}\|_k \int_0^\infty \frac{1}{(|x_N-y_N|^2+r^2)^{\frac{d_k+qs}{2}}} r^{d_k-2} dr \\
&= \|\mathbb{S}^{N-2}\|_k \frac{1}{|x_N-y_N|^{1+qs}} \int_0^\infty \frac{t^{d_k-2}}{(1+t^2)^{\frac{d_k+qs}{2}}} dt \\
&= \|\mathbb{S}^{N-2}\|_k \frac{1}{|x_N-y_N|^{1+qs}} \frac{\Gamma((d_k-1)/2)\Gamma((1+qs)/2)}{\Gamma((d_k+qs)/2)}.
\end{aligned} \tag{46}$$

Here the quantity $\|\mathbb{S}^{N-2}\|_k$ is the volume of the unit sphere in $\mathbb{R}^{N-1}$ with respect to the weighted measure $d\mu_k$ restricted to the sphere. Also by using the gamma function we obtain

$$\begin{aligned}
&\frac{1}{\Gamma((d_k+qs)/2)} \int_0^\infty s^{\frac{d_k+qs}{2}-1} e^{-s(|x_N-y_N|^2+|x'-y'|^2)} ds \\
&\qquad = \frac{1}{(|x_N-y_N|^2+|x'-y'|^2)^{\frac{d_k+qs}{2}}}.
\end{aligned} \tag{47}$$

Substitute the equations (45), (46) and (47) in (44) we get the integral

$$\iint\limits_{\mathbb{R}^N_+\times\mathbb{R}^N_+} G(x,y)d\mu_k(x)d\mu_k(y) = \|\mathbb{S}^{N-2}\|_k \frac{\Gamma((d_k-1)/2)\Gamma((1+qs)/2)}{\Gamma((d_k+qs)/2)}.$$

$$\times \int_{\mathbb{R}^N_+} u(x)^p \int_0^\infty \frac{|x_N^\alpha - y_N^\alpha|^p y_N^{\alpha p(p-1)}}{(x_N^{\alpha p}+y_N^{\alpha p})^p} \frac{dy_N d\mu_k(x)}{|x_N-y_N|^{1+qs}}.$$

Set $t = y_N/x_N$, then

$$\iint\limits_{\mathbb{R}^N_+\times\mathbb{R}^N_+} G(x,y)d\mu_k(x)d\mu_k(y) = \|\mathbb{S}^{N-2}\|_k \frac{\Gamma((d_k-1)/2)\Gamma((1+qs)/2)}{\Gamma((d_k+qs)/2)}$$

$$\times \int_{\mathbb{R}^N_+} u(x)^p \int_0^\infty \frac{|x_N^\alpha - y_N^\alpha|^p y_N^{\alpha p(p-1)}}{(x_N^{\alpha p}+y_N^{\alpha p})^p} \frac{dy_N d\mu_k(x)}{|x_N-y_N|^{1+qs}}$$

$$= I \int_{\mathbb{R}^N_+} \frac{u(x)^p}{x_N^{qs}} d\mu_k(x),$$

where

$$I = \|\mathbb{S}^{N-2}\|_k \frac{\Gamma((d_k-1)/2)\Gamma((1+qs)/2)}{\Gamma((d_k+qs)/2)} \int_0^\infty \frac{|1-t^\alpha|^p t^{\alpha p(p-1)+2\lambda_k+1}}{(1+t^{\alpha p})^p |1-t|^{1+qs}} dt.$$

Following the similar steps used in proving Lemma 4 we get

$$\iint\limits_{\mathbb{R}^N_+\times\mathbb{R}^N_+} G(x,y)d\mu_k(x)d\mu_k(y) = I \int_{\mathbb{R}^N_+} \frac{|v(x)|^p}{x_N^{qs+1-ps}}$$

$$= C \iint\limits_{\mathbb{R}^N\times\mathbb{R}^N} |v(x)-v(y)|^p K^\beta_{q,0}(x,y) d\mu_k(x) d\mu_k(y)$$

and the inequality (see the proof of Lemma 4 and the beginning of Section 5 for more understanding)

$$\iint\limits_{\mathbb{R}^N\times\mathbb{R}^N} |u(x)-u(y)|^p \Phi_{qs}(x,y) d\mu_k(x) d\mu_k(y) \leqslant C \iint\limits_{\mathbb{R}^N\times\mathbb{R}^N} K^\beta_{q,0}(x,y) d\mu_k(x) d\mu_k(y).$$

5.1. Proof of Theorem 2

We follow the similar steps of the proof of Theorem 1. As in that case we have two cases $p \geqslant 2$ and $p < 2$.

Case 1: $p \geqslant 2$

From [4], we have

$$\int_{\mathbb{R}^N_+}\int_{\mathbb{R}^N_+} |u(x)-u(y)|^p \Phi_{ps}(x,y) d\mu_k(x) d\mu_k(y) - C_{d_k,s,p} \int_{\mathbb{R}^N_+} \frac{|u(x)|^p}{x_N^{ps}} d\mu_k(x)$$

$$+c_p \int_{\mathbb{R}^N_+}\int_{\mathbb{R}^N_+} |v(x)-v(y)|^p \Phi_{ps}(x,y) \frac{d\mu_k(x)}{x_N^{(1-ps)/2}} \frac{d\mu_k(y)}{y_N^{(1-ps)/2}}.$$

But since $\Omega \subset \mathbb{R}^N_+$ bounded, we have $\Phi_{ps}(x,y) \geqslant C(\Omega)\Phi_{qs}(x,y)$ on Ω, and

$$\int_{\mathbb{R}^N_+}\int_{\mathbb{R}^N_+} |v(x)-v(y)|^p \Phi_{ps}(x,y) \frac{d\mu_k(x)}{x_N^{(d_k-ps)/2}} \frac{d\mu_k(y)}{y_N^{(d_k-ps)/2}} \geqslant C(\Omega) \iint\limits_{\Omega\times\Omega} |v(x)-v(y)|^p K^{\beta}_{q,0}(x,y) d\mu_k(x) d\mu_k(y).$$

The proof of Theorem 2 for $p \geqslant 2$ will be completed by applying Lemma 6.

Case 2: $1<p<2$

Let f_1 and f_2 be as in the proof of Lemma 6 and define D_1 and D_2 as in (32) just by replacing w by w_0. Now we have

$$\begin{aligned}
&\iint\limits_{D_1} f_1(x,y) d\mu_k(x) d\mu_k(y) + \iint\limits_{D_2} f_2(x,y) d\mu_k(x) d\mu_k(y) \\
&= C(\Omega) \iint\limits_{\Omega\times\Omega} |v(x)-v(y)|^p K^{\beta}_{q}(x,y) d\mu_k(x) d\mu_k(y) \\
&:= C(\Omega) H_{\Omega,0}(v).
\end{aligned}$$

A similar calculations from (33) to (37) yield

$$\begin{aligned}
&\iint\limits_{D_1} f_1(x,y) d\mu_k(x) d\mu_k(y) \\
&\leqslant C(\Omega) H^{\frac{2-p}{2}}_{\Omega,0}(v) \\
&\quad \times \left(\iint\limits_{\mathbb{R}^N_+\times\mathbb{R}^N_+} |u(x)-u(y)|^p \Phi_{ps}(x,y) d\mu_k(x) d\mu_k(y) - \Lambda^0_{d_k,s,p} \int_{\mathbb{R}^N} \frac{|u(x)|^p}{x_N^{ps}} d\mu_k(x) \right)^{\frac{p}{2}}.
\end{aligned} \tag{48}$$

Similarly for f_2

$$\begin{aligned}
&\iint\limits_{D_1} f_2(x,y) d\mu_k(x) d\mu_k(y) \\
&\leqslant C(\Omega) H^{\frac{2-p}{2}}_{\Omega,0}(v) \\
&\quad \times \left(\iint\limits_{\mathbb{R}^N_+\times\mathbb{R}^N_+} |u(x)-u(y)|^p \Phi_{ps}(x,y) d\mu_k(x) d\mu_k(y) - \Lambda^0_{d_k,s,p} \int_{\mathbb{R}^N} \frac{|u(x)|^p}{x_N^{ps}} d\mu_k(x) \right)^{\frac{p}{2}}.
\end{aligned} \tag{49}$$

Combining (48) and (49) we arrive at

$$H_{\Omega,0}(v) \leqslant C(\Omega) \iint\limits_{\mathbb{R}^N_+ \times \mathbb{R}^N_+} |u(x)-u(y)|^p \Phi_{ps}(x,y) d\mu_k(x) d\mu_k(y)$$
$$\times -\Lambda^0_{d_k,s,p} \int_{\mathbb{R}^N_+} \frac{|u(x)|^p}{x_N^{ps}} d\mu_k(x).$$

Putting both cases together we can write

$$\iint\limits_{\mathbb{R}^N_+ \times \mathbb{R}^N_+} |u(x)-u(y)|^p \Phi_{ps}(x,y) d\mu_k(x) d\mu_k(y) - \Lambda^0_{d_k,s,p} \int_{\mathbb{R}^N_+} \frac{|u(x)|^p}{x_N^{ps}} d\mu_k(x)$$
$$\geqslant C(\Omega) \iint\limits_{\Omega \times \Omega} |v(x)-v(y)|^p K^{\beta}_{q,0}(x,y) d\mu_k(x) d\mu_k(y). \tag{50}$$

A direct application of Lemma 6 and (50) we get the desired improved farctional Hardy inequality

$$\iint\limits_{\mathbb{R}^N_+ \times \mathbb{R}^N_+} |u(x)-u(y)|^p \Phi_{ps}(x,y) d\mu_k(x) d\mu_k(y) - \Lambda^0_{d_k,s,p} \int_{\mathbb{R}^N_+} \frac{|u(x)|^p}{x_N^{ps}} d\mu_k(x)$$
$$\geqslant C \iint\limits_{\Omega \times \Omega} |u(x)-u(y)|^p \Phi_{qs}(x,y) d\mu_k(x) d\mu_k(y).$$

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