INSTITUT DE FRANCE
Académie des sciences

Comptes Rendus
Mathématique

Vamsi Pritham Pingali

A note on Demailly's approach towards a conjecture of Griffiths

Volume 359, issue 4 (2021), p. 501-503

Published online: 17 June 2021

https://doi.org/10.5802/crmath.192

Les Comptes Rendus. Mathématique *sont membres du Centre Mersenne pour l'édition scientifique ouverte*
www.centre-mersenne.org
e-ISSN : 1778-3569

Comptes Rendus
Mathématique
2021, 359, nº 4, p. 501-503
https://doi.org/10.5802/crmath.192

Complex analysis and geometry / *Analyse et géométrie complexes*

A note on Demailly's approach towards a conjecture of Griffiths

Vamsi Pritham Pingali [a]

[a] Department of Mathematics, Indian Institute of Science, Bangalore 560012, India
E-mail: vamsipingali@iisc.ac.in

Abstract. We prove that a "cushioned" Hermitian–Einstein-type equation proposed by Demailly in an approach towards a conjecture of Griffiths on the existence of a Griffiths positively curved metric on a Hartshorne ample vector bundle, has an essentially unique solution when the bundle is stable. This result indicates that the proposed approach must be modified in order to attack the aforementioned conjecture of Griffiths.

Funding. This work is partially supported by grant F.510/25/CAS-II/2018(SAP-I) from UGC (Govt. of India), and a MATRICS grant MTR/2020/000100 from SERB (Govt. of India).

Manuscript received 4th February 2021, accepted 16th February 2021.

1. Introduction

The notion of ampleness/positivity is paramount in algebraic geometry. For a holomorphic line bundle, there is only one notion of differentio-geometric positivity, i.e., there is a smooth Hermitian metric whose curvature form is a Kähler form. By the Kodaira embedding theorem, it coincides with algebro-geometric ampleness. A holomorphic vector bundle E is said to be Hartshorne ample if $\mathcal{O}_{E^*}(1)$ is an ample line bundle over $\mathbb{P}(E^*)$. There is no unique differentio-geometric notion of positivity of curvature Θ of a smooth Hermitian metric h. There are several competing inequivalent notions. The most natural of these notions are *Griffiths positivity* ($\langle v, \sqrt{-1}\Theta v\rangle$ is a Kähler form for all $v \neq 0$), *Nakano positivity* (the bilinear form defined by $\sqrt{-1}\Theta$ on $T^{1,0}M \otimes E$ is positive-definite), and *dual-Nakano positivity* (the Hermitian holomorphic bundle (E^*, h^*) is Nakano *negative*). Nakano positivity and dual-Nakano positivity imply Griffiths positivity and all three of them imply Hartshorne ampleness. A famous conjecture of Griffiths [5] asks whether Hartshorne ample vector bundles admit Griffiths positively curved metrics. This conjecture is still open. However, a considerable amount of work has been done to provide evidence in its favour [1–3, 6, 8–10, 13].

Relatively recently, Demailly [3] proposed a programme to prove the aformentioned conjecture of Griffiths for a holomorphic rank$-r$ vector bundle E on a compact Kähler manifold (X, ω_0). In fact, if Demailly's method works, it will end up proving a stronger conjecture : Do Hartshorne-ample bundles admit dual-Nakano positively curved metrics? Demailly's approach involves solving a family (depending on a parameter $0 \leq t \leq 1$) of vector bundle Monge–Ampère equations (distinct from the one introduced in [11]) in conjunction with "cushioned" Hermitian–Einstein-type equations (see [3, Theorem 2.17]):

$$\det_{TX\otimes E^*}\left(\Theta^T_{h_t}+(1-t)\alpha\omega_0\otimes I_{E^*}\right)^{1/r}=f_t\frac{(\det h_0)^{\lambda}}{(\det h_t)^{\lambda}}\omega_0^n, \tag{1}$$

$$\left(\sqrt{-1}F_{h_t}-\frac{\sqrt{-1}}{r}\operatorname{tr}F_{h_t}\right)\omega_0^{n-1}=-\epsilon\frac{(\det h_0)^{\mu}}{(\det h)^{\mu}}\ln\left(\frac{hh_0^{-1}}{\det(hh_0^{-1})^{1/r}}\right)\omega_0^n, \tag{2}$$

where h_0 is a smooth background Hermitian metric, $\mu,\lambda\geq 0$ are fixed constants, $\alpha>0$ is a large enough constant so that $\Theta_{h_0}+\alpha\omega$ is dual-Nakano positively curved, and $f_t>0$ are smooth positive functions. We focus on the cushioned Hermitian–Einstein-type equation in the following theorem.

Theorem 1. *Let E be an ω_0-stable rank$-r$ holomorphic bundle on X. Let H_0 be a Hermitian–Einstein metric on E with respect to ω_0, that is, $\sqrt{-1}F_{H_0}\omega_0^{n-1}=\lambda\omega_0^n$. Let h be a smooth metric on E solving the following cushioned Hermitian–Einstein equation for given parameters $\epsilon\geq 0,\mu\geq 0$.*

$$\left(\sqrt{-1}F_{h}-\frac{\sqrt{-1}}{r}\operatorname{tr}F_{h}\right)\omega_0^{n-1}=-\epsilon\frac{(\det H_0)^{\mu}}{(\det h)^{\mu}}\ln\left(\frac{hH_0^{-1}}{\det(hH_0^{-1})^{1/r}}\right)\omega_0^n, \tag{3}$$

where h,H_0 are matrices (any holomorphic trivialisation will do). Then $h=H_0e^{-f}$ for some smooth function f.

As a result, if we consider the system of the vector bundle Monge–Ampère equation and the cushioned Hermitian–Einstein-type equation on an ω_0-stable ample E, and if solutions exist all the way till $t=1$, the final $t=1$ solution, by virtue of the fact that it satisfies the cushioned Hermitian–Einstein-type equation, has to be of the form H_0e^f. This condition might be a strong restriction (which is unlikely to be met owing to [7, 12] without a restriction on the second Chern character). On the other hand, if we replace ω_0 by say $(1-t)\omega_0+t\sqrt{-1}\operatorname{tr}(F_{h_t})$ (or the choice in [3, Section 2.19] for instance), the above argument will not be applicable and there might be some hope for the approach to yield an affirmative solution to the Griffiths Conjecture.

Acknowledgements

The author thanks Jean-Pierre Demailly for fruitful discussions and for encouraging me to write up this note.

2. Proof of uniqueness

In a holomorphic trivialisation, our conventions are : $\langle v,w\rangle_H=v^TH\bar{w}$, if g is an endomorphism then $g.s=[g]^T\vec{s}$, $\nabla s=ds+A^Ts$, $A=\partial HH^{-1}$, $F=dA-A\wedge A=\bar{\partial}A$, and $\nabla g=dg+[g,A]$.

The proof is motivated by a similar one by Donaldson for Riemann surfaces [4]. In general, $h=qH_0$ where q is some smooth H_0-Hermitian positive-definite endomorphism of E. We decompose q further as $q=e^{-f}g$ where $\det(g)=1$ and f is a smooth function. Thus, $F_h=F_{H_0}+\partial\bar{\partial}f+\bar{\partial}(\partial_0gg^{-1})$. The trace-free part of the curvature is $F_h^{\circ}=F_0^{\circ}+\bar{\partial}(\partial_0gg^{-1})$. Substituting these expressions in 3 and using the fact that H_0 is Hermitian–Einstein with respect to ω_0, we get

$$\sqrt{-1}\bar{\partial}(\partial_0gg^{-1})\omega_0^{n-1}=-\epsilon e^{r\mu f}\ln g\,\omega_0^n. \tag{4}$$

Now we compute

$$\begin{aligned}\frac{1}{2}\sqrt{-1}\bar{\partial}\partial\operatorname{tr}(g^2)\omega_0^{n-1}&=\sqrt{-1}\bar{\partial}\operatorname{tr}(g\partial_0g)\omega_0^{n-1}=\sqrt{-1}\operatorname{tr}(\bar{\partial}g\partial_0g)\omega_0^{n-1}+\sqrt{-1}\operatorname{tr}(g\bar{\partial}\partial_0g)\omega_0^{n-1}\\&=\sqrt{-1}\operatorname{tr}(\bar{\partial}g\partial_0g)\omega_0^{n-1}-\epsilon e^{r\mu f}\operatorname{tr}(g^2\ln g)\omega_0^n-\sqrt{-1}\operatorname{tr}(g\partial_0gg^{-1}\bar{\partial}g)\omega_0^{n-1}\\&\leq-\epsilon e^{r\mu f}\operatorname{tr}(g^2\ln g)\omega_0^n.\end{aligned} \tag{5}$$

Note that $\operatorname{tr}(g^2 \ln g) = \sum_i \lambda_i^2 \ln(\lambda_i)$ where $\lambda_i > 0$ are the eigenvalues of g such that $\lambda_1 \leq \lambda_2 \ldots$. The product of the λ_i is 1. Thus,

$$\sum_{i \mid \lambda_i < 1} |\ln(\lambda_i)| = \sum_{i \mid \lambda_i > 1} \ln(\lambda_i)$$

which implies that

$$\begin{aligned} \sum_{1 \leq i \leq p \mid \lambda_p \leq 1,\ \lambda_{p+1} > 1} \lambda_i^2 |\ln(\lambda_i)| \leq \sum_{1 \leq i \leq p \mid \lambda_p \leq 1,\ \lambda_{p+1} > 1} \lambda_p^2 |\ln(\lambda_i)| = \lambda_p^2 \sum_{i=p+1}^{n} \ln(\lambda_i) \\ \leq \sum_{i=p+1}^{n} \lambda_i^2 \ln(\lambda_i). \end{aligned} \tag{6}$$

Therefore, $\operatorname{tr}(g^2 \ln g) \geq 0$ and hence

$$\frac{1}{2}\sqrt{-1}\bar{\partial}\partial \operatorname{tr}(g^2)\omega_0^{n-1} \leq 0. \tag{7}$$

The strong maximum principle then implies that actually

$$\frac{1}{2}\sqrt{-1}\bar{\partial}\partial \operatorname{tr}(g^2)\omega_0^{n-1} = 0, \tag{8}$$

and hence $g = I$. □

References

[1] B. Berndtsson, "Curvature of vector bundles associated to holomorphic fibrations", *Ann. Math.* **169** (2009), no. 2, p. 531-560.

[2] F. Campana, H. Flenner, "A characterization of ample vector bundles on a curve", *Math. Ann.* **287** (1990), no. 4, p. 571-575.

[3] J.-P. Demailly, "Hermitian–Yang–Mills approach to the conjecture of Griffiths on the positivity of ample vector bundles", https://arxiv.org/abs/2002.02677, 2020.

[4] S. K. Donaldson, "A new proof of a theorem of Narasimhan and Seshadri", *J. Differ. Geom.* **2** (1983), p. 269-277.

[5] P. A. Griffiths, "Hermitian differential geometry, Chern classes and positive vector bundles", in *Global analysis*, Princeton University Press, 1969, p. 185-251.

[6] K. Liu, X. Sun, X. Yang, "Positivity and vanishing theorems for ample vector bundles", *J. Algebr. Geom.* **22** (2013), no. 2, p. 303-331.

[7] M. Lübke, "A note on positivity of Einstein bundles", *Indag. Math.* **2** (1991), no. 3, p. 311-318.

[8] C. Mourougane, S. Takayama, "Hodge metrics and positivity of direct images", *J. Reine Angew. Math.* **606** (2007), p. 167-178.

[9] P. Naumann, "arXiv preprint", https://arxiv.org/abs/1710.10034, 2017.

[10] V. P. Pingali, "Representability of Chern–Weil forms", *Math. Z.* **288** (2018), no. 1-2, p. 629-641.

[11] ———, "A vector bundle version of the Monge–Ampère equation", *Adv. Math.* **360** (2020), article no. 106921 (40 pages).

[12] M. Schneider, A. Tancredi, "Positive vector bundles on complex surfaces", *Manuscr. Math.* **50** (1985), no. 1, p. 133-144.

[13] H. Umemura, "Some results in the theory of vector bundles", *Nagoya Math. J.* **52** (1973), p. 97-128.