Super-Chandrasekhar limiting mass white dwarfs as emergent phenomena of noncommutative squashed fuzzy spheres

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Abstract

The indirect evidence for at least a dozen massive white dwarfs violating the Chandrasekhar mass-limit is considered to be one of the wonderful discoveries in astronomy for more than a decade. Researchers have already proposed a diverse amount of models to explain this astounding phenomenon. However, each of these models always carries some drawbacks. On the other hand, noncommutative geometry is one of the best replicas of quantum gravity, which is yet to be proved from observations. Madore introduced the idea of a fuzzy sphere to describe a formalism of noncommutative geometry. This article shows that the idea of a squashed fuzzy sphere can self-consistently explain the super-Chandrasekhar limiting mass white dwarfs. We further show that the length-scale beyond which the noncommutativity is prominent is an emergent phenomenon, and there is no prerequisite for an ad-hoc length-scale.

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1. INTRODUCTION

The end phase of a star possessing a mass $M \lesssim (10 \pm 2)M_\odot$ [1] is a white dwarf (WD), where the inward pressure due to gravity balances the outward pressure due to the degenerate electron gas, to maintain its stable equilibrium [2]. Chandrasekhar first proposed thereby a limiting mass of the WD and estimated the maximum mass of a stable non-rotating non-magnetized carbon-oxygen WD to be $\sim 1.4M_\odot$ [3]. Beyond this mass-limit, the pressure balance no longer sustains, and the WDs explode due to new fusion reactions to produce type Ia supernovae (SNeIa). However, recent observations from more than a dozen of over-luminous SNeIa suggest that the maximum mass of a WD could be as high as $\sim 2.8M_\odot$ [4–12], indicating a clear violation of the Chandrasekhar mass-limit. Moreover, various simulations have already ruled out the possibility for the existence of a double degenerate scenario for the generation of the over-luminous SNeIa, and $2.8M_\odot$ progenitor mass as such a double degenerate WD produces an off-center ignition and forms a neutron star rather than an over-luminous SNIa [13, 14]. All these indirect evidences suggest that conventional physics may significantly be violated inside the high-density regime of massive WDs.

Ostriker and Hartwick first showed that the rotation could alone increase the mass of a WD up to $\sim 1.8M_\odot$ [15], which, however, cannot explain the inferred significantly super-Chandrasekhar WDs. Thereafter, Mukhopadhyay and his collaborators showed that the high constant magnetic field or highly fluctuating magnetic field such that the average field is low could comprise super-Chandrasekhar WDs with mass up-to $\sim 2.6M_\odot$ due to the formation of Landau levels [16, 17]. However, such a high field may also destabilize the WD, if it is varying with radial coordinate, depending on the field geometry, as the magnetic field can change its shape and size (which is a classical effect of the magnetic field) [18–21]. As a result, having a high magnetic field inside the WDs is a question of debate over the years. Later, various researchers have proposed different models, such as modified gravity [22, 23], presence of charge in a WD [24], generalized Heisenberg uncertainty principle [25], lepton number violation [26], and many more to explain the super-Chandrasekhar WDs. We have earlier shown that in the presence of noncommutativity (NC), high-density WDs can have a mass up to $\sim 2.6M_\odot$ [27]. While deriving this mass-limit, we proposed based on Wigner's idea [28] that the effect of NC is significant, mostly at the center of a dense WD, where the inter-electron separation is less than the Compton wavelength.

Noncommutative geometry is a sub-class of quantum gravity that can explain various interesting phenomena ranging from the early universe cosmology to the inside of the event horizon of a black hole. However, there is no direct observational evidence for NC so far. In ordinary quantum mechanics (QM), the position variable ($\hat{x}$) does not commute with the corresponding momentum component ($\hat{p}$) and follows the Heisenberg's uncertainty principle, whereas various components of $\hat{x}$ and $\hat{p}$ commute among themselves. In noncommutative geometry, $\hat{x}$ and $\hat{p}$ even do not commute among themselves. One of the popular ways of proposing NC is just an ad-hoc consideration of $[\hat{x}_i, \hat{x}_j] = i\eta$ and $[\hat{p}_i, \hat{p}_j] = i\theta$, where η and θ are the noncommutative variables. Nicolini et al. [29] used this form of NC to show the shift of the event horizon for black holes. They also showed that this form of NC removes the essential singularity present at the black hole center. Similarly, other researchers have used different forms of noncommutative geometry to describe various problems of physics related to the fundamental length-scale, Berry curvature, Landau levels, etc. [30–35]. As mentioned above, for many of these theories, the basic assumption in the structure of NC among the position and momentum variables is quite ad-hoc. Madore [36] first introduced a self-consistent model of 3−dimensional NC, named fuzzy sphere, where the position variables follow the well-known angular momentum algebra of QM. This formalism was used by various researchers to obtain the thermodynamical properties of non-interacting degenerate electron gas [37, 38]. Andronache and Steinacker [39] later modified the idea of a fuzzy sphere by projecting it on an equatorial plane, named this configuration a squashed fuzzy sphere, and showed that this model of NC mimics the magnetic field in the non-relativistic limit. They also reckoned that this form of NC could quantize the plane by forming different energy levels in the semi-classical limit, which is similar to the Landau levels formed in the presence of a magnetic field in the non-relativistic limit.

In this paper, we use the idea of a squashed fuzzy sphere to apply it in a WD. This formalism is superior to the earlier one we used in [27] due to two main reasons. First, fuzzy sphere algebra is symmetric under rotation, and second, the squashed fuzzy sphere captures naturally expected commutative geometry at the boundary of the WD as the NC decreases from the center to the surface. We show that, by assuming the WD as a noncommutative fuzzy sphere, its mass increases significantly to form a super-Chandrasekhar limiting mass WD. Moreover, in this formalism, the length-scale of the system over which effect of NC is prominent depends on the Compton wavelength of an electron and the Planck scale; thereby, it turns out to be an emergent phenomenon. There is no need to pre-assume a length-scale for the prominence of NC. In the presence of NC, we show that fermions behave less like fermions, and hence, the effective pressure between them reduces, which allows more mass to accumulate.

The plan of this paper is as follows. In §2, we discuss the basic formalism of the squashed fuzzy sphere and thereby obtain the exact (i.e., applicable in both relativistic and non-relativistic regimes) energy dispersion relation for the squashed fuzzy sphere in §3. We compare this dispersion relation with the well-known energy dispersion relation for the Landau levels formed due to a magnetic field. Thereafter, in §4, we derive the equation of state (EoS) for the degenerate electrons in a squashed fuzzy sphere. In this section, we also discuss the emergence of an effective length-scale for the prominence of NC. Further, we use the degenerate equation of state in §5 to derive the mass-radius relation for the WD, and thereby to obtain its limiting mass. Moreover, we use the dispersion relation to discuss the neutron drip briefly in the presence of NC in §6 and to show the region on which this EoS is valid. Finally we end with conclusions in §7.

2. SQUASHED FUZZY SPHERE

A fuzzy sphere S_N^2, first introduced by Madore [36], is like an ordinary sphere S^2, except that its coordinates follow the angular momentum algebra of usual QM and, hence, they, in general, do not commute among themselves. In $\mathbb{R}^3$, a sphere of radius $r \in \mathbb{R}$ is defined as the set of points which follows

$$X_1^2 + X_2^2 + X_3^2 = r^2, \tag{2.1}$$

with X_1, X_2, X_3 being the ordinary cartesian coordinates of the points. In a fuzzy sphere, the coordinates X_i $(i = 1, 2, 3)$ follows

$$X_i = \kappa J_i, \tag{2.2}$$

where κ is the proportionality (scaling) constant and J_i are the generators of $\mathtt{SU}(2)$ group which follows the angular momentum algebra in an $N-$dimensional irreducible representation such that

$$J_1^2 + J_2^2 + J_3^2 = \frac{\hbar^2}{4}\left(N^2 - 1\right)\mathbb{I} = C_N \mathbb{I}, \tag{2.3}$$

with $C_N = \hbar^2 (N^2 - 1)/4$, $\hbar = h/2\pi$, h is the Planck constant and $\mathbb{I}$ is the $N-$dimensional identity matrix. Substituting J_i in terms of X_i and defining $k = \kappa r$, we obtain the following relations

$$\kappa = \frac{r}{\sqrt{C_N}} \quad \text{and} \quad k = \frac{r^2}{\sqrt{C_N}}. \tag{2.4}$$

Since the angular momentum algebra follows $[J_i, J_j] = i\hbar\epsilon_{ijk}J_k$, the coordinates of the fuzzy sphere needs to obey the following commutation relation

$$[X_i, X_j] = i\frac{k\hbar}{r}\epsilon_{ijk}X_k. \tag{2.5}$$

Moreover, in a fuzzy sphere, the fuzzy Laplacian is defined as [39]

$$\Box = \frac{1}{k^2}\sum_{i=1}^{3}[X_i, [X_i, .]], \tag{2.6}$$

which satisfies the following eigenvalue equation

$$\Box \hat{Y}_m^l = \frac{\hbar^2}{r^2} l(l+1)\hat{Y}_m^l, \tag{2.7}$$

where $l(l+1)$ are the eigenvalues and $\hat{Y}_m^l$ are the eigenfunctions of the fuzzy Laplacian with l taking all the integer values from 0 to $N-1$ and m taking all the integer values from $-l$ to l.

A squashed fuzzy sphere is interpreted as the configuration when all the points of a fuzzy sphere are projected on any of its equatorial planes. Note that this projection is not a stereographic projection. Figure 1 shows the projection of all the points of a fuzzy sphere on the $x-y$ equatorial plane. The upper hemisphere points are projected on the upper side of the equatorial plane and, the points of the lower hemisphere are projected on its lower side and, then, they are glued together. Writing X_3 in terms of X_1 and X_2 from Equation (2.1), and substituting it in Equation (2.5), the commutation relation for a squashed fuzzy sphere is modified as [39]

$$[X_1, X_2] = \pm i\frac{k\hbar}{r}\sqrt{r^2 - X_1^2 - X_2^2}. \tag{2.8}$$

The Laplacian for the squashed fuzzy sphere is given by [39]

$$\Box_s = \frac{1}{k^2}\sum_{i=1}^{2}[X_i, [X_i, .]], \tag{2.9}$$

which satisfies the following eigenvalue equation

$$\Box_s \hat{Y}_m^l = \frac{\hbar^2}{r^2}\left\{l(l+1) - m^2\right\}\hat{Y}_m^l, \tag{2.10}$$

where $l(l+1) - m^2$ are eigenvalues of the squashed fuzzy Laplacian.

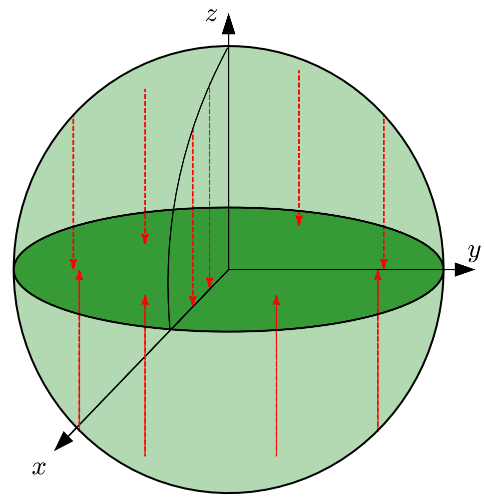


FIG. 1: An illustrative diagram of a squashed fuzzy sphere which is obtained by projecting all the points of the fuzzy sphere on an equatorial plane.

3. ENERGY DISPERSION RELATION IN A SQUASHED FUZZY SPHERE

The Dirac operator for the squashed fuzzy sphere is defined as [39]

$$\not{D} = \frac{c}{k}\left(\sigma_1 \otimes [X_1, .] + \sigma_2 \otimes [X_2, .]\right), \tag{3.1}$$

where c is the speed of light and $\sigma_1, \sigma_2, \sigma_3$ are the Pauli spin matrices. The eigenvalues of this Dirac operator gives the quantized energy in a squashed fuzzy sphere. Now, the square of the Dirac operator is given by

$$\not{D}^2 = \frac{c^2}{k^2}\sigma_i\sigma_j \otimes [X_i, [X_i, .]] \tag{3.2}$$

$$= c^2 \left\{ \Box \otimes \mathbb{I}_2 - \frac{1}{k^2}\left([X_3, .] + \frac{k\hbar}{2r}\sigma_3\right)^2 + \frac{\hbar^2}{4r^2} \right\}. \tag{3.3}$$

Hence, the eigenvalues of $\not{D}^2$ (which are the squares of energies) are given by

$$E^2 = \frac{\hbar^2 c^2}{r^2}\left\{ l(l+1) - \left(m \pm \frac{1}{2}\right)^2 + \frac{1}{4} \right\}. \tag{3.4}$$

Since σ_3 has the eigenvalues ± 1, hence the $+$ and $-$ correspond to the energies of the spin-up and spin-down particles respectively. From Equation (2.4), using the relation $k\hbar = 2r^2/\sqrt{N^2-1}$, the above equation reduces to

$$E^2 = \frac{2\hbar c^2}{k\sqrt{N^2-1}}\left\{ l(l+1) - m(m \pm 1) \right\}. \tag{3.5}$$

Moreover, the relations between the cartesian and spherical polar coordinates are $x = r\sin\theta\cos\phi$ and $y = r\sin\theta\sin\phi$ with radius r being fixed for a particular fuzzy sphere. Therefore, using Equation (2.8), the algebra of a squashed fuzzy sphere in spherical polar coordinates can be recast as

$$[\sin\theta\cos\phi, \sin\theta\sin\phi] = \pm i\frac{k\hbar}{r^2}\cos\theta. \tag{3.6}$$

It is evident that the NC in spherical coordinates is between $\theta-$ and $\phi-$coordinates only, while the $r-$coordinate remains free. In other words, the squashed fuzzy sphere actually provides a NC between its azimuthal and polar coordinates because the squashed plane in a fuzzy sphere can be any of its equatorial planes and there is no particular direction for it. Hence, an electron with mass m_e, moving in a squashed fuzzy sphere, does not experience the effect of NC along $r-$coordinate. Therefore, defining $N = l_{\max}+1$, $\tilde{L} = l_{\max} - l$, $\tilde{M} = l - m$ and $\tilde{M}' = l + m$, the exact energy dispersion relation for an electron, moving in a squashed fuzzy sphere, is given by

$$E^2 = p_r^2c^2 + m_e^2c^4\left[1 + \{l(l+1) - m(m\pm 1)\}\frac{\hbar^2}{m_e^2c^2r^2}\right] \tag{3.7}$$

$$= p_r^2c^2 + m_e^2c^4\left[1 + \{l(l+1) - m(m\pm 1)\}\frac{2\hbar}{m_e^2c^2k\sqrt{N^2-1}}\right] \tag{3.8}$$

$$= \begin{cases} p_r^2c^2 + m_e^2c^4\left[1 + \tilde{M}\frac{2N-2\tilde{L}-\tilde{M}-1}{\sqrt{N^2-1}}\frac{2\hbar}{m_e^2c^2k}\right] & \text{for spin-up electrons} \\ p_r^2c^2 + m_e^2c^4\left[1 + \tilde{M}'\frac{2N-2\tilde{L}-\tilde{M}'-1}{\sqrt{N^2-1}}\frac{2\hbar}{m_e^2c^2k}\right] & \text{for spin-down electrons,} \end{cases} \tag{3.9}$$

where p_r is the momentum of the electron in radial direction. From Equation (3.7), it is evident that for spin-up electrons, $m = l$ represents the ground level, whereas $m = l-1$ as well as $m = -l$ are the first energy levels, and so on. Similarly, for spin-down electrons, $m = -l$ is the ground level, whereas $m = -l+1$ and $m = l$ are the first energy levels. Hence, the ground level has always one pair of spin-up and spin-down electrons, whereas all the other energy levels comprise a couple of pairs of spin-up and spin-down electrons. Figure 2 shows the quantized energy levels in a noncommutative squashed fuzzy sphere for $N = 20$ and $N = 50$. From this figure, it is evident that spacing between the energy levels decreases towards the higher energy levels. Moreover, in the large N limit, the expression for energy dispersion relation (3.9) reduces to

$$E^2 = p_r^2c^2 + m_e^2c^4\left(1 + 2\nu\frac{2\hbar}{m_e^2c^2k}\right), \qquad \nu \in \mathbb{Z}^{0+} \tag{3.10}$$

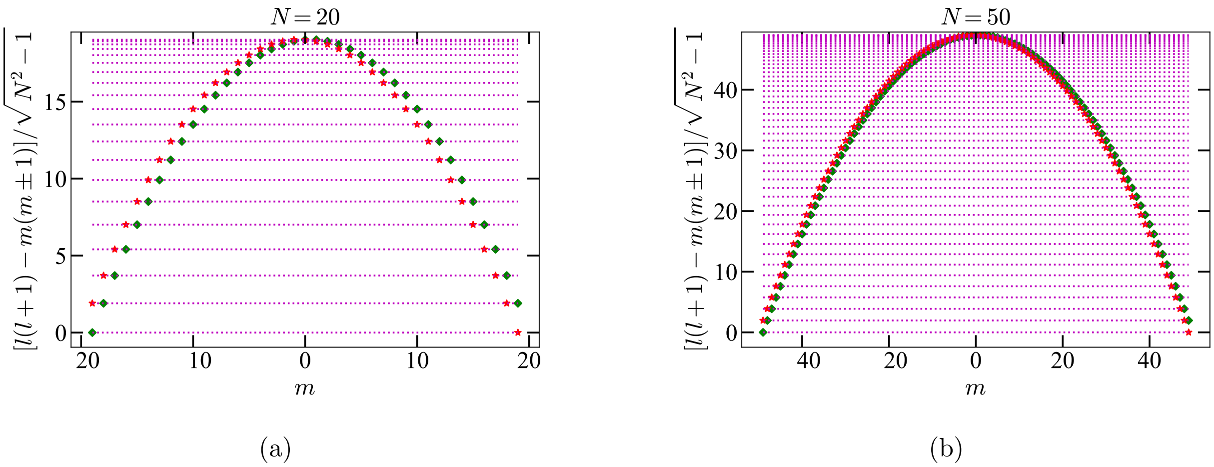


FIG. 2: Quantized energy levels in the noncommutative squashed fuzzy sphere. Here red stars represent the spin-up electrons and the green squares depict the spin-down electrons. It is evident that the ground level is occupied by one pair of spin-up and spin-down electrons and all the other higher energy levels have two pairs of them. Moreover, the energy gap decreases towards the higher energy levels.

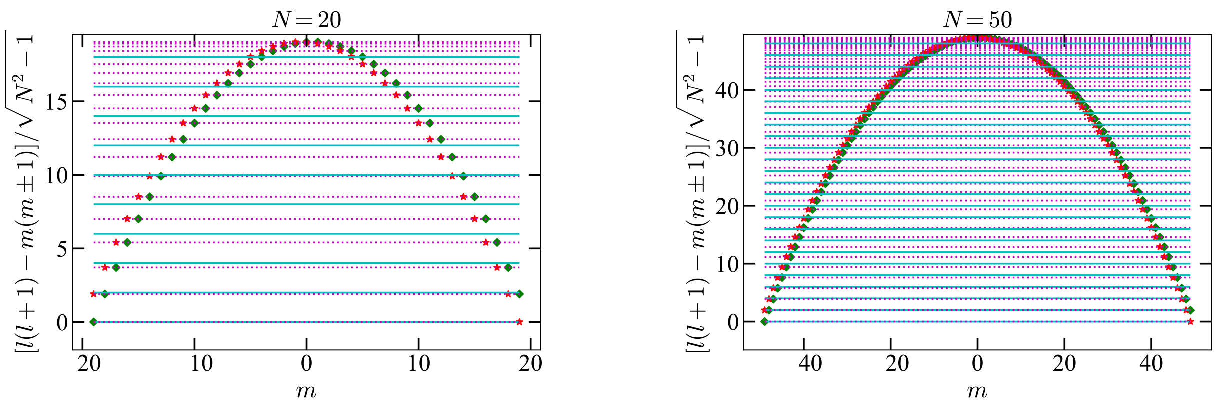
with $\nu = \tilde{M}$ for for spin-up and $\nu = \tilde{M}'$ for for spin-down electrons. This is the quantized energy for an electron in a squashed fuzzy sphere. Comparing it with the energy dispersion relation of Landau quantization due to the magnetic field, which is [40]

$$E^2 = p_z^2c^2 + m_e^2c^4\left(1 + 2\nu\frac{B}{B_c}\right), \qquad \nu \in \mathbb{Z}^{0+} \tag{3.11}$$

where p_z is the momentum of an electron along $z-$direction, B is the magnetic field along $z-$direction, and B_c is the critical magnetic field (Schwinger limit), given by $B_c = m_e^2c^3/\hbar e$ with e being the charge of an electron. Upon simplification, comparing equations (3.10) and (3.11), we obtain

$$B \equiv \frac{2c}{ek}. \tag{3.12}$$

From the above expression, it is evident that the term k^{-1} behaves as NC strength in a squashed fuzzy sphere. Andronache and Steinacker earlier obtained a similar relation between B and k in the case of non-relativistic squashed fuzzy sphere in NC [39]. Figure 3 shows the comparison of the approximated form of dispersion relation of Equation (3.10) with the exact expression from Equation (3.7) for $N = 20$ and $N = 50$. Comparing both the figures, it is evident that the more number of levels coincide when N is large. For $N = 20$, only the ground level and the first level match with the approximated levels, whereas, for $N = 50$, five levels coincide with the approximated levels. This shows a good agreement between equations (3.7) and (3.10) if $N \ggg 1$.



(a) (b)

FIG. 3: Comparison of the energy levels obtained from equations (3.7) and (3.10). Here the dotted magenta lines represent the exact energy levels, and the solid cyan lines show the approximated energy levels. It is evident that as N increases, more energy levels coincide with approximated energy levels. Here for $N = 20$, two levels coincide, whereas, for $N = 50$, five energy levels coincide.

Equation (3.10) provides the energy dispersion relation of one squashed fuzzy sphere inside which the strength of NC (k^{-1}) is constant. Let us now consider a series of concentric fuzzy spheres (each of them are squashed in a common equatorial plane) with the same values of N. This means that all these fuzzy spheres can be explained by the same $\mathtt{SU}(2)$ algebra. Moreover, from Equation (2.4), for a fixed N, we have $k \propto r^2$, i.e., k increases from the center to the surface; therefore, the strength of NC decreases from center to the surface. Hence the effective NC at a point of radius r, is due to the contributions from all the concentric spheres with radius more than r. As a result, effective k no longer remains constant inside the sphere, and the variation of k needs to be chosen so that k is minimum at the center and maximum at the surface. Figure 4 shows an illustrative diagram of the variation of k inside a fuzzy sphere. The effective NC at a point P is due to the contributions of NC from all the fuzzy spheres within which the point is enclosed. Since k^{-1} behaves as the strength of NC in a squashed fuzzy sphere, effective NC is maximum at the center, which gradually decreases outwards and becomes minimum (mathematically zero) at the surface.

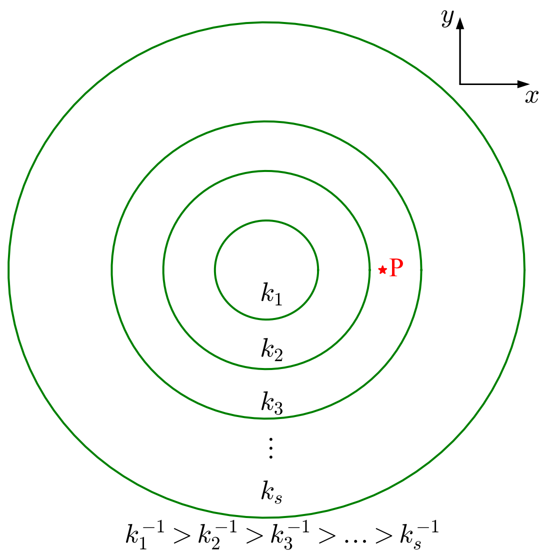


FIG. 4: Concentric fuzzy spheres with the same N, but effective k varies with the radius. Note that k^{-1} behaves as the strength of NC in a squashed fuzzy sphere, which decreases from the center to the surface.

4. DEGENERATE EQUATION OF STATE IN SQUASHED FUZZY SPHERE

for a given density, is higher as you find in figure 5. So hydrostatic equilibrium can be maintained to higher masses, preventing collapse Since in this paper, our primary focus is to apply the effect of NC in the squashed fuzzy sphere for WDs, we now discuss the EoS of degenerate electrons in the squashed fuzzy sphere. The grand canonical potential for this system is given by

$$\Omega = -\frac{4V}{\beta k h^2}\int_0^{p_{r,\max}} \mathrm{d}p_r \sum_{\nu=0}^{\nu_{\max}} g_\nu \ln\left(1+e^{\beta(\mu-E)}\right) \tag{4.1}$$

$$= -\frac{4V}{\beta k h^2}\int_0^{p_{r,\max}} \mathrm{d}p_r \sum_{\nu=0}^{\nu_{\max}} g_\nu \ln\left(1+ze^{-\beta E}\right), \tag{4.2}$$

where $z = e^{\beta\mu}$ is the fugacity, $\beta = 1/k_B T$, μ is the chemical potential, V is the volume of the system, k_B is the Boltzmann constant, T is the temperature, and g_ν is the degeneracy factor with $g_\nu = 2 - \delta_{\nu 0}$ where $\delta_{\mu\nu}$ being the Kronecker delta function. Note that V is the noncommutative volume, which is different from the classical volume. The relation between Ω and the number density of electrons (n_e) is given by

$$n_e = -\frac{\beta z}{V}\frac{\partial \Omega}{\partial z}. \tag{4.3}$$

Hence, the mass density (ρ) is given by

$$\rho = m_p \mu_e n_e, \tag{4.4}$$

where m_p is the mass of a proton and μ_e is the mean molecular weight per electron. Since in WDs, $T \ll T_F$ or $E \ll E_F$ with T_F and E_F respectively being the Fermi temperature and Fermi energy, they are usually assumed to be cold, and the electrons become degenerate. For cold WDs with $T \ll T_F$, we have $\mu \approx E_F$. With this approximation in the large thermodynamic limit, using equations (4.2), (4.3) and (4.4), we obtain

$$\rho \approx \frac{4 m_p \mu_e p_F}{k h^2} \sum_{\nu=0}^{\nu_{\max}} g_\nu = \frac{4 m_p \mu_e p_F}{k h^2} \left(2\nu_{\max} + 1\right), \tag{4.5}$$

where p_F is the Fermi momentum of the degenerate electrons in the squashed fuzzy sphere. Here $\nu_{\max}$ is the quantity that carries the information of the number of occupied energy levels. Now, defining $\theta_D = 2\hbar/m_e^2 c^2 k$ and using Equation (3.10), the relation between E_F and p_F is given by

$$E_F^2(\rho,\nu) = p_F^2(\rho) c^2 + m_e^2 c^4 \left\{1 + 2\nu\theta_D(\rho)\right\}. \tag{4.6}$$

Since $p_F^2 \geq 0$, using this equation, we obtain $E_F^2 \geq m_e^2 c^4 \left(1 + 2\nu\theta_D\right) \implies \nu \leq \left(E_F^2 - m_e^2 c^4\right)/2 m_e^2 c^4 \theta_D$. Hence, $\nu_{\max}$ is defined as the greatest integer less than or equal to $\left(E_{F,\max}^2 - m_e^2 c^4\right)/2 m_e^2 c^4 \theta_D$, i.e.

$$\nu_{\max} = \left[\frac{E_{F,\max}^2 - m_e^2 c^4}{2 m_e^2 c^4 \theta_D}\right] \tag{4.7}$$

$$\text{or, } E_{F,\max}^2 = m_e^2 c^4 \left(1 + 2\nu_{\max}\theta_D\right), \tag{4.8}$$

where $[\cdot]$ is the floor function. Let us now assume that $\theta_D \propto \rho^{2/3}$, such that the variation of k with respect to ρ is proposed to be

$$k = \xi \frac{\mu_e^{2/3} m_p^{2/3}}{h \rho^{2/3}}, \tag{4.9}$$

where ξ is a dimensionless proportionality constant. The power of ρ (which we assume to be 2/3) is chosen in such a way that at the low-density limit, the pressure-density relation follows the Chandrasekhar EoS for degenerate electrons. This equation provides the variation of k with ρ, as shown in Figure 4. Since $k^{-1} \propto \rho^{2/3}$, which means k is minimum at the center of the star where the density is maximum; it implies that the NC is maximum at the center. Towards the surface, density decreases gradually and, hence, k increases accordingly, producing a minimal NC towards the surface. Substituting this expression for k in Equation (4.5), we obtain

$$p_F = \frac{\xi h}{4\mu_e^{1/3} m_p^{1/3} (2\nu_{\max} + 1)} \rho^{1/3}. \tag{4.10}$$

From this equation, we obtain $p_F \propto \rho^{1/3}$, which Chandrasekhar also obtained in his theory [41] except that the proportionality constant is now different. Moreover, in the large thermodynamic limit (i.e., $E \ll E_F$), the pressure ($\mathcal{P}$) is given by [40]

$$\begin{aligned}\mathcal{P} &= \frac{2}{kh^2} \sum_{\nu=0}^{\nu_{\max}} g_\nu \left\{ p_F E_F - \left(m_e^2 c^3 + 2\nu \frac{2\hbar c}{k} \right) \ln \left(\frac{E_F + p_F c}{\sqrt{m_e^2 c^4 + 2\nu \frac{hc^2}{\pi k}}} \right) \right\} \\ &= \frac{2\rho^{2/3}}{\xi h \mu_e^{2/3} m_p^{2/3}} \sum_{\nu=0}^{\nu_{\max}} g_\nu \left\{ p_F E_F - \left(m_e^2 c^3 + 2\nu \frac{2\hbar c}{k} \right) \ln \left(\frac{E_F + p_F c}{\sqrt{m_e^2 c^4 + 2\nu \frac{hc^2}{\pi k}}} \right) \right\}. \end{aligned} \tag{4.11}$$

This equation for $\mathcal{P}$ combining with Equation (4.10), gives the complete EoS. Figure 5 shows the EoS for degenerate electrons in a squashed fuzzy sphere with various $\nu_{\max}$. The quantity ξ for each $\nu_{\max}$ is chosen so that at low density, the EoS merge to the Chandrasekhar EoS. It is evident that as $\nu_{\max}$ increases, noncommutative EoS merges with the Chandrasekhar EoS.

4.1. Scales of NC in squashed fuzzy sphere

The structure of NC in a squashed fuzzy sphere is essential to study the physics of a stellar object as NC becomes zero at its surface. In other words, since the density is minimal (mathematically zero) at the surface, NC effect should vanish, and the usual physics given by the QM must be restored there. Moreover, as mentioned above, k^{-1} is the strength of NC in a squashed fuzzy sphere. From Equation (2.4), it is evident that $r \propto \sqrt{k}$, which means that fuzziness is less at a bigger radius, and the usual QM dominates the underlying statistical mechanics. As a result, WDs with less central densities (which usually have bigger radii)

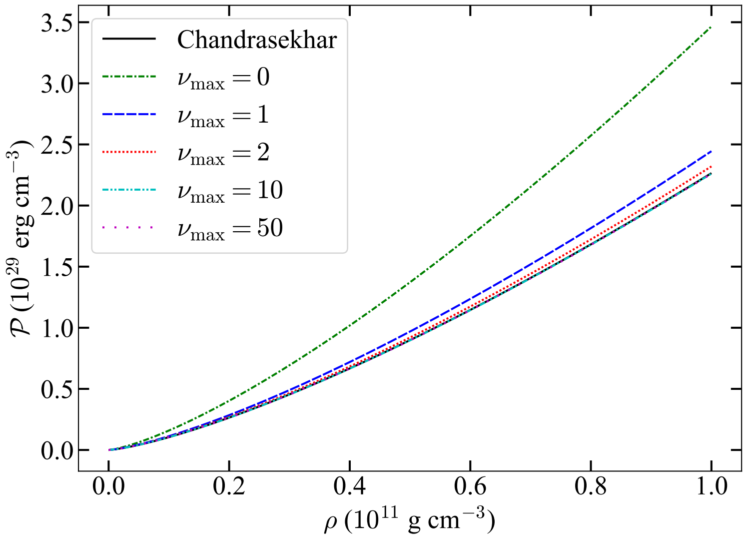


FIG. 5: EoS for degenerate electrons in a squashed fuzzy sphere with various $\nu_{\rm max}$ along with the Chandrasekhar EoS.

are expected to possess a larger value of k at the center than those with higher densities (or smaller radii).

We know that in the presence of a magnetic field, the effect of Landau quantization is prominent if $B \gtrsim B_c$, which is evident from Equation (3.11). Let us now examine the equivalent condition for the squashed fuzzy sphere algebra. Equation (3.10) can be recast to

$$E^2 = p_r^2 c^2 + m_e^2 c^4 \left(1 + 2\nu \frac{\lambda_e^2}{2\pi^2 \hbar k}\right), \qquad \nu \in \mathbb{Z}^{0+} \tag{4.12}$$

where λ_e is the Compton wavelength of an electron with $\lambda_e = h/m_e c \approx 2.426 \times 10^{-10}$ cm. Hence, the effect of NC is noticeable if $\lambda_e^2 \gtrsim 2\pi^2 \hbar k$. Moreover, we know that the characteristic length-scale of a system is the inter-particle separation, given by $\mathfrak{L} = (\rho/\mu_e m_p)^{-1/3}$. Substituting k as a function of ρ from Equation (4.9), we obtain the following relation

$$\mathfrak{L} \lesssim \frac{\lambda_e}{\sqrt{\pi \xi}} = \mathfrak{L}_{\rm eff}. \tag{4.13}$$

This is the condition for which the effect of NC to be significant in the case of degenerate electrons. It is evident from this equation that the effective length-scale for the prominence of NC is not only dependent on the Planck length, but it depends upon the system's scale also. This idea of length-scale can also be followed from the idea of Salecker and Wigner [28], which states that the new uncertainty in length-scale for a system has to be $\delta \sim \left(\mathfrak{L}\mathfrak{L}_p^2\right)^{1/3}$, where $\mathfrak{L}_p$ is the Planck length, and one needs to consider δ as the quantum measurement of length. Hence, for a WD, where $\mathfrak{L} \gg \mathfrak{L}_p$, the effective scale of NC turns out to be $\delta \gg \mathfrak{L}_p$. As a result, one can expect to observe a significant NC effect in a WD, even though the system's scale is far from the Planck scale. In the next section, we show that the effect of NC is prominent only in the highly-dense regime, whereas its effect is insignificant at the low-density and, hence, this formalism does not violate any of the observables in the low-density universe, such as the solar system.

This idea of NC's prominence depends on the system's length-scale reasonably overcomes one of the puzzles lying with the magnetic field. In the case of a magnetic field, as mentioned above, its quantum effect is prominent only if $B \gtrsim B_c = 4.414 \times 10^{13}$ G. However, we know that the magnetic field has both classical and quantum effects. Due to its classical effect, the magnetic field can change the WD's shape and size, thereby increasing its mass too. If a WD possesses a field significantly stronger than B_c, such a high field may also destabilize the WD, depending on the geometry, because of its high magnetic to gravitational energy ratio. Hence, the existence of a high field inside a WD is debatable. This problem can consistently be sorted out in the case of a noncommutative squashed fuzzy sphere. In our formalism of a squashed fuzzy sphere, since the NC is between the azimuthal and polar directions and it does not have any classical effect, the system's sphericity is not destroyed. Moreover, the length-scale for which NC has a noticeable effect is set-in by the theory itself and we do not need to assume an ad-hoc length-scale beforehand. Hence any further physical effect of NC will turn out to be an emergent phenomenon.

5. LIMITING MASS OF THE WD IN SQUASHED FUZZY SPHERE

The hydrostatic balance of any stellar object is obtained by simultaneously solving the pressure balance and mass balance equations (together known as the Tolman-Oppenheimer-Volkoff or TOV equations) along-with the EoS of the constituent particles. The TOV equations are given by [42]

$$\begin{aligned}
\frac{\mathrm{d}\mathcal{M}}{\mathrm{d}r} &= 4\pi r^2 \rho, \\
\frac{\mathrm{d}\mathcal{P}}{\mathrm{d}r} &= -\frac{G}{r^2}\left(\rho + \frac{\mathcal{P}}{c^2}\right)\left(\mathcal{M} + \frac{4\pi r^3 \mathcal{P}}{c^2}\right)\left(1 - \frac{2G\mathcal{M}}{c^2 r}\right)^{-1},
\end{aligned} \tag{5.1}$$

where $\mathcal{M}$ is the mass of the star inside a volume of radius r and G is Newton's gravitational constant. Figure 6 shows the mass–radius relation as well as the variation of the mass with respect to the central density ρ_c for the WDs which is obtained by solving the set of TOV equations (5.1) along with the EoS obtained in the previous section. For comparison, we also present Chandrasekhar's results therein. As mentioned in the §4, it is important that the effect of NC should be insignificant at lower densities and the usual statistical mechanics, governed by the usual QM, should prevail there.

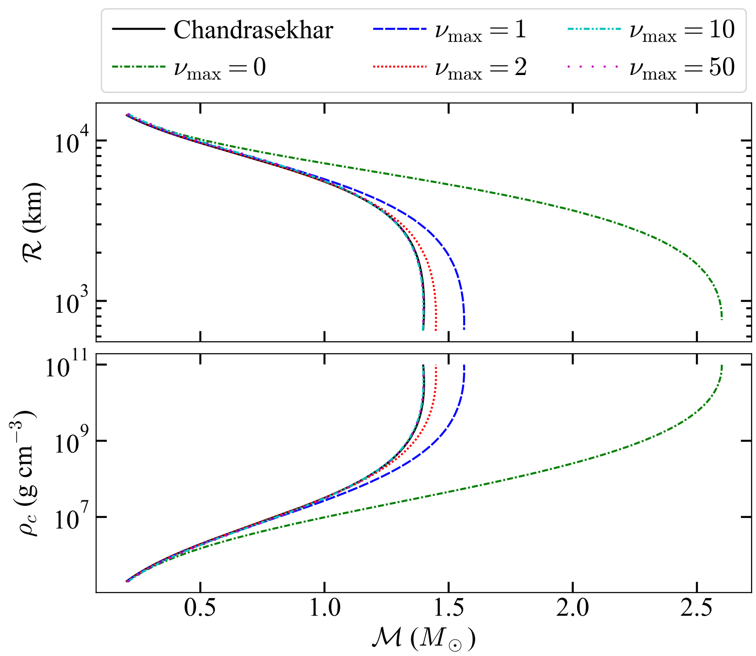


FIG. 6: Upper panel: The mass–radius relation, Lower panel: The variation of central density with the mass of the WD.

From Figure 6, it is evident that as the occupied energy levels (i.e., $\nu_{\rm max}$) increases, the effect of NC decreases, and eventually all the curves merge with the Chandrasekhar mass–radius curve. It can easily be understood from the simple length-scale idea mentioned in §4 4.1. To match our EoS at low-density with Chandrasekhar's, we find $\xi = 1.51$ for $\nu_{\rm max} = 0$ while $\xi = 156.56$ for $\nu_{\rm max} = 50$. Substituting these values in the relation (4.13), we find that the NC is prominent if $\mathfrak{L} \lesssim 1.11 \times 10^{-10}$ cm for $\nu_{\rm max} = 0$ whereas $\mathfrak{L} \lesssim 1.09 \times 10^{-11}$ cm for $\nu_{\rm max} = 50$. These length-scales correspond to $\rho \sim 2.4 \times 10^6$ g cm^{-3} and $\rho \sim 2.6 \times 10^9$ g cm^{-3}. Hence, for $\nu_{\rm max} = 0$, the effect of NC is significant in the regime of the WD with $\rho \gtrsim 2.4 \times 10^6$ g cm^{-3}, while the corresponding $\rho \gtrsim 2.6 \times 10^9$ g cm^{-3} for $\nu_{\rm max} = 50$. As a result, we do not observe any significant effect of NC in the WDs in the allowed density regime for $\nu_{\max} = 50$, and it continues to follow the Chandrasekhar's original mass–radius relation. Moreover, from equations (2.4) and (4.9), we obtain $N \sim 10^{37} - 10^{40}$ for different $\nu_{\max}$, which verifies the assumption of large N in Equation (3.10) does hold good.

6. NEUTRON DRIP IN THE PRESENCE OF NONCOMMUTATIVITY

We have obtained the modified dispersion relation in Equation (3.10) for a squashed fuzzy sphere. Since this is a different dispersion relation compared to that of the relativistic case, one can expect that the property of neutron drip alters in the presence of NC. A detailed discussion of neutron drip in the presence of a magnetic field is given by Vishal and Mukhopadhyay [43]. At the neutron drip density, protons and electrons combine to form neutrons, and they come out of the nucleus to form a neutron lattice. As a result, above this density, the electron degenerate pressure is no longer available, and the EoS obtained in §4 is no longer valid. The neutron drip density $\rho_{\rm drip}$ is given by

$$\rho_{\rm drip} = \frac{n_e M(A,Z)/Z + \epsilon_e - n_e m_e c^2}{c^2}, \tag{6.1}$$

where $M(A,Z)$ is the energy of a single ion with atomic number Z and mass number A. ϵ_e is the electron energy density at zero temperature, which is given by [40]

$$\epsilon_e = m_e c^2 \frac{4\pi\theta_D}{\lambda_e^3} \sum_{\nu=0}^{\nu_{\max}} g_\nu \left(1 + \nu\theta_D\right) \Psi\left(\frac{x_F(\nu)}{\sqrt{1+2\nu\theta_D}}\right), \tag{6.2}$$

where

$$x_F = \left(2230.31 - 2\nu\theta_D\right)^{1/2}, \quad \Psi(z) = \frac{1}{2} z\sqrt{1+z^2} + \frac{1}{2}\ln\left(z + \sqrt{1+z^2}\right).$$

Figure 7 shows the variation of $\rho_{\rm drip}$ with respect to θ_D. We notice that for θ_D below the linear regime, the drip density oscillates about the density obtained in the absence of NC which is $\sim 3 \times 10^{11}$ g cm^{-3}. It is also evident from the figure that the onset drip density increases linearly above a certain θ_D. This is because, beyond certain θ_D, the quantized electrons reside only in the ground state. To summarize, $\rho_{\rm drip}$ changes with the increase in NC and, hence, above this density, we can no longer use the EoS of degenerate electrons obtained in §4.

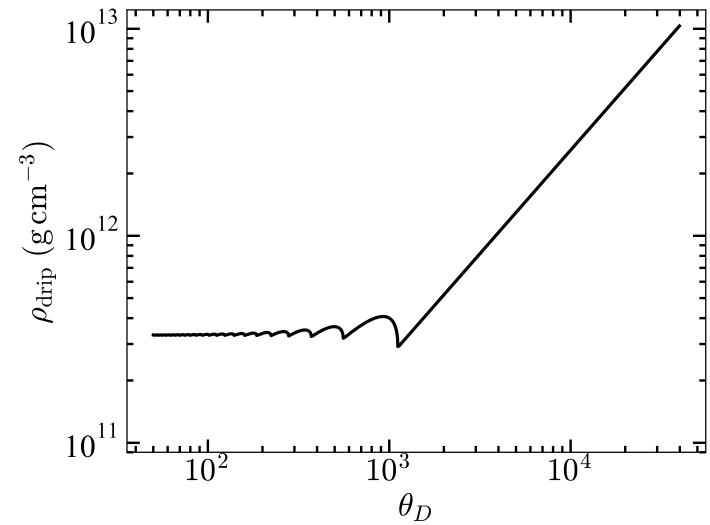


FIG. 7: Variation of neutron drip density as a function of θ_D.

7. CONCLUSIONS

NC is a fundamental property of matter and spacetime geometry. In the presence of NC, the statistical behaviour of the constituent particles changes. Because of this, in NC, the fermions behave less of conventional fermions. It is evident from Figure 2 that apart from the ground level, each energy levels comprise of two up-spin and two down-spin electrons. In the conventional picture of QM, we know that two electrons with the same spin cannot occupy a common energy state due to the Pauli's exclusion principle. However, since more than one electrons with the same spin occupy the common energy state in squashed fuzzy sphere NC, the effective repulsive pressure between the fermions decreases. Moreover, the Chandrasekhar mass-limit is the consequence of the competition between effective fermion pressure and gravitational attraction. If a fermion behaves less like a fermion (more like a boson), the inter-particle separation decreases, which generates a higher matter density tempting to accumulate more mass. Moreover, when the central density becomes $\sim 10^{8-10}$ g cm^{-3}, the inter-electron separation is less than the Compton wavelength of electrons, which triggers NC, and the core pressure increases to support more mass. However, fermion, losing its identity, leads to less repulsion among themselves, which further leads to less outward pressure to hold more mass. Combining all three effects, nevertheless, we obtain that the hydrostatic equilibrium is obtained at a higher mass for the highly dense WDs. Hence, they can collapse to a smaller size and extra mass can be accumulated to form super-Chandrasekhar WDs. In other words, in the presence of NC, for a given density, the degeneracy pressure is higher as shown in Figure 5. Hence, hydrostatic equilibrium can be maintained to higher masses, preventing collapse.

We previously investigated the effect of NC in a WD with the ad-hoc consideration of $[\hat{x}_i, \hat{x}_j] = i\eta_{ij}$ and $[\hat{p}_i, \hat{p}_j] = i\theta_{ij}$, and showed that it could lead to a super-Chandrasekhar WD if we assume that the NC is effective if the length-scale of the system is less than the Compton wavelength of electrons [27]. However, in the present exploration of NC through the squashed fuzzy sphere, the effect of NC's prominence is an emergent quantity through Equation (4.13), and there is no need to pre-assume any ad-hoc length-scale. It is essential to mention that our work is primarily based on semi-classical gravity, not quantum gravity and, hence, the scale of NC is expected to be different from that of the Planck scale. In the context of specific dynamics of quantum mechanics, gravity, and statistical mechanics, the NC scale gets complicated. From Equation (4.13), it is evident that if $\mathfrak{L} < \mathfrak{L}_{\text{eff}}$, the NC is efficient. Indeed, Salecker and Wigner already pointed it out as a quantum measurement of lengths [28]. Hence, the new uncertainty in length-scale appears to be $\delta \sim \left(\mathfrak{L}\mathfrak{L}_p^2\right)^{1/3}$. To summarize, the length-scale of uncertainty in a WD, which is a low energy system, depends both on the Planck length and length-scale of the system.

Earlier explorations for the formation of super-Chandrasekhar WDs in high magnetic fields and thereby the Landau levels was an interesting idea. However, we know that the magnetic field has not only microscopic effects but also macroscopic effects. As a result, in the presence of high varying magnetic fields, a WD would be deformed due to the Lorentz force of the magnetic field. Various explorations set limits on the maximum magnetic to gravitational energy ratio in a compact object [44]. Hence, the presence of a very high magnetic field inside a compact object is debatable. The idea of NC through the squashed fuzzy sphere fairly overcomes this problem as the fuzzy algebra's formalism is such that it does not affect the sphericity of the system. Moreover, from Equation (2.8), the structure of NC is such that at the surface, the NC is zero. Hence, the effective NC varies with the density (or, equivalently with distance from the center), and its effect is prominent if the inter-particle separation $\mathfrak{L} < \mathfrak{L}_{\text{eff}}$. Therefore, NC seems to be a better bet to explain the super-Chandrasekhar WDs as it does not possess any classical effect.

All the inferences of the super-Chandrasekhar WDs so far have been made indirectly, as none of them has been detected from direct observations. Similarly, there is no direct observational evidence for NC so far. Gravitational wave can be one of the prominent tools to detect such massive WDs directly. We earlier showed that WDs governed by noncommutative geometry can emit gravitational radiation for a long duration if they possess a minimal surface magnetic field [21]. In this way, one can estimate both the masses and radii of the WDs; thereby comparing with the theoretical mass-radius curves, we might have indirect observational evidence for NC.

ACKNOWLEDGMENTS

S. K. would like to thank F. G. Scholtz and H. C. Steinacker for their useful discussions on fuzzy sphere and underlying statistical mechanics. The authors would also like to thank V. P. Nair of The City College of New York for his useful comments on noncommutative algebra.

[1] G. R. Lauffer, A. D. Romero, and S. O. Kepler, MNRAS **480**, 1547 (2018), arXiv:1807.04774 [astro-ph.SR].

[2] S. L. Shapiro and S. A. Teukolsky, *Black holes, white dwarfs, and neutron stars: The physics of compact objects* (John Wiley and sons, 1983).

[3] S. Chandrasekhar, ApJ **74**, 81 (1931).

[4] D. A. Howell, M. Sullivan, P. E. Nugent, R. S. Ellis, A. J. Conley, D. Le Borgne, R. G. Carlberg, J. Guy, D. Balam, S. Basa, D. Fouchez, I. M. Hook, E. Y. Hsiao, J. D. Neill, R. Pain, K. M. Perrett, and C. J. Pritchet, Nature **443**, 308 (2006), astro-ph/0609616.

[5] M. Hicken, P. M. Garnavich, J. L. Prieto, S. Blondin, D. L. DePoy, R. P. Kirshner, and J. Parrent, ApJL **669**, L17 (2007), arXiv:0709.1501 [astro-ph].

[6] M. Yamanaka, K. S. Kawabata, K. Kinugasa, M. Tanaka, A. Imada, K. Maeda, K. Nomoto, A. Arai, S. Chiyonobu, Y. Fukazawa, O. Hashimoto, S. Honda, Y. Ikejiri, R. Itoh, Y. Kamata, N. Kawai, T. Komatsu, K. Konishi, D. Kuroda, H. Miyamoto, S. Miyazaki, O. Nagae, H. Nakaya, T. Ohsugi, T. Omodaka, N. Sakai, M. Sasada, M. Suzuki, H. Taguchi, H. Takahashi, H. Tanaka, M. Uemura, T. Yamashita, K. Yanagisawa, and M. Yoshida, ApJL **707**, L118 (2009), arXiv:0908.2059 [astro-ph.HE].

[7] F. Yuan, R. M. Quimby, J. C. Wheeler, J. Vinkó, E. Chatzopoulos, C. W. Akerlof, S. Kulkarni, J. M. Miller, T. A. McKay, and F. Aharonian, ApJ **715**, 1338 (2010), arXiv:1004.3329 [astro-ph.CO].

[8] M. Tanaka, K. S. Kawabata, M. Yamanaka, K. Maeda, T. Hattori, K. Aoki, K. Nomoto, M. Iye, T. Sasaki, P. A. Mazzali, and E. Pian, ApJ **714**, 1209 (2010), arXiv:0908.2057 [astro-ph.CO].

[9] R. A. Scalzo, G. Aldering, P. Antilogus, C. Aragon, S. Bailey, C. Baltay, S. Bongard, C. Buton, M. Childress, N. Chotard, Y. Copin, H. K. Fakhouri, A. Gal-Yam, E. Gangler, S. Hoyer, M. Kasliwal, S. Loken, P. Nugent, R. Pain, E. Pécontal, R. Pereira, S. Perlmutter, D. Rabinowitz, A. Rau, G. Rigaudier, K. Runge, G. Smadja, C. Tao, R. C. Thomas, B. Weaver, and C. Wu, ApJ **713**, 1073 (2010), arXiv:1003.2217.

[10] J. M. Silverman, M. Ganeshalingam, W. Li, A. V. Filippenko, A. A. Miller, and D. Poznanski, MNRAS **410**, 585 (2011), arXiv:1003.2417 [astro-ph.HE].

[11] S. Taubenberger, S. Benetti, M. Childress, R. Pakmor, S. Hachinger, P. A. Mazzali, V. Stanishev, N. Elias-Rosa, I. Agnoletto, F. Bufano, M. Ergon, A. Harutyunyan, C. Inserra, E. Kankare, M. Kromer, H. Navasardyan, J. Nicolas, A. Pastorello, E. Prosperi, F. Salgado, J. Sollerman, M. Stritzinger, M. Turatto, S. Valenti, and W. Hillebrandt, MNRAS **412**, 2735 (2011), arXiv:1011.5665 [astro-ph.SR].

[12] R. Scalzo, G. Aldering, P. Antilogus, C. Aragon, S. Bailey, C. Baltay, S. Bongard, C. Buton, A. Canto, F. Cellier-Holzem, M. Childress, N. Chotard, Y. Copin, H. K. Fakhouri, E. Gangler, J. Guy, E. Y. Hsiao, M. Kerschhaggl, M. Kowalski, P. Nugent, K. Paech, R. Pain, E. Pecontal, R. Pereira, S. Perlmutter, D. Rabinowitz, M. Rigault, K. Runge, G. Smadja, C. Tao, R. C. Thomas, B. A. Weaver, C. Wu, and T. Nearby Supernova Factory, ApJ **757**, 12 (2012), arXiv:1207.2695 [astro-ph.CO].

[13] H. Saio and K. Nomoto, ApJ **615**, 444 (2004), arXiv:astro-ph/0401141 [astro-ph].

[14] R. G. Martin, C. A. Tout, and P. Lesaffre, MNRAS **373**, 263 (2006), arXiv:astro-ph/0609192 [astro-ph].

[15] J. P. Ostriker and F. D. A. Hartwick, ApJ **153**, 797 (1968).

[16] A. Kundu and B. Mukhopadhyay, Modern Physics Letters A **27**, 1250084-1-1250084-15 (2012), arXiv:1204.1463 [astro-ph.SR].

[17] U. Das and B. Mukhopadhyay, Phys. Rev. Lett. **110**, 071102 (2013), arXiv:1301.5965 [astro-ph.SR].

[18] S. Subramanian and B. Mukhopadhyay, MNRAS **454**, 752 (2015), arXiv:1507.01606 [astro-ph.SR].

[19] S. Kalita and B. Mukhopadhyay, MNRAS **490**, 2692 (2019), arXiv:1905.02730 [astro-ph.HE].

[20] S. Kalita and B. Mukhopadhyay, IAU Symposium **357**, 79 (2020), arXiv:2001.10698 [astro-ph.HE].

[21] S. Kalita, B. Mukhopadhyay, T. Mondal, and T. Bulik, ApJ **896**, 69 (2020), arXiv:2004.13750 [astro-ph.HE].

[22] S. Kalita and B. Mukhopadhyay, J. Cosmology Astropart. Phys. **9**, 007 (2018), arXiv:1805.12550 [gr-qc].

[23] G. A. Carvalho, R. V. Lobato, P. H. R. S. Moraes, J. D. V. Arbañil, E. Otoniel, R. M. Marinho, and M. Malheiro, European Physical Journal C **77**, 871 (2017), arXiv:1706.03596 [gr-qc].

[24] H. Liu, X. Zhang, and D. Wen, Phys. Rev. D **89**, 104043 (2014), arXiv:1405.3774 [gr-qc].

[25] Y. C. Ong, J. Cosmology Astropart. Phys. **9**, 015 (2018), arXiv:1804.05176 [gr-qc].

[26] V. B. Belyaev, P. Ricci, F. Šimkovic, J. Adam, M. Tater, and E. Truhlík, Nuclear Physics A **937**, 17 (2015), arXiv:1212.3155 [nucl-th].

[27] S. Kalita, B. Mukhopadhyay, and T. R. Govindarajan, International Journal of Modern Physics D , IJMPD (accepted) (2021), arXiv:1912.00900 [gr-qc].

[28] H. Salecker and E. P. Wigner, Physical Review **109**, 571 (1958).

[29] P. Nicolini, A. Smailagic, and E. Spallucci, Physics Letters B **632**, 547 (2006), arXiv:gr-qc/0510112 [gr-qc].

[30] N. Seiberg and E. Witten, Journal of High Energy Physics **1999**, 032 (1999), arXiv:hep-th/9908142 [hep-th].

[31] G. Amelino-Camelia, Physics Letters B **510**, 255 (2001), arXiv:hep-th/0012238 [hep-th].

[32] J. Magueijo and L. Smolin, Phys. Rev. Lett. **88**, 190403 (2002), arXiv:hep-th/0112090 [hep-th].

[33] G. Amelino-Camelia, Nature **418**, 34 (2002), arXiv:gr-qc/0207049 [gr-qc].

[34] V. P. Nair, "Noncommutative Mechanics, Landau Levels, Twistors, and Yang-Mills Amplitudes," in *Lecture Notes in Physics, Berlin Springer Verlag*, Vol. 1, edited by S. Bellucci (Springer Berlin Heidelberg, Berlin, Heidelberg, 2006) p. 97.

[35] D. T. Son and N. Yamamoto, Phys. Rev. Lett. **109**, 181602 (2012), arXiv:1203.2697 [cond-mat.mes-hall].

[36] J. Madore, Classical and Quantum Gravity **9**, 69 (1992).

[37] N. Chandra, H. W. Groenewald, J. N. Kriel, F. G. Scholtz, and S. Vaidya, Journal of Physics A Mathematical General **47**, 445203 (2014), arXiv:1407.5857 [hep-th].

[38] F. G. Scholtz, J. N. Kriel, and H. W. Groenewald, Physical Review D **92**, 125013 (2015), arXiv:1508.05799 [hep-th].

[39] S. Andronache and H. C. Steinacker, Journal of Physics A Mathematical General **48**, 295401 (2015), arXiv:1503.03625 [hep-th].

[40] D. Lai and S. L. Shapiro, ApJ **383**, 745 (1991).

[41] S. Chandrasekhar, MNRAS **95**, 207 (1935).

[42] L. Ryder, *Introduction to General Relativity* (Cambridge University Press, 2009).

[43] M. V. Vishal and B. Mukhopadhyay, Phys. Rev. C **89**, 065804 (2014), arXiv:1403.0763 [astro-ph.SR].

[44] J. Braithwaite, MNRAS **397**, 763 (2009), arXiv:0810.1049.