



# $FI$ -modules over preadditive categories and torsion theories

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## ABSTRACT

We work with  $FI$ -modules over a small preadditive category  $\mathcal{R}$ , viewed as a ring with several objects. Our aim is to study torsion theories for  $FI$ -modules. We are especially interested in torsion theories on finitely generated  $FI$ -modules and the category of what we call “shift finitely generated”  $FI$ -modules. We also apply these methods to study inductive descriptions of  $FI$ -modules over  $\mathcal{R}$ .

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## 1. Introduction

Let  $FI$  be the category of finite sets and whose morphisms are injective maps. If  $R$  is a ring, an  $FI$ -module over  $R$  is a functor from  $FI$  to the category of  $R$ -modules. The notion of  $FI$ -modules was introduced by Church, Ellenberg and Farb in [7], with a view towards a deeper understanding of the Church-Farb theory [8] of representation stability for  $S_n$ -representations. This was further developed by Church, Ellenberg and Farb [10], [11], by Church, Ellenberg, Farb and Nagpal [9], by Church and Ellenberg [12], by Putman [27], by Putman and Sam [28], and by Sam and Snowden [30], [31]. Since then, a wide variety of results on  $FI$ -modules has been developed by numerous authors, with applications to algebraic topology, algebraic geometry and representation theory (see, for instance, [13], [22], [23], [25], [29]).

In this paper, our aim is to study torsion theories for  $FI$ -modules. We are especially interested in torsion theories on finitely generated  $FI$ -modules and the category of what we call “shift finitely generated”  $FI$ -modules (see Definition 3.9). We work with  $FI$ -modules over a small preadditive category  $\mathcal{R}$ , viewed as a ring with several objects in the sense of Mitchell [24]. As such, the category  $FI_{\mathcal{R}}$  of  $FI$ -modules over  $\mathcal{R}$  consists of functors from  $FI$  to the category  $Mod - \mathcal{R}$  of right modules over  $\mathcal{R}$ . We recall that a right module over  $\mathcal{R}$  is a functor from  $\mathcal{R}^{op}$  to the category of abelian groups. We mostly work with the case where

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$\mathcal{R}$  is such that the category  $Mod - \mathcal{R}$  is locally noetherian. When  $\mathcal{R}$  is abelian, we have studied in [2] how torsion theories on  $\mathcal{R}$  may be extended to certain classes of modules over  $\mathcal{R}$ .

We begin with a hereditary torsion theory  $(\mathcal{T}, \mathcal{F})$  on  $Mod - \mathcal{R}$ . We show that  $\tau$  induces a hereditary torsion class  $\overline{\mathcal{T}}$  on the subcategory  $FI_{\mathcal{R}}^{fg}$  of finitely generated  $FI$ -modules as well as a hereditary torsion class  $\overline{\mathcal{T}}^{sfg}$  on the category  $FI_{\mathcal{R}}^{sfg}$  of shift finitely generated  $FI$ -modules. We then extend  $\overline{\mathcal{T}}$  and  $\overline{\mathcal{T}}^{sfg}$  to Serre subcategories  $\hat{\mathcal{T}}$  and  $\tilde{\mathcal{T}}$  respectively of  $FI_{\mathcal{R}}$ . In other words, we have  $\hat{\mathcal{T}} \cap FI_{\mathcal{R}}^{fg} = \overline{\mathcal{T}}$  and  $\tilde{\mathcal{T}} \cap FI_{\mathcal{R}}^{sfg} = \overline{\mathcal{T}}^{sfg}$ . We then describe functors from  $FI_{\mathcal{R}}$  to  $\hat{\mathcal{T}}$ -closed and  $\tilde{\mathcal{T}}$ -closed objects of  $FI_{\mathcal{R}}$ . In fact, we show that an object in  $FI_{\mathcal{R}}$  is closed with respect to the Serre subcategory  $\hat{\mathcal{T}}$  if and only if it is closed with respect to  $\tilde{\mathcal{T}}$ . Finally, we apply these methods to study inductive descriptions of  $FI$ -modules over  $\mathcal{R}$ .

We begin in Section 2 with preliminary results on  $FI$ -modules over  $\mathcal{R}$ , extending those from [9, § 2.1]. In particular, we show that  $FI_{\mathcal{R}}$  is a Grothendieck category with a set of finitely generated projective generators. We also recall that when  $\mathcal{R}$  is such that  $Mod - \mathcal{R}$  is locally noetherian, the category  $FI_{\mathcal{R}}$  of  $FI$ -modules over  $\mathcal{R}$  becomes a locally noetherian category (see [9, Theorem A], [14], [28], [31] and [21, Theorem 9.1]).

For each  $n > 0$ , we set  $[n] := \{1, 2, \dots, n\}$ , while we take  $[0]$  to be the empty set. For any  $FI$ -module  $\mathcal{V} : FI \rightarrow Mod - \mathcal{R}$  and  $n \geq 0$ , we will often write  $\mathcal{V}([n])$  simply as  $\mathcal{V}_n$ . For each  $a \geq 0$ , the category  $FI_{\mathcal{R}}$  is equipped with a shift functor defined by setting

$$S^a : FI_{\mathcal{R}} \rightarrow FI_{\mathcal{R}} \quad S^a \mathcal{V}(S) := \mathcal{V}(S \sqcup [-a])$$

for each  $\mathcal{V} \in FI_{\mathcal{R}}$  and each finite set  $S$ , where  $[-a]$  is a fixed set of cardinality  $a$ . In [9, § 2.3], an  $FI$ -module  $V$  over a ring  $R$  is said to be torsion if it satisfies

$$V = \bigcup_{a \geq 0} Ker(X^a : V \mapsto S^a V) \quad (1.1)$$

Here,  $S^a$  is the  $a$ -th shift functor on  $FI$ -modules over  $R$  and  $X^a$  is the canonical morphism  $V \rightarrow S^a V$  induced by the inclusion  $T \hookrightarrow T \sqcup [-a]$  for each finite set  $T$ . In Section 3, we consider a torsion theory  $\tau = (\mathcal{T}, \mathcal{F})$  on  $Mod - \mathcal{R}$  and the subcategory of finitely generated  $FI$ -modules determined by setting

$$Ob(\overline{\mathcal{T}}) := \{\mathcal{V} \in Ob(FI_{\mathcal{R}}^{fg}) \mid \mathcal{V}_n \in \mathcal{T} \text{ for } n \gg 0\} \quad (1.2)$$

Our first result describes the induced torsion theory on  $FI_{\mathcal{R}}^{fg}$  and a formula for the torsion subobject of a finitely generated  $FI$ -module.

**Theorem 1.** (See 3.2 and 3.8.) *Let  $\mathcal{R}$  be a small preadditive category such that  $Mod - \mathcal{R}$  is locally noetherian. Let  $(\mathcal{T}, \mathcal{F})$  be a torsion theory on  $Mod - \mathcal{R}$ .*

(a) *Then,  $\overline{\mathcal{T}}$  is a torsion class in the category  $FI_{\mathcal{R}}^{fg}$  of finitely generated  $FI$ -modules over  $\mathcal{R}$ . Additionally, if  $\mathcal{T}$  is a hereditary torsion class, so is  $\overline{\mathcal{T}}$ .*

(b) *Suppose that  $\mathcal{T}$  is hereditary. Then, for any  $\mathcal{V} \in FI_{\mathcal{R}}^{fg}$ , the torsion subobject of  $\mathcal{V}$  with respect to the torsion class  $\overline{\mathcal{T}}$  is given by  $\overline{\mathcal{T}}(\mathcal{V})$ , where*

$$\overline{\mathcal{T}}(\mathcal{V})(S) := \operatorname{colim}_{a \geq 0} \lim (\mathcal{V}(S) \longrightarrow (S^a \mathcal{V})(S) \longleftarrow \mathcal{T}((S^a \mathcal{V})(S))) \quad (1.3)$$

for each finite set  $S$ .

In the category  $FI_{\mathcal{R}}$ , there are two kinds of finiteness conditions on objects. Other than finitely generated objects, there are the  $FI$ -modules that are generated in finite degree, i.e., generated by a (not necessarily

finite) collection of elements whose degree is bounded above (see Definition 2.2). The latter notion is related to the shift functor as follows (see Proposition 3.3 and Proposition 3.6)

$$\begin{aligned} \{\mathcal{V} \text{ is generated in finite degree}\} &\Rightarrow \{\mathbb{S}^d\mathcal{V} \text{ is generated in finite degree for each } d \geq 0\} \\ \{\mathcal{V} \text{ is generated in finite degree}\} &\Leftarrow \{\mathbb{S}^d\mathcal{V} \text{ is generated in finite degree for some } d \geq 0\} \end{aligned} \tag{1.4}$$

For finitely generated objects in  $FI_{\mathcal{R}}$ , we have (see Lemma 3.10)

$$\{\mathcal{V} \text{ is finitely generated}\} \Rightarrow \{\mathbb{S}^d\mathcal{V} \text{ is finitely generated for each } d \geq 0\} \tag{1.5}$$

However, if  $\mathcal{V}$  is an  $FI$ -module such that  $\mathbb{S}^d\mathcal{V}$  is finitely generated for some  $d \geq 0$ , it is not necessary that  $\mathcal{V}$  is finitely generated. This suggests that we should study a larger class of objects, related to finitely generated objects, for which conditions similar to (1.4) hold. We will say that an  $FI$ -module  $\mathcal{V}$  is shift finitely generated if there exists  $d \geq 0$  such that  $\mathbb{S}^d\mathcal{V}$  is finitely generated (see Definition 3.9). We denote by  $FI_{\mathcal{R}}^{sfg}$  the full subcategory of shift finitely generated  $FI$ -modules. It is clear that

$$\begin{aligned} \{\mathcal{V} \text{ is shift finitely generated}\} &\Rightarrow \{\mathbb{S}^d\mathcal{V} \text{ is shift finitely generated for each } d \geq 0\} \\ \{\mathcal{V} \text{ is shift finitely generated}\} &\Leftarrow \{\mathbb{S}^d\mathcal{V} \text{ is shift finitely generated for some } d \geq 0\} \end{aligned} \tag{1.6}$$

In other words, the objects in  $FI_{\mathcal{R}}^{sfg}$  are finitely generated up to some number of shifts. As described by Church and Ellenberg [12], there is an important analogy between  $FI$ -modules and the category  $Mod - \mathbb{C}[T]$  of graded modules over the univariate polynomial ring  $\mathbb{C}[T]$ . In this set of analogies (see [12, Table 1]), the shift functor on  $FI$ -modules corresponds to the grading shift on  $Mod - \mathbb{C}[T]$ . However, as also pointed out by Church and Ellenberg (see [12, Remark 2.4]), there are some distinctions between the two theories: while the grading shift functor on  $Mod - \mathbb{C}[T]$  is invertible, the shift functor on  $FI$ -modules is not. This makes it interesting to study the effect of the shift functor on generators of  $FI$ -modules. In the case of graded modules over  $\mathbb{C}[T]$ , we mention that an object  $M = \bigoplus_{n \in \mathbb{Z}} M_n$  is said to be quasi-finitely generated (see, for instance, [5, Definition 3.9.8], [18, Ex II.5.9]) if there exists some  $d$  such that  $\bigoplus_{n \geq d} M_n$  is finitely generated.

These graded modules which are finitely generated beyond some degree have also been studied by Serre [32, § 56].

We begin our study of shift finitely generated  $FI$ -modules by showing how a torsion theory on  $Mod - \mathcal{R}$  leads to a torsion theory on  $FI_{\mathcal{R}}^{sfg}$ . Given the torsion theory  $(\mathcal{T}, \mathcal{F})$  on  $Mod - \mathcal{R}$ , we now consider

$$Ob(\overline{\mathcal{T}}^{sfg}) := \{\mathcal{V} \in Ob(FI_{\mathcal{R}}^{sfg}) \mid \text{Every finitely generated } \mathcal{W} \subseteq \mathcal{V} \text{ lies in } \overline{\mathcal{T}}\} \tag{1.7}$$

The next result shows that  $FI_{\mathcal{R}}^{sfg}$  is a Serre subcategory of  $FI_{\mathcal{R}}$  and that the formula in (1.3) may be extended to describe the induced torsion theory on  $FI_{\mathcal{R}}^{sfg}$ . For this, we also obtain some intermediate results on torsion in locally noetherian Grothendieck categories.

**Theorem 2.** (See 3.11, 3.13 and 3.17.) *Suppose that  $Mod - \mathcal{R}$  is locally noetherian.*

(a) *Then, the full subcategory  $FI_{\mathcal{R}}^{sfg}$  given by*

$$Ob(FI_{\mathcal{R}}^{sfg}) := \{\mathcal{V} \in Ob(FI_{\mathcal{R}}) \mid \mathbb{S}^d\mathcal{V} \text{ is finitely generated for some } d \geq 0\} \tag{1.8}$$

*is a Serre subcategory of  $FI_{\mathcal{R}}$ , i.e., it is closed under extensions, quotients and subobjects.*

(b) If  $\tau = (\mathcal{T}, \mathcal{F})$  is a hereditary torsion theory on  $\text{Mod} - \mathcal{R}$ , then  $\overline{\mathcal{T}}^{sfg}$  is a hereditary torsion class in  $FI_{\mathcal{R}}^{sfg}$ . For any  $\mathcal{V} \in FI_{\mathcal{R}}^{sfg}$ , the torsion subobject of  $\mathcal{V}$  with respect to the torsion class  $\overline{\mathcal{T}}^{sfg}$  is given by

$$\overline{\mathcal{T}}^{sfg}(\mathcal{V})(S) := \text{colim}_{a \geq 0} \lim(\mathcal{V}(S) \longrightarrow (\mathbb{S}^a \mathcal{V})(S) \longleftarrow \mathcal{T}((\mathbb{S}^a \mathcal{V})(S))) \quad (1.9)$$

for each finite set  $S$ .

In the rest of this paper, we always suppose that  $\tau = (\mathcal{T}, \mathcal{F})$  is a hereditary torsion theory on  $\text{Mod} - \mathcal{R}$ . In Section 4, we consider the subcategory  $\hat{\mathcal{T}}$  of  $FI_{\mathcal{R}}$  defined by setting

$$\text{Ob}(\hat{\mathcal{T}}) := \{\mathcal{V} \in \text{Ob}(FI_{\mathcal{R}}) \mid \mathcal{V}_n \in \mathcal{T} \text{ for } n \gg 0\} \quad (1.10)$$

It is clear that we have  $\hat{\mathcal{T}} \cap FI_{\mathcal{R}}^{sfg} = \overline{\mathcal{T}}$ . While  $\hat{\mathcal{T}}$  is not a torsion class (it is not necessarily closed under direct sums), we observe that it is a Serre subcategory of  $FI_{\mathcal{R}}$ .

We recall that an object  $L$  in a Grothendieck category  $\mathcal{A}$  is said to be closed with respect to a Serre subcategory  $\mathcal{C} \subseteq \mathcal{A}$  if  $\text{Hom}(u, L) : \text{Hom}(B, L) \longrightarrow \text{Hom}(A, L)$  is an isomorphism for every  $u : A \longrightarrow B$  in  $\mathcal{A}$  such that  $\text{Ker}(u), \text{Coker}(u) \in \mathcal{C}$ . We first develop a general result (see Proposition 4.4) that in a locally noetherian Grothendieck category, it suffices to check this criterion with  $A, B$  finitely generated.

Corresponding to each  $\mathcal{V} \in FI_{\mathcal{R}}$ , we want to construct an object that is closed with respect to the Serre subcategory  $\hat{\mathcal{T}}$ . In other words, we will describe a functor from  $FI_{\mathcal{R}}$  taking values in the subcategory  $Cl(\hat{\mathcal{T}})$  of  $\hat{\mathcal{T}}$ -closed objects. For this, we first express  $\hat{\mathcal{T}}$  as a union  $\hat{\mathcal{T}} = \bigcup_{a \geq 0} \hat{\mathcal{T}}^a$  where

$$\text{Ob}(\hat{\mathcal{T}}^a) := \{\mathcal{V} \in \text{Ob}(FI_{\mathcal{R}}) \mid \mathcal{V}_n \in \mathcal{T} \text{ for all } n \geq a\} \quad \forall a \geq 0 \quad (1.11)$$

We will also need the functor  $\mathbb{E}_{\tau}$  which takes an  $FI$ -module  $\mathcal{V} : FI \longrightarrow \text{Mod} - \mathcal{R}$  to its composition with the torsion envelope in  $\text{Mod} - \mathcal{R}$ . We obtain the following result.

**Theorem 3.** (See 4.11 and 4.12.) Let  $\text{Mod} - \mathcal{R}$  be locally noetherian. Let  $\tau = (\mathcal{T}, \mathcal{F})$  be a hereditary torsion theory on  $\text{Mod} - \mathcal{R}$ . Then, we have a functor  $\mathbb{L}_{\tau} : FI_{\mathcal{R}} \longrightarrow Cl(\hat{\mathcal{T}})$  and a canonical morphism

$$l_{\tau}(\mathcal{V}) : \mathcal{V} \longrightarrow \mathbb{L}_{\tau}(\mathcal{V}) := \varinjlim_{k \geq 0} \mathbb{L}_{\tau}^k(\mathcal{V}) \quad (1.12)$$

for each  $\mathcal{V} \in FI_{\mathcal{R}}$ . Here,  $\mathbb{L}_{\tau}^k = \mathbb{T}^k \circ \mathbb{S}^k \circ \mathbb{E}_{\tau}$ , where  $\mathbb{T}^k : FI_{\mathcal{R}} \longrightarrow FI_{\mathcal{R}}$  is the right adjoint to the shift functor  $\mathbb{S}^k : FI_{\mathcal{R}} \longrightarrow FI_{\mathcal{R}}$ .

In Section 5, our aim is to prove a result similar to Theorem 3 for shift finitely generated objects by considering the subcategory

$$\text{Ob}(\tilde{\mathcal{T}}) := \{\mathcal{V} \in \text{Ob}(FI_{\mathcal{R}}) \mid \text{Every finitely generated } \mathcal{W} \subseteq \mathcal{V} \text{ lies in } \overline{\mathcal{T}}\} \quad (1.13)$$

which satisfies  $\tilde{\mathcal{T}} \cap FI_{\mathcal{R}}^{sfg} = \overline{\mathcal{T}}^{sfg}$ . Unlike in the case of  $\hat{\mathcal{T}}$  which is only a Serre subcategory, we will show that  $\tilde{\mathcal{T}}$  is actually a hereditary torsion class. Further, the  $\tilde{\mathcal{T}}$ -closed objects actually coincide with the  $\hat{\mathcal{T}}$ -closed objects. In other words, we have the following result.

**Theorem 4.** (See 5.4 and 5.8.) Let  $\text{Mod} - \mathcal{R}$  be a locally noetherian category and  $\tau = (\mathcal{T}, \mathcal{F})$  a hereditary torsion theory on  $\text{Mod} - \mathcal{R}$ . Then:

(a) The full subcategory  $\tilde{\mathcal{T}}$  is a hereditary torsion class.

(b) An object in  $FI_{\mathcal{R}}$  is closed with respect to  $\tilde{\mathcal{T}}$  if and only if it is closed with respect to  $\hat{\mathcal{T}}$ , i.e.,  $Cl(\hat{\mathcal{T}}) = Cl(\tilde{\mathcal{T}})$ . In particular, we have a functor  $\mathbb{L}_{\tau} : FI_{\mathcal{R}} \rightarrow Cl(\hat{\mathcal{T}}) = Cl(\tilde{\mathcal{T}})$ .

In Section 6, we begin by providing an inductive description for finitely generated  $FI$ -modules over  $\mathcal{R}$ . For this, we have to work with functors  $H_a : FI_{\mathcal{R}} \rightarrow FI_{\mathcal{R}}$ ,  $a \geq 0$ , which are defined as homology groups of a complex similar to the construction in [9, § 2.4]. The following result is analogous to [9, Theorem C].

**Theorem 5.** (See 6.4.) Suppose that  $Mod - \mathcal{R}$  is locally noetherian. Let  $\mathcal{V} \in FI_{\mathcal{R}}$  be a finitely generated object. Then, there exists  $N \geq 0$  such that

$$\operatorname{colim}_{\substack{T \subseteq S \\ |T| \leq N}} \mathcal{V}(T) = \mathcal{V}(S) \tag{1.14}$$

for each finite set  $S$ .

We conclude by proving a result similar to Theorem 5 for shift finitely generated objects in  $FI_{\mathcal{R}}$ . For this, we apply the methods developed in previous sections to the zero torsion class on  $Mod - \mathcal{R}$ . Our final result is as follows.

**Theorem 6.** (See 6.7 and 6.8.) Suppose that  $Mod - \mathcal{R}$  is locally noetherian. Let  $\mathcal{V} \in FI_{\mathcal{R}}$  be a shift finitely generated object.

(a) Fix  $a \geq 0$  and consider any finitely generated subobject  $\mathcal{W} \subseteq H_a(\mathcal{V})$ . Then, there exists  $N \geq 0$  such that  $\mathcal{W}_n = 0$  for all  $n \geq N$ .

(b) Let  $\mathcal{V} \in FI_{\mathcal{R}}$  be a shift finitely generated object such that  $H_0(\mathcal{V})$  and  $H_1(\mathcal{V})$  are finitely generated. Then, there exists  $N \geq 0$  such that

$$\operatorname{colim}_{\substack{T \subseteq S \\ |T| \leq N}} \mathcal{V}(T) = \mathcal{V}(S) \tag{1.15}$$

for each finite set  $S$ .

## 2. Finitely generated $FI$ -modules over rings with several objects

Let  $FI$  denote the category of finite sets and whose morphisms are injections. For each  $n > 0$ , we set  $[n] := \{1, 2, \dots, n\}$  while  $[0]$  is taken to be the empty set. Then,  $FI$  is equivalent to its full subcategory consisting of the objects  $[n]$  for  $n \geq 0$ . In particular, while  $FI$  is not a small category, we see that it is essentially small, i.e., equivalent to a small category.

We now let  $\mathcal{R}$  be a small preadditive category, viewed as a ring with several objects. Then, a (right)  $\mathcal{R}$ -module is a functor  $\mathcal{R}^{op} \rightarrow Ab$ , where  $Ab$  is the category of abelian groups. The category of right  $\mathcal{R}$ -modules will be denoted by  $Mod - \mathcal{R}$ . For each object  $r \in \mathcal{R}$ , we set  $H_r := \mathcal{R}(\_, r) : \mathcal{R}^{op} \rightarrow Ab$ .

It is well known (see, for instance, [16, § 1.4]) that  $Mod - \mathcal{R}$  is a locally finitely presented Grothendieck category, with the collection  $\{H_r\}_{r \in \mathcal{R}}$  being a family of finitely generated projective generators. We notice that since  $Mod - \mathcal{R}$  is a Grothendieck category, it is well-powered (see, for instance, [33, Proposition IV.6.6]), i.e., the collection of (equivalence classes of) subobjects of any  $V \in Mod - \mathcal{R}$  is a set.

**Definition 2.1.** Let  $\mathcal{R}$  be a small preadditive category. An  $FI$ -module over  $\mathcal{R}$  is a functor from  $FI$  to  $Mod - \mathcal{R}$ . For any such  $\mathcal{V} : FI \rightarrow Mod - \mathcal{R}$ , we set  $\mathcal{V}_n := \mathcal{V}([n])$  for each  $n \geq 0$ . For any morphism  $\phi : S \rightarrow T$  in  $FI$ , we denote by  $\phi_* : \mathcal{V}(S) \rightarrow \mathcal{V}(T)$  the induced morphism  $\mathcal{V}(\phi)$ .

The category of  $FI$ -modules over  $\mathcal{R}$  will be denoted by  $FI_{\mathcal{R}}$ .

Since  $FI$  is essentially small, it follows from [15, Theorem 14.2] that the category of  $FI$ -modules over  $\mathcal{R}$  is a Grothendieck category. In particular, for any morphism  $\tilde{f} : \mathcal{V} \rightarrow \mathcal{V}'$  in  $FI_{\mathcal{R}}$ , we have

$$\text{Ker}(\tilde{f})(S) = \text{Ker}(\tilde{f}(S) : \mathcal{V}(S) \rightarrow \mathcal{V}'(S)) \quad \text{Coker}(\tilde{f})(S) = \text{Coker}(\tilde{f}(S) : \mathcal{V}(S) \rightarrow \mathcal{V}'(S)) \quad (2.1)$$

for any  $S \in FI$ . In this paper, unless otherwise mentioned, by an  $FI$ -module, we will always mean an  $FI$ -module over  $\mathcal{R}$ . For any  $\mathcal{V} \in FI_{\mathcal{R}}$ , we set

$$\text{el}(\mathcal{V}) := \prod_{d \geq 0} \prod_{r \in \mathcal{R}} \mathcal{V}_d(r) \quad (2.2)$$

**Definition 2.2.** Fix  $d \geq 0$ . An  $FI$ -module  $\mathcal{V}$  is said to be generated in degree  $\leq d$  if there exists a (not necessarily finite) collection  $\{f_i | i \in I\} \subseteq \prod_{e \leq d} \prod_{r \in \mathcal{R}} \mathcal{V}_e(r)$  having the property that any subobject  $\mathcal{V}' \subseteq \mathcal{V}$  such that  $\{f_i | i \in I\} \subseteq \text{el}(\mathcal{V}')$  must satisfy  $\mathcal{V}' = \mathcal{V}$ .

We say that an  $FI$ -module over  $\mathcal{R}$  is generated in finite degree if it is generated in degree  $\leq d$  for some  $d \geq 0$ .

Given finite sets  $S, T \in FI$ , we will denote by  $(S, T)$  the set of injections  $S \hookrightarrow T$ , i.e., the morphisms from  $S$  to  $T$  in the category  $FI$ . For any  $r \in \mathcal{R}$  and  $d \geq 0$ , we now define  ${}_d\mathcal{M}_r \in FI_{\mathcal{R}}$  as follows:

$${}_d\mathcal{M}_r : FI \rightarrow \text{Mod} - \mathcal{R} \quad S \mapsto H_r^{([d], S)} \quad (2.3)$$

where  $H_r^{([d], S)}$  denotes the direct sum of copies of  $H_r$  indexed by the set  $([d], S)$ .

**Lemma 2.3.** Let  $\mathcal{V} \in FI_{\mathcal{R}}$ . For any  $d \geq 0$  and any  $r \in \mathcal{R}$ , we have a canonical isomorphism

$$FI_{\mathcal{R}}({}_d\mathcal{M}_r, \mathcal{V}) \cong \mathcal{V}_d(r) \quad (2.4)$$

of abelian groups.

**Proof.** By Yoneda Lemma, an element  $f \in \mathcal{V}_d(r)$  corresponds to a morphism  $f : H_r \rightarrow \mathcal{V}_d$  in  $\text{Mod} - \mathcal{R}$ . For any finite set  $S$ , we can take a direct sum of copies of  $f$  to obtain a morphism  $f^{([d], S)} : H_r^{([d], S)} \rightarrow \mathcal{V}_d^{([d], S)}$  in  $\text{Mod} - \mathcal{R}$ . Since  $\mathcal{V}$  is a covariant functor from  $FI$  to  $\text{Mod} - \mathcal{R}$ , each morphism  $\phi \in ([d], S)$  induces a morphism  $\mathcal{V}(\phi) : \mathcal{V}_d \rightarrow \mathcal{V}(S)$ . Together, these determine a morphism  $\mathcal{V}_d^{([d], S)} \rightarrow \mathcal{V}(S)$  from the direct sum  $\mathcal{V}_d^{([d], S)}$ . Composing with  $f^{([d], S)} : H_r^{([d], S)} \rightarrow \mathcal{V}_d^{([d], S)}$ , we obtain  $\tilde{f}(S) : {}_d\mathcal{M}_r(S) = H_r^{([d], S)} \rightarrow \mathcal{V}(S)$  in  $\text{Mod} - \mathcal{R}$ . Since these morphisms are functorial with respect to  $S \in FI$ , the element  $f \in \mathcal{V}_d(r)$  determines a morphism  $\tilde{f} : {}_d\mathcal{M}_r \rightarrow \mathcal{V}$  in  $FI_{\mathcal{R}}$ .

Conversely, suppose that we are given a morphism  $\tilde{f} : {}_d\mathcal{M}_r \rightarrow \mathcal{V}$  in  $FI_{\mathcal{R}}$ . In particular, this gives us a morphism  $\tilde{f}([d]) : {}_d\mathcal{M}_r([d]) = H_r^{([d], [d])} \rightarrow \mathcal{V}([d]) = \mathcal{V}_d$  in  $\text{Mod} - \mathcal{R}$ . Considering the identity morphism  $1_d \in ([d], [d])$  gives us an inclusion  $H_r \rightarrow H_r^{([d], [d])}$  which when composed with  $\tilde{f}([d])$  gives a morphism  $f : H_r \rightarrow \mathcal{V}_d$  in  $\text{Mod} - \mathcal{R}$ , i.e., an element  $f \in \mathcal{V}_d(r)$ . It may be easily verified that these two associations are inverse to each other, which proves the result.  $\square$

**Proposition 2.4.** (a) For  $d \geq 0$  and  $r \in \mathcal{R}$ , the object  ${}_d\mathcal{M}_r$  is a finitely generated object of  $FI_{\mathcal{R}}$ .

(b) The collection  $\{{}_d\mathcal{M}_r\}_{r \in \mathcal{R}, d \geq 0}$  is a set of generators for the Grothendieck category  $FI_{\mathcal{R}}$ .

(c) An object  $\mathcal{V}$  in  $FI_{\mathcal{R}}$  is finitely generated if and only if there is an epimorphism

$$\bigoplus_{i \in I} {}_{d_i}\mathcal{M}_{r_i} \rightarrow \mathcal{V} \quad (2.5)$$

for some finite collection  $\{(d_i, r_i)\}_{i \in I}$  with each  $d_i \geq 0$  and  $r_i \in \mathcal{R}$ .

(d) An object  $\mathcal{V}$  in  $FI_{\mathcal{R}}$  is finitely generated if and only if there is a finite collection  $\{f_1, \dots, f_k\} \subseteq \text{el}(\mathcal{V})$  such that any subobject  $\mathcal{V}' \hookrightarrow \mathcal{V}$  with  $\{f_1, \dots, f_k\} \subseteq \text{el}(\mathcal{V}')$  must satisfy  $\mathcal{V}' = \mathcal{V}$ .

(e) An object  $\mathcal{V}$  in  $FI_{\mathcal{R}}$  is generated in degree  $\leq d$  if and only if there is an epimorphism

$$\bigoplus_{i \in I} d_i \mathcal{M}_{r_i} \longrightarrow \mathcal{V} \tag{2.6}$$

for some collection  $\{(d_i, r_i)\}_{i \in I}$  with each  $0 \leq d_i \leq d$  and  $r_i \in \mathcal{R}$ .

**Proof.** We consider a filtered system  $\{\mathcal{W}_j\}_{j \in J}$  in  $FI_{\mathcal{R}}$  connected by monomorphisms and set  $\mathcal{W} := \varinjlim_{j \in J} \mathcal{W}_j$ .

By Lemma 2.3, any morphism  $\tilde{f} : d \mathcal{M}_r \longrightarrow \mathcal{W}$  corresponds to an element  $f \in \mathcal{W}_d(r) = \mathcal{W}([d])(r)$ . Since  $\mathcal{W}([d])(r) = \varinjlim_{j \in J} \mathcal{W}_j([d])(r)$ , we see that  $\tilde{f}$  must factor through  $\mathcal{W}_{j_0}$  for some  $j_0 \in J$ . This proves (a).

We have noted before that  $FI_{\mathcal{R}}$  is a Grothendieck category. We consider some  $\mathcal{V} \in FI_{\mathcal{R}}$  and some proper subobject  $\mathcal{V}' \subsetneq \mathcal{V}$ . Since the full subcategory of objects  $[n]$ ,  $n \geq 0$  forms a skeleton of  $FI$ , we must have some  $d \geq 0$  such that  $\mathcal{V}'_d \subsetneq \mathcal{V}_d$  and therefore some  $r \in \mathcal{R}$  such that  $\mathcal{V}'_d(r) \subsetneq \mathcal{V}_d(r)$ . Since  $FI_{\mathcal{R}}(d \mathcal{M}_r, \mathcal{V}) \cong \mathcal{V}_d(r)$  by Lemma 2.3, it follows that there exists a morphism  $d \mathcal{M}_r \longrightarrow \mathcal{V}$  in  $FI_{\mathcal{R}}$  which does not factor through  $\mathcal{V}'$ . It follows from [17, § 1.9] that  $\{d \mathcal{M}_r\}_{r \in \mathcal{R}, d \geq 0}$  is a set of generators for  $FI_{\mathcal{R}}$ . This proves (b). The “only if part” of (c) is clear from [17, Proposition 1.9.1]. The “if part” follows from the fact that a quotient of a finitely generated object is always finitely generated. Parts (d) and (e) also follow easily by using Lemma 2.3.  $\square$

Similar to [9, Definition 2.4], we now consider the following functor: for  $\mathcal{V} \in FI_{\mathcal{R}}$ , we define  $H_0(\mathcal{V}) : FI \longrightarrow \text{Mod} - \mathcal{R}$  by setting

$$\begin{aligned} H_0(\mathcal{V})(S) &:= \text{Coker} \left( \bigoplus \mathcal{V}(\phi) : \bigoplus_{\substack{\phi : T \hookrightarrow S \\ |T| < |S|}} \mathcal{V}(T) \longrightarrow \mathcal{V}(S) \right) \\ &= \mathcal{V}(S) / \left( \sum_{\substack{\phi : T \hookrightarrow S \\ |T| < |S|}} \text{Im}(\mathcal{V}(\phi) : \mathcal{V}(T) \longrightarrow \mathcal{V}(S)) \right) \end{aligned} \tag{2.7}$$

From (2.7), it is clear that there is a canonical epimorphism  $\mathcal{V} \longrightarrow H_0(\mathcal{V})$  and that for any  $\phi : T \longrightarrow S$  in  $FI$  with  $|T| < |S|$ , we have  $H_0(\mathcal{V})(\phi) = 0$ . It follows that the functor  $H_0$  is idempotent, i.e.,  $H_0^2 = H_0$ .

**Lemma 2.5.** (a) The functor  $H_0 : FI_{\mathcal{R}} \longrightarrow FI_{\mathcal{R}}$  preserves colimits.

(b) For any  $\mathcal{V} \in FI_{\mathcal{R}}$ , we have  $\mathcal{V} = 0$  if and only if  $H_0(\mathcal{V}) = 0$ .

(c) A morphism  $\tilde{f}$  in  $FI_{\mathcal{R}}$  is an epimorphism if and only if  $H_0(\tilde{f})$  is an epimorphism.

**Proof.** Part (a) follows from the fact that  $H_0$  is defined in (2.7) using cokernels. For (b), suppose we have  $\mathcal{V} \neq 0$  in  $FI_{\mathcal{R}}$  such that  $H_0(\mathcal{V}) = 0$ . Let  $n$  be the smallest integer  $\geq 0$  such that  $\mathcal{V}_n \neq 0$ . Then, for each finite set  $T$  with  $|T| < n$ , we have  $\mathcal{V}(T) = 0$  and it follows from (2.7) that  $H_0(\mathcal{V})_n = \mathcal{V}_n$ . This yields  $\mathcal{V}_n = 0$ , which is a contradiction. This proves (b). Part (c) follows by using (a) and applying the result of (b) to the cokernel of  $\tilde{f}$ .  $\square$

We now consider a functor

$$\text{Gr} : FI_{\mathcal{R}} \longrightarrow \text{Mod} - \mathcal{R} \quad \mathcal{V} \mapsto \bigoplus_{n \geq 0} \mathcal{V}_n \tag{2.8}$$



It is clear that  $Gr(\mathcal{V}) = 0$  if and only if  $\mathcal{V} = 0$ . We also note that  $Gr$  preserves cokernels, kernels and coproducts.

**Proposition 2.6.** *Let  $\mathcal{V} \in FI_{\mathcal{R}}$ . Then, the following are equivalent:*

- (a)  $\mathcal{V}$  is finitely generated in  $FI_{\mathcal{R}}$ .
- (b)  $H_0(\mathcal{V})$  is finitely generated in  $FI_{\mathcal{R}}$ .
- (c)  $Gr(H_0(\mathcal{V})) = \bigoplus_{n \geq 0} H_0(\mathcal{V})_n$  is finitely generated in  $Mod - \mathcal{R}$ .

**Proof.** (a)  $\Rightarrow$  (b): By definition,  $H_0(\mathcal{V})$  is a quotient of  $\mathcal{V}$ . If  $\mathcal{V}$  is finitely generated, so is its quotient  $H_0(\mathcal{V})$ .

(b)  $\Rightarrow$  (a): By Proposition 2.4(b), we know that the collection  $\{d_i \mathcal{M}_{r_i}\}_{i \in I}$  is a set of generators for the Grothendieck category  $FI_{\mathcal{R}}$ . This gives us an epimorphism

$$\bigoplus_{i \in I} d_i \mathcal{M}_{r_i} \longrightarrow \mathcal{V} \quad (2.9)$$

for some collection  $\{(d_i, r_i)\}_{i \in I}$  with each  $d_i \geq 0$  and  $r_i \in \mathcal{R}$ . By Lemma 2.5(c), applying the functor  $H_0$  induces an epimorphism  $\bigoplus_{i \in I} H_0(d_i \mathcal{M}_{r_i}) \longrightarrow H_0(\mathcal{V})$ . Since  $H_0(\mathcal{V})$  is finitely generated, it follows that there is a finite subset  $J \subseteq I$  such that  $\bigoplus_{i \in J} H_0(d_i \mathcal{M}_{r_i}) \longrightarrow H_0(\mathcal{V})$  is an epimorphism. Applying Lemma 2.5(c) again, we see that  $\bigoplus_{i \in J} d_i \mathcal{M}_{r_i} \longrightarrow \mathcal{V}$  is an epimorphism. Each  $d_i \mathcal{M}_{r_i}$  is finitely generated by Proposition 2.4(a) and since  $J$  is finite, the result follows.

(b)  $\Rightarrow$  (c): Since  $H_0(\mathcal{V})$  is finitely generated and  $H_0$  is an idempotent functor, it follows from Proposition 2.4(c) that there is an epimorphism  $\bigoplus_{i \in I} H_0(d_i \mathcal{M}_{r_i}) \longrightarrow H_0(\mathcal{V})$  for a finite set  $I$ . It suffices therefore to show that  $\bigoplus_{n \geq 0} H_0(d \mathcal{M}_r)_n$  is finitely generated in  $Mod - \mathcal{R}$  for each  $d \geq 0$  and each  $r \in \mathcal{R}$ . Since  $H_r$  is finitely generated in  $Mod - \mathcal{R}$  for each  $r \in \mathcal{R}$ , it is clear from the definitions in (2.3) and (2.7) that each  $H_0(d \mathcal{M}_r)_n$  is finitely generated in  $Mod - \mathcal{R}$ . We also notice that for  $n \geq d + 1$ , every morphism  $[d] \longrightarrow [n]$  in  $FI$  factors through a subset of  $[n]$  of cardinality  $\leq n - 1$ . The quotient in (2.7) now shows that  $H_0(d \mathcal{M}_r)_n = 0$  for  $n \geq d + 1$ . This proves the result.

(c)  $\Rightarrow$  (b): Since  $\{d_i \mathcal{M}_{r_i}\}_{i \in I}$  is a set of generators for the category  $FI_{\mathcal{R}}$ , we must have an epimorphism  $\bigoplus_{i \in I} d_i \mathcal{M}_{r_i} \longrightarrow H_0(\mathcal{V})$ . Since  $Gr(H_0(\mathcal{V}))$  is finitely generated, we can find some finite subset  $J \subseteq I$  such that  $Coker \left( \bigoplus_{i \in J} Gr(d_i \mathcal{M}_{r_i}) \longrightarrow Gr(H_0(\mathcal{V})) \right) = 0$ . It follows that  $\bigoplus_{i \in J} d_i \mathcal{M}_{r_i} \longrightarrow H_0(\mathcal{V})$  is an epimorphism.  $\square$

**Proposition 2.7.** *Let  $\mathcal{V} \in FI_{\mathcal{R}}$  and fix  $d \geq 0$ . Then, the following are equivalent:*

- (a)  $\mathcal{V}$  is generated in degree  $\leq d$ .
- (b)  $H_0(\mathcal{V})$  is generated in degree  $\leq d$ .
- (c)  $H_0(\mathcal{V})_n = 0$  for  $n > d$ .

**Proof.** (a)  $\Rightarrow$  (b): By definition,  $H_0(\mathcal{V})$  is a quotient of  $\mathcal{V}$ . Hence, this is clear from Proposition 2.4(e).

(b)  $\Rightarrow$  (c): By reasoning similar to the proof of (b)  $\Rightarrow$  (c) in Proposition 2.6, it suffices to show that  $H_0(d' \mathcal{M}_{r'})_n = 0$  when  $d' \leq d$  and  $n > d$ . This latter fact has also been established in the proof of Proposition 2.6.

(c)  $\Rightarrow$  (a): Since  $\{d_i \mathcal{M}_{r_i}\}_{i \in I}$  is a set of generators for the category  $FI_{\mathcal{R}}$ , we must have an epimorphism

$$\tilde{e} : \bigoplus_{i \in I} d_i \mathcal{M}_{r_i} \longrightarrow \mathcal{V} \quad (2.10)$$



We restrict to all pairs  $(d_i, r_i)_{i \in I}$  with  $d_i \leq d$  and consider the induced morphism

$$\tilde{f} : \bigoplus_{i \in I, d_i \leq d} d_i \mathcal{M}_{r_i} \longrightarrow \mathcal{V} \tag{2.11}$$

We claim that  $\tilde{f}$  is an epimorphism. By Lemma 2.5, it suffices to show that  $H_0(\tilde{f})$  is an epimorphism, i.e., each  $H_0(\tilde{f})_n$  is an epimorphism. For  $n > d$  we have

$$H_0(\tilde{f})_n : \bigoplus_{i \in I, d_i \leq d} H_0(d_i \mathcal{M}_{r_i})_n \longrightarrow H_0(\mathcal{V})_n = 0 \tag{2.12}$$

which must be an epimorphism. We now consider  $n \leq d$  and examine the epimorphism

$$H_0(\tilde{e})_n : \bigoplus_{i \in I} H_0(d_i \mathcal{M}_{r_i})_n \longrightarrow H_0(\mathcal{V})_n \tag{2.13}$$

induced by (2.10). By definition,  $H_0(d_i \mathcal{M}_{r_i})_n$  is a quotient of  $H_{r_i}^{([d_i], [n])}$ . For any  $d_i > d$ , we must therefore have  $H_0(d_i \mathcal{M}_{r_i})_n = 0$  since  $n \leq d < d_i$ . Hence,  $H_0(\tilde{f})_n = H_0(\tilde{e})_n$  is an epimorphism for  $n \leq d$  and the result follows.  $\square$

We now recall some generalities on objects in a Grothendieck abelian category  $\mathcal{A}$  that we will use throughout this paper (see, for instance, [1], [33] and [26]).

(a) An object  $X$  in  $\mathcal{A}$  is said to be finitely generated if the functor  $Hom_{\mathcal{A}}(X, \_ ) : \mathcal{A} \longrightarrow \mathbf{Ab}$  preserves filtered colimits of monomorphisms.

(b) An object  $X$  in  $\mathcal{A}$  is said to be finitely presented if the functor  $Hom_{\mathcal{A}}(X, \_ ) : \mathcal{A} \longrightarrow \mathbf{Ab}$  preserves filtered colimits.

(c) An object  $Y$  in  $\mathcal{A}$  is said to be noetherian if every subobject is finitely generated.

(d) The category  $\mathcal{A}$  is said to be locally noetherian if it has a set of noetherian generators.

In a locally noetherian Grothendieck category  $\mathcal{A}$ , the finitely generated objects coincide with the finitely presented objects (see, for instance, [26, Chapter 5.8]) as well as with the noetherian objects. Further, the full subcategory of finitely generated objects in  $\mathcal{A}$  forms an abelian category, which we denote by  $\mathcal{A}^{fg}$ .

We conclude this section by recalling the following result.

**Theorem 2.8.** (See [9, Theorem A], [14], [28], [31] and [21, Theorem 9.1].) *Let  $\mathcal{A}$  be a locally noetherian Grothendieck category. Then, the category  $Fun(FI, \mathcal{A})$  of functors from  $FI$  to  $\mathcal{A}$  is locally noetherian.*

In particular, if  $\mathcal{R}$  is a small preadditive category such that  $Mod - \mathcal{R}$  is locally noetherian, it follows from Theorem 2.8 that the category  $FI_{\mathcal{R}}$  is locally noetherian. In that case, if  $\mathcal{V}$  is a finitely generated  $FI$ -module over  $\mathcal{R}$ , any submodule of  $\mathcal{V}$  is finitely generated.

### 3. Torsion theories and the positive shift functor

We recall that a torsion theory  $\tau = (\mathcal{T}, \mathcal{F})$  on an abelian category  $\mathcal{A}$  consists of a pair of full and replete subcategories  $\mathcal{T}$  and  $\mathcal{F}$  of  $\mathcal{A}$  such that  $Hom_{\mathcal{A}}(T, F) = 0$  for any  $T \in \mathcal{T}$ ,  $F \in \mathcal{F}$  and for any object  $X \in \mathcal{A}$  there exists a short exact sequence

$$0 \longrightarrow T(X) \longrightarrow X \longrightarrow F(X) \longrightarrow 0$$

with  $T(X) \in \mathcal{T}$ ,  $F(X) \in \mathcal{F}$  (see, for instance, [4, § I.1]).

Accordingly, we start with a torsion theory  $\tau = (\mathcal{T}, \mathcal{F})$  on  $Mod - \mathcal{R}$ . Let  $\mathcal{R}$  be a small preadditive category such that  $Mod - \mathcal{R}$  is locally noetherian. From Theorem 2.8, we know that the category  $FI_{\mathcal{R}}$  is locally noetherian. We will show how to extend  $\tau$  to a torsion theory  $(\overline{\mathcal{T}}, \overline{\mathcal{F}})$  on the abelian category  $FI_{\mathcal{R}}^{fg}$  of finitely generated  $FI$ -modules over  $\mathcal{R}$ . We let  $\overline{\mathcal{T}}$  be the full subcategory of  $FI_{\mathcal{R}}^{fg}$  defined by setting

$$Ob(\overline{\mathcal{T}}) := \{\mathcal{V} \in Ob(FI_{\mathcal{R}}^{fg}) \mid \mathcal{V}_n \in \mathcal{T} \text{ for } n \gg 0\} \tag{3.1}$$

We need to show that  $\overline{\mathcal{T}}$  is a torsion class in  $FI_{\mathcal{R}}^{fg}$ . If an abelian category is complete and cocomplete, it is well known (see, for instance, [4, § I.1]) that any full subcategory closed under quotients, extensions and arbitrary coproducts must be a torsion class. However,  $FI_{\mathcal{R}}^{fg}$  being the subcategory of finitely generated  $FI$ -modules, does not contain arbitrary coproducts. As such, in order to identify torsion classes in  $FI_{\mathcal{R}}^{fg}$ , we will use the following simple result from [3].

**Proposition 3.1.** (See [3, Proposition 4.8].) *Let  $\mathcal{B}$  be an abelian category such that every object in  $\mathcal{B}$  is noetherian. Let  $\mathcal{C} \subseteq \mathcal{B}$  be a full and replete subcategory that is closed under extensions and quotients. Let  $\mathcal{C}^\perp \subseteq \mathcal{B}$  be the full subcategory given by*

$$Ob(\mathcal{C}^\perp) := \{N \in \mathcal{B} \mid Hom_{\mathcal{B}}(C, N) = 0 \text{ for all } C \in \mathcal{C}\}$$

*Then  $(\mathcal{C}, \mathcal{C}^\perp)$  is a torsion pair on  $\mathcal{B}$ .*

**Proposition 3.2.** *Let  $\mathcal{R}$  be a small preadditive category such that  $Mod - \mathcal{R}$  is locally noetherian. Let  $\mathcal{T}$  be a torsion class on  $Mod - \mathcal{R}$ . Then,  $\overline{\mathcal{T}}$  is a torsion class in the category  $FI_{\mathcal{R}}^{fg}$  of finitely generated  $FI$ -modules over  $\mathcal{R}$ . Additionally, if  $\mathcal{T}$  is a hereditary torsion class, so is  $\overline{\mathcal{T}}$ .*

**Proof.** Since  $FI_{\mathcal{R}}$  is locally noetherian, it follows that every object in the category  $FI_{\mathcal{R}}^{fg}$  is noetherian. Applying Proposition 3.1, it suffices to show that  $\overline{\mathcal{T}}$  is closed under extensions and quotients. Accordingly, if  $0 \rightarrow \mathcal{V}' \rightarrow \mathcal{V} \rightarrow \mathcal{V}'' \rightarrow 0$  is a short exact sequence with  $\mathcal{V}', \mathcal{V}'' \in \overline{\mathcal{T}}$ , we can choose  $N$  large enough so that  $\mathcal{V}'_n, \mathcal{V}''_n \in \mathcal{T}$  for all  $n > N$ . We have short exact sequences

$$0 \rightarrow \mathcal{V}'_n \rightarrow \mathcal{V}_n \rightarrow \mathcal{V}''_n \rightarrow 0 \tag{3.2}$$

Since  $\mathcal{T}$  is closed under extensions, it now follows that  $\mathcal{V}_n \in \mathcal{T}$  for all  $n > N$ . Hence,  $\mathcal{V} \in \overline{\mathcal{T}}$ .

On the other hand, if  $\mathcal{V}' \rightarrow \mathcal{V}$  is an epimorphism with  $\mathcal{V}' \in \overline{\mathcal{T}}$ , we know that since  $\mathcal{T}$  is closed under quotients, we must have  $\mathcal{V}_n \in \mathcal{T}$  for  $n \gg 0$ . This gives  $\mathcal{V} \in \overline{\mathcal{T}}$ . By similar reasoning, it is clear that if  $\mathcal{T}$  is a hereditary torsion class (i.e., closed under subobjects), so is  $\overline{\mathcal{T}}$ .  $\square$

Given  $\mathcal{V} \in FI_{\mathcal{R}}^{fg}$ , we would like to obtain an explicit description for its torsion subobject in  $\overline{\mathcal{T}}$ . For this, we will need to consider (positive) ‘shift functors’ on the category  $FI_{\mathcal{R}}$  in a manner analogous to [9, § 2.1]. For each  $a \geq 0$ , we fix a set  $[-a]$  of cardinality  $a$ . Then, the category  $FI$  is equipped with a shift functor

$$\mathbb{S}^a : FI \rightarrow FI \quad S \mapsto S \sqcup [-a] \tag{3.3}$$

formed by taking the disjoint union with  $[-a]$ . For a morphism  $\phi : S \rightarrow T$  in  $FI$ ,  $\mathbb{S}^a(\phi)$  is obtained by extending  $\phi$  with the identity on  $[-a]$ . Then,  $\mathbb{S}^a$  induces a “positive shift functor” on  $FI_{\mathcal{R}}$ , which we continue to denote by  $\mathbb{S}^a$

$$\mathbb{S}^a : FI_{\mathcal{R}} \rightarrow FI_{\mathcal{R}} \quad \mathcal{V} \mapsto \mathcal{V} \circ \mathbb{S}^a \tag{3.4}$$

It is immediate that  $\mathbb{S}^a$  preserves all limits and colimits. It is also clear that  $\mathbb{S}^a$  does not depend on the choice of the set  $[-a]$  of cardinality  $a$ . Before we proceed further, we will collect some basic properties of the functor  $\mathbb{S}^a$ .

**Proposition 3.3.** *Fix  $a \geq 0$ . If  $\mathcal{V} \in FI_{\mathcal{R}}$  is generated in degree  $\leq d$ , then  $\mathbb{S}^a(\mathcal{V})$  is also generated in degree  $\leq d$ .*

**Proof.** Since  $\mathbb{S}^a$  preserves coproducts and epimorphisms, it follows from the ‘if and only if’ condition in Proposition 2.4(e) that it suffices to prove the result for  $\mathcal{V} = {}_{d'}\mathcal{M}_r$  with  $d' \leq d$ .

Given a finite set  $S$ , we notice easily that

$$([d'], \mathbb{S}^a(S)) = \bigcup_{j=0}^a ([d' - j], S) \times ([j], [-a]) \tag{3.5}$$

Therefore, we obtain

$$\mathbb{S}^a({}_{d'}\mathcal{M}_r) = \bigoplus_{j=0}^a {}_{d'-j}\mathcal{M}_r^{([j], [-a])} \quad \square \tag{3.6}$$

**Corollary 3.4.** *For any  $a \geq 0$ ,  $d \geq 0$  and  $r \in \mathcal{R}$ , we have  $\mathbb{S}^a({}_d\mathcal{M}_r) = {}_d\mathcal{M}_r \oplus {}_d\mathcal{N}_{r,a}$ , where  ${}_d\mathcal{N}_{r,a}$  is finitely generated in degree  $\leq d - 1$ .*

**Proof.** This is clear from Proposition 2.4 and the expression in (3.6).  $\square$

**Lemma 3.5.** *Fix  $a \geq 0$ . Then, for every  $\mathcal{V} \in FI_{\mathcal{R}}$  and every  $n \geq 0$ , there is an epimorphism  $H_0(\mathbb{S}^a(\mathcal{V}))_n \rightarrow H_0(\mathcal{V})_{n+a}$  in  $Mod - \mathcal{R}$ .*

**Proof.** By the definition in (2.7), we have

$$\begin{aligned} H_0(\mathbb{S}^a(\mathcal{V}))_n &= \text{Coker} \left( \bigoplus_{\substack{\phi : T \hookrightarrow [n] \\ |T| < n}} \mathbb{S}^a(\mathcal{V})(T) \rightarrow \mathbb{S}^a(\mathcal{V})_n \right) \\ &= \text{Coker} \left( \bigoplus_{\substack{\phi \sqcup 1_{[-a]} : (T \sqcup [-a]) \hookrightarrow [n+a] \\ |T| < n}} \mathcal{V}(T \sqcup [-a]) \rightarrow \mathcal{V}_{n+a} \right) \end{aligned} \tag{3.7}$$

The morphism  $\bigoplus_{\substack{\phi \sqcup 1_{[-a]} : (T \sqcup [-a]) \hookrightarrow [n+a] \\ |T| < n}} \mathcal{V}(T \sqcup [-a]) \rightarrow \mathcal{V}_{n+a}$  appearing in (3.7) factors through the canonical morphism  $\bigoplus_{\substack{\phi : T \hookrightarrow [n+a] \\ |T| < n+a}} \mathcal{V}(T) \rightarrow \mathcal{V}_{n+a}$  which gives us a factorization

$$\mathcal{V}_{n+a} \rightarrow H_0(\mathbb{S}^a(\mathcal{V}))_n \rightarrow H_0(\mathcal{V})_{n+a} \tag{3.8}$$

of the canonical epimorphism  $\mathcal{V}_{n+a} \rightarrow H_0(\mathcal{V})_{n+a}$ . It follows that  $H_0(\mathbb{S}^a(\mathcal{V}))_n \rightarrow H_0(\mathcal{V})_{n+a}$  is an epimorphism.  $\square$

**Proposition 3.6.** *Fix  $a \geq 0$ . Suppose that  $\mathcal{V} \in FI_{\mathcal{R}}$  is such that  $\mathbb{S}^a(\mathcal{V})$  is generated in degree  $\leq d$ . Then,  $\mathcal{V}$  is generated in degree  $\leq a + d$ .*

**Proof.** From Lemma 3.5, it is clear that  $H_0(S^a(\mathcal{V}))_n = 0 \Rightarrow H_0(\mathcal{V})_{n+a} = 0$ . The result is now a consequence of the equivalent statements in Proposition 2.7.  $\square$

We now return to the torsion theory  $\tau = (\mathcal{T}, \mathcal{F})$  on  $Mod - \mathcal{R}$ . For any object  $P$  in  $Mod - \mathcal{R}$ , we denote its torsion subobject by  $\mathcal{T}(P)$ . For any  $\mathcal{V} \in FI_{\mathcal{R}}$  and any finite set  $S$ , the canonical inclusion  $S \hookrightarrow S \sqcup [-a]$  induces a morphism  $\mathcal{V}(S) \rightarrow \mathcal{V}(S \sqcup [-a])$  and hence a morphism  $\psi_a^{\mathcal{V}} : \mathcal{V} \rightarrow S^a \mathcal{V}$  in  $FI_{\mathcal{R}}$ . We now set

$$\overline{\mathcal{T}}(\mathcal{V})(S) := \operatorname{colim}_{a \geq 0} \lim \left( \mathcal{V}(S) \xrightarrow{\psi_a^{\mathcal{V}}(S)} (S^a \mathcal{V})(S) \longleftarrow \mathcal{T}((S^a \mathcal{V})(S)) \right) \tag{3.9}$$

for each finite set  $S$ . It is clear that  $\overline{\mathcal{T}}(\mathcal{V})$  is an  $FI$ -module and that  $\overline{\mathcal{T}}(\mathcal{V}) \subseteq \mathcal{V}$ .

**Lemma 3.7.** *Let  $\mathcal{R}$  be such that  $Mod - \mathcal{R}$  is locally noetherian. Let  $\tau = (\mathcal{T}, \mathcal{F})$  be a hereditary torsion theory on  $Mod - \mathcal{R}$ . Then, for any  $\mathcal{V} \in FI_{\mathcal{R}}^{fg}$ , the subobject  $\overline{\mathcal{T}}(\mathcal{V})$  belongs to the torsion class  $\overline{\mathcal{T}}$ .*

**Proof.** Since  $\mathcal{V}$  is finitely generated and  $\overline{\mathcal{T}}(\mathcal{V}) \subseteq \mathcal{V}$ , we know that  $\overline{\mathcal{T}}(\mathcal{V})$  is finitely generated and hence noetherian. As such, the increasing chain appearing in the definition of  $\overline{\mathcal{T}}(\mathcal{V})$  in (3.9) must be stationary. In other words, we can find  $a > 0$  such that

$$\begin{aligned} \overline{\mathcal{T}}(\mathcal{V})(S) &= \lim \left( \mathcal{V}(S) \xrightarrow{\psi_a^{\mathcal{V}}(S)} \mathcal{V}(S \sqcup [-a]) \longleftarrow \mathcal{T}(\mathcal{V}(S \sqcup [-a])) \right) \\ &= \lim \left( \mathcal{V}(S) \xrightarrow{\psi_b^{\mathcal{V}}(S)} \mathcal{V}(S \sqcup [-b]) \longleftarrow \mathcal{T}(\mathcal{V}(S \sqcup [-b])) \right) \end{aligned} \tag{3.10}$$

for every  $b \geq a$  and every finite set  $S$ . For the sake of convenience, we put  $\mathcal{W} := \overline{\mathcal{T}}(\mathcal{V})$ . The morphism  $\mathcal{W}(S) \rightarrow \mathcal{V}(S) \xrightarrow{\psi_b^{\mathcal{V}}(S)} \mathcal{V}(S \sqcup [-b])$  factors through  $\psi_b^{\mathcal{W}} : \mathcal{W}(S) \rightarrow \mathcal{W}(S \sqcup [-b])$  as well as the subobject  $\mathcal{T}(\mathcal{V}(S \sqcup [-b])) \subseteq \mathcal{V}(S \sqcup [-b])$ . Since  $\mathcal{W}(S \sqcup [-b]) \subseteq \mathcal{V}(S \sqcup [-b])$  and  $\tau$  is hereditary, it follows that

$$\operatorname{Im}(\psi_b^{\mathcal{W}}(S) : \mathcal{W}(S) \rightarrow \mathcal{W}(S \sqcup [-b])) \in \mathcal{T} \tag{3.11}$$

We now consider any morphism  $\phi : S \rightarrow S'$  in  $FI$  with  $|S'| - |S| = b \geq a$ . Choosing a bijection between  $S \sqcup [-b]$  and  $S'$ , we obtain a commutative diagram

$$\begin{array}{ccc} & \mathcal{W}(S \sqcup [-b]) & \\ \psi_b^{\mathcal{W}}(S) \nearrow & & \searrow \cong \\ \mathcal{W}(S) & \xrightarrow{\mathcal{W}(\phi)} & \mathcal{W}(S') \end{array} \tag{3.12}$$

Combining (3.11) and (3.12), we see that

$$\operatorname{Im}(\mathcal{W}(\phi) : \mathcal{W}(S) \rightarrow \mathcal{W}(S')) \in \mathcal{T} \quad \forall \phi : S \rightarrow S', |S'| - |S| = b \geq a \tag{3.13}$$

Since  $\mathcal{W}$  is finitely generated, we can choose some  $d$  such that  $\mathcal{W}$  is finitely generated in degree  $d$ . By Proposition 2.7, we see that  $H_0(\mathcal{W})_n = 0$  for  $n > d$ . From the definition of  $H_0(\mathcal{W})$  in (2.7), it is clear that

$$\mathcal{W}_n = \left( \sum_{\substack{\phi : S \hookrightarrow [n] \\ |S| \leq d}} \operatorname{Im}(\mathcal{W}(\phi) : \mathcal{W}(S) \rightarrow \mathcal{W}_n) \right) \quad \forall n > d \tag{3.14}$$

Combining (3.13) and (3.14), we see that for  $n > a + d$ , we must have  $\mathcal{W}_n \in \mathcal{T}$ . Hence,  $\mathcal{W} \in \overline{\mathcal{T}}$ .  $\square$

**Theorem 3.8.** Let  $\mathcal{R}$  be such that  $\text{Mod} - \mathcal{R}$  is locally noetherian. Let  $\tau = (\mathcal{T}, \mathcal{F})$  be a hereditary torsion theory on  $\text{Mod} - \mathcal{R}$ . Then, for any  $\mathcal{V} \in FI_{\mathcal{R}}^{fg}$ , the torsion subobject of  $\mathcal{V}$  with respect to the torsion class  $\overline{\mathcal{T}}$  is given by  $\overline{\mathcal{T}}(\mathcal{V})$ .

**Proof.** We set  $\mathcal{W} = \overline{\mathcal{T}}(\mathcal{V})$  and maintain the notation from the proof of Lemma 3.7. Then, we have  $a > 0$  such that

$$\begin{array}{ccc} \mathcal{W}(S) & \longrightarrow & \mathcal{T}(\mathcal{V}(S \sqcup [-b])) \\ \downarrow & & \downarrow \\ \mathcal{V}(S) & \xrightarrow{\psi_b^{\mathcal{V}}(S)} & \mathcal{V}(S \sqcup [-b]) \end{array} \tag{3.15}$$

is a fiber square for each  $b \geq a$  and each finite set  $S$ . Now let  $\mathcal{W}' \subseteq \mathcal{V}$  be such that  $\mathcal{W}' \in \overline{\mathcal{T}}$ . Then, there exists  $N$  such that  $\mathcal{W}'_n \in \mathcal{T}$  for all  $n \geq N$ . For  $n \geq N + a$ , we consider the commutative diagram

$$\begin{array}{ccc} \mathcal{W}'(S) & \xrightarrow{\psi_n^{\mathcal{W}'}(S)} & \mathcal{W}'(S \sqcup [-n]) \\ \downarrow & & \downarrow \\ \mathcal{V}(S) & \xrightarrow{\psi_n^{\mathcal{V}}(S)} & \mathcal{V}(S \sqcup [-n]) \end{array} \tag{3.16}$$

Then  $\mathcal{W}'(S \sqcup [-n]) \in \mathcal{T}$  and it follows that the composed morphism  $\mathcal{W}'(S) \rightarrow \mathcal{V}(S \sqcup [-n])$  appearing in (3.16) factors through  $\mathcal{T}(\mathcal{V}(S \sqcup [-n]))$ . From the fiber square (3.15), it now follows that the inclusion  $\mathcal{W}'(S) \hookrightarrow \mathcal{V}(S)$  factors through a morphism  $\mathcal{W}'(S) \rightarrow \mathcal{W}(S)$ . It follows that  $\mathcal{W}' \subseteq \mathcal{W}$ .

We have shown in Proposition 3.2 that  $\overline{\mathcal{T}}$  is a torsion class in  $FI_{\mathcal{R}}^{fg}$ . From Lemma 3.7 we already know that  $\mathcal{W} \in \overline{\mathcal{T}}$ . The reasoning above shows that  $\mathcal{W} = \overline{\mathcal{T}}(\mathcal{V})$  contains all torsion subobjects of  $\mathcal{V}$  and the result follows.  $\square$

We will now apply similar methods to study torsion theories in the subcategory of what we call “shift finitely generated  $FI$ -modules.”

**Definition 3.9.** Let  $\mathcal{V} \in FI_{\mathcal{R}}$ . Then, we will say that  $\mathcal{V}$  is shift finitely generated if there exists  $d \geq 0$  such that  $S^d \mathcal{V}$  is finitely generated. The full subcategory of shift finitely generated objects will be denoted by  $FI_{\mathcal{R}}^{sfg}$ .

**Lemma 3.10.** Let  $\mathcal{V} \in FI_{\mathcal{R}}$ .

- (a) If  $d \geq 0$  is such that  $S^d \mathcal{V}$  is finitely generated, so is  $S^e \mathcal{V}$  for any  $e \geq d$ .
- (b) If  $\mathcal{V}$  is shift finitely generated, so is  $S^a \mathcal{V}$  for any  $a \geq 0$ .
- (c) Any  $\mathcal{V} \in FI_{\mathcal{R}}^{sfg}$  is generated in finite degree.

**Proof.** Since  $S^d \mathcal{V}$  is finitely generated, we can choose an epimorphism of the form  $\bigoplus_{i=1}^k d_i \mathcal{M}_{r_i} \rightarrow S^d \mathcal{V}$ . For  $e \geq d$ , this induces an epimorphism  $S^{e-d} \left( \bigoplus_{i=1}^k d_i \mathcal{M}_{r_i} \right) \rightarrow S^e \mathcal{V}$ . From Corollary 3.4, we know that each  $S^{e-d} d_i \mathcal{M}_{r_i}$  is finitely generated. This proves (a). The result of (b) is clear from (a). For (c), we proceed as follows: if  $S^d \mathcal{V}$  is finitely generated, it follows from Proposition 2.6 that  $H_0(S^d \mathcal{V})_n = 0$  for  $n \gg 0$ . Then, the epimorphism  $H_0(S^d \mathcal{V})_n \rightarrow H_0(\mathcal{V})_{n+d}$  in Lemma 3.5 shows that  $H_0(\mathcal{V})_m = 0$  for  $m \gg 0$ . It now follows from Proposition 2.7 that  $\mathcal{V}$  is generated in finite degree.  $\square$

**Proposition 3.11.** *Suppose that  $Mod - \mathcal{R}$  is locally noetherian. Then  $FI_{\mathcal{R}}^{sfg}$  is a Serre subcategory of  $FI_{\mathcal{R}}$ , i.e., it is closed under subobjects, quotients and extensions.*

**Proof.** Let  $0 \rightarrow \mathcal{V}' \rightarrow \mathcal{V} \rightarrow \mathcal{V}'' \rightarrow 0$  be a short exact sequence in  $FI_{\mathcal{R}}$ . Since  $\mathbb{S}$  is exact, this gives a short exact sequence

$$0 \rightarrow \mathbb{S}^d \mathcal{V}' \rightarrow \mathbb{S}^d \mathcal{V} \rightarrow \mathbb{S}^d \mathcal{V}'' \rightarrow 0 \tag{3.17}$$

in  $FI_{\mathcal{R}}$  for each  $d \geq 0$ . Since  $Mod - \mathcal{R}$  is locally noetherian, it is clear from (3.17) and Theorem 2.8 that  $FI_{\mathcal{R}}^{sfg}$  is closed under quotients and subobjects. It remains to show that  $FI_{\mathcal{R}}^{sfg}$  is closed under extensions. We suppose that  $\mathcal{V}', \mathcal{V}'' \in FI_{\mathcal{R}}^{sfg}$  and choose  $d$  large enough so that  $\mathbb{S}^d \mathcal{V}'$  and  $\mathbb{S}^d \mathcal{V}''$  are finitely generated.

Consequently, we can choose epimorphisms  $\mathcal{P} \rightarrow \mathbb{S}^d \mathcal{V}'$  and  $\mathcal{Q} \rightarrow \mathbb{S}^d \mathcal{V}''$ , where  $\mathcal{P}$  and  $\mathcal{Q}$  are finite direct sums of the family  $\{d\mathcal{M}_r\}_{r \in \mathcal{R}, d \geq 0}$  of generators of  $FI_{\mathcal{R}}$ . Using Lemma 2.3, each object in  $\{d\mathcal{M}_r\}_{r \in \mathcal{R}, d \geq 0}$  is projective and hence the epimorphism  $\mathcal{Q} \rightarrow \mathbb{S}^d \mathcal{V}''$  lifts to a morphism  $\mathcal{Q} \rightarrow \mathbb{S}^d \mathcal{V}$ . It may be easily verified that the induced morphism  $\mathcal{P} \oplus \mathcal{Q} \rightarrow \mathbb{S}^d \mathcal{V}$  is an epimorphism and the result follows.  $\square$

Our next objective is to define a torsion theory on  $FI_{\mathcal{R}}^{sfg}$  starting from a torsion theory  $\tau = (\mathcal{T}, \mathcal{F})$  on  $Mod - \mathcal{R}$ . For this, we need to identify the torsion objects in  $FI_{\mathcal{R}}^{sfg}$ . For finitely generated objects in  $FI_{\mathcal{R}}$ , we already have that

$$Ob(\overline{\mathcal{T}}) := \{\mathcal{V} \in Ob(FI_{\mathcal{R}}^{sfg}) \mid \mathcal{V}_n \in \mathcal{T} \text{ for } n \gg 0\} \tag{3.18}$$

as defined in (3.1). In the case of  $FI_{\mathcal{R}}^{sfg}$ , we cannot proceed directly as in the proof of Proposition 3.2, because every object in  $FI_{\mathcal{R}}^{sfg}$  is not necessarily noetherian. For a hereditary torsion theory  $\tau = (\mathcal{T}, \mathcal{F})$ , we now set

$$Ob(\overline{\mathcal{T}}^{sfg}) := \{\mathcal{V} \in Ob(FI_{\mathcal{R}}^{sfg}) \mid \text{Every finitely generated } \mathcal{W} \subseteq \mathcal{V} \text{ lies in } \overline{\mathcal{T}}\} \tag{3.19}$$

Accordingly, we set

$$Ob(\overline{\mathcal{F}}^{sfg}) := \{\mathcal{V} \in Ob(FI_{\mathcal{R}}^{sfg}) \mid Hom(\mathcal{W}, \mathcal{V}) = 0 \text{ for every } \mathcal{W} \in \overline{\mathcal{T}}^{sfg}\} \tag{3.20}$$

We will now show that  $(\overline{\mathcal{T}}^{sfg}, \overline{\mathcal{F}}^{sfg})$  defines a torsion theory on  $FI_{\mathcal{R}}^{sfg}$ .

**Lemma 3.12.** *Suppose that  $Mod - \mathcal{R}$  is locally noetherian and let  $\tau = (\mathcal{T}, \mathcal{F})$  be a hereditary torsion theory on  $Mod - \mathcal{R}$ . For  $\mathcal{V} \in FI_{\mathcal{R}}^{sfg}$ , let  $\mathcal{I}$  be the sum of all finitely generated subobjects of  $\mathcal{V}$  which lie in  $\overline{\mathcal{T}}$ . Then,  $\mathcal{I} \in \overline{\mathcal{T}}^{sfg}$ .*

**Proof.** Let  $\mathcal{I} = \sum_{i \in I} \mathcal{I}_i$ , where  $\{\mathcal{I}_i\}_{i \in I}$  is the collection of all finitely generated subobjects of  $\mathcal{V}$  which lie in  $\overline{\mathcal{T}}$ . Then, we can express  $\mathcal{I}$  as the filtered colimit  $\mathcal{I} = \varinjlim_{J \in Fin(I)} \sum_{j \in J} \mathcal{I}_j$ , where  $Fin(I)$  is the collection of finite subsets of  $I$ . We now consider some finitely generated  $\mathcal{I}' \subseteq \mathcal{I}$ . Then, there exists some finite  $J \subseteq I$  such that  $\mathcal{I}' \subseteq \sum_{j \in J} \mathcal{I}_j$ , i.e., we have a monomorphism  $\mathcal{I}' \hookrightarrow Im \left( \bigoplus_{j \in J} \mathcal{I}_j \rightarrow \mathcal{V} \right)$ . Since  $\mathcal{T}$  is a hereditary torsion class, it follows from Proposition 3.2 that  $\overline{\mathcal{T}}$  is closed under extensions, quotients and subobjects. Since each  $\mathcal{I}_j \in \overline{\mathcal{T}}$ , it is now clear that  $\mathcal{I}' \in \overline{\mathcal{T}}$ . Hence,  $\mathcal{I} \in \overline{\mathcal{T}}^{sfg}$ .  $\square$

**Proposition 3.13.** *Suppose that  $Mod - \mathcal{R}$  is locally noetherian and let  $\tau = (\mathcal{T}, \mathcal{F})$  be a hereditary torsion theory on  $Mod - \mathcal{R}$ . Then  $\overline{\mathcal{T}}^{sfg}$  is a hereditary torsion class in  $FI_{\mathcal{R}}^{sfg}$ .*

**Proof.** From the definition in (3.19), it is clear that  $\overline{\mathcal{T}}^{sfg}$  is closed under subobjects. We take  $\mathcal{V} \in FI_{\mathcal{R}}^{sfg}$  and let  $\mathcal{T} \subseteq \mathcal{V}$  be as in the proof of Lemma 3.12. From Lemma 3.12, we know that  $\mathcal{T} \in \overline{\mathcal{T}}^{sfg}$ . It suffices therefore to show that  $\mathcal{V}/\mathcal{T} \in \overline{\mathcal{T}}^{sfg}$ .

We consider therefore a morphism  $f : \mathcal{X} \rightarrow \mathcal{V}/\mathcal{T}$  with  $\mathcal{X} \in \overline{\mathcal{T}}^{sfg}$ . Since  $FI_{\mathcal{R}}$  is locally finitely generated, we can show that  $f = 0$  by verifying that  $f' = f|_{\mathcal{X}'} : \mathcal{X}' \rightarrow \mathcal{V}/\mathcal{T}$  is zero for every finitely generated  $\mathcal{X}' \subseteq \mathcal{X}$ . First, we note that we can write  $Im(f' : \mathcal{X}' \rightarrow \mathcal{V}/\mathcal{T}) = \mathcal{Y}/\mathcal{T}$  where  $\mathcal{T} \subseteq \mathcal{Y} \subseteq \mathcal{V}$ . Since  $\mathcal{X} \in \overline{\mathcal{T}}^{sfg}$ , it follows that  $\mathcal{X}' \in \overline{\mathcal{T}}$ . Since  $\overline{\mathcal{T}}$  is closed under quotients, it follows that  $\mathcal{Y}/\mathcal{T} \in \overline{\mathcal{T}}$ .

We now consider a finitely generated subobject  $\mathcal{Z} \subseteq \mathcal{Y}$ . Since  $\overline{\mathcal{T}}$  is closed under subobjects, we get  $\mathcal{Y}/\mathcal{T} \supseteq (\mathcal{Z} + \mathcal{T})/\mathcal{T} = \mathcal{Z}/(\mathcal{Z} \cap \mathcal{T}) \in \overline{\mathcal{T}}$ . On the other hand, since  $FI_{\mathcal{R}}$  is locally noetherian, we know that  $\mathcal{Z} \cap \mathcal{T} \subseteq \mathcal{Z}$  must be finitely generated. Now since  $\mathcal{Z} \cap \mathcal{T} \subseteq \mathcal{T}$  and  $\mathcal{T} \in \overline{\mathcal{T}}^{sfg}$ , it follows that  $\mathcal{Z} \cap \mathcal{T} \in \overline{\mathcal{T}}$ . Since  $\overline{\mathcal{T}}$  is closed under extensions, the short exact sequence

$$0 \rightarrow \mathcal{Z} \cap \mathcal{T} \rightarrow \mathcal{Z} \rightarrow \mathcal{Z}/\mathcal{Z} \cap \mathcal{T} \rightarrow 0 \tag{3.21}$$

gives  $\mathcal{Z} \in \overline{\mathcal{T}}$ . From the definition of  $\mathcal{T}$ , it now follows that  $\mathcal{Z} \subseteq \mathcal{T}$ . Again since  $FI_{\mathcal{R}}$  is locally finitely generated, this gives  $\mathcal{Y} \subseteq \mathcal{T}$ . Hence,  $f' = 0$ . This proves the result.  $\square$

We will now compute an expression for the torsion submodule  $\overline{\mathcal{T}}^{sfg}(\mathcal{V})$  of  $\mathcal{V} \in FI_{\mathcal{R}}^{sfg}$ . This will be done in several steps. We denote by  $fg(\mathcal{V})$  the collection of finitely generated subobjects of  $\mathcal{V}$ .

**Proposition 3.14.** *Suppose that  $Mod - \mathcal{R}$  is locally noetherian and let  $\tau = (\mathcal{T}, \mathcal{F})$  be a hereditary torsion theory on  $Mod - \mathcal{R}$ .*

- (a) *Suppose  $\mathcal{V} \in FI_{\mathcal{R}}^{fg}$ . Then,  $\overline{\mathcal{T}}(\mathcal{V}) = \overline{\mathcal{T}}^{sfg}(\mathcal{V})$ .*
- (b) *For  $\mathcal{V} \in FI_{\mathcal{R}}^{sfg}$ , we have  $\overline{\mathcal{T}}^{sfg}(\mathcal{V}) = \varinjlim_{\mathcal{V}' \in fg(\mathcal{V})} \overline{\mathcal{T}}^{sfg}(\mathcal{V}') = \varinjlim_{\mathcal{V}' \in fg(\mathcal{V})} \overline{\mathcal{T}}(\mathcal{V}')$ .*

**Proof.** (a) It is immediate that  $\overline{\mathcal{T}}(\mathcal{V}) \subseteq \overline{\mathcal{T}}^{sfg}(\mathcal{V})$ . Since  $\mathcal{V}$  is finitely generated, we know that  $\overline{\mathcal{T}}^{sfg}(\mathcal{V})$  is finitely generated. Since  $\overline{\mathcal{T}}^{sfg}(\mathcal{V}) \in \overline{\mathcal{T}}^{sfg}$ , it follows that  $\overline{\mathcal{T}}^{sfg}(\mathcal{V}) \in \overline{\mathcal{T}}$ . Hence,  $\overline{\mathcal{T}}^{sfg}(\mathcal{V}) \subseteq \overline{\mathcal{T}}(\mathcal{V})$  and the result follows.

(b) It is clear that  $\varinjlim_{\mathcal{V}' \in fg(\mathcal{V})} \overline{\mathcal{T}}^{sfg}(\mathcal{V}') \subseteq \overline{\mathcal{T}}^{sfg}(\mathcal{V})$ . Conversely, we consider any finitely generated subobject  $\mathcal{W} \subseteq \overline{\mathcal{T}}^{sfg}(\mathcal{V})$ . Then,  $\mathcal{W} \in \overline{\mathcal{T}}$  and hence  $\mathcal{W} = \overline{\mathcal{T}}(\mathcal{W}) = \overline{\mathcal{T}}^{sfg}(\mathcal{W})$ . It follows that  $\mathcal{W} \subseteq \varinjlim_{\mathcal{V}' \in fg(\mathcal{V})} \overline{\mathcal{T}}^{sfg}(\mathcal{V}')$ .

Since  $FI_{\mathcal{R}}$  is locally finitely generated, we get  $\overline{\mathcal{T}}^{sfg}(\mathcal{V}) \subseteq \varinjlim_{\mathcal{V}' \in fg(\mathcal{V})} \overline{\mathcal{T}}^{sfg}(\mathcal{V}')$  and the result follows.  $\square$

**Lemma 3.15.** *Let  $\mathcal{B}$  be a locally noetherian Grothendieck category and suppose that  $t = (\mathbf{T}, \mathbf{F})$  is a hereditary torsion theory on  $\mathcal{B}$ . Let  $M \in \mathcal{B}$  and let  $i : N \hookrightarrow M$  be a subobject satisfying the following properties:*

- (a)  $N \in \mathbf{T}$ .
- (b) *If  $T' \in \mathbf{T}$  is a finitely generated object, any morphism  $f' : T' \rightarrow M$  factors through  $N$ .*  
*Then,  $N$  is the torsion subobject of  $M$ , i.e.,  $N = \mathbf{T}(M)$ .*

**Proof.** We consider  $T \in \mathbf{T}$  and a morphism  $f : T \rightarrow M$ . Let  $fg(T)$  be the collection of finitely generated subobjects of  $T$ . For any  $T' \in fg(T)$ , the induced map  $f' = f|_{T'} : T' \rightarrow M$  factors through some  $g' : T' \rightarrow N$  as  $f' = i \circ g'$ . Since  $i$  is a monomorphism, this  $g'$  is necessarily unique.

If  $j : T' \hookrightarrow T''$  is an inclusion with  $T', T'' \in fg(T)$ , we notice that  $i \circ g' = f' = f'' \circ j = i \circ g'' \circ j$ . Since  $i$  is a monomorphism, this gives  $g' = g'' \circ j$ . These maps  $\{g' : T' \rightarrow N\}_{T' \in fg(T)}$  together induce a



morphism from the colimit  $g : T = \varinjlim_{T' \in fg(T)} T' \rightarrow N$ . We notice that  $(i \circ g)|_{T'} = i \circ g' = f' = f|_{T'}$  for each  $T' \in fg(T)$ . Since  $T = \varinjlim_{T' \in fg(T)} T'$ , it now follows that  $i \circ g = f$ . This proves the result.  $\square$

**Proposition 3.16.** *Let  $\mathcal{B}$  be a locally noetherian Grothendieck category and suppose that  $t = (\mathbf{T}, \mathbf{F})$  is a hereditary torsion theory on  $\mathcal{B}$ . Let  $\{M_i\}_{i \in I}$  be a filtered system of objects of  $\mathcal{B}$ . Then, we have*

$$\varinjlim_{i \in I} \mathbf{T}(M_i) = \mathbf{T} \left( \varinjlim_{i \in I} M_i \right) \tag{3.22}$$

**Proof.** Since torsion classes are always closed under colimits, we know that  $\varinjlim_{i \in I} \mathbf{T}(M_i) \in \mathbf{T}$ . Considering the monomorphisms  $\mathbf{T}(M_i) \hookrightarrow M_i$ , the filtered colimit induces an inclusion  $\varinjlim_{i \in I} \mathbf{T}(M_i) \hookrightarrow \varinjlim_{i \in I} M_i$ . We now consider a finitely generated object  $T' \in \mathbf{T}$  along with a morphism  $f' : T' \rightarrow \varinjlim_{i \in I} M_i$ . Since  $\mathcal{B}$  is locally noetherian,  $T'$  is also finitely presented. It follows that  $f'$  factors through some  $g' : T' \rightarrow M_{i_0}$ . Since  $T' \in \mathbf{T}$ ,  $g'$  factors uniquely through the torsion subobject  $\mathbf{T}(M_{i_0})$ . Hence,  $f' : T' \rightarrow \varinjlim_{i \in I} M_i$  factors through  $\varinjlim_{i \in I} \mathbf{T}(M_i)$ . The result now follows from Lemma 3.15.  $\square$

**Theorem 3.17.** *Let  $\mathcal{R}$  be such that  $Mod - \mathcal{R}$  is locally noetherian. Let  $\tau = (\mathcal{T}, \mathcal{F})$  be a hereditary torsion theory on  $Mod - \mathcal{R}$ . For any  $\mathcal{V} \in FI_{\mathcal{R}}^{sfg}$ , the torsion subobject of  $\mathcal{V}$  with respect to the torsion class  $\overline{\mathcal{T}}^{sfg}$  is given by*

$$\overline{\mathcal{T}}^{sfg}(\mathcal{V})(S) := \operatorname{colim}_{a \geq 0} \lim \left( \mathcal{V}(S) \xrightarrow{\psi_a^{\mathcal{V}}(S)} (\mathbb{S}^a \mathcal{V})(S) \longleftarrow \mathcal{T}((\mathbb{S}^a \mathcal{V})(S)) \right) \tag{3.23}$$

for each finite set  $S$ .

**Proof.** From Proposition 3.14, we know that

$$\overline{\mathcal{T}}^{sfg}(\mathcal{V}) = \varinjlim_{\mathcal{V}' \in fg(\mathcal{V})} \overline{\mathcal{T}}(\mathcal{V}') \tag{3.24}$$

Since each  $\mathcal{V}' \in fg(\mathcal{V})$  lies in  $FI_{\mathcal{R}}^{fg}$ , it follows from Theorem 3.8 that

$$\overline{\mathcal{T}}(\mathcal{V}')(S) := \operatorname{colim}_{a \geq 0} \lim \left( \mathcal{V}'(S) \xrightarrow{\psi_a^{\mathcal{V}'}(S)} (\mathbb{S}^a \mathcal{V}')(S) \longleftarrow \mathcal{T}((\mathbb{S}^a \mathcal{V}')(S)) \right) \tag{3.25}$$

for each finite set  $S$ . Since  $Mod - \mathcal{R}$  is locally noetherian and  $\mathcal{V} = \varinjlim_{\mathcal{V}' \in fg(\mathcal{V})} \mathcal{V}'$ , it now follows from Proposition 3.16 that

$$\mathcal{T}((\mathbb{S}^a \mathcal{V})(S)) = \varinjlim_{\mathcal{V}' \in fg(\mathcal{V})} \mathcal{T}((\mathbb{S}^a \mathcal{V}')(S)) \tag{3.26}$$

for each  $a \geq 0$ . The result of (3.23) is now clear from (3.24), (3.25), (3.26) and the fact that filtered colimits commute with finite limits.  $\square$

#### 4. Torsion closed FI-modules

We continue with  $\mathcal{R}$  being a small preadditive category such that  $Mod - \mathcal{R}$  is locally noetherian. Given a torsion theory  $\tau = (\mathcal{T}, \mathcal{F})$  on  $Mod - \mathcal{R}$ , we have described the induced torsion class  $\overline{\mathcal{T}}$  on finitely

generated  $FI$ -modules. In this section, we will always suppose that  $\mathcal{T}$  is a hereditary torsion class. Then, from Proposition 3.2, we know that  $\overline{\mathcal{T}}$  is a hereditary torsion class on  $FI_{\mathcal{R}}^{fg}$ .

We now consider the full subcategories  $\hat{\mathcal{T}}$  and  $\{\hat{\mathcal{T}}^a\}_{a \geq 0}$  of  $FI_{\mathcal{R}}$  determined by setting

$$\begin{aligned} Ob(\hat{\mathcal{T}}) &:= \{\mathcal{V} \in Ob(FI_{\mathcal{R}}) \mid \mathcal{V}_n \in \mathcal{T} \text{ for } n \gg 0\} \\ Ob(\hat{\mathcal{T}}^a) &:= \{\mathcal{V} \in Ob(FI_{\mathcal{R}}) \mid \mathcal{V}_n \in \mathcal{T} \text{ for all } n \geq a\} \quad \forall a \geq 0 \end{aligned} \tag{4.1}$$

It is also clear that we have a filtration

$$\hat{\mathcal{T}}^0 \subseteq \hat{\mathcal{T}}^1 \subseteq \hat{\mathcal{T}}^2 \subseteq \dots \quad \hat{\mathcal{T}} = \bigcup_{a \geq 0} \hat{\mathcal{T}}^a \tag{4.2}$$

We observe that each  $\hat{\mathcal{T}}^a$  is closed under extensions, quotients, subobjects and coproducts, making it a hereditary torsion class in the category  $FI_{\mathcal{R}}$  (see, for instance, [4, § I.1]). However, we notice that  $\hat{\mathcal{T}}$  need not be a torsion class in  $FI_{\mathcal{R}}$ , because it may not contain arbitrary coproducts. In fact,  $\hat{\mathcal{T}}$  is only a Serre subcategory, i.e., it is closed under extensions, subobjects and quotients. The purpose of this section is to construct functors from  $FI_{\mathcal{R}}$  to  $\hat{\mathcal{T}}^a$ -closed objects and to  $\hat{\mathcal{T}}$ -closed objects of  $FI_{\mathcal{R}}$ . For this, we will first develop some general results on locally noetherian Grothendieck categories.

**Definition 4.1.** (See, for instance, [19, Definition III.2.2].) Let  $\mathcal{A}$  be a Grothendieck category and let  $\mathcal{C}$  be a Serre subcategory. Then:

- (1) A morphism  $u : A \rightarrow B$  in  $\mathcal{A}$  is said to be a  $\mathcal{C}$ -isomorphism if both  $Ker(u)$  and  $Coker(u)$  lie in  $\mathcal{C}$ .
- (2) An object  $L$  in  $\mathcal{A}$  is said to be  $\mathcal{C}$ -closed if for every  $\mathcal{C}$ -isomorphism  $u : A \rightarrow B$  in  $\mathcal{A}$ , the induced morphism  $Hom(u, L) : Hom(B, L) \rightarrow Hom(A, L)$  is an isomorphism.
- (3) A morphism  $f : A \rightarrow A_{\mathcal{C}}$  in  $\mathcal{A}$  is said to be a  $\mathcal{C}$ -envelope if  $f$  is a  $\mathcal{C}$ -isomorphism and  $A_{\mathcal{C}}$  is  $\mathcal{C}$ -closed.

Let  $\mathcal{A}$  be a Grothendieck category. From now onwards, for  $X \in \mathcal{A}$ , we will denote by  $fg(X)$  the set of its finitely generated subobjects.

**Lemma 4.2.** Let  $\mathcal{A}$  be a locally noetherian Grothendieck category and  $\mathcal{C}$  be a Serre subcategory. Suppose that an object  $L \in \mathcal{A}$  has the following property: for any  $\mathcal{C}$ -isomorphism  $u' : A' \rightarrow B'$  with  $A', B'$  finitely generated, the induced morphism  $Hom(u', L) : Hom(B', L) \rightarrow Hom(A', L)$  is an isomorphism.

Then, for any  $\mathcal{C}$ -isomorphism  $u : A \rightarrow B$  that is an epimorphism in  $\mathcal{A}$ , the induced morphism  $Hom(u, L) : Hom(B, L) \rightarrow Hom(A, L)$  is an isomorphism.

**Proof.** We consider a  $\mathcal{C}$ -isomorphism  $u : A \rightarrow B$  that is an epimorphism in  $\mathcal{A}$ . By definition,  $Coker(u) = 0$  and  $Ker(u) \in \mathcal{C}$ . Let  $A' \subseteq A$  be a finitely generated subobject and let  $u' : A' \rightarrow B$  denote the restriction of  $u$  to  $A'$ . Set  $B' := Im(u') \subseteq B$ . Since  $A'$  is finitely generated, so is its quotient  $B'$ . Then,  $u' : A' \rightarrow B'$  satisfies  $Coker(u') = 0$  and  $Ker(u') \subseteq Ker(u) \in \mathcal{C}$ . Since  $\mathcal{C}$  is a Serre subcategory, we get  $Ker(u') \in \mathcal{C}$ . It follows that  $u'$  is a  $\mathcal{C}$ -isomorphism.

Using the given property of  $L$ , we now obtain that the induced morphism  $Hom(u', L) : Hom(B', L) \rightarrow Hom(A', L)$  is an isomorphism. Since  $\mathcal{A}$  is locally finitely generated, we know that  $A$  is the filtered colimit over all  $A' \in fg(A)$ . Since  $u$  is an epimorphism, we know that  $B$  is the filtered colimit over the corresponding objects  $\{B' = Im(u|_{A'} : A' \rightarrow B)\}_{A' \in fg(A)}$ . It follows that

$$Hom(u, L) : Hom(B, L) = \varprojlim_{A' \in fg(A)} Hom(B', L) \xrightarrow{\cong} \varprojlim_{A' \in fg(A)} Hom(A', L) = Hom(A, L) \tag{4.3}$$

is an isomorphism. This proves the result.  $\square$

**Lemma 4.3.** *Let  $\mathcal{A}$  be a locally noetherian Grothendieck category and  $\mathcal{C}$  be a Serre subcategory. Suppose that an object  $L \in \mathcal{A}$  has the following property: for any  $\mathcal{C}$ -isomorphism  $u' : A' \rightarrow B'$  with  $A', B'$  finitely generated, the induced morphism  $Hom(u', L) : Hom(B', L) \rightarrow Hom(A', L)$  is an isomorphism.*

*Then, for any  $\mathcal{C}$ -isomorphism  $u : A \rightarrow B$  that is a monomorphism in  $\mathcal{A}$ , the induced morphism  $Hom(u, L) : Hom(B, L) \rightarrow Hom(A, L)$  is an isomorphism.*

**Proof.** We consider a  $\mathcal{C}$ -isomorphism  $u : A \rightarrow B$  that is a monomorphism in  $\mathcal{A}$ . Then, by definition,  $Ker(u) = 0$  and  $Coker(u) \in \mathcal{C}$ . Let  $B' \subseteq B$  be a finitely generated subobject and let  $u' : A' := A \times_B B' \rightarrow B'$  be the pullback of  $u$  along  $B' \hookrightarrow B$ . Clearly,  $Ker(u') = 0$ . Since  $\mathcal{A}$  is locally noetherian and  $A' \subseteq B'$ , it follows that  $A'$  is finitely generated.

We now note that all the squares in the following diagram are pullback squares.

$$\begin{array}{ccccc}
 A' & \xrightarrow{u'} & B' & \longrightarrow & 0 \\
 \downarrow & & \downarrow & & \downarrow \\
 A & \xrightarrow{u} & B & \longrightarrow & B/B'
 \end{array} \tag{4.4}$$

It follows that  $A' = Ker(A \xrightarrow{u} B \rightarrow B/B')$  and hence we have a monomorphism  $A/A' \rightarrow B/B'$ . A simple application of Snake Lemma to the following diagram

$$\begin{array}{ccccccc}
 0 & \longrightarrow & A' & \longrightarrow & A & \longrightarrow & A/A' \longrightarrow 0 \\
 & & u' \downarrow & & u \downarrow & & \downarrow \\
 0 & \longrightarrow & B' & \longrightarrow & B & \longrightarrow & B/B' \longrightarrow 0
 \end{array} \tag{4.5}$$

now gives us the exact sequence  $0 \rightarrow Coker(u') \rightarrow Coker(u)$ . Since  $\mathcal{C}$  is a Serre subcategory, we now get  $Coker(u') \in \mathcal{C}$ . Using the given property of  $L$ , we now obtain that the induced morphism  $Hom(u', L) : Hom(B', L) \rightarrow Hom(A', L)$  is an isomorphism. Since  $\mathcal{A}$  is locally finitely generated, we know that  $B$  is the filtered colimit over all  $B' \in fg(B)$ . We notice that  $A$  is the filtered colimit over the corresponding objects  $\{A' = A \times_B B'\}_{B' \in fg(B)}$ . It follows that

$$Hom(u, L) : Hom(B, L) = \varinjlim_{B' \in fg(B)} Hom(B', L) \xrightarrow{\cong} \varinjlim_{B' \in fg(B)} Hom(A', L) = Hom(A, L) \tag{4.6}$$

is an isomorphism. This proves the result.  $\square$

**Proposition 4.4.** *Let  $\mathcal{A}$  be a locally noetherian Grothendieck category and  $\mathcal{C}$  be a Serre subcategory. For an object  $L \in \mathcal{A}$ , the following are equivalent.*

(1) *The object  $L$  is  $\mathcal{C}$ -closed, i.e., for any  $\mathcal{C}$ -isomorphism  $u : A \rightarrow B$ , the induced morphism  $Hom(u, L) : Hom(B, L) \rightarrow Hom(A, L)$  is an isomorphism.*

(2) *For any  $\mathcal{C}$ -isomorphism  $u' : A' \rightarrow B'$  with  $A', B'$  finitely generated, the induced morphism  $Hom(u', L) : Hom(B', L) \rightarrow Hom(A', L)$  is an isomorphism.*

**Proof.** We only need to show that (2)  $\Rightarrow$  (1). Let  $u : A \rightarrow B$  be a  $\mathcal{C}$ -isomorphism. Then, we can factor  $u$  uniquely as  $A \xrightarrow{f} C \xrightarrow{g} B$  where  $f$  is an epimorphism and  $g$  is a monomorphism. We notice that

$$Ker(f) = Ker(u) \in \mathcal{C} \quad Coker(f) = 0 \quad Ker(g) = 0 \quad Coker(g) = Coker(u) \in \mathcal{C} \tag{4.7}$$

and hence both  $f$  and  $g$  are  $\mathcal{C}$ -isomorphisms. The result is now clear from Lemma 4.2 and Lemma 4.3.  $\square$

We now return to  $FI$  modules over  $\mathcal{R}$  along with a hereditary torsion theory  $\tau = (\mathcal{T}, \mathcal{F})$  on  $Mod - \mathcal{R}$ . Accordingly, there is a functor  $\mathbb{E}_\tau : Mod - \mathcal{R} \rightarrow Mod - \mathcal{R}$  that takes any  $V \in Mod - \mathcal{R}$  to its torsion envelope  $\mathbb{E}_\tau(V)$ . We refer the reader to [16, Theorem 2.5] for the explicit construction of this functor. Since  $Mod - \mathcal{R}$  is a locally finitely presented Grothendieck category, we note that hereditary torsion classes in  $Mod - \mathcal{R}$  are the same as localizing subcategories of  $Mod - \mathcal{R}$  (see, for instance, [6, Theorem 1.13.5]).

By abuse of notation, we will also denote by  $\mathbb{E}_\tau$  the functor given by

$$\mathbb{E}_\tau : FI_{\mathcal{R}} \rightarrow FI_{\mathcal{R}} \quad \mathbb{E}_\tau(\mathcal{V})(S) := \mathbb{E}_\tau(\mathcal{V}(S)) \tag{4.8}$$

for any  $\mathcal{V} \in FI_{\mathcal{R}}$  and any finite set  $S$ . The canonical morphisms  $\mathcal{V}(S) \rightarrow \mathbb{E}_\tau(\mathcal{V}(S))$  together induce a morphism  $i_\tau(\mathcal{V}) : \mathcal{V} \rightarrow \mathbb{E}_\tau(\mathcal{V})$  in  $FI_{\mathcal{R}}$ . We also observe that from (4.8) it is clear that

$$\mathbb{E}_\tau \mathbb{S}^a(\mathcal{V}) = \mathbb{S}^a \mathbb{E}_\tau(\mathcal{V}) \quad \forall a \geq 0 \tag{4.9}$$

We will denote by  $Cl(\mathcal{T})$  (resp.  $Cl(\hat{\mathcal{T}}^a)$ ,  $Cl(\hat{\mathcal{T}})$ ) the full subcategory of  $Mod - \mathcal{R}$  (resp.  $FI_{\mathcal{R}}$ ) consisting of closed objects with respect to the Serre subcategory  $\mathcal{T} \subseteq Mod - \mathcal{R}$  (resp.  $\hat{\mathcal{T}}^a, \hat{\mathcal{T}} \subseteq FI_{\mathcal{R}}$ ).

**Lemma 4.5.** *Let  $\mathcal{L} \in FI_{\mathcal{R}}$  be such that  $\mathcal{L}(S)$  is  $\mathcal{T}$ -closed for each finite set  $S$ . Then,  $\mathcal{L}$  is  $\hat{\mathcal{T}}^0$ -closed.*

**Proof.** Let  $u : \mathcal{A} \rightarrow \mathcal{B}$  be a  $\hat{\mathcal{T}}^0$ -isomorphism in  $FI_{\mathcal{R}}$ . Then, by definition, we have  $Ker(u), Coker(u) \in \hat{\mathcal{T}}^0$ , i.e., for each finite set  $S$ , we must have  $Ker(u(S)), Coker(u(S)) \in \mathcal{T}$ . We consider a morphism  $f : \mathcal{B} \rightarrow \mathcal{L}$  in  $FI_{\mathcal{R}}$ . If  $f \circ u = 0$ , it follows that  $f(S) \circ u(S) = 0$  for each  $S \in FI$ . Since each  $\mathcal{L}(S)$  is  $\mathcal{T}$ -closed, we know that  $Hom(u(S), \mathcal{L}(S)) : Hom(\mathcal{B}(S), \mathcal{L}(S)) \rightarrow Hom(\mathcal{A}(S), \mathcal{L}(S))$  is an isomorphism. This gives  $f(S) = 0$  for each  $S \in FI$ , i.e.,  $f = 0$ .

On the other hand, consider a morphism  $g : \mathcal{A} \rightarrow \mathcal{L}$  in  $FI_{\mathcal{R}}$ . Each  $\mathcal{L}(S)$  is  $\mathcal{T}$ -closed, which gives us a unique morphism  $f(S) : \mathcal{B}(S) \rightarrow \mathcal{L}(S)$  such that  $g(S) = f(S) \circ u(S)$ . We claim that  $\{f(S)\}_{S \in FI}$  gives a morphism  $f : \mathcal{B} \rightarrow \mathcal{L}$ , i.e., for any  $\phi : S \rightarrow T$  in  $FI$ , we have  $\mathcal{L}(\phi) \circ f(S) = f(T) \circ \mathcal{B}(\phi) : \mathcal{B}(S) \rightarrow \mathcal{L}(T)$ . For this, we notice that

$$f(T) \circ \mathcal{B}(\phi) \circ u(S) = f(T) \circ u(T) \circ \mathcal{A}(\phi) = \mathcal{L}(\phi) \circ f(S) \circ u(S) : \mathcal{A}(S) \rightarrow \mathcal{L}(T) \tag{4.10}$$

Since  $u(S)$  is a  $\mathcal{T}$ -isomorphism and  $\mathcal{L}(T)$  is  $\mathcal{T}$ -closed, we must have an isomorphism  $Hom(\mathcal{B}(S), \mathcal{L}(T)) \rightarrow Hom(\mathcal{A}(S), \mathcal{L}(T))$ . From (4.10), it is now clear that  $\mathcal{L}(\phi) \circ f(S) = f(T) \circ \mathcal{B}(\phi)$ .

We have now shown that the induced morphism  $FI_{\mathcal{R}}(u, \mathcal{L}) : FI_{\mathcal{R}}(\mathcal{B}, \mathcal{L}) \rightarrow FI_{\mathcal{R}}(\mathcal{A}, \mathcal{L})$  is both a monomorphism and an epimorphism, i.e., an isomorphism. This proves the result.  $\square$

**Proposition 4.6.** *Let  $\mathcal{L} \in FI_{\mathcal{R}}$ . Then,  $\mathcal{L}$  is  $\hat{\mathcal{T}}^0$ -closed if and only if  $\mathcal{L}(S)$  is  $\mathcal{T}$ -closed for each finite set  $S$ .*

**Proof.** Since  $\mathcal{T}$  is a localizing subcategory of  $Mod - \mathcal{R}$ , the functor  $\mathbb{E}_\tau : Mod - \mathcal{R} \rightarrow Cl(\mathcal{T})$  is left adjoint to the inclusion  $Cl(\mathcal{T}) \rightarrow Mod - \mathcal{R}$ . The “unit” of this adjunction gives a canonical morphism  $V \rightarrow \mathbb{E}_\tau(V)$  for each  $V \in Mod - \mathcal{R}$ . Taken together, such maps induce a canonical morphism  $i_\tau : \mathcal{L} \rightarrow \mathbb{E}_\tau(\mathcal{L})$  for each  $\mathcal{L} \in FI_{\mathcal{R}}$ .

From the construction in (4.8), it is clear that  $i_\tau(S) : \mathcal{L}(S) \rightarrow \mathbb{E}_\tau(\mathcal{L})(S)$  is a  $\mathcal{T}$ -isomorphism in  $Mod - \mathcal{R}$  for each finite set  $S$ . In other words,  $Ker(i_\tau(S)), Coker(i_\tau(S)) \in \mathcal{T}$ . Hence,  $Ker(i_\tau), Coker(i_\tau) \in \hat{\mathcal{T}}^0$  and it follows that  $i_\tau$  is a  $\hat{\mathcal{T}}^0$ -isomorphism. From Lemma 4.5 and the definition in (4.8), it is clear that  $\mathbb{E}_\tau(\mathcal{L})$  is  $\hat{\mathcal{T}}^0$ -closed. By Definition 4.1, it follows that  $i_\tau : \mathcal{L} \rightarrow \mathbb{E}_\tau(\mathcal{L})$  is a  $\hat{\mathcal{T}}^0$ -envelope for  $\mathcal{L}$ .

If we now suppose that  $\mathcal{L}$  is  $\hat{\mathcal{T}}^0$ -closed and  $\hat{\mathcal{T}}^0$  is a hereditary torsion class, it follows from the uniqueness of the  $\hat{\mathcal{T}}^0$ -envelope that  $i_\tau : \mathcal{L} \rightarrow \mathbb{E}_\tau(\mathcal{L})$  is an isomorphism. In particular, it follows that  $\mathcal{L}(S) \cong \mathbb{E}_\tau(\mathcal{L}(S))$  is  $\mathcal{T}$ -closed in  $Mod - \mathcal{R}$  for each finite set  $S$ . This proves the “only if” part of the result. The “if part” is clear from Lemma 4.5.  $\square$

From the definition of the subcategories  $\{\hat{\mathcal{T}}^a\}_{a \geq 0}$ , it is clear that we have a descending filtration

$$Cl(\hat{\mathcal{T}}^0) \supseteq Cl(\hat{\mathcal{T}}^1) \supseteq \dots \quad \dots \supseteq Cl(\hat{\mathcal{T}}^a) \supseteq Cl(\hat{\mathcal{T}}^{a+1}) \supseteq \dots \tag{4.11}$$

In order to obtain functors going in the other direction, we will need to use the right adjoint of the shift functor  $\mathbb{S} := \mathbb{S}^1$ .

**Lemma 4.7.** *For each  $a \geq 0$ , the functor  $\mathbb{S}^a : FI_{\mathcal{R}} \rightarrow FI_{\mathcal{R}}$  has a right adjoint  $\mathbb{T}^a : FI_{\mathcal{R}} \rightarrow FI_{\mathcal{R}}$ .*

**Proof.** Since  $FI_{\mathcal{R}}$  is a Grothendieck category and  $\mathbb{S} = \mathbb{S}^1 : FI_{\mathcal{R}} \rightarrow FI_{\mathcal{R}}$  preserves colimits, it follows (see, for instance, [20, Theorem 8.3.27]) that it must have a right adjoint  $\mathbb{T} = \mathbb{T}^1 : FI_{\mathcal{R}} \rightarrow FI_{\mathcal{R}}$ . Then, for each  $a \geq 0$ ,  $\mathbb{T}^a$  is a right adjoint of  $\mathbb{S}^a$ .  $\square$

**Lemma 4.8.** *Let  $k \geq 0$  and let  $u : \mathcal{A} \rightarrow \mathcal{B}$  be a  $\hat{\mathcal{T}}^k$ -isomorphism. Then, for each  $0 \leq a \leq k$ , the induced morphism  $\mathbb{S}^a(u) : \mathbb{S}^a(\mathcal{A}) \rightarrow \mathbb{S}^a(\mathcal{B})$  is a  $\hat{\mathcal{T}}^{k-a}$ -isomorphism.*

**Proof.** For any finite set  $S$ , we know that

$$\begin{aligned} Ker(\mathbb{S}^a(u))(S) &= \mathbb{S}^a(Ker(u))(S) = Ker(u)(S \sqcup [-a]) \\ Coker(\mathbb{S}^a(u))(S) &= \mathbb{S}^a(Coker(u))(S) = Coker(u)(S \sqcup [-a]) \end{aligned} \tag{4.12}$$

Since  $u : \mathcal{A} \rightarrow \mathcal{B}$  is a  $\hat{\mathcal{T}}^k$ -isomorphism, it is clear from (4.12) that when  $|S| \geq k - a$ , both  $Ker(\mathbb{S}^a(u))(S)$ ,  $Coker(\mathbb{S}^a(u))(S) \in \mathcal{T}$ . The result follows.  $\square$

Before we proceed further, we record here the following observation about the functor  $\mathbb{T}$ .

**Proposition 4.9.** *Let  $a, d \geq 0$  and  $r \in \mathcal{R}$ . Then, for any  $\mathcal{V} \in FI_{\mathcal{R}}$ ,  $\mathcal{V}_d(r)$  is a direct summand of  $\mathbb{T}^a(\mathcal{V})_d(r)$ .*

**Proof.** From Lemma 2.3, we know that  $\mathbb{T}^a(\mathcal{V})_d(r) = FI_{\mathcal{R}}(d\mathcal{M}_r, \mathbb{T}^a(\mathcal{V}))$  for any  $d \geq 0$  and  $r \in \mathcal{R}$ . Using the adjoint pair  $(\mathbb{S}^a, \mathbb{T}^a)$  and Corollary 3.4, we obtain

$$\begin{aligned} \mathbb{T}^a(\mathcal{V})_d(r) &= FI_{\mathcal{R}}(d\mathcal{M}_r, \mathbb{T}^a(\mathcal{V})) = FI_{\mathcal{R}}(\mathbb{S}^a(d\mathcal{M}_r), \mathcal{V}) = FI_{\mathcal{R}}(d\mathcal{M}_r \oplus d\mathcal{N}_{r,a}, \mathcal{V}) \\ &= \mathcal{V}_d(r) \oplus FI_{\mathcal{R}}(d\mathcal{N}_{r,a}, \mathcal{V}) \quad \square \end{aligned}$$

**Proposition 4.10.** *For any  $a, b \geq 0$ , the right adjoint  $\mathbb{T}^a : FI_{\mathcal{R}} \rightarrow FI_{\mathcal{R}}$  restricts to a functor  $\mathbb{T}^a : Cl(\hat{\mathcal{T}}^b) \rightarrow Cl(\hat{\mathcal{T}}^{a+b})$ .*

**Proof.** We consider some  $\mathcal{L} \in Cl(\hat{\mathcal{T}}^b)$  and  $u : \mathcal{A} \rightarrow \mathcal{B}$  in  $FI_{\mathcal{R}}$  that is a  $\hat{\mathcal{T}}^{a+b}$ -isomorphism. We consider the commutative diagram:

$$\begin{CD} FI_{\mathcal{R}}(\mathcal{B}, \mathbb{T}^a \mathcal{L}) @>{FI_{\mathcal{R}}(u, \mathbb{T}^a \mathcal{L})}>> FI_{\mathcal{R}}(\mathcal{A}, \mathbb{T}^a \mathcal{L}) \\ @V{\cong}VV @VV{\cong}V \\ FI_{\mathcal{R}}(\mathbb{S}^a(\mathcal{B}), \mathcal{L}) @>{FI_{\mathcal{R}}(\mathbb{S}^a(u), \mathcal{L})}>> FI_{\mathcal{R}}(\mathbb{S}^a(\mathcal{A}), \mathcal{L}) \end{CD} \tag{4.13}$$

Here, it follows from Lemma 4.8 that the lower horizontal arrow is an isomorphism. This proves the result.  $\square$

We can now give functors that explicitly construct objects in  $Cl(\hat{\mathcal{T}}^k)$ .

**Proposition 4.11.** Let  $\tau = (\mathcal{T}, \mathcal{F})$  be a hereditary torsion theory on  $Mod - \mathcal{R}$ . Then, we have functors

$$\mathbb{L}_\tau^k := \mathbb{T}^k \circ \mathbb{S}^k \circ \mathbb{E}_\tau : FI_{\mathcal{R}} \longrightarrow Cl(\hat{\mathcal{T}}^k) \quad \forall k \geq 0 \tag{4.14}$$

Additionally, there are canonical morphisms of functors  $l_\tau^k : Id \longrightarrow \mathbb{L}_\tau^k$  such that  $l_\tau^{k+1} = c_\tau^k \circ l_\tau^k$ , where  $c_\tau^k : \mathbb{L}_\tau^k = \mathbb{T}^k \circ \mathbb{S}^k \circ \mathbb{E}_\tau \longrightarrow \mathbb{L}_\tau^{k+1} = \mathbb{T}^k \circ (\mathbb{T} \circ \mathbb{S}) \circ \mathbb{S}^k \circ \mathbb{E}_\tau$  is induced by the counit corresponding to the adjoint pair  $(\mathbb{S}, \mathbb{T})$ .

**Proof.** For any  $\mathcal{V} \in FI_{\mathcal{R}}$ , it is clear from Proposition 4.6 that  $\mathbb{E}_\tau(\mathcal{V}) \in Cl(\hat{\mathcal{T}}^0)$ . From (4.9), it is now clear that  $\mathbb{S}^k(\mathbb{E}_\tau(\mathcal{V})) = \mathbb{E}_\tau(\mathbb{S}^k(\mathcal{V})) \in Cl(\hat{\mathcal{T}}^0)$ . It now follows from Proposition 4.10 that  $\mathbb{L}_\tau^k(\mathcal{V}) = \mathbb{T}^k \circ \mathbb{S}^k \circ \mathbb{E}_\tau(\mathcal{V}) \in Cl(\hat{\mathcal{T}}^k)$ . We recall that we have a canonical morphism  $i_\tau(\mathcal{V}') : \mathcal{V}' \longrightarrow \mathbb{E}_\tau(\mathcal{V}')$  for each  $\mathcal{V}' \in FI_{\mathcal{R}}$ . Using the adjunctions  $(\mathbb{S}, \mathbb{T})$ ,  $(\mathbb{S}^k, \mathbb{T}^k)$  and  $(\mathbb{S}^{k+1}, \mathbb{T}^{k+1})$ , we now have a commutative diagram

$$\begin{array}{ccc} \mathcal{V} & & \\ \downarrow l_\tau^k(\mathcal{V}) & \searrow l_\tau^{k+1}(\mathcal{V}) & \\ \mathbb{L}_\tau^k(\mathcal{V}) = \mathbb{T}^k \mathbb{S}^k \mathbb{E}_\tau(\mathcal{V}) & \xrightarrow{c_\tau^k(\mathcal{V})} & \mathbb{T}^k(\mathbb{T} \circ \mathbb{S}) \mathbb{S}^k \mathbb{E}_\tau(\mathcal{V}) = \mathbb{L}_\tau^{k+1}(\mathcal{V}) \end{array}$$

The result follows.  $\square$

We are now ready to construct a functor that gives objects that are closed with respect to  $\hat{\mathcal{T}}$ .

**Theorem 4.12.** Let  $Mod - \mathcal{R}$  be locally noetherian. Let  $\tau = (\mathcal{T}, \mathcal{F})$  be a hereditary torsion theory on  $Mod - \mathcal{R}$ . Then, we have a functor  $\mathbb{L}_\tau : FI_{\mathcal{R}} \longrightarrow Cl(\hat{\mathcal{T}})$  and a canonical morphism

$$l_\tau(\mathcal{V}) : \mathcal{V} \longrightarrow \mathbb{L}_\tau(\mathcal{V}) := \varinjlim_{k \geq 0} \mathbb{L}_\tau^k(\mathcal{V}) \tag{4.15}$$

for each  $\mathcal{V} \in FI_{\mathcal{R}}$ .

**Proof.** For  $\mathcal{V} \in FI_{\mathcal{R}}$ , it follows from Proposition 4.11 that the morphisms  $l_\tau^k(\mathcal{V}) : \mathcal{V} \longrightarrow \mathbb{L}_\tau^k(\mathcal{V})$  combine to give a morphism  $l_\tau(\mathcal{V}) : \mathcal{V} \longrightarrow \mathbb{L}_\tau(\mathcal{V}) = \varinjlim_{k \geq 0} \mathbb{L}_\tau^k(\mathcal{V})$ . We need to check that  $\mathbb{L}_\tau(\mathcal{V})$  is  $\hat{\mathcal{T}}$ -closed. For

this, we will show that for any  $\hat{\mathcal{T}}$ -isomorphism  $u : \mathcal{A} \longrightarrow \mathcal{B}$ , the induced morphism  $FI_{\mathcal{R}}(u, \mathbb{L}_\tau(\mathcal{V})) : FI_{\mathcal{R}}(\mathcal{B}, \mathbb{L}_\tau(\mathcal{V})) \longrightarrow FI_{\mathcal{R}}(\mathcal{A}, \mathbb{L}_\tau(\mathcal{V}))$  is an isomorphism.

Using Proposition 4.4, we may restrict ourselves to the case where  $\mathcal{A}, \mathcal{B}$  are finitely generated. Since  $FI_{\mathcal{R}}$  is a locally noetherian category, it follows that  $\mathcal{A}, \mathcal{B}$  are also finitely presented, i.e., the functors  $FI_{\mathcal{R}}(\mathcal{A}, \_)$ ,  $FI_{\mathcal{R}}(\mathcal{B}, \_)$  preserve filtered colimits. We now consider the morphism

$$FI_{\mathcal{R}}(u, \mathbb{L}_\tau(\mathcal{V})) : FI_{\mathcal{R}}(\mathcal{B}, \mathbb{L}_\tau(\mathcal{V})) = \varinjlim_{k \geq 0} FI_{\mathcal{R}}(\mathcal{B}, \mathbb{L}_\tau^k(\mathcal{V})) \longrightarrow \varinjlim_{k \geq 0} FI_{\mathcal{R}}(\mathcal{A}, \mathbb{L}_\tau^k(\mathcal{V})) = FI_{\mathcal{R}}(\mathcal{A}, \mathbb{L}_\tau(\mathcal{V}))$$

Since  $u : \mathcal{A} \longrightarrow \mathcal{B}$  is a  $\hat{\mathcal{T}}$ -isomorphism, we can choose  $N$  large enough so that  $Ker(u)(S), Coker(u)(S) \in \mathcal{T}$  for finite sets  $S$  of cardinality  $\geq N$ . Hence,  $u : \mathcal{A} \longrightarrow \mathcal{B}$  is a  $\hat{\mathcal{T}}^k$ -isomorphism for each  $k \geq N$ . Since  $\mathbb{L}_\tau^k(\mathcal{V})$  is  $\hat{\mathcal{T}}^k$ -closed, it follows that  $FI_{\mathcal{R}}(\mathcal{B}, \mathbb{L}_\tau^k(\mathcal{V})) \longrightarrow FI_{\mathcal{R}}(\mathcal{A}, \mathbb{L}_\tau^k(\mathcal{V}))$  is an isomorphism for each  $k \geq N$ . The result is now clear.  $\square$

### 5. The torsion class $\tilde{\mathcal{T}}$ and its closed objects

We continue with  $Mod-\mathcal{R}$  being a locally noetherian category and  $\tau = (\mathcal{T}, \mathcal{F})$  being a hereditary torsion theory on  $Mod-\mathcal{R}$ . In Section 3, we used this torsion theory to construct a torsion class  $\overline{\mathcal{T}}$  in the category  $FI_{\mathcal{R}}^{fg}$  of finitely generated modules. In Section 4, we considered the Serre subcategory  $\hat{\mathcal{T}} \subseteq FI_{\mathcal{R}}$  which was constructed so that  $\hat{\mathcal{T}} \cap FI_{\mathcal{R}}^{fg} = \overline{\mathcal{T}}$ .

Additionally, in Section 3, we had also used the torsion theory  $\tau = (\mathcal{T}, \mathcal{F})$  to construct a torsion class  $\overline{\mathcal{T}}^{sfg}$  in the category  $FI_{\mathcal{R}}^{sfg}$  of shift finitely generated modules. As such, in this section, we will define a full subcategory  $\tilde{\mathcal{T}} \subseteq FI_{\mathcal{R}}$  such that  $\tilde{\mathcal{T}} \cap FI_{\mathcal{R}}^{sfg} = \overline{\mathcal{T}}^{sfg}$ . For this, we define:

$$Ob(\tilde{\mathcal{T}}) := \{\mathcal{V} \in Ob(FI_{\mathcal{R}}) \mid \text{Every finitely generated } \mathcal{W} \subseteq \mathcal{V} \text{ lies in } \overline{\mathcal{T}}\} \tag{5.1}$$

The subcategory  $\hat{\mathcal{T}}$  considered in Section 4 was a Serre subcategory. Hence, we would expect that its counterpart  $\tilde{\mathcal{T}}$  defined in (5.1) is also a Serre subcategory. We will now show that  $\tilde{\mathcal{T}}$  satisfies an even stronger property, i.e., it is a hereditary torsion class.

**Lemma 5.1.** *The subcategory  $\tilde{\mathcal{T}}$  is closed under extensions.*

**Proof.** Suppose that we have a short exact sequence

$$0 \longrightarrow \mathcal{V}' \longrightarrow \mathcal{V} \longrightarrow \mathcal{V}'' \longrightarrow 0$$

in  $FI_{\mathcal{R}}$  with  $\mathcal{V}', \mathcal{V}'' \in \tilde{\mathcal{T}}$ . We consider a finitely generated subobject  $\mathcal{W} \subseteq \mathcal{V}$ . This gives two short exact sequences fitting into the commutative diagram

$$\begin{array}{ccccccc} 0 & \longrightarrow & \mathcal{W} \cap \mathcal{V}' & \longrightarrow & \mathcal{W} & \longrightarrow & \mathcal{W}/(\mathcal{W} \cap \mathcal{V}') = (\mathcal{W} + \mathcal{V}')/\mathcal{V}' & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & \mathcal{V}' & \longrightarrow & \mathcal{V} & \longrightarrow & \mathcal{V}'' = \mathcal{V}/\mathcal{V}' & \longrightarrow & 0 \end{array} \tag{5.2}$$

Since  $\mathcal{W}$  is finitely generated, so is the subobject  $\mathcal{W} \cap \mathcal{V}'$  and the quotient  $\mathcal{W}/(\mathcal{W} \cap \mathcal{V}')$ . The vertical maps in (5.2) are all monomorphisms. Since  $\mathcal{V}', \mathcal{V}'' \in \tilde{\mathcal{T}}$ , we can find  $N$  large enough so that  $(\mathcal{W} \cap \mathcal{V}')_n, (\mathcal{W}/(\mathcal{W} \cap \mathcal{V}'))_n \in \mathcal{T}$  for  $n \geq N$ . Then,  $\mathcal{W}_n \in \mathcal{T}$  for all  $n \geq N$ . Hence,  $\mathcal{V} \in \tilde{\mathcal{T}}$ .  $\square$

**Lemma 5.2.** *The subcategory  $\tilde{\mathcal{T}}$  contains all coproducts.*

**Proof.** Let  $\{\mathcal{V}_i\}_{i \in I}$  be a family of objects in  $\tilde{\mathcal{T}}$ . Using Lemma 5.1, we know that every finite direct sum of objects from  $\{\mathcal{V}_i\}_{i \in I}$  lies in  $\tilde{\mathcal{T}}$ . We note that  $\bigoplus_{i \in I} \mathcal{V}_i$  is equal to the filtered colimit of  $\bigoplus_{j \in J} \mathcal{V}_j$  taken over all finite subsets  $J \subseteq I$ . Then, if  $\mathcal{W} \subseteq \bigoplus_{i \in I} \mathcal{V}_i$  is a finitely generated object, we can find some finite subset  $J \subseteq I$  such that  $\mathcal{W} \subseteq \bigoplus_{j \in J} \mathcal{V}_j$ . It follows that  $\mathcal{W}_n \in \mathcal{T}$  for  $n \gg 0$ . This proves the result.  $\square$

**Lemma 5.3.** *The subcategory  $\tilde{\mathcal{T}}$  is closed under quotients.*

**Proof.** We consider an epimorphism  $f : \mathcal{V} \longrightarrow \mathcal{W}$  in  $FI_{\mathcal{R}}$  with  $\mathcal{V} \in \tilde{\mathcal{T}}$ . We consider a finitely generated subobject  $\mathcal{W}' \subseteq \mathcal{W}$ . The finitely generated subobjects of  $f^{-1}(\mathcal{W}') \subseteq \mathcal{V}$  form a filtered system and hence their images in  $\mathcal{W}'$  form a filtered system of subobjects whose union is  $\mathcal{W}'$ . As such, we can find some finitely generated subobject  $\mathcal{V}' \subseteq f^{-1}(\mathcal{W}') \subseteq \mathcal{V}$  such that  $f|_{\mathcal{V}'} : \mathcal{V}' \longrightarrow \mathcal{W}'$  is an epimorphism. Since  $\mathcal{V} \in \tilde{\mathcal{T}}$ , we know that  $\mathcal{V}'_n \in \mathcal{T}$  for  $n \gg 0$ . Then,  $\mathcal{W}'_n \in \mathcal{T}$  for  $n \gg 0$ . This proves the result.  $\square$



**Proposition 5.4.** *Let  $Mod - \mathcal{R}$  be a locally noetherian category and  $\tau = (\mathcal{T}, \mathcal{F})$  a hereditary torsion theory on  $Mod - \mathcal{R}$ . Then, the full subcategory  $\tilde{\mathcal{T}} \subseteq FI_{\mathcal{R}}$  defined by setting*

$$Ob(\tilde{\mathcal{T}}) := \{ \mathcal{V} \in Ob(FI_{\mathcal{R}}) \mid \text{Every f.g. } \mathcal{W} \subseteq \mathcal{V} \text{ satisfies } \mathcal{W}_n \in \mathcal{T} \text{ for } n \gg 0 \} \tag{5.3}$$

*is a hereditary torsion class.*

**Proof.** From the definition, it is clear that  $\tilde{\mathcal{T}}$  is closed under subobjects. From Lemma 5.1, Lemma 5.2 and Lemma 5.3 we know respectively that  $\tilde{\mathcal{T}}$  is closed under extensions, coproducts and quotients. Since  $FI_{\mathcal{R}}$  is a Grothendieck category, it now follows from [4, I.1] that  $\tilde{\mathcal{T}}$  is a hereditary torsion class.  $\square$

Now that we know  $\tilde{\mathcal{T}}$  is a hereditary torsion class, our aim is to describe closed objects with respect to  $\tilde{\mathcal{T}}$  as well as a functor from  $FI_{\mathcal{R}}$  to  $Cl(\tilde{\mathcal{T}})$ . This will be done in several steps.

For any  $\mathcal{V} \in FI_{\mathcal{R}}$  and any  $n \geq 0$ , we now denote by  $fg_n^\tau(\mathcal{V})$  the collection of all finitely generated subobjects  $\mathcal{W}' \subseteq \mathcal{V}$  such that  $\mathcal{W}'_m \in \mathcal{T}$  for every  $m \geq n$ . Clearly,  $fg_n^\tau(\mathcal{V}) \subseteq \hat{\mathcal{T}}^n$ .

**Proposition 5.5.** *Let  $Mod - \mathcal{R}$  be a locally noetherian category and  $\tau = (\mathcal{T}, \mathcal{F})$  a hereditary torsion theory on  $Mod - \mathcal{R}$ . Every  $\mathcal{V} \in \tilde{\mathcal{T}}$  is equipped with an increasing filtration*

$$F^0\mathcal{V} \subseteq F^1\mathcal{V} \subseteq \dots \subseteq \mathcal{V} \tag{5.4}$$

*with  $F^n\mathcal{V} \in \hat{\mathcal{T}}^n$  for each  $n \geq 0$ .*

**Proof.** We let  $F^n\mathcal{V}$  be the sum of all  $\mathcal{W} \in fg_n^\tau(\mathcal{V})$ . Since  $\mathcal{V} \in \tilde{\mathcal{T}}$ , every finitely generated subobject  $\mathcal{W} \subseteq \mathcal{V}$  lies in  $fg_n^\tau(\mathcal{V})$  for some  $n \geq 0$ . Since  $\mathcal{V}$  is the sum of its finitely generated subobjects, it follows that we have a filtration as in (5.4) whose union is  $\mathcal{V}$ .

It remains to show that  $F^n\mathcal{V} \in \hat{\mathcal{T}}^n$  for each  $n \geq 0$ . We note that any finite sum  $\sum_{j \in J} \mathcal{W}_j$  of objects in  $fg_n^\tau(\mathcal{V})$  is a quotient of the direct sum  $\bigoplus_{j \in J} \mathcal{W}_j$  and hence lies in  $fg_n^\tau(\mathcal{V})$ . We also observe that  $F^n\mathcal{V}$  is the colimit of  $\sum_{j \in J} \mathcal{W}_j$  as  $J$  varies over all finite collections of objects in  $fg_n^\tau(\mathcal{V})$ . Since  $\mathcal{T}$  is closed under colimits (being a torsion class), the result follows.  $\square$

**Lemma 5.6.** *Let  $u : \mathcal{A} \rightarrow \mathcal{B}$  be an epimorphism in  $FI_{\mathcal{R}}$  with  $Ker(u) = \mathcal{K} \in \tilde{\mathcal{T}}$ . Let  $\mathcal{L} \in Cl(\hat{\mathcal{T}})$ . Then, the induced morphism  $FI_{\mathcal{R}}(u, \mathcal{L}) : FI_{\mathcal{R}}(\mathcal{B}, \mathcal{L}) \rightarrow FI_{\mathcal{R}}(\mathcal{A}, \mathcal{L})$  is an isomorphism.*

**Proof.** Using Proposition 5.5, we can consider a filtration  $\mathcal{K}^0 \subseteq \mathcal{K}^1 \subseteq \dots$  on  $\mathcal{K} \in \tilde{\mathcal{T}}$  with each  $\mathcal{K}^i \in \hat{\mathcal{T}}^i$ . Since  $\hat{\mathcal{T}}^i \subseteq \hat{\mathcal{T}}$ , we know that  $\mathcal{L} \in Cl(\hat{\mathcal{T}})$  lies in  $Cl(\hat{\mathcal{T}}^i)$  for each  $i$ . It follows therefore that we have an isomorphism  $FI_{\mathcal{R}}(\mathcal{A}/\mathcal{K}^i, \mathcal{L}) \rightarrow FI_{\mathcal{R}}(\mathcal{A}, \mathcal{L})$  for each  $i$ . We know that  $\mathcal{B} = \mathcal{A}/\mathcal{K}$ . Taking limits, we therefore obtain an isomorphism

$$FI_{\mathcal{R}}(u, \mathcal{L}) : FI_{\mathcal{R}}(\mathcal{B}, \mathcal{L}) = FI_{\mathcal{R}}(\varinjlim_{i \geq 0} \mathcal{A}/\mathcal{K}^i, \mathcal{L}) = \varinjlim_{i \geq 0} FI_{\mathcal{R}}(\mathcal{A}/\mathcal{K}^i, \mathcal{L}) \xrightarrow{\cong} FI_{\mathcal{R}}(\mathcal{A}, \mathcal{L}) \quad \square$$

**Lemma 5.7.** *Let  $u : \mathcal{A} \rightarrow \mathcal{B}$  be a monomorphism in  $FI_{\mathcal{R}}$  with  $Coker(u) = \mathcal{C} \in \tilde{\mathcal{T}}$ . Let  $\mathcal{L} \in Cl(\hat{\mathcal{T}})$ . Then, the induced morphism  $FI_{\mathcal{R}}(u, \mathcal{L}) : FI_{\mathcal{R}}(\mathcal{B}, \mathcal{L}) \rightarrow FI_{\mathcal{R}}(\mathcal{A}, \mathcal{L})$  is an isomorphism.*

**Proof.** Using Proposition 5.5, we can consider a filtration  $\mathcal{C}^0 \subseteq \mathcal{C}^1 \subseteq \dots$  on  $\mathcal{C} \in \tilde{\mathcal{T}}$  with each  $\mathcal{C}^i \in \hat{\mathcal{T}}^i$ . This corresponds to a filtration  $\mathcal{B}^0 \subseteq \mathcal{B}^1 \subseteq \dots$  on  $\mathcal{B}$  such that  $\mathcal{B}^i/\mathcal{A} = \mathcal{C}^i$ .

Since  $\hat{\mathcal{T}}^i \subseteq \hat{\mathcal{T}}$ , we know that  $\mathcal{L} \in Cl(\hat{\mathcal{T}})$  lies in  $Cl(\hat{\mathcal{T}}^i)$  for each  $i$ . It follows therefore that we have an isomorphism  $FI_{\mathcal{R}}(\mathcal{B}^i, \mathcal{L}) \rightarrow FI_{\mathcal{R}}(\mathcal{A}, \mathcal{L})$  for each  $i$ . Taking limits, we therefore obtain an isomorphism

$$FI_{\mathcal{R}}(u, \mathcal{L}) : FI_{\mathcal{R}}(\mathcal{B}, \mathcal{L}) = FI_{\mathcal{R}}(\lim_{i \geq 0} \mathcal{B}^i, \mathcal{L}) = \lim_{i \geq 0} FI_{\mathcal{R}}(\mathcal{B}^i, \mathcal{L}) \xrightarrow{\cong} FI_{\mathcal{R}}(\mathcal{A}, \mathcal{L}) \quad \square$$

**Theorem 5.8.** *Let  $Mod - \mathcal{R}$  be a locally noetherian category and  $\tau = (\mathcal{T}, \mathcal{F})$  a hereditary torsion theory on  $Mod - \mathcal{R}$ . Then,  $Cl(\hat{\mathcal{T}}) = Cl(\tilde{\mathcal{T}})$ . In particular, we have a functor  $\mathbb{L}_{\tau} : FI_{\mathcal{R}} \rightarrow Cl(\hat{\mathcal{T}}) = Cl(\tilde{\mathcal{T}})$ .*

**Proof.** From the definitions in (4.1) and (5.1), we know that  $\hat{\mathcal{T}} \subseteq \tilde{\mathcal{T}}$ , whence it follows that  $Cl(\hat{\mathcal{T}}) \supseteq Cl(\tilde{\mathcal{T}})$ . We now consider a morphism  $u : \mathcal{A} \rightarrow \mathcal{B}$  in  $FI_{\mathcal{R}}$  which is  $\tilde{\mathcal{T}}$ -closed. Then,  $u$  may be expressed as the composition

$$u : \mathcal{A} \xrightarrow{u'} \mathcal{A}/Ker(u) \xrightarrow{u''} \mathcal{B} \tag{5.5}$$

where  $u'$  is an epimorphism with  $Ker(u') \in \tilde{\mathcal{T}}$  and  $u''$  is a monomorphism with  $Coker(u'') \in \tilde{\mathcal{T}}$ . Let  $\mathcal{L} \in Cl(\hat{\mathcal{T}})$ . From Lemma 5.6 and Lemma 5.7, it follows that  $FI_{\mathcal{R}}(u', \mathcal{L})$  and  $FI_{\mathcal{R}}(u'', \mathcal{L})$  are both isomorphisms. Hence,  $FI_{\mathcal{R}}(u, \mathcal{L})$  is an isomorphism. Hence,  $\mathcal{L} \in Cl(\tilde{\mathcal{T}})$ . This proves the result.  $\square$

**6. The functors  $H_a$  and properties of finitely and shift finitely generated modules**

We return to the general case, i.e.,  $\mathcal{R}$  is a small preadditive category, but  $Mod - \mathcal{R}$  is not necessarily noetherian. Fix  $a \geq 0$ . Let  $\mathcal{V} \in FI_{\mathcal{R}}$ . In a manner similar to [9, § 2], we define the functor

$$\mathbb{B}^{-a} : FI_{\mathcal{R}} \rightarrow FI_{\mathcal{R}} \quad \mathbb{B}^{-a}(\mathcal{V})(S) := \bigoplus_{\phi : [a] \hookrightarrow S} \mathcal{V}(S, \phi) = \bigoplus_{\phi : [a] \hookrightarrow S} \mathcal{V}(S - \phi[a]) \tag{6.1}$$

It is clear from (6.1) that  $\mathbb{B}^{-a}(\mathcal{V}) \in FI_{\mathcal{R}}$ . For each  $a \geq 1$ , we consider the set  $\{s_i : [a - 1] \rightarrow [a]\}_{1 \leq i \leq a}$  of standard order-preserving injections, where the image of  $s_i$  misses  $i$ . Then, for any  $\phi : [a] \rightarrow S$ , we have  $S - \phi\{a\} \subsetneq S - \phi \circ s_i\{a - 1\}$  which induces a morphism  $d_i(S, \phi) : \mathcal{V}(S - \phi\{a\}) \rightarrow \mathcal{V}(S - \phi \circ s_i\{a - 1\})$ . Taking the alternating sum  $\sum_{i=1}^a (-1)^i d_i$  of these maps in the usual manner, we obtain a complex

$$\mathbb{B}^{-*}(\mathcal{V}) : \dots \rightarrow \mathbb{B}^{-a}(\mathcal{V}) \rightarrow \mathbb{B}^{-(a-1)}(\mathcal{V}) \rightarrow \dots \rightarrow \mathbb{B}^{-1}(\mathcal{V}) \rightarrow \mathbb{B}^0(\mathcal{V}) = \mathcal{V} \rightarrow 0 \tag{6.2}$$

Let  $S_a$  be the permutation group on  $a$  objects and consider the group ring  $\mathbb{Z}[S_a]$ . We consider the small preadditive category  $\mathcal{R}[S_a]$  defined by setting  $Ob(\mathcal{R}) = Ob(\mathcal{R}[S_a])$  and

$$\mathcal{R}[S_a](r, r') := \mathcal{R}(r, r') \otimes_{\mathbb{Z}} \mathbb{Z}[S_a] \tag{6.3}$$

The composition in  $\mathcal{R}[S_a]$  is the usual composition in  $\mathcal{R}$  extended by the multiplication in  $\mathbb{Z}[S_a]$ . Given a morphism  $f \cdot \sigma \in \mathcal{R}[S_a](r, r')$ , i.e.,  $f \in \mathcal{R}(r, r')$  and  $\sigma \in S_a$ , we notice that we have a map

$$\mathcal{V}(S, \phi)(r') = \mathcal{V}(S - \phi[a])(r') \xrightarrow{\mathcal{V}(S - \phi[a])(f)} \mathcal{V}(S - \phi[a])(r) = \mathcal{V}(S - \phi \circ \sigma[a])(r) = \mathcal{V}(S, \phi \circ \sigma)(r) \tag{6.4}$$

for each  $\phi : [a] \rightarrow S$  in  $FI$ . Using the maps in (6.4), it may be easily verified that  $\mathbb{B}^{-a}$  may be treated as a functor  $\mathbb{B}^{-a} : FI_{\mathcal{R}} \rightarrow FI_{\mathcal{R}[S_a]}$ . On the other hand, the canonical  $\mathcal{R}-\mathcal{R}$ -bimodule given by morphism spaces in  $\mathcal{R}$  may be extended to a left  $\mathcal{R}[S_a]$  right  $\mathcal{R}$ -module:

$$\begin{aligned} H_{\mathcal{R}}^{\varepsilon} : \mathcal{R}^{op} \otimes \mathcal{R}[S_a] &\rightarrow Ab & (r', r) &\mapsto \mathcal{R}(r', r) \\ H_{\mathcal{R}}^{\varepsilon}(f_1, f_2 \cdot \sigma) : H_{\mathcal{R}}^{\varepsilon}(r', r) &\rightarrow H_{\mathcal{R}}^{\varepsilon}(r''', r'') & f &\mapsto (-1)^{sgn(\sigma)} f_2 \circ f \circ f_1 \end{aligned} \tag{6.5}$$

Here  $sgn(\sigma)$  is the sign of the permutation in  $\mathbb{Z}_2$ . This allows us to define a functor

$$\tilde{\mathbb{B}}^{-a} : FI_{\mathcal{R}} \longrightarrow FI_{\mathcal{R}} \quad \tilde{\mathbb{B}}^{-a}(\mathcal{V})(S) := \mathbb{B}^{-a}(\mathcal{V})(S) \otimes_{\mathcal{R}[S_a]} H_{\mathcal{R}}^{\varepsilon} \tag{6.6}$$

We observe that  $\tilde{\mathbb{B}}^{-a}(\mathcal{V})(S) \in Mod - \mathcal{R}$  is a direct sum of all  $\mathcal{V}(T)$  as  $T$  varies over all the distinct subsets of  $S$  such that  $|T| = |S| - a$ . It may be verified by direct computation that the complex in (6.2) descends to a complex

$$\tilde{\mathbb{B}}^{-*}(\mathcal{V}) : \quad \dots \longrightarrow \tilde{\mathbb{B}}^{-a}(\mathcal{V}) \longrightarrow \tilde{\mathbb{B}}^{-(a-1)}(\mathcal{V}) \longrightarrow \dots \longrightarrow \tilde{\mathbb{B}}^{-1}(\mathcal{V}) \longrightarrow \tilde{\mathbb{B}}^0(\mathcal{V}) = \mathcal{V} \longrightarrow 0 \tag{6.7}$$

For  $\mathcal{V} \in FI_{\mathcal{R}}$ , we set  $H_a(\mathcal{V}) := H^{-a}(\tilde{\mathbb{B}}^{-*}(\mathcal{V})) \in FI_{\mathcal{R}}$ . In particular, it is easy to observe that  $\mathbb{B}^{-a}({}_{a+d}\mathcal{M}_r) = {}_{a+d}\mathcal{M}_r$ . Since  $\mathbb{B}^{-a}$  is exact, it follows that for any  $\mathcal{V} \in FI_{\mathcal{R}}^{fg}$ , the object  $\mathbb{B}^{-a}(\mathcal{V})$  is finitely generated. Further, since  $\tilde{\mathbb{B}}^{-a}(\mathcal{V})$  is a quotient of  $\mathbb{B}^{-a}(\mathcal{V})$ , it follows that  $\tilde{\mathbb{B}}^{-a}(\mathcal{V}) \in FI_{\mathcal{R}}^{fg}$ .

**Proposition 6.1.** *Let  $\mathcal{V} \in FI_{\mathcal{R}}$ . Fix a finite set  $S$  and consider the colimit  $\mathop{\text{colim}}_{T \subsetneq S} \mathcal{V}(T)$  taken over all proper subsets of  $S$  ordered by inclusion. Then, we have*

$$H^0(\tilde{\mathbb{B}}^{-*}(\mathcal{V}))(S) = \text{Coker} \left( \mathop{\text{colim}}_{T \subsetneq S} \mathcal{V}(T) \longrightarrow \mathcal{V}(S) \right) = H_0(\mathcal{V})(S) \tag{6.8}$$

$$H^{-1}(\tilde{\mathbb{B}}^{-*}(\mathcal{V}))(S) = \text{Ker} \left( \mathop{\text{colim}}_{T \subsetneq S} \mathcal{V}(T) \longrightarrow \mathcal{V}(S) \right) = H_1(\mathcal{V})(S) \tag{6.9}$$

**Proof.** For any injection  $\phi : T' \hookrightarrow S$  with  $|T'| < |S|$ , it is obvious that  $\phi$  factors through a proper subset of  $S$ . Comparing with the definition in (2.7), we see that  $H_0(\mathcal{V}) = \text{Coker} \left( \mathop{\text{colim}}_{T \subsetneq S} \mathcal{V}(T) \longrightarrow \mathcal{V}(S) \right)$ . From the discussion above, we know that  $\tilde{\mathbb{B}}^{-1}(\mathcal{V})(S) \in Mod - \mathcal{R}$  is a direct sum of all  $\mathcal{V}(T)$  as  $T$  varies over all the distinct subsets of  $S$  such that  $|T| = |S| - 1$ . Since the inclusion of any proper subset of  $S$  factors through a subset of size  $|S| - 1$ , we also observe that  $H^0(\tilde{\mathbb{B}}^{-*}(\mathcal{V}))(S) = \text{Coker}(\tilde{\mathbb{B}}^{-1}(\mathcal{V})(S) \longrightarrow \mathcal{V}(S)) = \text{Coker} \left( \mathop{\text{colim}}_{T \subsetneq S} \mathcal{V}(T) \longrightarrow \mathcal{V}(S) \right)$ . This proves (6.8).

To prove (6.9), we proceed as follows: for each subset  $T \subseteq S$  of cardinality  $|S| - 2$ , there are exactly two subsets  $T_1, T_2 \subseteq S$  each of cardinality  $|S| - 1$  such that  $T \subseteq T_1, T_2$ . This induces maps  $\mathcal{V}(T) \longrightarrow \mathcal{V}(T_1)$  and  $\mathcal{V}(T) \longrightarrow \mathcal{V}(T_2)$ . We now observe that

$$\mathop{\text{colim}}_{T \subsetneq S} \mathcal{V}(T) = \text{Coeq} \left( \bigoplus_{\substack{T \subsetneq S \\ |T| = |S| - 2}} \mathcal{V}(T) \rightrightarrows \bigoplus_{\substack{T \subsetneq S \\ |T| = |S| - 1}} \mathcal{V}(T) \right) \tag{6.10}$$

From the definition of the differential in the complex in (6.7), it is clear that the expression in (6.10) is identical to  $\text{Coker}(\tilde{\mathbb{B}}^{-2}(\mathcal{V}) \longrightarrow \tilde{\mathbb{B}}^{-1}(\mathcal{V}))(S)$ . It follows that

$$H^{-1}(\tilde{\mathbb{B}}^{-*}(\mathcal{V}))(S) = \text{Ker}(\text{Coker}(\tilde{\mathbb{B}}^{-2}(\mathcal{V}) \longrightarrow \tilde{\mathbb{B}}^{-1}(\mathcal{V}))(S) \longrightarrow \mathcal{V}(S)) = \text{Ker} \left( \mathop{\text{colim}}_{T \subsetneq S} \mathcal{V}(T) \longrightarrow \mathcal{V}(S) \right) \quad \square$$

**Proposition 6.2.** *Let  $\mathcal{V} \in FI_{\mathcal{R}}$ . Then, for each  $a \geq 0$ , the canonical map  $H_a(\mathcal{V}) \longrightarrow S^1 H_a(\mathcal{V})$  is zero.*

**Proof.** We consider the system of maps

$$\{G^{-b} : \mathbb{B}^{-b}\mathcal{V} \longrightarrow S^1 \mathbb{B}^{-b-1}\mathcal{V}\}_{b \geq 0} \tag{6.11}$$

defined as follows: for a finite set  $S$  and a map  $\phi : [b] \longrightarrow S$ , denote by  $\bar{\phi} : [b+1] \longrightarrow S \sqcup [-1]$  the map given by

$$\bar{\phi}(i) = \begin{cases} * & \text{if } i = 1 \\ \phi(i - 1) & \text{otherwise} \end{cases} \tag{6.12}$$

where  $[-1]$  has been chosen to be the single element set  $\{*\}$ . Then, the identifications

$$\mathcal{V}(S, \phi) = \mathcal{V}(S - \phi[b]) \xrightarrow{=} \mathcal{V}(S \sqcup [-1] - \bar{\phi}[b + 1]) = \mathcal{V}(S \sqcup [-1], \bar{\phi}) \tag{6.13}$$

combine to determine the map  $G^{-b}(S) : \mathbb{B}^{-b}\mathcal{V}(S) \rightarrow \mathbb{B}^{-b-1}\mathcal{V}(S \sqcup [-1]) = \mathbb{S}^1\mathbb{B}^{-b-1}\mathcal{V}(S)$ . As in the proof of [9, Proposition 2.25], it may be verified that the maps  $G^{-b}$  induce a homotopy equivalence between the zero map and the canonical map  $\widetilde{\mathbb{B}}^{-*}(\mathcal{V}) \rightarrow \mathbb{S}^1\widetilde{\mathbb{B}}^{-*}(\mathcal{V})$ .  $\square$

**Proposition 6.3.** *Suppose that  $\text{Mod} - \mathcal{R}$  is locally noetherian. Let  $\mathcal{V} \in FI_{\mathcal{R}}$  be a finitely generated object. Then, for each  $a \geq 0$ , there exists  $N \geq 0$  such that  $H_a(\mathcal{V})_n = 0$  for all  $n \geq N$ .*

**Proof.** We have explained before that if  $\mathcal{V} \in FI_{\mathcal{R}}$  is finitely generated,  $\widetilde{\mathbb{B}}^{-a}(\mathcal{V})$  is finitely generated. Since  $FI_{\mathcal{R}}$  is locally noetherian, it follows that  $H_a(\mathcal{V})$  is also finitely generated. We consider the trivial torsion theory  $\tau_0$  on  $\text{Mod} - \mathcal{R}$  whose torsion class is 0. Using Proposition 3.2, this induces a torsion class on  $FI_{\mathcal{R}}^{fg}$  whose torsion class  $\overline{\mathcal{T}}_0$  is given by

$$\text{Ob}(\overline{\mathcal{T}}_0) := \{\mathcal{V} \in \text{Ob}(FI_{\mathcal{R}}^{fg}) \mid \mathcal{V}_n = 0 \text{ for } n \gg 0\} \tag{6.14}$$

Using Theorem 3.8, we know that the torsion subobject of  $H_a(\mathcal{V})$  is given by

$$\begin{aligned} \overline{\mathcal{T}}_0(H_a(\mathcal{V}))(S) &= \text{colim}_{b \geq 0} \lim \left( H_a(\mathcal{V})(S) \xrightarrow{\psi_b^{H_a(\mathcal{V})}(S)} \mathbb{S}^b H_a(\mathcal{V})(S) \longleftarrow \mathcal{T}_0(\mathbb{S}^b H_a(\mathcal{V})(S)) \right) \\ &= \text{colim}_{b \geq 1} \lim \left( H_a(\mathcal{V})(S) \xrightarrow{\psi_b^{H_a(\mathcal{V})}(S)} \mathbb{S}^b H_a(\mathcal{V})(S) \longleftarrow 0 \right) \end{aligned} \tag{6.15}$$

From Proposition 6.2 and the expression in (6.15), it now follows that  $\overline{\mathcal{T}}_0(H_a(\mathcal{V}))(S) = H_a(\mathcal{V})(S)$ . Hence,  $H_a(\mathcal{V}) \in \overline{\mathcal{T}}_0$  and the result follows.  $\square$

We now have an analogue of [9, Theorem C].

**Theorem 6.4.** *Suppose that  $\text{Mod} - \mathcal{R}$  is locally noetherian. Let  $\mathcal{V} \in FI_{\mathcal{R}}$  be a finitely generated object. Then, there exists  $N \geq 0$  such that*

$$\text{colim}_{\substack{T \subseteq S \\ |T| \leq N}} \mathcal{V}(T) = \mathcal{V}(S) \tag{6.16}$$

for each finite set  $S$ .

**Proof.** Using Proposition 6.3, we can choose  $N \geq 1$  such that  $H_0(\mathcal{V})_n = H_1(\mathcal{V})_n = 0$  for all  $n \geq N$ . It is clear that (6.16) holds for all  $S$  such that  $|S| \leq N$ . We consider a set  $S$  with  $|S| > N$  and suppose that (6.16) holds for all finite sets  $U$  of cardinality  $< |S|$ . We observe that

$$\text{colim}_{\substack{T \subseteq S \\ |T| \leq N}} \mathcal{V}(T) = \text{colim}_{U \subsetneq S} \text{colim}_{\substack{T \subseteq U \\ |T| \leq N}} \mathcal{V}(T) \tag{6.17}$$

Since each  $U$  appearing in (6.17) has cardinality  $< |S|$ , we have

$$\text{colim}_{\substack{T \subseteq S \\ |T| \leq N}} \mathcal{V}(T) = \text{colim}_{U \subsetneq S} \mathcal{V}(U) \tag{6.18}$$

Finally since  $|S| > N$ , we know that  $H_0(\mathcal{V})(S) = H_1(\mathcal{V})(S) = 0$ . The result is now clear from the expressions in Proposition 6.1.  $\square$

So far in this section, we have used the properties of  $H_a(\mathcal{V})$  for  $\mathcal{V}$  finitely generated. We will now consider the objects  $H_a(\mathcal{V})$  when  $\mathcal{V}$  is shift finitely generated.

**Lemma 6.5.** *Let  $\mathcal{V} \in FI_{\mathcal{R}}$  be shift finitely generated. Then, for any  $a \geq 0$ ,  $\mathbb{B}^{-a}(\mathcal{V})$  is also shift finitely generated.*

**Proof.** Since  $\mathcal{V} \in FI_{\mathcal{R}}^{sfg}$ , we choose  $d \geq 0$  such that  $\mathbb{S}^d\mathcal{V}$  is finitely generated. We choose  $e \geq a + d$ . We will show that  $\mathbb{S}^e\mathbb{B}^{-a}(\mathcal{V})$  is finitely generated. For any finite set  $T$ , we see that

$$\begin{aligned}
 \mathbb{S}^e\mathbb{B}^{-a}(\mathcal{V})(T) &= \mathbb{B}^{-a}\mathcal{V}(T \sqcup [-e]) \\
 &= \bigoplus_{\phi: [a] \rightarrow T \sqcup [-e]} \mathcal{V}(T \sqcup [-e] - \phi[a]) \\
 &= \bigoplus_{j=0}^a \left( \bigoplus_{\substack{\phi: [a] \rightarrow T \sqcup [-e] \\ \text{Im}(\phi) \cap T = a-j}} \mathcal{V}(T \sqcup [-e] - \phi[a]) \right) = \bigoplus_{j=0}^a \left( \bigoplus_{\substack{\phi = (\phi', \phi'') \\ \phi': [a-j] \rightarrow T \\ \phi'': [j] \rightarrow [-e]}} \mathcal{V}(T \sqcup [-e] - \phi[a]) \right)^{\oplus \binom{a}{j}} \\
 &= \bigoplus_{j=0}^a \left( \bigoplus_{\substack{\phi': [a-j] \rightarrow T \\ \phi'': [j] \rightarrow [-e]}} \mathcal{V}((T - \phi'[a-j]) \sqcup ([-e] - \phi''[j])) \right)^{\oplus \binom{a}{j}} \tag{6.19} \\
 &= \bigoplus_{j=0}^a \left( \bigoplus_{\phi': [a-j] \rightarrow T} \mathcal{V}((T - \phi'[a-j]) \sqcup [-(e-j)]) \right)^{\oplus \binom{a}{j} \cdot \binom{[j], [e]}{[j], [e]}} \\
 &= \bigoplus_{j=0}^a \left( \bigoplus_{\phi': [a-j] \rightarrow T} \mathbb{S}^{e-j}\mathcal{V}(T - \phi'[a-j]) \right)^{\oplus \binom{a}{j} \cdot \binom{[j], [e]}{[j], [e]}} \\
 &= \bigoplus_{j=0}^a (\mathbb{B}^{-(a-j)}(\mathbb{S}^{e-j}\mathcal{V})(T))^{\oplus \binom{a}{j} \cdot \binom{[j], [e]}{[j], [e]}}
 \end{aligned}$$

Since  $e \geq a + d$ , we know that  $e - j \geq d$  for each  $0 \leq j \leq a$ . Hence, each  $\mathbb{S}^{e-j}\mathcal{V}$  appearing in the direct sum in (6.19) is finitely generated. Then, each  $\mathbb{B}^{-(a-j)}(\mathbb{S}^{e-j}\mathcal{V})$  is finitely generated and it is now clear from (6.19) that  $\mathbb{S}^e\mathbb{B}^{-a}(\mathcal{V})$  is finitely generated.  $\square$

**Proposition 6.6.** *Let  $Mod - \mathcal{R}$  be locally noetherian. Let  $\mathcal{V} \in FI_{\mathcal{R}}$  be shift finitely generated. Then, for any  $a \geq 0$ ,  $H_a(\mathcal{V})$  is also shift finitely generated.*

**Proof.** From Lemma 6.5, we know that  $\mathbb{B}^{-a}(\mathcal{V})$  is also shift finitely generated. We have shown in Proposition 3.11 that  $FI_{\mathcal{R}}^{sfg}$  is a Serre subcategory. From the definitions, it is now clear that  $\widetilde{\mathbb{B}}^{-a}(\mathcal{V})$  and hence  $H_a(\mathcal{V})$  lie in  $FI_{\mathcal{R}}^{sfg}$ .  $\square$

**Theorem 6.7.** *Suppose that  $Mod - \mathcal{R}$  is locally noetherian. Let  $\mathcal{V} \in FI_{\mathcal{R}}$  be a shift finitely generated object. Fix  $a \geq 0$  and consider any finitely generated subobject  $\mathcal{W} \subseteq H_a(\mathcal{V})$ . Then, there exists  $N \geq 0$  such that  $\mathcal{W}_n = 0$  for all  $n \geq N$ .*

**Proof.** Since  $\mathcal{V} \in FI_{\mathcal{R}}^{sfg}$ , we know from Proposition 6.6 that  $H_a(\mathcal{V})$  is shift finitely generated. We consider the trivial torsion theory  $\tau_0$  on  $Mod - \mathcal{R}$  whose torsion class is 0. Using Proposition 3.13, this induces a torsion class on  $FI_{\mathcal{R}}^{sfg}$  whose torsion class  $\overline{\mathcal{T}}_0^{sfg}$  is given by

$$Ob(\overline{\mathcal{T}}_0^{sfg}) := \{\mathcal{V} \in Ob(FI_{\mathcal{R}}^{sfg}) \mid \text{Every finitely generated } \mathcal{W} \subseteq \mathcal{V} \text{ satisfies } \mathcal{W}_n = 0 \text{ for } n \gg 0\} \tag{6.20}$$

Using Theorem 3.17, we know that the torsion subobject of  $H_a(\mathcal{V})$  is given by

$$\begin{aligned} \overline{\mathcal{T}}_0^{sfg}(H_a(\mathcal{V}))(S) &= \operatorname{colim}_{b \geq 0} \operatorname{lim} \left( H_a(\mathcal{V})(S) \xrightarrow{\psi_b^{H_a(\mathcal{V})}(S)} \mathbb{S}^b H_a(\mathcal{V})(S) \longleftarrow \mathcal{T}_0(\mathbb{S}^b H_a(\mathcal{V})(S)) \right) \\ &= \operatorname{colim}_{b \geq 1} \operatorname{lim} \left( H_a(\mathcal{V})(S) \xrightarrow{\psi_b^{H_a(\mathcal{V})}(S)} \mathbb{S}^b H_a(\mathcal{V})(S) \longleftarrow 0 \right) \end{aligned} \tag{6.21}$$

From Proposition 6.2 and the expression in (6.21), it now follows that  $\overline{\mathcal{T}}_0^{sfg}(H_a(\mathcal{V}))(S) = H_a(\mathcal{V})(S)$ . Hence,  $H_a(\mathcal{V}) \in \overline{\mathcal{T}}_0^{sfg}$  and the result follows.  $\square$

We conclude with the following result.

**Corollary 6.8.** *Suppose that  $Mod - \mathcal{R}$  is locally noetherian. Let  $\mathcal{V} \in FI_{\mathcal{R}}$  be a shift finitely generated object such that  $H_0(\mathcal{V})$  and  $H_1(\mathcal{V})$  are finitely generated. Then, there exists  $N \geq 0$  such that*

$$\operatorname{colim}_{\substack{T \subseteq S \\ |T| \leq N}} \mathcal{V}(T) = \mathcal{V}(S) \tag{6.22}$$

for each finite set  $S$ .

**Proof.** Since  $H_0(\mathcal{V})$  and  $H_1(\mathcal{V})$  are finitely generated, it follows from Theorem 6.7 that there exists  $N \geq 0$  such that  $H_0(\mathcal{V})_n = H_1(\mathcal{V})_n = 0$  for all  $n \geq N$ . The rest of the proof now follows in a manner similar to that of Theorem 6.4: it is clear that (6.22) holds for all  $S$  such that  $|S| \leq N$ . We consider a set  $S$  with  $|S| > N$  and suppose that (6.22) holds for all finite sets  $U$  of cardinality  $< |S|$ . We observe that

$$\operatorname{colim}_{\substack{T \subseteq S \\ |T| \leq N}} \mathcal{V}(T) = \operatorname{colim}_{U \subsetneq S} \operatorname{colim}_{\substack{T \subseteq U \\ |T| \leq N}} \mathcal{V}(T) \tag{6.23}$$

Since each  $U$  appearing in (6.23) has cardinality  $< |S|$ , we have

$$\operatorname{colim}_{\substack{T \subseteq S \\ |T| \leq N}} \mathcal{V}(T) = \operatorname{colim}_{U \subsetneq S} \mathcal{V}(U) \tag{6.24}$$

Finally since  $|S| > N$ , we know that  $H_0(\mathcal{V})(S) = H_1(\mathcal{V})(S) = 0$ . The result is now clear from the expressions in Proposition 6.1.  $\square$

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