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ON IRREDUCIBLE SUPERSINGULAR REPRESENTATIONS OF $\mathrm{GL}_2(F)$

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Let F be a nonarchimedean local field of residual characteristic $p > 3$ and residue degree $f > 1$. We study a certain type of diagram, called *cyclic diagrams*, and use them to show that the universal supersingular modules of $\mathrm{GL}_2(F)$ admit infinitely many nonisomorphic irreducible admissible quotients.

Introduction

Let F be a nonarchimedean local field of residual characteristic p and residue degree f. Fix a uniformizer $\varpi \in F$. The theory of smooth representations of reductive F-groups on $\overline{\mathbb{F}}_p$-vector spaces has its origins in the paper of Barthel and Livné [1994] in which they classify all smooth irreducible representations of $\mathrm{GL}_2(F)$ with central characters except *supersingular* representations. The first examples of supersingular representations of $\mathrm{GL}_2(F)$ were constructed by Paškūnas [2004] using equivariant coefficient systems on the Bruhat–Tits tree, or equivalently, using *diagrams* . Let K, Z and N denote, respectively, the standard maximal compact subgroup, the center and the normalizer of the standard Iwahori subgroup I of $\mathrm{GL}_2(F)$ so that the stabilizer of the standard vertex of the tree is KZ and that of the standard edge is N. A diagram is a finite data of a smooth KZ-representation D_0, a smooth N-representation D_1 and an IZ-equivariant map $D_1 \to D_0$. This data can be glued together (in a noncanonical way) to obtain smooth representations of $\mathrm{GL}_2(F)$ inside some injective envelopes.

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Breuil and Paškūnas [2012] developed the theory of diagrams further and constructed irreducible supersingular representations of $\mathrm{GL}_2(\mathbb{Q}_{p^f})$ with prescribed K-socles from certain indecomposable (but not irreducible) diagrams. Here, $\mathbb{Q}_{p^f}$ is the degree f unramified extension of $\mathbb{Q}_p$. Their results, in particular, imply that $\mathrm{GL}_2(\mathbb{Q}_{p^f})$, with $f > 1$, has infinitely many irreducible admissible supersingular representations on which p acts trivially, unlike $\mathrm{GL}_2(\mathbb{Q}_p)$ which has only finitely many such representations. Since the diagrams considered by them are not irreducible, the irreducibility of the corresponding representations of $\mathrm{GL}_2(\mathbb{Q}_{p^f})$ depends on certain computations with Witt vectors which do not extend to a ramified F or to an F of positive characteristic. In this note, we focus on irreducible diagrams in order to construct irreducible supersingular representations of $\mathrm{GL}_2(F)$ for all local fields F.

The complexity of supersingular representations of $\mathrm{GL}_2(F)$ for $f > 1$ can already be seen in the complexity of classifying irreducible diagrams for $f > 1$. To this end, we consider a particular type of irreducible diagrams which are rigid enough. We call them *cyclic diagrams*. These are irreducible diagrams on direct sums of extensions of weights such that the action of $\left(\begin{smallmatrix} 0 & 1 \\ \varpi & 0 \end{smallmatrix}\right)$ permutes characters cyclically. We show that cyclic diagrams exist for all $\mathrm{GL}_2(F)$ and the D_0 of any cyclic diagram has more than 2 irreducible subquotients if $f > 1$ (see Theorem 1.6 and Remark 1.2). As a result, when $f > 1$, a family of cyclic diagrams parametrized by $\overline{\mathbb{F}}_p^{\times}$ gives rise to infinitely many nonisomorphic irreducible admissible supersingular representations of $\mathrm{GL}_2(F)$ with trivial ϖ-action (see Theorem 3.2). This implies that, for all local fields F with $f > 1$, the universal supersingular modules of $\mathrm{GL}_2(F)$ have infinitely many nonisomorphic irreducible admissible quotients (see Corollary 3.3). While Corollary 3.3 follows from the main results in [Breuil and Paškūnas 2012] for $F = \mathbb{Q}_{p^f}$, it is a new result, to our knowledge, for F ramified over $\mathbb{Q}_p$ and for F of positive characteristic.

We conclude by mentioning a recent note by Wu [2021] in the similar spirit in which he gives a uniform proof of the fact that the universal supersingular modules of $\mathrm{GL}_2(F)$ are not admissible for any p-adic field $F \neq \mathbb{Q}_p$ by showing that the supersingular representations are not of finite presentations.

Notation and convention. Let $p > 3$ be a prime number. Let F be a nonarchimedean local field of residual characteristic p and residue degree f. Let $\mathcal{O} \subseteq F$ be the valuation ring with a uniformizer ϖ. Let $\overline{\mathbb{F}}_p$ be the algebraic closure of the finite field $\mathbb{F}_{p^f}$ of size p^f. Fix an embedding $\mathbb{F}_{p^f} \hookrightarrow \overline{\mathbb{F}}_p$. Let $G = \mathrm{GL}_2(F)$, $K = \mathrm{GL}_2(\mathcal{O})$, $\Gamma = \mathrm{GL}_2(\mathbb{F}_{p^f})$ and Z be the center of G. Let B and U be the subgroups of Γ consisting of the upper triangular matrices and the upper triangular unipotent matrices, respectively. Let I and $I(1)$ be the preimages of B and U, respectively, under the reduction modulo ϖ map $K \twoheadrightarrow \Gamma$. The subgroups I and $I(1)$ of K are the Iwahori and the pro-p Iwahori subgroup of K, respectively. The normalizer N of I in G is a subgroup generated by I and $\Pi = \left(\begin{smallmatrix} 0 & 1 \\ \varpi & 0 \end{smallmatrix}\right)$. Note that N is also the normalizer of $I(1)$ in G. Let $K(1)$ denote the kernel of the map $K \twoheadrightarrow \Gamma$, i.e., first principal congruence subgroup of K. Unless stated otherwise, all representations considered in this note are on $\overline{\mathbb{F}}_p$-vector spaces.

A *weight* is an irreducible representation of Γ. Any weight is of the form

$$\Big(\bigotimes_{j=0}^{f-1} \mathrm{Sym}^{r_j}\, \overline{\mathbb{F}}_p^2 \circ \Phi^j\Big) \otimes \det{}^m$$

for some integers $0 \le r_0, \dots, r_{f-1} \le p-1$ and $0 \le m \le p^f - 2$, where $\Phi : \Gamma \to \Gamma$ is the automorphism induced by the Frobenius map $\alpha \mapsto \alpha^p$ on $\mathbb{F}_{p^f}$ and $\det : \Gamma \to \mathbb{F}_{p^f}^\times$ is the determinant character. We denote such a weight by $\boldsymbol{r} \otimes \det^m$, where $\boldsymbol{r}$ is the f-tuple $(r_0, \dots, r_{f-1})$ of integers. Let $\sigma = \boldsymbol{r} \otimes \det^m$ be a weight; its subspace σ^U of U-fixed vectors is 1-dimensional and stable under the action of B because B normalizes U. The resulting B-character, denoted by $\chi(\sigma)$, sends $\left(\begin{smallmatrix} a & b \\ 0 & d \end{smallmatrix}\right) \in B$ to $a^r (ad)^m$, where $r = \sum_{j=0}^{f-1} r_j p^j$. Any B-character valued in $\overline{\mathbb{F}}_p^\times$ factors through the quotient B/U which is identified with the subgroup of diagonal matrices in B by the section $B/U \to B$, $\left(\begin{smallmatrix} a & 0 \\ 0 & d \end{smallmatrix}\right)U \mapsto \left(\begin{smallmatrix} a & 0 \\ 0 & d \end{smallmatrix}\right)$. For a B-character χ, let χ^s be the inflation to B of the conjugation-by-s character $t \mapsto \chi(sts^{-1})$ on B/U, where $s = \left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right)$. We say that a weight is *generic* if it is not equal to $(0, 0, \dots, 0) \otimes \det^m$ or $(p-1, p-1, \dots, p-1) \otimes \det^m$ for any m. The map $\sigma \mapsto \chi(\sigma)$ gives a bijection from the set of generic weights to the set of B-characters χ such that $\chi \neq \chi^s$. If σ is a generic weight, let us denote by $\sigma^{[s]}$ the generic weight corresponding to the character $\chi(\sigma)^s$. For $\sigma = \boldsymbol{r} \otimes \det^m$, $\sigma^{[s]} = (p-1-r_0, \dots, p-1-r_{f-1}) \otimes \det^{m+r}$. We refer the reader to [Barthel and Livné 1994, §1] for all nontrivial assertions in this paragraph.

Given two weights σ and τ, let $E(\sigma, \tau)$ be the unique nonsplit Γ-extension $0 \to \sigma \to E(\sigma, \tau) \to \tau \to 0$ if it exists [Breuil and Paškūnas 2012, Corollary 5.6]. We also denote $E(\sigma, \tau)$ by $\sigma — \tau$. A finite-dimensional representation of Γ is said to be *multiplicity-free* if its Jordan–Hölder factors are multiplicity-free. For any group H, the socle and the cosocle of an H-representation π are denoted by $\mathrm{soc}_H\, \pi$ and $\mathrm{cosoc}_H\, \pi$, respectively.

Note that a weight is a smooth irreducible representation of K (respectively, of KZ) and a B-character is a smooth I-character (respectively, IZ-character) via the map $K \twoheadrightarrow \Gamma$ (respectively, $KZ \twoheadrightarrow \Gamma$). In fact, the weights exhaust all smooth irreducible representations of K (respectively, of KZ such that ϖ acts trivially).

1. Cyclic modules

We are interested in the following type of representations of Γ:

Definition 1.1. A finite-dimensional representation D_0 of Γ is called a *cyclic module* of Γ if there exists a finite set $\{\sigma_1, \sigma_2, \dots, \sigma_n\}$ of distinct generic weights such that $E(\sigma_i, \sigma_{i-1}^{[s]})$ exists for all $1 \le i \le n$, $D_0 = \bigoplus_{i=1}^n E(\sigma_i, \sigma_{i-1}^{[s]})$ and $D_0^U = \bigoplus_{i=1}^n E(\sigma_i, \sigma_{i-1}^{[s]})^U = \bigoplus_{i=1}^n \chi(\sigma_i) \oplus \chi(\sigma_{i-1})^s$ with the convention $\sigma_0 = \sigma_n$.

If $D_0 = \bigoplus_{i=1}^n E(\sigma_i, \sigma_{i-1}^{[s]})$ is a cyclic module of Γ, then, by Frobenius reciprocity, there is a nonzero map $\operatorname{Ind}_B^\Gamma \chi(\sigma_{i-1})^s \to E(\sigma_i, \sigma_{i-1}^{[s]})$ for all $1 \le i \le n$. Since the principal series representation $\operatorname{Ind}_B^\Gamma \chi(\sigma_{i-1})^s$ has cosocle $\sigma_{i-1}^{[s]}$, and $\sigma_i \neq \sigma_{i-1}^{[s]}$, the map $\operatorname{Ind}_B^\Gamma \chi(\sigma_{i-1})^s \to E(\sigma_i, \sigma_{i-1}^{[s]})$ is surjective, and hence σ_i belongs to the first graded piece $\operatorname{gr}^1_{\mathrm{cosoc}}\big(\operatorname{Ind}_B^\Gamma \chi(\sigma_{i-1})^s\big)$ of the cosocle filtration of $\operatorname{Ind}_B^\Gamma \chi(\sigma_{i-1})^s$ for all $1 \le i \le n$.

Remark 1.2. If $D_0 = \bigoplus_{i=1}^n E(\sigma_i, \sigma_{i-1}^{[s]})$ is a cyclic module of Γ with $n = 1$, i.e., $D_0 = E(\sigma, \sigma^{[s]})$, then the surjective map $\operatorname{Ind}_B^\Gamma \chi(\sigma)^s \to E(\sigma, \sigma^{[s]})$ is actually an isomorphism: if the kernel is nonzero, then it has socle σ because $\operatorname{soc}_\Gamma \operatorname{Ind}_B^\Gamma \chi(\sigma)^s = \sigma$. But σ also occurs in the image as a subquotient, which contradicts the fact that a principal series is multiplicity-free [Breuil and Paškūnas 2012, Lemma 2.2]. Therefore $\operatorname{Ind}_B^\Gamma \chi(\sigma)^s \cong E(\sigma, \sigma^{[s]})$, and this forces $\Gamma = \mathrm{GL}_2(\mathbb{F}_p)$ by [Breuil and Paškūnas 2012, Theorem 2.4]. In fact, any cyclic module of $\mathrm{GL}_2(\mathbb{F}_p)$ is a principal series representation. Indeed, if $\Gamma = \mathrm{GL}_2(\mathbb{F}_p)$ and $E(\sigma, \tau)$ is a nonsplit Γ-extension between generic weights σ and τ such that $E(\sigma, \tau)^U = \chi(\sigma) \oplus \chi(\tau)$, then $\tau = \sigma^{[s]}$, and thus $E(\sigma, \tau) = \operatorname{Ind}_B^\Gamma \chi(\sigma)^s$ [Breuil and Paškūnas 2012, Corollary 5.6 (i) and Proposition 4.13 or Corollary 14.10].

In order to construct cyclic modules of $\Gamma = \mathrm{GL}_2(\mathbb{F}_{p^f})$ for $f > 1$, we take a closer look at the weights appearing in the first graded pieces of cosocle filtrations of principal series. Let x be a formal variable, and let $\mathbb{Z} \pm x := \{n \pm x : n \in \mathbb{Z}\}$ denote the set of linear polynomials in x having integral coefficients with leading coefficient ± 1. Let $(\mathbb{Z} \pm x)^f$ be the set of f-tuples of polynomials in $\mathbb{Z} \pm x$. For $\boldsymbol{\lambda} = (\lambda_0(x), \dots, \lambda_{f-1}(x)) \in (\mathbb{Z} \pm x)^f$ and $\boldsymbol{r} \in \mathbb{Z}^f$, let $\boldsymbol{\lambda}(\boldsymbol{r}) := \big(\lambda_0(r_0), \lambda_1(r_1), \dots, \lambda_{f-1}(r_{f-1})\big) \in \mathbb{Z}^f$. Recall the polynomial $e(\boldsymbol{\lambda}) \in \mathbb{Z} \oplus \bigoplus_{j=0}^{f-1} \mathbb{Z} x_j$ associated to $\boldsymbol{\lambda} \in (\mathbb{Z} \pm x)^f$ in [Breuil and Paškūnas 2012, §2]:

$$e(\boldsymbol{\lambda})(x_0, \dots, x_{f-1}) := \begin{cases} \frac{1}{2}\Big(\sum\limits_{j=0}^{f-1} p^j (x_j - \lambda_j(x_j))\Big), & \text{if } \lambda_{f-1}(x_{f-1}) \in \{x_{f-1}, x_{f-1} - 1\}, \\ \frac{1}{2}\Big(p^f - 1 + \sum\limits_{j=0}^{f-1} p^j (x_j - \lambda_j(x_j))\Big), & \text{otherwise.} \end{cases}$$

For each $f > 1$, let $\boldsymbol{\mu} \in (\mathbb{Z} \pm x)^f$ be the f-tuple of polynomials defined by

$$\begin{aligned} \mu_0(x) &:= x - 1, \\ \mu_1(x) &:= p - 2 - x, \\ \mu_j(x) &:= p - 1 - x, \quad \text{for } 2 \le j \le f - 1. \end{aligned} \tag{1.3}$$

Let $g \in S_f$ be the cyclic permutation of an f-tuple mapping its j-th entry to the $(j+1)$-th entry and the last entry to the first one. If $\sigma = \boldsymbol{\lambda}(\boldsymbol{r}) \otimes \eta$ is a generic weight of $\Gamma = \mathrm{GL}_2(\mathbb{F}_{p^f})$ for some determinant-power character η and $f > 1$, then $\mathrm{gr}^1_{\mathrm{cosoc}}\big(\mathrm{Ind}_B^\Gamma \chi(\sigma)^s\big)$ consists of f number of weights which can be described by the set

$$\big\{(g^i\boldsymbol{\mu})(\boldsymbol{\lambda}(\boldsymbol{r})) \otimes \det{}^{e(g^i\boldsymbol{\mu})(\boldsymbol{\lambda}(\boldsymbol{r}))}\eta : 0 \le i \le f-1\big\},$$

see [Breuil and Paškūnas 2012, Theorem 2.4].

For $\boldsymbol{\lambda} = (\lambda_0(x), \dots, \lambda_{f-1}(x))$ and $\boldsymbol{\lambda}' = (\lambda'_0(x), \dots, \lambda'_{f-1}(x)) \in (\mathbb{Z} \pm x)^f$, let

$$\boldsymbol{\lambda} \circ \boldsymbol{\lambda}' := \big(\lambda_0(\lambda'_0(x)), \dots, \lambda_{f-1}(\lambda'_{f-1}(x))\big) \in (\mathbb{Z} \pm x)^f.$$

Define an integer l to be equal to f (respectively, $2f$) if f is odd (respectively, even). Let

$$\begin{aligned}\boldsymbol{\mu}^{(0)} &:= (x, x, \dots, x),\\ \boldsymbol{\mu}^{(k)} &:= g^{k-1}\boldsymbol{\mu} \circ g^{k-2}\boldsymbol{\mu} \circ \cdots \circ g\boldsymbol{\mu} \circ \boldsymbol{\mu}, \quad \text{for all } 1 \le k \le l.\end{aligned}$$

For $\boldsymbol{r} \in \mathbb{Z}^f$, let

$$\begin{aligned}e_0(\boldsymbol{r}) &:= 0,\\ e_k(\boldsymbol{r}) &:= \sum_{j=0}^{k-1} e(g^j\boldsymbol{\mu})(\boldsymbol{\mu}^{(j)}(\boldsymbol{r})) \in \mathbb{Z}, \quad \text{for all } 1 \le k \le l.\end{aligned}$$

Lemma 1.4. (1) *We have* $\boldsymbol{\mu}^{(l)} = \boldsymbol{\mu}^{(0)} = (x, x, \dots, x)$ *in* $(\mathbb{Z} \pm x)^f$.

(2) *The f-tuples* $\boldsymbol{\mu}^{(1)}, \boldsymbol{\mu}^{(2)}, \dots, \boldsymbol{\mu}^{(l)}$ *are all distinct.*

(3) *The integer $e_l(\boldsymbol{r})$ is independent of $\boldsymbol{r}$ and is* 0 *modulo* $p^f - 1$.

Proof. (1) It follows from the definition of $\boldsymbol{\mu}^{(k)}$ that $\boldsymbol{\mu}^{(k)} = g^{k-1}\boldsymbol{\mu} \circ \boldsymbol{\mu}^{(k-1)}$ for all $1 \le k \le l$. Hence, for $1 \le k \le l$,

$$\mu_j^{(k)}(x) = \begin{cases} \mu_j^{(k-1)}(x) - 1 & \text{if } j \equiv 1-k \bmod f,\\ p-2-\mu_j^{(k-1)}(x) & \text{if } j \equiv 2-k \bmod f,\\ p-1-\mu_j^{(k-1)}(x) & \text{otherwise.}\end{cases} \tag{1.5}$$

It is now easy to check using (1.5) that for each j, $\mu_j^{(l)}(x) = x$.

(2) Let us assign to $\boldsymbol{\mu}^{(k)}$ an element $\boldsymbol{m}^{(k)} \in (\mathbb{Z}/2\mathbb{Z})^f$ by the rule that its j-th entry $m_j^{(k)}$ is 0 if and only if the sign of x in $\mu_j^{(k)}(x)$ is $+$. Here, $(\mathbb{Z}/2\mathbb{Z})^f$ is the direct sum of f copies of the group $\mathbb{Z}/2\mathbb{Z}$ of order 2 and has a natural action of $\langle g\rangle$ by group automorphisms. We show that the elements $\boldsymbol{m}^{(1)}, \boldsymbol{m}^{(2)}, \dots, \boldsymbol{m}^{(l)}$ are all distinct in $(\mathbb{Z}/2\mathbb{Z})^f$, which then implies part (2). We have $\boldsymbol{m}^{(1)} = (0, 1, 1, \dots, 1)$

and $\boldsymbol{m}^{(k)} = g^{k-1}\boldsymbol{m}^{(1)} + \boldsymbol{m}^{(k-1)}$ for $k > 1$, because $\boldsymbol{\mu}^{(k)} = g^{k-1}\boldsymbol{\mu}^{(1)} \circ \boldsymbol{\mu}^{(k-1)}$. Suppose $\boldsymbol{m}^{(k_1)} = \boldsymbol{m}^{(k_2)}$ for some $1 \le k_1 < k_2 \le l$. Then

$$\boldsymbol{m}^{(k_1)} = \boldsymbol{m}^{(k_2)} = g^{k_2-1}\boldsymbol{m}^{(1)} + g^{k_2-2}\boldsymbol{m}^{(1)} + \cdots + g^{k_1}\boldsymbol{m}^{(1)} + \boldsymbol{m}^{(k_1)}.$$

This gives that

$$g^{k_1+(k_2-k_1-1)}\boldsymbol{m}^{(1)} + g^{k_1+(k_2-k_1-2)}\boldsymbol{m}^{(1)} + \cdots + g^{k_1}\boldsymbol{m}^{(1)} = (0, 0, \ldots, 0).$$

The action of g^{-k_1} on both sides then gives

$$g^{k_2-k_1-1}\boldsymbol{m}^{(1)} + g^{k_2-k_1-2}\boldsymbol{m}^{(1)} + \cdots + \boldsymbol{m}^{(1)} = (0, 0, \ldots, 0),$$

i.e., $\boldsymbol{m}^{(k_2-k_1)} = (0, 0, \ldots, 0)$. This is a contradiction because $k_2 - k_1 < l$ and for any $l' < l$, $\boldsymbol{m}^{(l')} \neq (0, 0, \ldots, 0)$. The latter fact can be easily checked by looking at $m_0^{(l')}$ and $m_1^{(l')}$. One has $m_0^{(l')} \neq m_1^{(l')}$ for $l' < l$ except when $l = 2f$ and $l' = f$, in which case $m_0^{(l')} = m_1^{(l')} = 1$.

(3) Let us first consider f to be odd (so $l = f$). Expanding the expression for $e_l(\boldsymbol{r})$ and rearranging the terms, one gets

$$e_l(\boldsymbol{r}) = c + \sum_{k=1}^{f-1} \mu_0^{(k)}(r_0) + p \sum_{k=0}^{f-2} \mu_1^{(k)}(r_1) + p^2 \sum_{k=-1}^{f-3} \mu_2^{(k)}(r_2) + \cdots + p^{f-1} \sum_{k=-(f-2)}^{0} \mu_{f-1}^{(k)}(r_{f-1}),$$

where c is the constant term of the polynomial

$$e(g^{f-1}\boldsymbol{\mu}) + e(g^{f-2}\boldsymbol{\mu}) + \cdots + e(g\boldsymbol{\mu}) + e(\boldsymbol{\mu}),$$

and $k = -n$ for positive n means $k = f - n$ in the summation $\sum_k$. Using (1.5), one checks that each summand $\sum_k \mu_j^{(k)}(r_j)$ above (with appropriate lower and upper limit) is independent of r_j and equals $\frac{1}{2}(f-1)(p-1) - 1$. We leave it to the reader to check that $c \equiv (p^f - 1)/(p-1) \bmod p^f - 1$. Therefore,

$$e_l(\boldsymbol{r}) = \frac{p^f - 1}{p - 1}\left(\frac{f-1}{2}(p-1)\right) \equiv 0 \bmod p^f - 1.$$

The proof for even f is similar. In this case, $c \equiv 2\big((p^f - 1)/(p-1)\big) \bmod p^f - 1$, and $\sum_k \mu_j^{(k)}(r_j) = 2\big(\frac{1}{2}(f-1)(p-1) - 1\big)$ for all j. Therefore, $e_l(\boldsymbol{r})$ is again 0 modulo $p^f - 1$. $\square$

Theorem 1.6. *The group Γ admits a multiplicity-free cyclic module D_0.*

Proof. The case $f = 1$ is treated in Remark 1.2. Let $f > 1$. The proof is constructive. Start with a weight $\sigma_0 := \boldsymbol{r} \otimes \eta$ of Γ for some $1 \le r_0, \ldots, r_{f-1} \le p - 3$ and for some determinant-power character η. Observe that $\sigma_0 := \boldsymbol{\mu}^{(0)}(\boldsymbol{r}) \otimes \det^{e_0(\boldsymbol{r})}\eta$. Let

$$\sigma_k := \boldsymbol{\mu}^{(k)}(\boldsymbol{r}) \otimes \det{}^{e_k(\boldsymbol{r})}\eta$$

for all $1 \leq k \leq l$. We claim that the set $\{\sigma_1, \sigma_2, \dots, \sigma_l\}$ is the required set to construct a cyclic module. Using (1.5), one checks that

$$\mu_j^{(k)}(x) \in \{x, x-1, x+1, p-2-x, p-3-x, p-1-x\}$$

for all $1 \leq k \leq l$ and $0 \leq j \leq f-1$. Since $p > 3$, this means that the weights $\sigma_1, \sigma_2, \dots, \sigma_l$ are well defined. Further, by Lemma 1.4 and its proof, one sees that the weights $\sigma_1, \sigma_2, \dots, \sigma_l$ are all distinct generic weights and $\sigma_l = \sigma_0$. Now let $1 \leq k \leq l$. We know that the weights appearing in $\mathrm{gr}^1_{\mathrm{cosoc}}\big(\mathrm{Ind}_B^\Gamma \chi(\sigma_{k-1})^s\big)$ are

$$\big\{(g^i \boldsymbol{\mu})(\boldsymbol{\mu}^{(k-1)}(\boldsymbol{r})) \otimes \det{}^{e(g^i\boldsymbol{\mu})(\boldsymbol{\mu}^{(k-1)}(\boldsymbol{r}))} \det{}^{e_{k-1}(\boldsymbol{r})}\eta : 0 \leq i \leq f-1\big\}.$$

In particular, $\mathrm{gr}^1_{\mathrm{cosoc}}\big(\mathrm{Ind}_B^\Gamma \chi(\sigma_{k-1})^s\big)$ contains the weight

$$(g^{k-1}\boldsymbol{\mu})(\boldsymbol{\mu}^{(k-1)}(\boldsymbol{r})) \otimes \det{}^{e(g^{k-1}\boldsymbol{\mu})(\boldsymbol{\mu}^{(k-1)}(\boldsymbol{r}))} \det{}^{e_{k-1}(\boldsymbol{r})}\eta = \boldsymbol{\mu}^{(k)}(\boldsymbol{r}) \otimes \det{}^{e_k(\boldsymbol{r})}\eta = \sigma_k.$$

Since $\mathrm{gr}^0_{\mathrm{cosoc}}\big(\mathrm{Ind}_B^\Gamma \chi(\sigma_{k-1})^s\big) = \mathrm{cosoc}_\Gamma \mathrm{Ind}_B^\Gamma \chi(\sigma_{k-1})^s = \sigma_{k-1}^{[s]}$ is simple, $E(\sigma_k, \sigma_{k-1}^{[s]})$ exists and is equal to the unique quotient of $\mathrm{Ind}_B^\Gamma \chi(\sigma_{k-1})^s$ with socle σ_k. Since $(\mathrm{Ind}_B^\Gamma \chi(\sigma_{k-1})^s)^U = \chi(\sigma_{k-1}) \oplus \chi(\sigma_{k-1})^s$, we have $E(\sigma_k, \sigma_{k-1}^{[s]})^U = \chi(\sigma_k) \oplus \chi(\sigma_{k-1})^s$. Therefore, it follows that $D_0 := \bigoplus_{k=1}^l E(\sigma_k, \sigma_{k-1}^{[s]})$ is a cyclic module of Γ and has socle of length l.

It remains to show that D_0 is multiplicity-free. By definition, $\mathrm{soc}_\Gamma D_0$ is multiplicity-free. Thus also $\sigma_{k_1}^{[s]} \neq \sigma_{k_2}^{[s]}$ for any $k_1 \neq k_2$, $1 \leq k_1, k_2 \leq l$, because $(\sigma^{[s]})^{[s]} = \sigma$. Now, if $\sigma_{k_1} = \sigma_{k_2-1}^{[s]}$ for some $1 \leq k_1, k_2 \leq l$, then there is a nonsplit Γ-extension between σ_{k_1} and σ_{k_2}. Consider the elements $\boldsymbol{m}^{(k_1)}$ and $\boldsymbol{m}^{(k_2)}$ of $(\mathbb{Z}/2\mathbb{Z})^f$ assigned to $\boldsymbol{\mu}^{(k_1)}$ and $\boldsymbol{\mu}^{(k_2)}$, respectively, in the proof of Lemma 1.4. By [Breuil and Paškūnas 2012, Lemma 5.6 (i)], the number of 1s in $\boldsymbol{m}^{(k_1)}$ and $\boldsymbol{m}^{(k_2)}$ have different parity. However, if f is odd, then one checks that the number of 1s in $\boldsymbol{m}^{(k)}$ is always even for all $1 \leq k \leq l$ implying that $\sigma_{k_1} \neq \sigma_{k_2-1}^{[s]}$ for any $1 \leq k_1, k_2 \leq l$. If f is even, then it is not true that the number of 1s in $\boldsymbol{m}^{(k)}$ is always either even or odd, and it is a priori possible that $\sigma_k = \sigma_{k+f}^{[s]}$, because $\boldsymbol{m}^{(k)} + \boldsymbol{m}^{(k+f)} = (1, 1, \dots, 1)$. However, using (1.5), one explicitly checks that $\sigma_k \neq \sigma_{k+f}^{[s]}$ for any $1 \leq k \leq l$. $\square$

Remark 1.7. When f is odd, the argument given in the proof of Theorem 1.6 shows that any cyclic module of Γ is multiplicity-free. This is not true when f is even (see the next remark). We further point out that the definition of the f-tuple $\boldsymbol{\mu}$ is not canonical. Any other cyclic permutation of $\boldsymbol{\mu}$ also gives rise to a cyclic module of Γ by the same construction as above. We expect that all multiplicity-free cyclic modules of Γ are obtained in this way, and thus any multiplicity-free cyclic module of Γ has socle of length l.

Example 1.8. The construction in the proof of Theorem 1.6 produces following multiplicity-free cyclic modules for $f = 2, 3$. The weights are written without their twists by determinant-power characters.

$$\begin{aligned} f=2: \quad D_0 = {} & (r_0-1,\, p-2-r_1) — (p-1-r_0,\, p-1-r_1) \\ & \oplus (p-1-r_0,\, p-3-r_1) — (p-r_0,\, r_1+1) \\ & \oplus (p-2-r_0,\, r_1+1) — (r_0,\, r_1+2) \\ & \oplus (r_0,\, r_1) — (r_0+1,\, p-2-r_1). \end{aligned}$$

$$\begin{aligned} f=3: \quad D_0 = {} & (r_0-1,\, p-2-r_1,\, p-1-r_2) — (p-1-r_0,\, p-1-r_1,\, p-1-r_2) \\ & \oplus (p-1-r_0,\, r_1+1,\, p-2-r_2) — (p-r_0,\, r_1+1,\, r_2) \\ & \oplus (r_0,\, r_1,\, r_2) — (r_0,\, p-2-r_1,\, r_2+1). \end{aligned}$$

Remark 1.9. Let $\mathbb{Q}_{p^f}$ denote the degree f unramified extension of $\mathbb{Q}_p$. The multiplicity-free cyclic module of $\mathrm{GL}_2(\mathbb{F}_{p^2})$ (respectively, $\mathrm{GL}_2(\mathbb{F}_{p^3})$) in Example 1.8 occurs as a submodule of $D_0(\rho)$ of a Diamond diagram associated to an irreducible (respectively, reducible split) generic Galois representation ρ of $\mathbb{Q}_{p^2}$ (respectively, $\mathbb{Q}_{p^3}$) (see [Breuil and Paškūnas 2012, §14]).

Schein [2022] constructed irreducible supersingular representations of $G = \mathrm{GL}_2(F)$ with K-socles compatible with Serre's weight conjecture for a ramified p-adic field F of residue degree 2. His construction is based on constructing cyclic modules of $\mathrm{GL}_2(\mathbb{F}_{p^2})$ with prescribed socles. The involved cyclic modules of $\mathrm{GL}_2(\mathbb{F}_{p^2})$ have socles of lengths $> l$ and are not multiplicity-free (see [Schein 2022, Example 3.9]).

2. Cyclic diagrams

We recall the definition of a diagram from [Breuil and Paškūnas 2012, §9]. A *diagram* (of G) is a data (D_0, D_1, r) consisting of a smooth KZ-representation D_0, a smooth N-representation D_1 and an IZ-equivariant map $r : D_1 \to D_0$. A diagram (D_0, D_1, r) is called a *basic* 0-*diagram* if ϖ acts trivially on D_0 and D_1, and the map r induces an isomorphism $D_1 \cong D_0^{I(1)}$ of IZ-representations. Now, let $D_0 = \bigoplus_{i=1}^n E(\sigma_i, \sigma_{i-1}^{[s]})$ be a multiplicity-free cyclic module of Γ. Viewing D_0 as a smooth KZ-representation via $KZ \twoheadrightarrow \Gamma$ with trivial ϖ-action, $D_1 := D_0^{I(1)} = D_0^U$ can be equipped with a smooth N-action by defining the action of $\Pi : \chi(\sigma_i) \to \chi(\sigma_i)^s$ to be multiplication by a scalar $t_i \in \overline{\mathbb{F}}_p^\times$ for all i after choosing bases. This defines a unique N-action on D_1 such that ϖ-acts trivially and gives a basic 0-diagram (D_0, D_1, can) where $\mathrm{can} : D_1 \hookrightarrow D_0$ is the canonical inclusion.

Definition 2.1. A basic 0-diagram (D_0, D_1, can) on a multiplicity-free cyclic module D_0 is called a *cyclic diagram*.

Note that a cyclic diagram exists for all G by Theorem 1.6.

Lemma 2.2. *Let* (D_0, D_1, can) *be a cyclic diagram on a cyclic module* $D_0 = \bigoplus_{i=1}^n E(\sigma_i, \sigma_{i-1}^{[s]})$, *and let* $\Pi : \chi(\sigma_i) \to \chi(\sigma_i)^s$ *be given by multiplication by a scalar* $t_i \in \overline{\mathbb{F}}_p^\times$ *for all* $1 \le i \le n$. *Then*

(1) (D_0, D_1, can) *is irreducible, and*

(2) *the isomorphism class of* (D_0, D_1, can) *is determined by the product* $t_1 \cdots t_n \in \overline{\mathbb{F}}_p^\times$.

Proof. (1) Let $V \subseteq D_0$ be a nonzero KZ-subrepresentation such that $V^{I(1)}$ is stable under the action of N. Then, for some $1 \le i \le n$, V contains σ_i and thus also contains $\chi(\sigma_i)$. Since $\Pi(\chi(\sigma_i)) = \chi(\sigma_i)^s$, V contains $\chi(\sigma_i)^s$. By Frobenius reciprocity, there is a nonzero map $\mathrm{Ind}_B^\Gamma \chi(\sigma_i)^s \to V$ whose image is $E(\sigma_{i+1}, \sigma_i^{[s]})$. Thus $E(\sigma_{i+1}, \sigma_i^{[s]}) \subseteq V$. Continuing in this way, we get that $D_0 = \bigoplus_{i=1}^n E(\sigma_i, \sigma_{i-1}^{[s]}) \subseteq V$. Hence $V = D_0$.

(2) Let $D = (D_0, D_1, \mathrm{can})$, and let D' be a diagram isomorphic to D. Then D' is also a cyclic diagram on the cyclic module D_0. Let $\Pi : \chi(\sigma_i) \to \chi(\sigma_i)^s$ in D' be given by multiplication by a scalar $t_i' \in \overline{\mathbb{F}}_p^\times$ for all $1 \le i \le n$. As the diagrams D and D' are isomorphic, there is an isomorphism $\varphi : D_0 \to D_0$ of KZ-representations such that $\varphi(\Pi v) = \Pi\varphi(v)$ for all $v \in D_1$. Since D_0 is multiplicity-free, an easy application of Schur's lemma gives

$$\mathrm{End}_{KZ}(D_0) = \mathrm{End}_\Gamma(D_0) \cong \mathrm{End}_\Gamma(E(\sigma_1, \sigma_n^{[s]})) \times \cdots \times \mathrm{End}_\Gamma(E(\sigma_n, \sigma_{n-1}^{[s]})) \cong \overline{\mathbb{F}}_p^n.$$

So, if the isomorphism φ corresponds to $(a_1, \dots, a_n) \in (\overline{\mathbb{F}}_p^\times)^n$, then $(a_1, \dots, a_n)$ satisfies $a_i = a_{i-1} t_{i-1}' (t_{i-1})^{-1}$ for all $1 \le i \le n$. This implies that $t_1 t_2 \cdots t_n = t_1' t_2' \cdots t_n'$. On the other hand, if $t_1 t_2 \cdots t_n = t_1' t_2' \cdots t_n'$, then the scalar multiplications by $a_i = \prod_{j=1}^{i-1} t_j' t_j^{-1}$ on $E(\sigma_i, \sigma_{i-1}^{[s]})$ with $a_1 = 1$ give an isomorphism of cyclic diagrams on D_0. See also [Dotto and Le 2021, Proposition 4.4]. □

For a cyclic diagram $D = (D_0, D_1, \mathrm{can})$, we introduce the notation $t(D) = t_1 t_2 \cdots t_n$ for later use. With this notation, Lemma 2.2 (2) says that the map $D \mapsto t(D)$ gives a bijection between the set of isomorphism classes of cyclic diagrams on D_0 and $\overline{\mathbb{F}}_p^\times$.

3. Supersingular representations

We now use cyclic diagrams to show that $G = \mathrm{GL}_2(F)$ admits infinitely many smooth admissible irreducible supersingular representations when F has residue degree $f > 1$. It uses the following key theorem of Breuil and Paškūnas:

Theorem 3.1. *Let* (D_0, D_1, r) *be a basic* 0*-diagram such that* $D_0^{K(1)}$ *is finite-dimensional. Then there exists a smooth admissible representation* π *of* G *on which* ϖ *acts trivially, and such that*:

(1) *one has the inclusion* $(D_0, D_1, r) \subseteq (\pi|_{KZ}, \pi|_N, \mathrm{id})$ *of diagrams*,

(2) *π is generated as a G-representation by D_0 and*

(3) $\mathrm{soc}_\Gamma D_0 = \mathrm{soc}_K D_0 = \mathrm{soc}_K \pi$.

Moreover, if (D_0, D_1, r) is irreducible, then any such G-representation π is irreducible.

Proof. The first part is essentially proved in [Breuil and Paškūnas 2012, Theorem 9.8]. See also the proof of [Breuil and Paškūnas 2012, Theorem 19.8 (i)]. The proof of the irreducibility of π is given in unpublished lecture notes of Breuil [Breuil 2007, Proposition 5.11]. We reproduce it here: let $\pi' \subseteq \pi$ be a nonzero G-subrepresentation. Then $\pi' \cap D_0$ is a nonzero KZ-subrepresentation of D_0 by (3), and $(\pi' \cap D_0)^{I(1)} = \pi' \cap D_1$ is stable under the action of Π. Hence, $(\pi' \cap D_0, (\pi' \cap D_0)^{I(1)}, \mathrm{can})$ is a nonzero subdiagram of (D_0, D_1, r). By the irreducibility of (D_0, D_1, r), we get that $\pi' \cap D_0 = D_0$. Hence, $\pi' = \pi$ using (2). □

When F has residue degree 1, the cyclic diagrams are the basic 0-diagrams on principal series representations of $\mathrm{GL}_2(\mathbb{F}_p)$ (Theorem 1.6), and thus Theorem 3.1 applied to cyclic diagrams gives rise to irreducible (ramified) principal series representations of G (see [Breuil and Paškūnas 2012, §10]). In contrast, when F has residue degree $f > 1$, Theorem 3.1 applied to cyclic diagrams gives rise to irreducible supersingular representations of G as we shall see now. Recall from [Barthel and Livné 1994] that a smooth irreducible representation π of G with central character is a quotient of $\pi(\sigma, \lambda, \chi) := (\mathrm{ind}_{KZ}^G \sigma/(T - \lambda)) \otimes (\chi \circ \det)$ for some weight σ, some $\lambda \in \overline{\mathbb{F}}_p^\times$ and some smooth character $\chi : F^\times \to \overline{\mathbb{F}}_p^\times$. Here, $\mathrm{ind}_{KZ}^G \sigma$ is the compactly induced representation with ϖ acting trivially on σ and $T \in \mathrm{End}_G(\mathrm{ind}_{KZ}^G \sigma)$ is the distinguished Hecke operator. By definition, π is supersingular if it is a quotient of some $\pi(\sigma, 0, \chi)$. The representations $\pi(\sigma, 0, \chi)$ are called *universal supersingular modules*.

Theorem 3.2. *Let F be a nonarchimedean local field of residue degree $f > 1$. Then the group G admits infinitely many nonisomorphic smooth admissible irreducible supersingular representations on which ϖ acts trivially. Further, all these representations have the same K-socle.*

Proof. We use the existence of multiplicity-free cyclic modules from Theorem 1.6 to construct a family of cyclic diagrams of G. Let D_0 be a multiplicity-free cyclic module constructed in Theorem 1.6 and for each $t \in \overline{\mathbb{F}}_p^\times$, let $D(t) = (D_0, D_1, \mathrm{can})$ be a cyclic diagram on D_0 such that $t(D(t)) = t$. By Theorem 3.1, there is a smooth admissible representation $\pi(t)$ (fix one for each $D(t)$) of G with trivial action of ϖ such that $D(t) \subseteq (\pi(t)|_{KZ}, \pi(t)|_N, \mathrm{can})$, D_0 generates $\pi(t)$ as a G-representation and $\mathrm{soc}_K D_0 = \mathrm{soc}_K \pi(t)$. We claim that $\{\pi(t)\}_{t \in \overline{\mathbb{F}}_p^\times}$ is the desired family of representations of G. Indeed, by Lemma 2.2 (1) and Theorem 3.1, each $\pi(t)$ is an irreducible G-representation.

Suppose there is an isomorphism $\varphi : \pi(t) \xrightarrow{\sim} \pi(t')$ of G-representations for $t \neq t'$. It restricts to an isomorphism $\varphi : D_0 \xrightarrow{\sim} D_0$ of KZ-representations, because $\operatorname{soc}_K D_0 = \operatorname{soc}_K \pi(t) = \operatorname{soc}_K \pi(t')$ and because D_0 is multiplicity-free. This gives rise to an isomorphism $D(t) \cong D(t')$ of cyclic diagrams which contradicts Lemma 2.2 (2). Thus $\pi(t)$ and $\pi(t')$ are not isomorphic for $t \neq t'$.

It remains to show that each $\pi(t)$ is supersingular. Let $\sigma \in \operatorname{soc}_K \pi(t)$. Then $\operatorname{Hom}_G(\operatorname{ind}_{KZ}^G \sigma, \pi(t)) = \operatorname{Hom}_K(\sigma, \pi(t)^{K(1)})$ is a nonzero finite-dimensional $\overline{\mathbb{F}}_p$-vector space, because $\pi(t)$ is admissible. Hence, $\operatorname{Hom}_G(\operatorname{ind}_{KZ}^G \sigma, \pi(t))$ contains a nonzero eigenvector for the action of Hecke operator T with eigenvalue, let's say, λ. As $\pi(t)$ is irreducible, it follows that $\pi(t)$ is a quotient of $\pi(\sigma, \lambda, 1)$. If $\lambda \neq 0$, then by [Barthel and Livné 1994, Lemma 28 and Theorem 33], we have $\dim_{\overline{\mathbb{F}}_p} \pi(t)^{I(1)} \leq 2$. However, as $f > 1$, $\operatorname{soc}_K D_0$ is not irreducible, and thus $\dim_{\overline{\mathbb{F}}_p} D_0^{I(1)} = \dim_{\overline{\mathbb{F}}_p} D_1 \geq 4$ (in fact, $\dim_{\overline{\mathbb{F}}_p} D_1 = 2l$). But this implies that $\dim_{\overline{\mathbb{F}}_p} \pi(t)^{I(1)} > 2$, because $\pi(t)$ contains D_0. So we get a contradiction. Thus, $\lambda = 0$ and $\pi(t)$ is supersingular. □

Recall from [Barthel and Livné 1994, Corollary 31] that $\pi(\sigma, \lambda, \chi)$ has a unique (admissible) irreducible quotient for $\lambda \neq 0$. However, for $\lambda = 0$, we have the following result as an immediate corollary of Theorem 3.2:

Corollary 3.3. *Let F be a nonarchimedean local field of residue degree $f > 1$. Then the universal supersingular module $\pi(\sigma, 0, \chi)$ of G has infinitely many nonisomorphic admissible irreducible quotients for any given weight $\sigma = \boldsymbol{r} \otimes \eta$, with $1 \leq r_0, \ldots, r_{f-1} \leq p-3$ and any smooth character χ.*

Proof. As in the proof of Theorem 3.2, consider a family $\{D(t)\}_{t \in \overline{\mathbb{F}}_p^\times}$ of cyclic diagrams on a cyclic module D_0 from Theorem 1.6 whose socle contains the given weight σ, and let $\{\pi(t)\}_{t \in \overline{\mathbb{F}}_p^\times}$ be a corresponding family of smooth admissible irreducible supersingular G-representations. By the proof of Theorem 3.2, each $\pi(t)$ occurs as a quotient of $\pi(\sigma, 0, 1)$. So the corollary holds for $\pi(\sigma, 0, 1)$, and hence also for its smooth twist $\pi(\sigma, 0, \chi)$. □

Remark 3.4. If $F = \mathbb{Q}_{p^f}$ with $f > 1$, then the recent works of Le [2019] and Ghate and Sheth [2020] show that the universal supersingular modules of G also admit infinitely many nonisomorphic nonadmissible irreducible quotients.

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