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ORIGINAL PAPER

Birkhoff-James orthogonality and its local symmetry in some sequence spaces

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Abstract
We study Birkhoff-James orthogonality and its local symmetry in some sequence spaces namely ℓ_p, for $1 \leq p \leq \infty$, $p \neq 2$, c, c_0 and c_{00}. Using the characterization of the local symmetry of Birkhoff-James orthogonality, we characterize isometries of each of these spaces onto itself and obtain the Banach-Lamperti theorem for onto operators on the sequence spaces.

Keywords Birkhoff-James orthogonality · Smooth points · Left-symmetric points · Right-symmetric points · Onto isometries · Ultrafilters

Mathematics Subject Classification Primary 46B45 · Secondary 46B20 · 46B25

1 Introduction

In recent times, symmetry of Birkhoff-James orthogonality has been a topic of considerable interest [1, 8, 9, 12–14, 19]. It is now well known that the said symmetry plays an important role in the study of the geometry of Banach spaces. The present article aims to explore Birkhoff-James orthogonality and its local symmetry in some well studied sequence spaces. As an outcome of our exploration, we acquire the Banach-Lamperti Theorem [15] for onto

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operators on some classical sequence spaces by characterizing the onto isometries of the same. We would like to mention that recently such a study has been carried out in the context of ℓ_p^n spaces for $1 \leq p \leq \infty$, $p \neq 2$ in [3]. It should also be noted that Birkhoff-James orthogonality is closely related to the norm derivatives and their various properties, which find applications in understanding the geometry of Banach spaces. Recently, such a study has been carried out in [22], where the notion of local smoothness induced by the norm derivatives has been completely characterized. We refer the readers to some of the related works [4, 5, 25] for more information in this context.

Let us now establish the relevant notations and terminologies to be used throughout the article. Denote the scalar field $\mathbb{R}$ or $\mathbb{C}$ by $\mathbb{K}$ and recall the sign function $\operatorname{sgn} : \mathbb{K} \to \mathbb{K}$, given by

$$\operatorname{sgn}(x) = \begin{cases} \frac{x}{|x|}, & x \neq 0, \\ 0, & x = 0. \end{cases}$$

Consider a normed linear space $\mathbb{X}$ over $\mathbb{K}$ and denote its continuous dual by $\mathbb{X}^*$. Let $J(x)$ denote the collection of all support functionals of a non-zero $x \in \mathbb{X}$, i.e.,

$$J(x) := \{f \in \mathbb{X}^* : \|f\| = 1,\ |f(x)| = \|x\|\}. \tag{1.1}$$

A non-zero element $x \in \mathbb{X}$ is said to be *smooth* if $J(x)$ is singleton.

Given $x, y \in \mathbb{X}$, x is said to be *Birkhoff-James orthogonal* to y [2], denoted by $x \perp_B y$, if

$$\|x + \lambda y\| \geq \|x\|, \quad \textit{for all } \lambda \in \mathbb{K}.$$

James proved in [11] that $x \perp_B y$ if and only if $x = 0$ or there exists $f \in J(x)$ such that $f(y) = 0$. In the same article he also proved that a non-zero $x \in \mathbb{X}$ is smooth if and only if Birkhoff-James orthogonality is right additive at x, i.e.,

$$x \perp_B y,\ x \perp_B z \ \Rightarrow\ x \perp_B (y + z), \quad \textit{for every } y, z \in \mathbb{X}.$$

Birkhoff-James orthogonality is not symmetric in general, i.e., $x \perp_B y$ does not necessarily imply that $y \perp_B x$. In fact, James proved in [10] that Birkhoff-James orthogonality is symmetric in a normed linear space of dimension higher than 2 if and only if the space is an inner product space. However, the importance of studying the local symmetry of Birkhoff-James orthogonality in describing the geometry of normed linear spaces has been illustrated in [3, Theorem 2.11], [18, Corollary 2.3.4.]. Let us recall the following definition in this context from [17], which is of paramount importance in our present study.

Definition 1 An element x of a normed linear space $\mathbb{X}$ is said to be *left-symmetric* (*resp. right-symmetric*) if

$$x \perp_B y \ \Rightarrow\ y \perp_B x \ \ (\textit{resp. } y \perp_B x \ \Rightarrow\ x \perp_B y),$$

for every $y \in \mathbb{X}$.

The left-symmetric and the right-symmetric points of ℓ_p^n spaces where $1 \leq p \leq \infty$, $p \neq 2$, were characterized in [3]. Here we take a step forward towards generalizing these results in the following sequence spaces: ℓ_p, for $1 \leq p \leq \infty$ and $p \neq 2$, c, c_0 and c_{00}. Characterizations of the smooth points, the left-symmetric points and the right-symmetric points of a given Banach space are of paramount importance in understanding the geometry of the Banach space. We refer the readers to [1, 8, 9, 12–14, 19–21, 23, 24] for some prominent work in this direction.

The local symmetry of Birkhoff-James orthogonality in a Banach space also plays an important role in determining the isometric isomorphisms on the space. Let us observe that Corollary 2.3.4. of [18] in this regard can be stated in the following generalized form:

Corollary 1.1 *Let $\mathbb{X}$ and $\mathbb{Y}$ be two normed linear spaces and let $T : \mathbb{X} \to \mathbb{Y}$ be an onto linear isometry. Then $x \in \mathbb{X}$ is left-symmetric (resp. right-symmetric) if and only if $T(x) \in \mathbb{Y}$ is left-symmetric (resp. right-symmetric).*

This result is used for proving the Banach-Lamperti Theorem for onto operators on the sequence spaces, i.e., for the case where the measure space is $\mathbb{N}$ equipped with the counting measure by finding the onto isometries of ℓ_p, for $1 \leq p \leq \infty$ and $p \neq 2$. We also do the same for the spaces c, c_0 and c_{00} as a direct consequence of the results characterizing the local symmetry of Birkhoff-James orthogonality in these spaces. It can be noted that Lamperti's idea in [15] uses the concept of convexity, concavity and Radon-Nikodym derivatives along with the properties of the integral involved in the definition of the L_p norm and therefore cannot be generalized in case of $p = \infty$. Our approach using the local symmetry of Birkhoff-James orthogonality however, has no such restrictions and hence is applied for the $p = \infty$ case as well.

In the first section we completely characterize Birkhoff-James orthogonality in ℓ_∞ over $\mathbb{K}$ and then characterize the left-symmetric and the right-symmetric points of the space. As a corollary of our results, we obtain characterizations of Birkhoff-James orthogonality and the left-symmetric and the right-symmetric points in c, c_0 and c_{00}. Using Corollary 1.1, we find the isometries of each of these spaces onto itself.

In the second and third sections we obtain the same characterizations in ℓ_1 and ℓ_p spaces for $1 < p < \infty$ and $p \neq 2$ respectively. Observe that the $p = 2$ case is trivial since ℓ_2 is a Hilbert space. We also find all the isometries of these spaces onto themselves using Corollary 1.1.

Since we are proving the Banach-Lamperti Theorem for onto operators on the sequence spaces by this isometry characterization, we define signed permutation operators on $\mathbb{K}^{\mathbb{N}}$, the vector space of all sequences in $\mathbb{K}$.

Definition 2 A map $T : \mathbb{K}^{\mathbb{N}} \to \mathbb{K}^{\mathbb{N}}$ is said to be a *signed permutation operator* if there exists a bijection $\sigma : \mathbb{N} \to \mathbb{N}$ such that

$$T(x) = \left(c_n x_{\sigma(n)}\right), \quad x = (x_n)_{n \in \mathbb{N}} \in \mathbb{K}^{\mathbb{N}},$$

where $|c_n| = 1$, for every $n \in \mathbb{N}$.

2 Geometry of ℓ_∞

In this section, we characterize Birkhoff-James orthogonality between two elements of ℓ_∞ and then obtain characterizations of the smooth points, the left-symmetric points and the right-symmetric points of the space. To serve our purpose, we review some basic facts about the convergence of a $\mathbb{K}$-valued sequence under an ultrafilter on $\mathbb{N}$. A detailed treatment on ultrafilters can be found in [6, 7].

2.1 Ultrafilters on $\mathbb{N}$ and convergence of sequences under them

We begin by recalling a few definitions.

Definition 3 (*Filters and Ultrafilters*) A non-empty subset $\mathcal{F}$ of the power set of a non-empty set X is said to be a *filter* on X if

(i) $\emptyset \notin \mathcal{F}$.
(ii) $A,\ B \in \mathcal{F} \ \ \Rightarrow \ \ A \cap B \in \mathcal{F}$.
(iii) $A \in \mathcal{F}$ *and* $A \subset B \ \ \Rightarrow B \in \mathcal{F}$.

A filter $\mathcal{U}$ on X is said to be an *ultrafilter* on X if any filter on X containing $\mathcal{U}$ must be $\mathcal{U}$.

Note that a filter $\mathcal{U}$ is an ultrafilter if and only if for every $A \subset X$, $A \in \mathcal{U}$ or $X \backslash A \in \mathcal{U}$ if and only if for every $n \in \mathbb{N}$ and $A_1, A_2, \ldots A_n \subset X$, $X = \bigcup_{i=1}^{n} A_i$ implies $A_i \in \mathcal{U}$, for some $1 \leq i \leq n$. An ultrafilter $\mathcal{U}$ on X is called a *principal ultrafilter* if there exists $x \in X$ such that $\{x\} \in \mathcal{U}$. An ultrafilter which is not a principal ultrafilter is called a *free ultrafilter*.

We also recall the definition of a filter base.

Definition 4 A non-empty subset $\mathcal{B}$ of the power set of a non-empty set X is said to be a filter base if

(i) $\emptyset \notin \mathcal{B}$.
(ii) If $A,\ B \in \mathcal{B}$, there exists $C \in \mathcal{B}$ such that $C \subset A \cap B$.

Note that every filter base $\mathcal{B}$ is contained in a unique minimal filter given by $\{A \subset X : B \subset A,\ \textit{for some}\ B \in \mathcal{B}\}$. Since every filter is contained in some ultrafilter by Zorn's lemma, every filter base is also contained in some ultrafilter.

We now focus on the case $X = \mathbb{N}$. Recall the definition of convergence along a filter:

Definition 5 (*Convergence along a filter*) Let $(x_n)_{n\in\mathbb{N}}$ be a sequence in $\mathbb{K}$ and let $\mathcal{F}$ be a filter on $\mathbb{N}$. Then we say x_n converges to some point $x_0 \in \mathbb{K}$, denoted by $\lim\limits_{\mathcal{F}} x_n$, under $\mathcal{F}$ if for every $\epsilon > 0$,

$$\{n \in \mathbb{N} : |x_n - x_0| < \epsilon\} \in \mathcal{F}.$$

Let us state a few well-known results pertaining to the convergence of a sequence under a filter without proof.

Theorem 2.1 *Let $\mathcal{F}$ is a filter on $\mathbb{N}$ and $(x_n)_{n\in\mathbb{N}}$ and $(y_n)_{n\in\mathbb{N}}$ be two sequences in $\mathbb{K}$. Then the following hold true:*

(i) $\lim_{\mathcal{F}} x_n$*, if exists, is unique.*
(ii) If $x = \lim_{\mathcal{F}} x_n$*, and* $f : \mathbb{K} \to \mathbb{K}$ *is continuous, then* $f(x) = \lim_{\mathcal{F}} f(x_n)$.
(iii) If $x = \lim_{\mathcal{F}} x_n$ *and* $y = \lim_{\mathcal{F}} y_n$*, then* $x + \lambda y = \lim_{\mathcal{F}} (x_n + \lambda y_n)$*, for any* $\lambda \in \mathbb{K}$*. Also,* $xy = \lim_{\mathcal{F}} x_n y_n$ *and* $\frac{x}{y} = \lim_{\mathcal{F}} \frac{x_n}{y_n}$ *if* $y_n \neq 0 \neq y$*, for every* $n \in \mathbb{N}$.
(iv) If $\mathcal{F}$ is an ultrafilter and $(x_n)_{n\in\mathbb{N}}$ is bounded, then $\lim_{\mathcal{F}} x_n$ *exists.*

It is trivial to see that if $\mathcal{U}$ is the principal ultrafilter containing $\{N\}$, for some $N \in \mathbb{N}$, $\lim\limits_{\mathcal{U}} x_n = x_N$. We establish the following result pertaining to the limit of a bounded sequence under any free ultrafilter.

Proposition 2.2 *Let $(x_n)_{n\in\mathbb{N}}$ and $(y_n)_{n\in\mathbb{N}}$ be two bounded sequences in $\mathbb{K}$. Then the following hold:*

(i) $x_0 = \lim_{\mathcal{U}} x_n$*, for some free ultrafilter $\mathcal{U}$ on $\mathbb{N}$ if and only if x_0 is a subsequential limit of x_n.*
(ii) $x_0 = \lim_{\mathcal{U}} x_n$ *and* $y_0 = \lim_{\mathcal{U}} y_n$*, for some free ultrafilter $\mathcal{U}$ on $\mathbb{N}$ if and only if there exists an increasing sequence $(n_k)_{k\in\mathbb{N}}$ on $\mathbb{N}$ such that $x_{n_k} \to x_0$ and $y_{n_k} \to y_0$ as $k \to \infty$.*

Proof Recall that an ultrafilter is free if and only if it contains no finite subset of $\mathbb{N}$.

(i) We first prove the necessity. Suppose $x_0 = \lim_{\mathcal{U}} x_n$, for some free ultrafilter $\mathcal{U}$ on $\mathbb{N}$. If x_0 is not a subsequential limit of x_n, there exists $\delta > 0$ such that $\{n \in \mathbb{N} : |x_n - x_0| < \delta\}$ is finite which is a contradiction since $\mathcal{U}$ is a free ultrafilter.

Now, assume that $(n_k)_{k\in\mathbb{N}}$ is an increasing sequence in $\mathbb{N}$ such that $x_{n_k} \to x_0$, for some $x_0 \in \mathbb{K}$ and consider the following set:

$$\{A \subset \mathbb{N} : \{n_k : k \in \mathbb{N}\}\backslash A \text{ is finite}\}. \tag{2.1}$$

Observe that the set defined by (2.1) is a filter base and therefore is contained in some ultrafilter $\mathcal{U}$ on $\mathbb{N}$. Clearly, no finite subset of $\mathbb{N}$ is an element of $\mathcal{U}$ and hence $\mathcal{U}$ is a free ultrafilter. Since, $\lim_{\mathcal{U}} x_n = x_0$, the sufficiency is established.

(ii) The sufficiency can be proved like (i). To prove the necessity, let there be no sequence $(n_k)_{k\in\mathbb{N}}$ in $\mathbb{N}$ such that $x_{n_k} \to x_0$ and $y_{n_k} \to y_0$. Then there exist $\delta_1, \delta_2 > 0$ such that

$$\{n \in \mathbb{N} : |x_n - x_0| < \delta_1\} \cap \{n \in \mathbb{N} : |y_n - y_0| < \delta_2\} \text{ is finite},$$

a contradiction since $\mathcal{U}$ is a free ultrafilter and therefore contains no finite subset of $\mathbb{N}$. This proves the necessity.

2.2 Birkhoff-James orthogonality and smoothness of a point in ℓ_∞

We begin this sub-section by recalling a few known results:

Theorem 2.3 *The space ℓ_∞ is isometrically isomorphic to $C(\beta\mathbb{N})$, the Banach space of all $\mathbb{K}$-valued continuous functions on $\beta\mathbb{N}$ equipped with the supremum norm, where $\beta\mathbb{N}$ denotes the Stone-Čech compactification of $\mathbb{N}$. Recalling the homeomorphism between $\beta\mathbb{N}$ and the space of all ultrafilters on $\mathbb{N}$ equipped with the Stone topology, one can explicitly write down such an isometric isomorphism $T : \ell_\infty \to C(\beta\mathbb{N})$, given by*

$$T\left((x_n)_{n\in\mathbb{N}}\right)(\mathcal{U}) = \lim_{\mathcal{U}} x_n, \quad \mathcal{U} \text{ an ultrafilter on } \mathbb{N}. \tag{2.2}$$

Also, since $\beta\mathbb{N}$ is compact Hausdorff, by an application of the Riesz representation Theorem in measure theory, we have the following result:

Theorem 2.4 *The dual space of $C(\beta\mathbb{N})$ is isometrically isomorphic to the space of all regular $\mathbb{K}$-valued Borel measures on $\beta\mathbb{N}$ equipped with the total variation norm and the functional corresponding to a regular $\mathbb{K}$-valued Borel measure μ acting on $C(\beta\mathbb{N})$ is given by*

$$\mu : f \mapsto \int_{\beta\mathbb{N}} f d\mu, \quad f \in C(\beta\mathbb{N}).$$

We note that by Theorem 2.3 and Theorem 2.4, ℓ_∞^* is isometrically isomorphic to the space of all regular $\mathbb{K}$-valued Borel measures on $\beta\mathbb{N}$ equipped with the total variation norm. We begin by characterizing the support functionals of a non-zero $f \in C(\beta\mathbb{N})$ and introduce the following definition in this regard.

Definition 6 For a given $f \in C(\beta\mathbb{N})$, we define M_f to be the collection of all the points in $\beta\mathbb{N}$ where f attains its norm, i.e.,

$$M_f := \{\mathcal{U} \in \beta\mathbb{N} : |f(\mathcal{U})| = \|f\|\}.$$

Using the above, we now characterize $J(f)$, (see (1.1) for definition) for a non-zero $f \in C(\beta\mathbb{N})$.

Theorem 2.5 *Let $f \in C(\beta\mathbb{N})$ be non-zero. Then $\mu \in J(f)$, for some regular $\mathbb{K}$-valued Borel measure μ if and only if*

$$|\mu|\left(\beta\mathbb{N} \setminus M_f\right) = 0, \quad |\mu|(M_f) = 1 \ \textit{ and } \ d\mu(\mathcal{U}) = \overline{\operatorname{sgn}(f(\mathcal{U}))}d|\mu|(\mathcal{U}),$$

for almost every $\mathcal{U} \in M_f$ with respect to the measure μ, where $|\mu|$ denotes the total variation of μ.

Proof The sufficiency follows by elementary computations. Now, if $\mu \in J(f)$, then $d\mu(\mathcal{U}) = e^{i\theta(\mathcal{U})}d|\mu|(\mathcal{U})$, for some measurable function $\theta : \beta\mathbb{N} \to \mathbb{R}$. Note that

$$\|f\| = \int_{\beta\mathbb{N}} f(\mathcal{U})e^{i\theta(\mathcal{U})}d|\mu|(\mathcal{U}) \leq \int_{\beta\mathbb{N}} |f(\mathcal{U})|d|\mu|(\mathcal{U}) \leq \|f\|.$$

Hence equality must hold in both the inequalities involved. Equality in the second inequality implies that $|f(\mathcal{U})| = \|f\|$, for almost every $\mathcal{U} \in \beta\mathbb{N}$, giving $|\mu|\left(\beta\mathbb{N}\setminus M_f\right) = 0$ (and hence, $|\mu|(M_f) = 1$) and equality in the first inequality gives that for almost every $\mathcal{U} \in M_f$, with respect to μ,

$$f(\mathcal{U})e^{i\theta(\mathcal{U})} = \|f\| \quad \Rightarrow \quad e^{i\theta(\mathcal{U})} = \overline{\operatorname{sgn}(f(\mathcal{U}))}.$$

We now come to our characterization of Birkhoff-James orthogonality in ℓ_∞.

Theorem 2.6 *Let $x = (x_n)_{n\in\mathbb{N}}$ and $y = (y_n)_{n\in\mathbb{N}}$ be two elements of ℓ_∞. Then the following are equivalent:*

(i) $x \perp_B y$.
(ii) $0 \in \operatorname{conv}\{\lim_{\mathcal{U}} \overline{x_n}y_n : \lim_{\mathcal{U}}|x_n| = \|x\|\}$.
(iii) *0 lies in the convex hull of* $\{\overline{x_n}y_n : |x_n| = \|x\|\}$ *and* $\{\lim_{k\to\infty} \overline{x_{n_k}}y_{n_k} : n_k$ *is an increasing sequence in* $\mathbb{N}$, $\lim_{k\to\infty}\left|x_{n_k}\right| = \|x\|$, y_{n_k} *converge as* $k \to \infty\}$

Proof The result holds trivially if $x = 0$. Hence, we assume that $x \neq 0$.

$(i) \Leftrightarrow (ii)$

If $0 \in \operatorname{conv}\{\lim_{\mathcal{U}} \overline{x_n}y_n : \lim_{\mathcal{U}}|x_n| = \|x\|\}$, then there exist ultrafilters $\mathcal{U}_i$ on $\mathbb{N}$ and $\lambda_i \in [0, 1]$, for $1 \leq i \leq m$ such that $\sum_{i=1}^m \lambda_i = 1$ and

$$\sum_{i=1}^m \lambda_i \lim\nolimits_{\mathcal{U}_i} \overline{x_n}y_n = 0, \quad \lim\nolimits_{\mathcal{U}_i}|x_n| = \|x\|, \ \textit{ for every } 1 \leq i \leq m.$$

Consider the functional $\Psi : \ell_\infty \to \mathbb{K}$, given by

$$\Psi\left((z_n)_{n\in\mathbb{N}}\right) = \sum_{i=1}^n \lambda_i \lim\nolimits_{\mathcal{U}_i} \overline{\operatorname{sgn}(x_n)}z_n, \quad (z_n)_{n\in\mathbb{N}} \in \ell_\infty.$$

Then clearly Ψ has norm 1 and $\Psi(x) = \|x\|$. Hence Ψ is a support functional of x that annihilates y and establishes the sufficiency.

Now, let us assume that $x \perp_B y$. Then by Theorem 2.3, we have the two maps $f, g : \beta\mathbb{N} \to \mathbb{K}$, given by

$$f(\mathcal{U}) := \lim_{\mathcal{U}} x_n, \quad g(\mathcal{U}) := \lim_{\mathcal{U}} y_n, \quad \mathcal{U} \in \beta\mathbb{N},$$

satisfying $f \perp_B g$ in $C(\beta\mathbb{N})$. Observe that $M_f = \{\mathcal{U} \in \beta\mathbb{N} : \lim_{\mathcal{U}}|x_n| = \|x\|\}$. Now, by Theorem 2.5, there must exist a regular positive Borel measure ν on M_f with $\nu(M_f) = 1$ such that

$$\int_{M_f} \overline{sgn(f(\mathcal{U}))}g(\mathcal{U})d\nu(\mathcal{U}) = 0 \quad \Leftrightarrow \quad \int_{M_f} \overline{f(\mathcal{U})}g(\mathcal{U})d\nu(\mathcal{U}) = 0.$$

Let Λ be the collection of all regular positive Borel measures μ on M_f with $\mu(M_f) = 1$. Consider the map $\Phi : \Lambda \to \mathbb{K}$, given by

$$\Phi(\mu) := \int_{M_f} \overline{f(\mathcal{U})} g(\mathcal{U}) d\mu(\mathcal{U}), \quad \mu \in \Lambda.$$

Then clearly $\Phi(\Lambda)$ must be a convex subset of $\mathbb{K}$ since Λ is convex. Observe that by Theorem 2.5, Λ is the collection of all support functionals of $|f| \in C(\beta\mathbb{N})$ and hence is weak* compact by the Banach-Alaoglu Theorem. Since the map Φ is continuous under the weak* topology, $\Phi(\Lambda)$ will be compact. Hence by the Krein-Milman Theorem, $\Phi(\Lambda)$ must be the closed convex hull of its extreme points.

We claim that the only extreme points of the set $\Phi(\Lambda)$ are of the form $\overline{f(\mathcal{U})} g(\mathcal{U})$, for some $\mathcal{U} \in M_f$. Let $\Phi(\mu)$ be an extreme point of $\Phi(\Lambda)$ where μ is not a Dirac delta measure at any point on M_f. Clearly, if $\overline{f} g$ is constant at every point of the support of μ, then $\Phi(\mu) = \overline{f(\mathcal{U})} g(\mathcal{U})$, for any $\mathcal{U}$ in the support of μ. Therefore, let us assume that $\mathcal{U}, \mathcal{V}$ are two points in the support of μ with $\overline{f(\mathcal{U})} g(\mathcal{U}) \neq \overline{f(\mathcal{V})} g(\mathcal{V})$. Fix $0 < \epsilon < \frac{1}{2} |\overline{f(\mathcal{U})} g(\mathcal{U}) - \overline{f(\mathcal{V})} g(\mathcal{V})|$ and set:

$$G_\epsilon := \left\{ \mathcal{W} \in M_f : \left| \overline{f(\mathcal{U})} g(\mathcal{U}) - \overline{f(\mathcal{W})} g(\mathcal{W}) \right| < \epsilon \right\}.$$

Then G_ϵ is an open subset of M_f containing $\mathcal{U}$ and $M_f \backslash G_\epsilon$ contains a neighbourhood of $\mathcal{V}$ in M_f. Hence $\mu(G_\epsilon), \mu(M_f \backslash G_\epsilon) > 0$. Now since μ can be written as a convex combination of $\frac{1}{\mu(G_\epsilon)} \mu|_{G_\epsilon}$ and $\frac{1}{\mu(M_f \backslash G_\epsilon)} \mu|_{M_f \backslash G_\epsilon}$,

$$\Phi(\mu) = \frac{1}{\mu(G_\epsilon)} \int_{G_\epsilon} \overline{f(\mathcal{W})} g(\mathcal{W}) d\mu(\mathcal{W}), \quad \textit{for every } 0 < \epsilon < \frac{1}{2} |\overline{f(\mathcal{U})} g(\mathcal{U}) - \overline{f(\mathcal{V})} g(\mathcal{V})|,$$

as $\Phi(\mu)$ is an extreme point of $\Phi(\Lambda)$. Hence,

$$\begin{aligned} \left| \Phi(\mu) - \overline{f(\mathcal{U})} g(\mathcal{U}) \right| &= \frac{1}{\mu(G_\epsilon)} \left| \int_{G_\epsilon} \left(\overline{f(\mathcal{W})} g(\mathcal{W}) - \overline{f(\mathcal{U})} g(\mathcal{U}) \right) d\mu(\mathcal{W}) \right| \\ &\leq \frac{1}{\mu(G_\epsilon)} \int_{G_\epsilon} \left| \overline{f(\mathcal{W})} g(\mathcal{W}) - \overline{f(\mathcal{U})} g(\mathcal{U}) \right| d\mu\,(\mathcal{W}) \leq \epsilon, \end{aligned}$$

giving $\Phi(\mu) = \overline{f(\mathcal{U})} g(\mathcal{U})$ since $0 < \epsilon < \frac{1}{2} |\overline{f(\mathcal{U})} g(\mathcal{U}) - \overline{f(\mathcal{V})} g(\mathcal{V})|$ is arbitrary.

Therefore,

$$0 \in \overline{\text{conv}} \left\{ \lim_{\mathcal{U}} \overline{x_n} y_n : \lim_{\mathcal{U}} |x_n| = \|x\| \right\}, \quad \textit{as} \quad \Phi(\nu) = 0.$$

Clearly, since M_f is compact, $\{\lim_{\mathcal{U}} \overline{x_n} y_n : \lim_{\mathcal{U}} |x_n| = \|x\|\}$ is also compact. We now prove that the convex hull of a compact subset of $\mathbb{K}$ is closed. Indeed, using Caratheodory's theorem, for any element x in $\text{conv}(K)$, where $K \subset \mathbb{K}$ is compact, there exist $x_1, x_2, x_3 \in K$ and $\lambda_1, \lambda_2, \lambda_3 \in [0, 1]$ such that

$$\sum_{i=1}^{3} \lambda_i x_i = x \quad \textit{and} \quad \sum_{i=1}^{3} \lambda_i = 1.$$

Hence if $x_0 \in \overline{\text{conv}(K)}$, there exist sequences $x_i^{(n)} \in K$ and $\lambda_i^{(n)} \in [0, 1]$, for $1 \leq i \leq 3$ and $n \in \mathbb{N}$ such that

$$\lim_{n \to \infty} \sum_{i=1}^{3} \lambda_i^{(n)} x_i^{(n)} = x_0, \quad \textit{and} \quad \sum_{i=1}^{3} \lambda_i^{(n)} = 1, \quad \textit{for every } n \in \mathbb{N}.$$

Since K and $[0, 1]$ are compact, considering a convergent subsequence of all the six sequences and passing on to the limits, we have $x_0 \in \operatorname{conv}(K)$. Combining all the results, we therefore obtain

$$0 \in \operatorname{conv}\left\{\lim_{\mathcal{U}} \overline{x_n} y_n : \lim_{\mathcal{U}} |x_n| = \|x\|\right\}.$$

$(ii) \Leftrightarrow (iii)$ This is immediate from Proposition 2.2 and the fact that if $\mathcal{U}$ is a principal ultrafilter containing $\{N\}$, for some $N \in \mathbb{N}$, then $\lim_{\mathcal{U}} x_n = x_N$, for every sequence $(x_n)_{n\in\mathbb{N}}$ in ℓ_∞.

We record the characterization of Birkhoff-James orthogonality in ℓ_∞, for the real case as a corollary below.

Corollary 2.7 *Let $x = (x_n)_{n\in\mathbb{N}}$ and $y = (y_n)_{n\in\mathbb{N}}$ be two elements of ℓ_∞ over $\mathbb{R}$. Then $x \perp_B y$ if and only if any of the following holds true:*

(i) There is an increasing sequence of natural numbers $(n_k)_{k\in\mathbb{N}}$ such that $\left|x_{n_k}\right| \to \|x\|$ and $x_{n_k} y_{n_k} \to 0$ as $k \to \infty$.
(ii) There are increasing sequences of natural numbers $(n_k)_{k\in\mathbb{N}}$ and $(m_k)_{k\in\mathbb{N}}$ such that $\left|x_{n_k}\right| \to \|x\|$ and $\left|x_{m_k}\right| \to \|x\|$ as $k \to \infty$ with $x_{n_k} y_{n_k} \geq 0 \geq x_{m_k} y_{m_k}$, for every $k \in \mathbb{N}$.
(iii) There is an increasing sequence of natural numbers $(n_k)_{k\in\mathbb{N}}$ such that $\left|x_{n_k}\right| \to \|x\|$ as $k \to \infty$ and a natural number N such that $|x_N| = \|x\|$ and $x_N y_N$ and $x_{n_k} y_{n_k}$ are of different signs for every $k \in \mathbb{N}$.
(iv) There are $N, M \in \mathbb{N}$ such that $|x_N| = \|x\| = |x_M|$ and $x_N y_N \geq 0 \geq x_M y_M$.

We conclude this section by characterizing the smooth points of ℓ_∞.

Theorem 2.8 *Let $x = (x_n)_{n\in\mathbb{N}}$ be a non-zero element in ℓ_∞. Then x is a smooth point if and only if there is no subsequence of $(|x_n|)_{n\in\mathbb{N}}$ that converges to $\|x\|$ and there exists a unique $N \in \mathbb{N}$ such that $|x_N| = \|x\|$.*

Proof We first prove the sufficiency. Suppose, there is no subsequence of $(|x_n|)_{n\in\mathbb{N}}$ that converges to $\|x\|$ and there exists a unique $N \in \mathbb{N}$ such that $|x_N| = \|x\|$. Then it follows from Theorem 2.6 that $x \perp_B y$, for some $y = (y_n)_{n\in\mathbb{N}} \in \ell_\infty$, if and only if $y_N = 0$. Consequently, $x \perp_B y$ and $x \perp_B z$ implies $x \perp_B (y + z)$, proving x is smooth.

Again, if $x \in \ell_\infty$ is smooth, then the function $f : \beta\mathbb{N} \to \mathbb{K}$, given by

$$f(\mathcal{U}) := \lim_{\mathcal{U}} x_n, \quad \mathcal{U} \in \beta\mathbb{N},$$

is smooth in $C(\beta\mathbb{N})$ (Theorem 2.3). Therefore, if $\mathcal{U}, \mathcal{V} \in M_f$, then $\Psi, \Phi : C(\beta\mathbb{N}) \to \mathbb{K}$, given by

$$\Psi(g) := \overline{\operatorname{sgn}(f(\mathcal{U}))} g(\mathcal{U}), \quad \Phi(g) := \overline{\operatorname{sgn}(f(\mathcal{V}))} g(\mathcal{V}), \quad g \in C(\beta\mathbb{N}),$$

are two distinct support functionals of f because there exists a continuous map $F : \beta\mathbb{N} \to [0, 1]$ such that $F(\mathcal{U}) = 0$ and $F(\mathcal{V}) = 1$, since $\beta\mathbb{N}$ is compact Hausdorff. Hence M_f must be singleton.

Now, if $\mathcal{U} \in M_f$ is a free ultrafilter, then by Theorem 2.2, there exists an increasing sequence $(n_k)_{k\in\mathbb{N}}$ of natural numbers such that $\left|x_{n_k}\right| \to \|x\|$ as $k \to \infty$. Consider the following two collections:

$$\{A \subset \mathbb{N} : \{n_{2k} : k \in \mathbb{N}\} \setminus A \text{ is finite }\} \text{ and } \{A \subset \mathbb{N} : \{n_{2k-1} : k \in \mathbb{N}\} \setminus A \text{ is finite }\}.$$

Clearly, both of these sets are filter bases on $\mathbb{N}$ and are therefore contained in free ultrafilters. Also note that since the first collection contains $\{n_{2k} : k \in \mathbb{N}\}$ and the second collection contains $\mathbb{N}\setminus\{n_{2k} : k \in \mathbb{N}\}$, the two ultrafilters must be distinct. However, the sequence $(|x_n|)$ has the same limit under both of the ultrafilters. Hence if M_f is singleton, it can contain only a principle ultrafilter and thus the necessity follows. □

2.3 Local symmetry of Birkhoff-James orthogonality in ℓ_∞

In this sub-section we characterize the left-symmetric and the right-symmetric points of the space ℓ_∞.

Theorem 2.9 *The only non-zero left-symmetric points of ℓ_∞ are scalar multiples of e_n for $n \in \mathbb{N}$, where e_n denotes the sequence having the n-th term 1 and the rest of the terms 0.*

Proof Since $\lim_{\mathcal{U}} e_n = 0$, for every ultrafilter other than the principal ultrafilter containing $\{n\}$, by Theorem 2.6, $e_n \perp_B x$, for some $x = (x_n)_{n\in\mathbb{N}} \in \ell_\infty$ if and only if $x_n = 0$. Hence for $\lambda \in \mathbb{K}$,

$$\|x + \lambda e_n\| = \max\{|\lambda|, \|x\|\} \geq \|x\|,$$

and this proves the necessity.

Now, let $x = (x_n)_{n\in\mathbb{N}}$ be a left-symmetric point of ℓ_∞. If $\lim_{\mathcal{U}}|x_n| = \|x\|$, for some free ultrafilter $\mathcal{U}$ on $\mathbb{N}$, then set $y = (y_n)_{n\in\mathbb{N}} \in \ell_\infty$, given by $y_n := \frac{1}{n}\operatorname{sgn}(x_n)$. Then clearly, $\lim_{\mathcal{U}} \overline{x_n} y_n = 0$, since $\mathcal{U}$ is a free ultrafilter. Hence, $x \perp_B y$. However, by Theorem 2.6, for some $z = (z_n)_{n\in\mathbb{N}} \in \ell_\infty$, $y \perp_B z$ if and only if $z_N = 0$ where $N = \min\{n \in \mathbb{N} : x_n \neq 0\}$. Thus $y \not\perp_B x$, implying that $\lim_{\mathcal{U}}|x_n| \neq \|x\|$, for any free ultrafilter $\mathcal{U}$ on $\mathbb{N}$.

Hence, there exists $N \in \mathbb{N}$ such that $|x_N| = \|x\|$. Now, suppose that $x_M \neq 0$, for some $M \neq N \in \mathbb{N}$. Then clearly, setting $y = e_M$, yields that $x \perp_B y$ and $y \not\perp_B x$, by Theorem 2.6, which establishes the necessity. □

Now, from Corollary 1.1, if $T : \ell_\infty \to \ell_\infty$ is an onto linear isometry, then for every $n \in \mathbb{N}$, $T(e_n) = ce_m$, for some $m \in \mathbb{N}$ and some unimodular constant c. Also, since T is onto, T is invertible and T^{-1} is also an onto isometry and hence T must be a signed permutation operator. Note that this extends the result of Lamperti [15] on L_p spaces for $1 \leq p < \infty$ to the $p = \infty$ case for onto operators with the measure space as $\mathbb{N}$ under counting measure. We record this result as a corollary.

Corollary 2.10 *Let $T : \ell_\infty \to \ell_\infty$ be an onto linear isometry. Then T must be a signed permutation operator.*

We conclude this sub-section by characterizing the right-symmetric points of ℓ_∞.

Theorem 2.11 *$x = (x_n)_{n\in\mathbb{N}} \in \ell_\infty$ is a right-symmetric point if and only if $|x_n| = \|x\|$ for every $n \in \mathbb{N}$.*

Proof Note that if $|x_n| = \|x\|$, for every $n \in \mathbb{N}$, then $\lim_{\mathcal{U}}|x_n| = \|x\|$, for every ultrafilter $\mathcal{U}$ on $\mathbb{N}$ and hence by Theorem 2.6, $y \perp_B x$, for $y = (y_n)_{n\in\mathbb{N}} \in \ell_\infty$ if and only if

$$\begin{aligned} & 0 \in \operatorname{conv}\left\{\lim_{\mathcal{U}} \overline{y_n} x_n : \lim_{\mathcal{U}}|y_n| = \|y\|\right\} \\ \Rightarrow\ & 0 \in \operatorname{conv}\left\{\lim_{\mathcal{U}} \overline{x_n} y_n : \lim_{\mathcal{U}}|x_n| = \|x\|\right\} \end{aligned}$$

$$\Rightarrow \quad x \perp_B y.$$

Hence the sufficiency is clear and we therefore focus on proving the necessity.

Now, if $x \neq 0$ is right-symmetric and $|x_N| < \|x\|$ for some $N \in \mathbb{N}$, set $y = (y_n)_{n\in\mathbb{N}} \in \ell_\infty$ given by

$$y_n := \begin{cases} \operatorname{sgn}(x_n), & n \neq N, \\ -\operatorname{sgn}(x_N), & n = N \text{ and if } x_N \neq 0, \\ 1, & n = N \text{ and if } x_N = 0. \end{cases}$$

Then clearly, by Theorem 2.6, $y \perp_B x$. However,

$$\overline{x_n} y_n := \begin{cases} |x_n|, & n \neq N, \\ -|x_N|, & n = N. \end{cases}$$

Hence, if $\mathcal{U}$ is an ultrafilter not containing $\{N\}$,

$$\lim_{\mathcal{U}} \overline{x_n} y_n = \lim_{\mathcal{U}} |x_n|. \tag{2.3}$$

Therefore, since $|x_N| < \|x\|$, the limit of $|x_n|$ under the principal ultrafilter containing $\{N\}$ is not $\|x\|$ and hence by (2.3) and Theorem 2.6, $x \not\perp_B y$ establishing the necessity.

2.4 Geometry of c, c_0 and c_{00}

Recall that c, c_0 and c_{00} are the collections of all convergent sequences, convergent to zero sequences and eventually zero sequences respectively. Let us denote the limit of a sequence $x = (x_n)_{n\in\mathbb{N}} \in c$ by $\lim x$. Since all the three spaces are subspaces of ℓ_∞, two elements in any of these spaces are Birkhoff-James orthogonal if and only if they are Birkhoff-James orthogonal in ℓ_∞. Keeping this fundamental principle in mind we observe that if $x = (x_n)_{n\in\mathbb{N}} \in c$, $\lim_{\mathcal{U}} x_n = \lim x$ and if $x \in c_0$, $\lim_{\mathcal{U}} x_n = 0$, for every free ultrafilter $\mathcal{U}$ on $\mathbb{N}$. Thus we obtain the following result from Theorem 2.6.

Theorem 2.12 *Suppose that $x = (x_n)_{n\in\mathbb{N}}$ and $y = (y_n)_{n\in\mathbb{N}}$ are two sequences in $\mathbb{K}$. Then*
(i) If $x, y \in c$, then $x \perp_B y$ if and only if

$$0 \in \operatorname{conv}\left(\{\overline{x_n} y_n : |x_n| = \|x\|\} \cup \{\lim \overline{x} y\}\right), \quad \text{if } \lim|x| = \|x\|,$$
$$\text{and } \quad 0 \in \operatorname{conv}\{\overline{x_n} y_n : |x_n| = \|x\|\}, \quad \text{if } \lim|x| \neq \|x\|.$$

In particular, for $\mathbb{K} = \mathbb{R}$, $x \perp_B y$ if and only if any of the following is true:

1. *There exists $N \in \mathbb{N}$ such that $|x_N| = \|x\|$ and $y_N = 0$ or $\lim|x| = \|x\|$ and $\lim y = 0$.*
2. *There exists $N \in \mathbb{N}$ such that $|x_N| = \lim|x| = \|x\|$ and $x_N y_N$ and $\lim xy$ are of different signs.*
3. *There exist $N, M \in \mathbb{N}$ such that $|x_N| = |x_M| = \|x\|$ and $x_N y_N < 0 < x_M y_M$.*

(ii) If $x, y \in c_0$ or c_{00}, $x \perp_B y$ if and only if

$$0 \in \operatorname{conv}\{\overline{x_n} y_n : |x_n| = \|x\|\}.$$

In particular, for $\mathbb{K} = \mathbb{R}$, $x \perp_B y$ if and only if one of the following holds:

1. *There exists $N \in \mathbb{N}$ such that $|x_N| = \|x\|$ and $y_N = 0$.*
2. *There exist $N, M \in \mathbb{N}$ such that $|x_N| = |x_M| = \|x\|$ and $x_N y_N < 0 < x_M y_M$.*

The characterization of the smooth points of these spaces requires a little bit more work.

Theorem 2.13 *(i)* $x = (x_n)_{n\in\mathbb{N}} \in c$ *is smooth if and only if* $|x_n| < \|x\|$, *for every* $n \in \mathbb{N}$, *or there exists a unique* $N \in \mathbb{N}$, *such that* $|x_N| = \|x\|$ *and* $\lim|x| \neq \|x\|$.

(ii) $x = (x_n)_{n\in\mathbb{N}} \in c_0$ *or* c_{00} *is smooth if and only if there exists a unique* $N \in \mathbb{N}$ *such that* $|x_N| = \|x\|$.

Proof (i) If there are distinct $N, M \in \mathbb{N}$ such that $|x_N| = |x_M| = \|x\|$, then the two functionals $\Psi, \Phi : c \to \mathbb{K}$ given by

$$\Psi(y) := \overline{\operatorname{sgn}(x_N)}y_N, \quad \Phi(y) := \overline{\operatorname{sgn}(x_M)}y_M, \quad y = (y_n)_{n\in\mathbb{N}} \in c,$$

are distinct support functionals of x. Again if $|x_N| = \lim|x| = \|x\|$, for some $N \in \mathbb{N}$, then $\Psi', \Phi' : c \to \mathbb{K}$ given by

$$\Psi'(y) := \overline{\operatorname{sgn}(x_N)}y_N, \quad \Phi'(y) := \lim_{n\to\infty} \overline{\operatorname{sgn}(x_n)}y_n, \quad y = (y_n)_{n\in\mathbb{N}} \in c,$$

are two distinct support functionals of x, establishing the necessity.

Now, by Theorem 2.12, if there exists a unique $N \in \mathbb{N}$ with $|x_N| = \|x\|$ and $\lim|x| < \|x\|$, then $x \perp_B y$, for some $y = (y_n)_{n\in\mathbb{N}} \in c$ if and only if $y_N = 0$ and clearly, Birkhoff-James orthogonality is right additive at x. Again if $|x_n| < \|x\|$, for every $n \in \mathbb{N}$, then $\lim|x| = \|x\|$ and again by Theorem 2.12, $x \perp_B y$, for some $y \in c$ if and only if $\lim y = 0$ proving the right additivity of Birkhoff-James orthogonality at x and the sufficiency is established.

(ii) The result for c_0 and c_{00} follows from part (i) of this theo and the observation that for $x \in c_0$ or c_{00}, $\lim|x| = 0 \neq \|x\|$ unless $x = 0$.

Using these two results and going through the proofs of Theorem 2.9, Theorem 2.11 and Corollary 2.10, we obtain the following result.

Theorem 2.14 *1. The left-symmetric points of each of these spaces are* λe_n *for* $n \in \mathbb{N}$ *and* $\lambda \in \mathbb{K}$.

2. *The right-symmetric points of* c *are the sequences* $x = (x_n)_{n\in\mathbb{N}}$ *such that* $|x_n| = \|x\|$, *for every* $n \in \mathbb{N}$ *and the spaces* c_0 *and* c_{00} *have no non-zero right-symmetric point.*
3. *The linear isometries of each of these spaces onto itself are the signed permutations operators.*

3 Geometry of ℓ_1

In this section, we characterize Birkhoff-James orthogonality and its local symmetry in ℓ_1. We begin with a known result.

Theorem 3.1 *The dual space of* ℓ_1 *is isometrically isomorphic to* ℓ_∞ *with the functional* Ψ_a *corresponding to an element* $a = (a_n)_{n\in\mathbb{N}} \in \ell_\infty$, *given by*

$$\Psi_a(x) = \sum_{n=1}^{\infty} a_n x_n, \quad x = (x_n)_{n\in\mathbb{N}} \in \ell_1.$$

In order to avoid confusion, we denote the norm of an element in ℓ_1 by $\|.\|_1$ and the norm of an element in its dual, identified with the ℓ_∞ space by $\|.\|_\infty$. We come to the following preliminary lemma before giving a complete characterization of Birkhoff-James orthogonality in ℓ_1.

Lemma 3.2 *Let $x = (x_n)_{n\in\mathbb{N}} \in \ell_1$ be non-zero. Then $a = (a_n)_{n\in\mathbb{N}} \in \ell_\infty$ is a support functional of x if and only if $a_n = \overline{\operatorname{sgn}(x_n)}$ if $x_n \neq 0$ and $|a_n| \leq 1$ if $x_n = 0$.*

Proof The sufficiency can be established by elementary calculations. For the necessity, note that

$$\|x\|_1 = \sum_{n=1}^{\infty} a_n x_n = \sum_{x_n\neq 0} a_n x_n \leq \sum_{a_n\neq 0} |a_n x_n| \leq \|a\|_\infty \sum_{x_n\neq 0} |x_n| = \|x\|_1.$$

Hence, equality holds in both the inequalities, giving $|a_n| = \|a\|_\infty = 1$ and $a_n x_n = |a_n x_n|$, i.e., $a_n = \overline{\operatorname{sgn}(x_n)}$, for every $n \in \mathbb{N}$ with $x_n \neq 0$. Also, since $\|a\|_\infty = 1$, $|a_n| \leq 1$, for $n \in \mathbb{N}$ with $x_n = 0$.

We now come to the characterization of Birkhoff-James orthogonality in ℓ_1.

Theorem 3.3 *Let $x = (x_n)_{n\in\mathbb{N}}$ and $y = (y_n)_{n\in\mathbb{N}} \in \ell_1$. Then $x \perp_B y$ if and only if*

$$\left|\sum_{n=1}^{\infty} \overline{\operatorname{sgn}(x_n)} y_n\right| \leq \sum_{x_n=0} |y_n|, \tag{3.1}$$

where sum over an empty set is defined to be 0.

Proof If $x \perp_B y$, then there exists $a = (a_n)_{n\in\mathbb{N}} \in \ell_\infty$ with $\|a\|_\infty = 1$ such that $\Psi_a \in J(x)$ and $\Psi_a(y) = 0$. This yields

$$\left|\sum_{x_n\neq 0} \overline{\operatorname{sgn}(x_n)} y_n\right| \leq \sum_{x_n=0} |a_n y_n| \leq \sum_{x_n=0} |y_n|,$$

proving the necessity.

Again, if (3.1) holds, set $k \in \mathbb{K}$, $|k| \leq 1$ such that

$$\sum_{n=1}^{\infty} \overline{\operatorname{sgn}(x_n)} y_n = k\left(\sum_{x_n=0} |y_n|\right).$$

Consider $a = (a_n)_{n\in\mathbb{N}}$ given by

$$a_n := \begin{cases} \overline{\operatorname{sgn}(x_n)}, & a_n \neq 0, \\ -k\,\overline{\operatorname{sgn}(y_n)}, & a_n = 0. \end{cases}$$

Then clearly $a \in \ell_\infty$ and by Lemma 3.2, $\Psi_a \in J(x)$. Since $\Psi_a(y) = 0$, the sufficiency follows.

As a corollary to the above result, we mention the case when $\mathbb{K} = \mathbb{R}$.

Corollary 3.4 *Suppose that $x = (x_n)_{n\in\mathbb{N}}$ and $y = (y_n)_{n\in\mathbb{N}} \in \ell_1$ are two members of ℓ_1 over $\mathbb{R}$. Let $\mathcal{N}_1 := \{n \in \mathbb{N} : x_n y_n > 0\}$, $\mathcal{N}_2 := \{n \in \mathbb{N} : x_n y_n < 0\}$ and $\mathcal{N}_0 := \{n \in \mathbb{N} : x_n = 0\}$. Then $x \perp_B y$ if and only if*

$$\left|\sum_{n\in\mathcal{N}_1} |y_n| - \sum_{n\in\mathcal{N}_2} |y_n|\right| \leq \sum_{n\in\mathcal{N}_0} |y_n|.$$

We now come to the characterization of the smooth points of ℓ_1.

Theorem 3.5 *The smooth points of ℓ_1 are the sequences having no zero term.*

Proof Suppose $x = (x_n)_{n\in\mathbb{N}} \in \ell_1$ has no zero term. Then by Theorem 3.3, if $x \perp_B y$ and $x \perp_B z$ where $y = (y_n)_{n\in\mathbb{N}}, z = (z_n)_{n\in\mathbb{N}} \in \ell_1$, then

$$\sum_{n=1}^{\infty} \overline{\operatorname{sgn}(x_n)} y_n = \sum_{n=1}^{\infty} \overline{\operatorname{sgn}(x_n)} y_n = 0.$$

Hence

$$\sum_{n=1}^{\infty} \overline{\operatorname{sgn}(x_n)}(y_n + z_n) = 0 \quad \Rightarrow \quad x \perp_B (y + z),$$

proving x is smooth.

If $x = (x_n)_{n\in\mathbb{N}} \in \ell_1$ has a zero term x_N, for some $N \in \mathbb{N}$, by Lemma 3.2, we obtain that $a = (a_n)_{n\in\mathbb{N}}$ and $b = (b_n)_{n\in\mathbb{N}} \in \ell_\infty$, given by

$$a_n := \begin{cases} \overline{\operatorname{sgn}(x_n)}, & x_n \neq 0, \\ 0, & x_n = 0, \end{cases} \qquad \textit{and} \;\; b_n := \begin{cases} \overline{\operatorname{sgn}(x_n)}, & x_n \neq 0, \\ 0, & x_n = 0 \textit{ and } n \neq N, \\ 1, & n = N. \end{cases}$$

are two distinct support functionals of x. Hence x cannot be smooth.

We finally come to the characterization of local symmetry of Birkhoff-James orthogonality in ℓ_1.

Theorem 3.6 *No non-zero point of ℓ_1 is left-symmetric.*

Proof Suppose that $x = (x_n)_{n\in\mathbb{N}} \in \ell_1$ is non-zero. If there exists $N \in \mathbb{N}$ such that $x_N = 0$, then setting $a = (a_n)_{n\in\mathbb{N}}$, given by $y_n := \operatorname{sgn}(x_n)\frac{1}{2^n}$, we have $\|y\|_1 < \infty$. Thus considering $z := y + 2\|y\|_1 e_N$, Theorem 3.3 implies $x \perp_B z$ and $z \not\perp_B x$. If $x_n \neq 0$, for every $n \in \mathbb{N}$, then there exists $M \in \mathbb{N}$ such that

$$\sum_{n=1}^{M} |x_n| \neq \sum_{n=M+1}^{\infty} |x_n|.$$

Set $y = (y_n)_{n\in\mathbb{N}}$, given by,

$$y_n := \begin{cases} \operatorname{sgn}(x_n), & 1 \leq n \leq M, \\ -\operatorname{sgn}(x_n)\frac{M}{2^{n-M}}, & n \geq M + 1. \end{cases}$$

Then $y \in \ell_1$. Again, by Theorem 3.3, $x \perp_B y$ and $y \not\perp_B x$ and hence x is not a left-symmetric point.

We now come to the characterization of the right-symmetric points of ℓ_1 and thereby find the collection of onto isometries of the space.

Theorem 3.7 *The only right-symmetric points of ℓ_1 are scalar multiples of e_n, for $n \in \mathbb{N}$.*

Proof Observe that for $y = (y_n)_{n\in\mathbb{N}} \in \ell_1$, by Theorem 3.3, $y \perp_B \lambda e_n$, for some $\lambda \neq 0$ if and only if $y_n = 0$ and hence $\lambda e_n \perp_B y$. Again if $x = (x_n)_{\mathbb{N}} \in \ell_1$ has $x_n, x_m \neq 0$, for $n \neq m$, we consider $r \in \mathbb{N}$ such that

$$0 < |x_r| \leq \sum_{k \neq r} |x_k|.$$

Hence by Theorem 3.3, $x \not\perp_B e_r$ but $e_r \perp_B x$.

We use this result to characterize the onto isometries of ℓ_1 with the help of Corollary 1.1 and thereby establish the Banach-Lamperti theorem for onto operators on ℓ_1.

Corollary 3.8 *Let $T : \ell_1 \to \ell_1$ be an onto linear isometry. Then T must be a signed permutation operator.*

4 Geometry of ℓ_p, for $1 < p < \infty, p \neq 2$

In this final section, we characterize Birkhoff-James orthogonality and its local symmetry in ℓ_p, for $p \in (1, \infty)\backslash\{2\}$. We begin with a well-known result.

Theorem 4.1 *The dual space of ℓ_p is isometrically isomorphic to ℓ_q where $\frac{1}{p} + \frac{1}{q} = 1$ with the functional Ψ_a corresponding to $a = (a_n)_{n\in\mathbb{N}} \in \ell_q$, given by*

$$\Psi_a(x) = \sum_{n=1}^{\infty} a_n x_n, \quad x = (x_n)_{n\in\mathbb{N}} \in \ell_p.$$

Hence, as was done in the case of ℓ_1, we continue to denote the norm of an element of ℓ_p by $\|.\|_p$ and the norm of an element in the dual of ℓ_p, identified with ℓ_q, by $\|.\|_q$. To begin with, we note down a corollary of this theorem pertaining to the characterization of the support functional of an element of ℓ_p.

Corollary 4.2 *Let $a = (a_n)_{n\in\mathbb{N}} \in \ell_q$ be a support functional of $x = (x_n)_{n\in\mathbb{N}} \in \ell_p \setminus \{0\}$. Then $a_n = \frac{1}{\|x\|_p^{p-1}} \overline{\operatorname{sgn}(x_n)} |x_n|^{p-1}$, $n \in \mathbb{N}$.*

The proof of this result involves elementary computations and the equality criteria of Holder's inequality. Observe also that the corollary establishes the uniqueness of the support functional of any non-zero element in the space, thereby proving its smoothness.

We now come to our characterization of Birkhoff-James orthogonality in ℓ_p which follows as a direct consequence of Corollary 4.2 and James' characterization of Birkhoff-James orthogonality.

Theorem 4.3 *Let $x = (x_n)_{n\in\mathbb{N}}$ and $y = (y_n)_{n\in\mathbb{N}} \in \ell_p$. Then $x \perp_B y$ if and only if*

$$\sum_{n=1}^{\infty} \overline{\operatorname{sgn}(x_n)} |x_n|^{p-1} y_n = 0.$$

We now characterize the local symmetry of Birkhoff-James orthogonality in ℓ_p.

Theorem 4.4 *$x = (x_n)_{n\in\mathbb{N}} \in \ell_p$ is a left-symmetric point if and only if x is a non-zero right-symmetric point if and only if $x = \lambda e_N$, for some $N \in \mathbb{N}$, $c \in \mathbb{K}$ or $x = \lambda_1 e_N + \lambda_2 e_M$, for some $N, M \in \mathbb{N}$ and $\lambda_1, \lambda_2 \in \mathbb{K}$, $|\lambda_1| = |\lambda_2|$.*

Proof We first characterize the left-symmetric points. Let $x \neq 0$. The sufficiency can be verified by elementary computations. Now, suppose $x_N, x_M \neq 0$ and $|x_N| \neq |x_M|$, for some $N, M \in \mathbb{N}$. Set $y = \operatorname{sgn}(x_N)|x_M|^{p-1} e_N - \operatorname{sgn}(x_M)|x_N|^{p-1} e_M$ and note that by Theorem 4.4, $x \perp_B y$. However,

$$\sum_{n=1}^{\infty} \overline{\operatorname{sgn}(y_n)} |y_n|^{p-1} x_n = |x_N x_M| \left(|x_N|^{p^2-2p} - |x_M|^{p^2-2p} \right) \neq 0,$$

since $p \neq 2$, proving $y \not\perp_B x$. Again if $x_N, x_M, x_K \neq 0$, we may assume that $|x_N| = |x_M| = |x_K|$, as otherwise, by the previous argument, x cannot be left-symmetric. But then, setting $y = \operatorname{sgn}(x_N)e_N - \frac{1}{2}(\operatorname{sgn}(x_M)e_M - \operatorname{sgn}(x_K)e_K)$ clearly yields $x \perp_B y$ and $y \not\perp_B x$ by Theorem 4.4.

Now, Proposition 2.1 of [16] states that in a smooth, strictly convex space, a point is left-symmetric if and only if it is right-symmetric. Hence the proof for the right-symmetric case follows from the left-symmetric case.

As a consequence of this result, we characterize all the onto isometries $T : \ell_p \to \ell_p$ and prove the Banach-Lamperti Theorem for onto operators on ℓ_p.

Theorem 4.5 *Let $T : \ell_p \to \ell_p$ be an onto linear isometry. Then T must be a signed permutation operator.*

Proof Observe that T and T^{-1} are both onto isometries and hence $T(x)$ and $T^{-1}(x)$ are both left-symmetric points of ℓ_p, for any $x \in \ell_p$ left-symmetric, by Corollary 1.1. Hence it is sufficient to show that for every $n \in \mathbb{N}$, there exists $m \in \mathbb{N}$ such that $T(e_n) = \lambda e_m$, for some $\lambda \in \mathbb{K}$ with $|\lambda| = 1$. Suppose by contradiction, $T(e_n) \neq \lambda e_m$, for any $m \in \mathbb{N}$ and $|\lambda| = 1$. Then $T(e_n) = \frac{1}{2^{\frac{1}{p}}}(\lambda_1 e_i + \lambda_2 e_j)$, for $i \neq j \in \mathbb{N}$ and $\lambda_1, \lambda_2 \in \mathbb{K}$, $|\lambda_1| = |\lambda_2| = 1$. Now, $T^{-1}\left(\frac{1}{2^{\frac{1}{p}}}(\lambda_1 e_i - \lambda_2 e_j)\right)$ is a left-symmetric point other than any unimodular multiple of e_n and hence must be λe_m for some $m \neq n$ and $|\lambda| = 1$ or $\frac{1}{2^{\frac{1}{p}}}(\nu_1 e_k + \nu_2 e_l)$, for some $k \neq l \in \mathbb{N}$ and $|\nu_1| = |\nu_2| = 1$. In either case, we clearly obtain that $T^{-1}(e_i)$ is not a left-symmetric point, which establishes the desired contradiction.

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