BLOCH'S FORMULA FOR 0-CYCLES WITH MODULUS AND HIGHER DIMENSIONAL CLASS FIELD THEORY

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Abstract. We prove Bloch's formula for the Chow group of 0-cycles with modulus on a smooth quasi-projective surface over a field. We use this formula to give a simple proof of the rank one case of a conjecture of Deligne and Drinfeld on lisse $\overline{\mathbb{Q}}_\ell$-sheaves. This was originally solved by Kerz and Saito in characteristic $\neq 2$.

CONTENTS

1. INTRODUCTION

Let X be a normal and complete variety over a finite field k. Let $U \subset X$ be a quasi-projective open subvariety which is smooth and whose complement is the support of an effective Cartier divisor on X. Let $(V_Z)_Z$ be a family of semisimple lisse $\overline{\mathbb{Q}}_\ell$-sheaves on the normalizations Z^N, where Z runs through all closed integral curves on U. The family $(V_Z)_Z$ is said to be a 2-skeleton sheaf if for two curves $Z_1 \neq Z_2$, the sheaves V_{Z_1} and V_{Z_2} become isomorphic up to semi-simplification after pull-back to the support of $Z_1^N \times_U Z_2^N$.

Let $\psi_Z \colon \overline{Z}^N \to \overline{Z}$ be the normalization of the closure $\overline{Z}$ of an integral curve $Z \subset U$ in X and let $Z_\infty = \psi_Z^{-1}(\overline{Z} \smallsetminus U)$. The family $(V_Z)_Z$ is said to have bounded ramification if there exists an effective Cartier divisor $D \subset X$ supported on $X \smallsetminus U$ such that for all integral curves $Z \subset U$, one has the inequality of Cartier divisors $\sum_{y \in Z_\infty} \mathrm{ar}_y(V_Z)[y] \leq \psi_Z^*(D)$ on $\overline{Z}^N$, where $\mathrm{ar}_y(V_Z)$ is the local Artin conductor of V_Z at y (see [Ser68]).

Motivated by the work of Drinfeld [Dri12], a conjecture of Deligne [EK12] states that given a 2-skeleton sheaf $(V_Z)_Z$ with bounded ramification, there exists a lisse $\overline{\mathbb{Q}}_\ell$-sheaf V on U such that $V|_{Z^N} \cong V_Z$ after semi-simplification for all integral curves $Z \subset U$.

Let $\mathrm{CH}_0(X|D)$ denote the Chow group of 0-cycles on X with modulus (see [BS19] or [KS16]) for any effective Cartier divisor $D \subset X$. Let $C(U) = \varprojlim_{D \subset X \smallsetminus U} \mathrm{CH}_0(X|D)$, where D runs through all effective Cartier divisors supported on $X \smallsetminus U$. Let $\pi_1^{\mathrm{ab}}(U)$ denote the abelianized étale fundamental group of U.

Kerz and Saito [KS16] showed that $C(U)$ is independent of the choice of the compactification X and that there is a reciprocity map $\rho_U : C(U) \to \pi_1^{\mathrm{ab}}(U)$. This induces a map $\rho_U^0 : C(U)^0 \to \pi_1^{\mathrm{ab}}(U)^0$ on the degree zero parts (see 8.3). The rank one case of the above conjecture of Deligne is then a direct consequence of another conjecture, namely, the map ρ_U^0 is an isomorphism of topological pro-finite abelian groups. Kerz and Saito proved this latter conjecture when $\mathrm{char}(k) \neq 2$ through a series of several highly non-trivial reductions.

In this paper, we give a simple and independent proof of the theorem of Kerz and Saito, including the missing characteristic 2 case. As a byproduct, we obtain a simple proof of the rank one case of Deligne's conjecture in all positive characteristics. We deduce these results by proving Bloch's formula for $\mathrm{CH}_0(X|D)$ when X is a smooth quasi-projective surface over an arbitrary base field k.

1.1. **The main results.** We now state our precise results. For a local ring A and an ideal $I \subset A$, let $K_*^M(A, I)$ denote the relative Milnor K-theory (e.g., see [KS86, § 1.3]). Let A^h denote the Henselization of A with respect to its maximal ideal. For a closed immersion of Noetherian schemes $D \subset X$ defined by the sheaf of ideals $\mathcal{I}_D$, we let $\mathcal{K}_{n,X}^M$ denote the Zariski (resp. Nisnevich) sheaf on X whose stalk at a point $x \in X$ is the relative Milnor K-group $K_n^M(\mathcal{O}_{X,x}, \mathcal{I}_{D,x})$ (resp. $K_n^M(\mathcal{O}_{X,x}^h, \mathcal{I}_{D,x}^h)$)

Theorem 1.1. *Let X be a smooth quasi-projective surface over a field k and let $D \subset X$ be an effective Cartier divisor. Then there are canonical isomorphisms*

(1.1) $$\rho_{X|D} : \mathrm{CH}_0(X|D) \xrightarrow{\cong} H_{\mathrm{zar}}^2(X, \mathcal{K}_{2,(X,D)}^M) \xrightarrow{\cong} H_{\mathrm{nis}}^2(X, \mathcal{K}_{2,(X,D)}^M).$$

When $D = 0$, Theorem 1.1 is classical and was first proven by Bloch [Blo74] for Quillen K-theory and by Kato [Kat86] for Milnor K-theory. For general D, Theorem 1.1 was proven earlier by Binda and Krishna [BK18, Theorem 1.8] when k is algebraically closed. When X is affine and k is perfect, this theorem was proven by Gupta and Krishna [GK20b, Theorem 1.4]. We shall refer to this theorem as the Bloch-Kato formula.

Combining Theorem 1.1 with an induction argument due to Kerz and Saito [KS16] and the Kato-Saito class field theory [KS86], we obtain the following.

Theorem 1.2. *Let U be a smooth quasi-projective variety of dimension $d \geq 1$ over a finite field k. Suppose there is an open immersion $U \subset X$ such that X is normal and proper over k and $(X \smallsetminus U)_{\mathrm{red}}$ is the support of an effective Cartier divisor on X. Then the reciprocity map*

$$\rho_U^0 : C(U)^0 \to \pi_1^{\mathrm{ab}}(U)^0$$

is an isomorphism of pro-finite topological abelian groups.

Since the rank one lisse $\overline{\mathbb{Q}}_\ell$-sheaves on U are same as the characters of the pro-finite group $\pi_1^{\mathrm{ab}}(U)$, the following is an immediate consequence of Theorem 1.2 and the Pontryagin duality theorem for pro-finite groups.

Corollary 1.3. *The rank one case of Deligne's conjecture is true for any smooth quasi-projective variety U over a finite field.*

We list some more applications of Theorem 1.1. Apart from its application to higher dimensional class field theory and solution of the rank one case of Deligne's conjecture, Theorem 1.1 has further applications. It is used in [BK21b] to prove a restriction isomorphism for the relative 0-cycle group on a regular and flat projective scheme over a henselian discrete valuation ring with arbitrary reduction. Theorem 1.1 is used in [BK21a] to give an explicit comparison between the motivic cohomology (in the sense of Suslin-Voevodsky) and the Levine-Weibel Chow group of a normal crossing scheme. Finally, it forms a crucial step in the recent proof of Bloch's formula for the Chow group of 0-cycles with modulus in higher dimensions in [GK20c].

1.2. **Outline of proofs.** Let U be as in Theorem 1.2. Kerz and Saito [KS16] defined a reciprocity map $\rho_U \colon C(U) \to \pi_1^{\mathrm{ab}}(U)$, and the goal is to show that this is an isomorphism on the degree zero parts. As a first step, it is possible to use an alteration (in the sense of de Jong) to assume that the compactification X of U is smooth and projective, with complement $(X \smallsetminus U)_{\mathrm{red}}$ the support of a strict normal crossing divisor. Next, using a generalized version of the Bertini theorem of Poonen [Poo04] and the Lefschetz hyperplane theorem of Kerz and Saito [KS14], one reduces to the case when $d \le 2$. This is the main point, and it is the case for which we provide a new and simple proof in this paper.

The $d = 1$ case is classical. Our strategy for proving Theorem 1.2 in $d = 2$ case is the following. We define a cycle class map $\rho_{X|D} \colon \mathrm{CH}_0(X|D) \to H^2_{\mathrm{nis}}(X, \mathcal{K}^M_{2,(X,D)})$ for any effective Cartier divisor $D \subset X$ supported on $X \smallsetminus U$. Theorem 1.1 says that this map is an isomorphism. Taking the limit over all such $D \subset X$, we get an isomorphism $\widetilde{\rho}_U \colon C(U) \to \varprojlim_D H^2_{\mathrm{nis}}(X, \mathcal{K}^M_{2,(X,D)})$. Let us denote the limit on the right side as $C^{\mathrm{KS}}(U)$.

Kato and Saito [KS86] constructed a reciprocity isomorphism $\widehat{\rho}^0 \colon C^{\mathrm{KS}}(U)^0 \xrightarrow{\cong} \pi_1^{\mathrm{ab}}(U)^0$. We thus have maps

$$C(U)^0 \xrightarrow{\widetilde{\rho}^0_U} C^{\mathrm{KS}}(U)^0 \xrightarrow{\widehat{\rho}^0} \pi_1^{\mathrm{ab}}(U)^0.$$

It is easy to check that the composite map is the reciprocity map ρ^0_U of Kerz and Saito. The map $\widehat{\rho}^0$ was shown to be an isomorphism by Kato and Saito [KS86]. This provides a proof of Theorem 1.2.

The heart of this paper is the proof of Theorem 1.1. This proof has two main ingredients: (1) the fundamental exact sequence of [BK18] which relates the 0-cycles group with modulus on smooth varieties with a modified version of the Levine-Weibel 0-cycle group (referred to as the lci Chow group) on a singular variety, and (2) a proof of Bloch's formula for the modified version of the Levine-Weibel 0-cycle group. For (2), we use the existence of a good theory of pull-back and push-forward for the modified Levine-Weibel Chow group and a pro-ℓ extension trick to reduce it to the case when the ground field is infinite.

When the ground field is algebraically closed, Levine [Lev85b] constructed a geometric theory of Chow ring on singular varieties. As a consequence, he showed that for a reduced quasi-projective surface X over an algebraically closed field, the cycle class map from the Levine-Weibel Chow group to $K_0(X)$ is injective. Even as Levine's complete proof is yet unpublished, a published account of the dimension two case of his proof is available in [BS98]. A major part of this paper is devoted to showing that Levine's program can be carried out for surfaces over an arbitrary infinite field. In particular, the injectivity of the cycle class map (for surfaces) holds over any infinite field. A known relation between the Levine-Weibel and the lci Chow groups (shown in [BK18]) then shows that this holds for the latter group too. The injectivity of the cycle class map implies Bloch's formula.

In § 2, we recall definitions of all cycle groups and recollect some results from [BK18] that play key roles in our proofs. In § 3, we construct the cycle class map and the Bloch-Kato map

which are used in the proof of Bloch's formula. We prove some results regarding these maps which are later used in the main proofs. The key results in this section are Theorems 3.6 and 3.7. The next four sections form the heart of the proofs, where the goal is to carry out Levine's strategy over an arbitrary infinite fields. Here, the main idea is solely due to Levine. However, due to the complications arising from the arbitrariness of the ground field, we need to include new arguments at several places to ensure that Levine's results remain valid over such fields. In § 8, we finish the proofs of our main results.

2. The Chow group of 0-cycles

In this section, we fix our notations and recall the definitions of various 0-cycle groups that we use in our proofs. We also prove some results relating these groups.

2.1. **Notations.** For a field k, we shall let $\mathbf{Sch}_k$ denote the category of quasi-projective schemes over k. We let $\mathbf{Sm}_k$ denote the subcategory of $\mathbf{Sch}_k$ consisting of smooth schemes over k. For $X \in \mathbf{Sch}_k$, we let X_{reg} denote the largest open subscheme of X which is regular. We let X_{sing} denote the complement of X_{reg} in X with its reduced induced closed subscheme structure. Recall here the convention that if $\dim(X) = d$, then X_{sing} contains every point of X which lies on an irreducible component of X of dimension less than d. We shall let X_{sm} denote the set of points in X where the map $X \to \mathrm{Spec}\,(k)$ is smooth. It is clear by the definition of smooth morphisms that $X_{\mathrm{sm}} \subset X$ is open. We let $X_{\mathrm{nsm}} = X \smallsetminus X_{\mathrm{sm}}$ with the reduced induced closed subscheme structure. Note that $X_{\mathrm{sm}} \subset X_{\mathrm{reg}}$ and hence $X_{\mathrm{sing}} \subset X_{\mathrm{nsm}}$.

For $X, Y \in \mathbf{Sch}_k$, we shall write the product $X \times_k Y$ as $X \times Y$. If $X_1 \times \cdots \times X_n$ is a product of schemes, then p_i will denote the projection from this product to its i-th factor (unless we use a specific notation in some context). We shall often write $X \times \mathbb{P}_k^n$ as $\mathbb{P}_X^n$. If $z \in Z := X \times Y$ is a closed point, then it is not true in general that z is uniquely determined by its projections $p_1(z) \in X$ and $p_2(z) \in Y$. However, this is indeed true if z happens to be a k-rational point of $X \times Y$. In this case, we can uniquely write $z = (p_1(z), p_2(z)) \in X \times Y$. More generally, if S is a scheme and $X, Y \in \mathbf{Sch}_S$, then the canonical map $(X \times_S Y)(S) \to X(S) \times Y(S)$ of sets is bijective. This fact will be used frequently in this paper. If $f: X' \to X$ is a morphism in $\mathbf{Sch}_k$, then a rational fiber of f will mean the scheme-theoretic fiber over a k-rational point of X.

We shall let $\mathcal{Z}(X)$ denote the free abelian group generated by integral closed subschemes of X. If $f: X' \to X$ is a proper map and $V \subset X'$ is an integral closed subscheme, then we shall let $f_*(V) \in \mathcal{Z}(X)$ denote the push-forward on X of the cycle $[V]$ in the sense of [Ful98].

If $X \in \mathbf{Sch}_k$ and $V_1, V_2 \subset X$ are two irreducible closed subschemes of X, then we shall say that they intersect properly if $\mathrm{codim}_X(V_1 \cap V_2) \geq \mathrm{codim}_X(V_1) + \mathrm{codim}_X(V_2)$. If V_1 or V_2 is not necessarily irreducible, then we shall say that they intersect properly if every irreducible component of V_1 intersects every irreducible component of V_2 properly. If V_1, V_2 are two closed subschemes intersecting properly, we shall let $V_1 \cdot V_2$ be their intersection product as a cycle on X in the sense of [Ful98, Chapter 6]. We shall use this intersection product only in the case when $V_1 \cap V_2 \subset X_{\mathrm{reg}}$. We shall let $\mathrm{codim}_X(\varnothing) = \infty$. If X is integral and $f \in k(X)^\times$ is a rational function, we shall let $\mathrm{div}(f)$ denote the cycle associated to f in $\mathcal{Z}(X)$ as in [Ful98].

2.2. **The Levine-Weibel Chow group of a surface.** Since we use the Levine-Weibel Chow group [LW85] only for surfaces, we do not recall its definition in full generality. In the case of surfaces, we shall use the following variant of the Levine-Weibel Chow group which was defined by Levine [Lev85b]. It was shown in *ibid.* that this variant is actually isomorphic to the Levine-Weibel Chow group, but we shall have no occasion to use this comparison. We fix an arbitrary field k.

Definition 2.1. Let X be a reduced quasi-projective surface over k and let $Y \subset X$ be a nowhere dense closed subscheme containing X_{sing}. Let $\mathcal{Z}_0(X, Y)$ be the free abelian group on the set of closed points lying in $X \smallsetminus Y$. Let $\mathcal{R}_0^L(X, Y)$ denote the subgroup of $\mathcal{Z}_0(X, Y)$ generated by cycles $C_0 - C_\infty \in \mathcal{Z}_0(X, Y)$ such that the following hold:

There exists a closed subscheme $C \subset X \times \mathbb{P}_k^1$ of pure dimension one such that

(1) $C \cap \mathbb{P}_Y^1$ is finite and $C \cap (Y \times \{0, \infty\}) = \varnothing$.
(2) $C \cap (X \times \{0, \infty\})$ is finite.
(3) The projection map $C \to \mathbb{P}_k^1$ is flat over a neighborhood of $\{0, \infty\}$.
(4) If $p_1 : \mathbb{P}_X^1 \to X$ is the projection, then $(p_1)_*(C \cdot (X \times \{0\})) = C_0$ and $(p_1)_*(C \cdot (X \times \{\infty\})) = C_\infty$.
(5) The ideal of C in $\mathbb{P}_X^1$ is a complete intersection in the local ring of every point $x \in C \cap \mathbb{P}_Y^1$.

We define $\mathrm{CH}_0^L(X, Y)$ to be the quotient $\frac{\mathcal{Z}_0(X,Y)}{\mathcal{R}_0^L(X,Y)}$. When $Y = X_{\mathrm{sing}}$, we write $\mathrm{CH}_0^L(X, Y)$ simply as $\mathrm{CH}_0^L(X)$. In this paper, we shall refer to $\mathrm{CH}_0^L(X)$ as the 'Levine-Weibel Chow group' of X. We shall call a subscheme C as above a *Cartier curve* relative to Y. A Cartier curve in general will mean the one relative to X_{sing}.

In the sequel, we shall use the notation $\mathrm{CH}_0^{\mathrm{LW}}(X, Y)$ for the original definition of the Chow group of 0-cycles given by Levine and Weibel [LW85]. The notion of "Cartier curves" was introduced in [LW85, Definition 1.2]: we shall reserve the term 'Levine-Weibel Cartier curves' for those defined in [LW85].

The following lemma allows us to assume C to be reduced in the definition of the Levine-Weibel Chow group if k is infinite.

Lemma 2.2. *Assume that k is infinite. Let the pair (X, Y) be as in Definition 2.1. Then $\mathcal{R}_0^L(X, Y)$ is the subgroup of $\mathcal{Z}_0(X, Y)$ generated by cycles $C_0 - C_\infty \in \mathcal{Z}_0(X, Y)$, where C is a reduced Cartier curve in $\mathbb{P}_X^1$. If X is integral, we can assume C to be integral.*

Proof. Let $C \subset \mathbb{P}_X^1$ be a Cartier curve. Then it is also a Levine-Weibel Cartier curve on $\mathbb{P}_X^1$. In the latter case, it was shown in [Lev87, Lemma 1.4] that we can find reduced Levine-Weibel Cartier curves $C_1, \ldots, C_r$ on $\mathbb{P}_X^1$ which miss $Y \times \{0, \infty\}$ (since its codimension in $\mathbb{P}_X^1$ is at least two, see [ESV99, Lemma 1.3]) such that we have

$$C \cdot (X \times \{0\}) - C \cdot (X \times \{\infty\}) = \sum_{i=1}^{r} \left[C_i \cdot (X \times \{0\}) - C_i \cdot (X \times \{\infty\}) \right]$$

in $\mathcal{Z}_0(\mathbb{P}_X^1, \mathbb{P}_Y^1)$. In particular, we get

$$C_0 - C_\infty = \sum_{i=1}^{r} \left[(p_1)_*(C_i \cdot (X \times \{0\})) - (p_1)_*(C_i \cdot (X \times \{\infty\})) \right] = \sum_{i=1}^{r} \left[(C_i)_0 - (C_i)_\infty \right].$$

Since each C_i is a Cartier curve in our sense and is reduced, we are done. If X was integral, it was shown in [Lev87, Lemma 1.4] that we could have chosen each C_i to be integral. This finishes the proof. $\square$

Lemma 2.3. *Assume that $C \subset \mathbb{P}_X^1$ is a reduced Cartier curve. We can then find another reduced Cartier curve $C' \subset \mathbb{P}_X^1$ such that the projection $C' \to X$ is finite and $C_0 - C_\infty = C_0' - C_\infty'$.*

Proof. Write $C = C_1 \cup \ldots \cup C_r$ as union of irreducible components. Suppose that all the components C_i for $1 \leq i \leq s$ are such that the composition $C_i \to X$ is finite and $p_1(C_i) = x_i$ is a closed point of X for $s < i \leq r$. Note that p_1 is projective. Let C_i be one of these non-finite

components. Then condition (1) in Definition 2.1 implies that $x_i \in X_{\text{reg}}$. Moreover, we have

$$(p_1)_*(C_i \cdot (X \times \{0\})) - (p_1)_*(C_i \cdot (X \times \{\infty\})) = (p_1)_*(C_i \times_{\mathbb{P}^1} \{0\} - C_i \times_{\mathbb{P}^1} \{\infty\}) = 0$$

as 0-cycles on X_{reg}. Note that the above intersection makes sense, thanks to conditions (1) and (2) in Definition 2.1.

We let $C' := \overline{C \setminus (C_{s+1} \cup \cdots \cup Cr)}$ (where the closure is taken in $\mathbb{P}^1_X$) with the reduced induced closed subscheme structure. Since x_i is a closed point of X_{reg} for every $s < i \leq r$ and C is reduced, it follows that the schemes C and C' agree in an open neighborhood of $\mathbb{P}^1_{X_{\text{sing}}}$. We conclude that C' is a reduced Cartier curve on $\mathbb{P}^1_X$, the projection $p_1 \colon C' \to X$ is finite and $C_0 - C_\infty = C'_0 - C'_\infty$.   □

Note that the above proof shows that an integral Cartier curve C is either finite over X or $C_0 - C_\infty = 0$ in $\mathcal{Z}_0(X, Y)$. Combining Lemmas 2.2 and 2.3, we therefore get the following.

Corollary 2.4. *Assume that k is infinite. Let the pair (X, Y) be as in Definition 2.1. Then $\mathcal{R}_0^L(X, Y)$ is the subgroup of $\mathcal{Z}_0(X, Y)$ generated by cycles $C_0 - C_\infty \in \mathcal{Z}_0(X, Y)$, where C is a reduced Cartier curve in $\mathbb{P}^1_X$ which is finite over X. If X is integral, we can assume C to be integral.*

2.3. The lci Chow group of a singular scheme.

Let k be any field. The lci Chow group of a singular scheme was introduced in [BK18]. We recall it here. Let C be a reduced equidimensional curve over k and let $k(C)$ denote its ring of total quotients. Let $C_1, \ldots, C_r$ be the irreducible components of C. Then $k(C)$ is the product of the quotient fields of all C_i's. For $f \in k(C)^\times$, we can therefore write $f = (f_i) \in \prod\limits_{i=1}^{r} k(C_i)^\times$. We let $\text{div}(f)$ be the sum $\sum\limits_{i=1}^{r} \text{div}(f_i) \in \mathcal{Z}(C)$. If $Z \subset C$ is a finite closed subset, we let $\mathcal{O}_{C,Z}$ denote the semi-local ring of C at Z. Since $\mathcal{O}_{C,Z}^\times \subset k(C)^\times$, the cycle $\text{div}(f) \in \mathcal{Z}(C)$ makes sense for any $f \in \mathcal{O}_{C,Z}^\times$. For such finite set Z, we let $\mathcal{Z}_0(C, Z)$ denote the subgroup of $\mathcal{Z}(C)$ generated by closed points away from Z.

Let X be a reduced quasi-projective scheme over k and let $Y \subset X$ be a nowhere dense closed subscheme containing X_{sing}.

Definition 2.5. Let C be a reduced curve in $\mathbf{Sch}_k$ and let $\nu \colon C \to X$ be a finite morphism. We say that $\nu \colon (C, Z) \to (X, Y)$ is *a good curve relative to Y* if Z is a closed subscheme of C such that the following hold.

(1) No component of C is contained in Z.
(2) $\nu^{-1}(Y) \subseteq Z$.
(3) ν is a local complete intersection morphism at every point $x \in C$ such that $\nu(x) \in Y$.

Given any good curve (C, Z) relative to Y, we can consider the push-forward (see § 2.1) $\mathcal{Z}_0(C, Z) \xrightarrow{\nu_*} \mathcal{Z}_0(X, Y)$. We let $\mathcal{R}_0(C, Z, X)$ be the subgroup of $\mathcal{Z}_0(X, Y)$ generated by the set $\{\nu_*(\text{div}(f)) | f \in \mathcal{O}_{C,Z}^\times\}$. We write $\mathcal{R}_0(X, Y)$ for the subgroup of $\mathcal{Z}_0(X, Y)$ defined as the image of the map

$$(2.1) \qquad \bigoplus_{\nu:(C,Z)\to(X,Y) \text{ good}} \mathcal{R}_0(C, Z, X) \to \mathcal{Z}_0(X, Y),$$

where the index set runs over the set of good curves relative to (X, Y). We define the *Chow group of 0-cycles on X (relative to Y)* to be the quotient

$$(2.2) \qquad\qquad \text{CH}_0(X, Y) = \frac{\mathcal{Z}_0(X, Y)}{\mathcal{R}_0(X, Y)}.$$

If $Y = X_{\mathrm{sing}}$, we shall write $\mathrm{CH}_0(X, Y)$ simply as $\mathrm{CH}_0(X)$. To avoid any conflict of notations, we shall denote the Chow group of n-dimensional cycles on X modulo rational equivalence in the sense of [Ful98, Chapter 1] by $\mathrm{CH}_n^F(X)$. To distinguish $\mathrm{CH}_0(X, Y)$ from $\mathrm{CH}_0^L(X, Y)$, we shall often refer to it as the 'lci Chow group' of (X, Y). The relation between $\mathrm{CH}_0^L(X, Y)$ and $\mathrm{CH}_0(X, Y)$ is given by the following lemma.

Lemma 2.6. *Let k be an infinite field. Let X be a reduced quasi-projective surface over k and let $Y \subset X$ be a nowhere dense closed subscheme containing X_{sing}. Then the identity map of $\mathcal{Z}_0(X, Y)$ induces a canonical surjection*

$$\phi_{(X,Y)} \colon \mathrm{CH}_0^L(X, Y) \twoheadrightarrow \mathrm{CH}_0(X, Y).$$

Proof. Let $C \subset \mathbb{P}_X^1$ be a Cartier curve relative to Y and let $Z = \mathbb{P}_Y^1 \cap C$. We can assume that C is reduced and finite over X by Corollary 2.4. It is then easy to see that (C, Z) is a good curve relative to Y via the composite map $\nu \colon C \hookrightarrow \mathbb{P}_X^1 \xrightarrow{p_X} X$. Furthermore, if we let $f \colon C \to \mathbb{P}_k^1$ be the projection map, then it defines an element of $k(C)$. The conditions (1) and (2) of Definition 2.1 imply in fact that $f \in \mathcal{O}_{C,Z}^{\times}$. It is clear that $\nu_*(\mathrm{div}(f)) = C_0 - C_\infty$. Hence, $C_0 - C_\infty \in \mathcal{R}_0(X, Y)$. This finishes the proof. $\square$

The lci Chow group $\mathrm{CH}_0(X)$ possesses a reasonable theory of pull-back and push-forward maps. This property will play a very important role in the proofs of our main theorems. In contrast, the Levine-Weibel Chow group $\mathrm{CH}_0^L(X)$ is not known to have any good theory of push-forward maps induced by finite maps between two varieties. A particular result that we shall use in the main proofs is the following. We let (X, Y) be as in § 2.3. For any field extension $k \hookrightarrow k'$, we let $X' = X_{k'} := X \otimes_k k'$ and $Y' = Y_{k'} := Y \otimes_k k'$. We let $\mathrm{pr}_{k'/k} \colon X' \to X$ denote the projection map.

Proposition 2.7. *Let $k \hookrightarrow k'$ be a separable algebraic (possibly infinite) extension of fields. Then the following hold.*

(1) *There exists a pull-back map $\mathrm{pr}_{k'/k}^* \colon \mathrm{CH}_0(X, Y) \to \mathrm{CH}_0(X', Y')$.*

(2) *If there exists a sequence of separable algebraic field extensions $k = k_0 \subset k_1 \subset \cdots \subset k'$ with $k' = \cup_i k_i$ such that $X_i := X_{k_i}$ and $Y_i := Y_{k_i}$ for each $i \geq 1$, then we have $\varinjlim_i \mathrm{CH}_0(X_i, Y_i) \xrightarrow{\cong} \mathrm{CH}_0(X', Y')$.*

(3) *If $k \hookrightarrow k'$ is finite, then there exists a push-forward $(\mathrm{pr}_{k'/k})_* \colon \mathrm{CH}_0(X', Y') \to \mathrm{CH}_0(X, Y)$ such that $(\mathrm{pr}_{k'/k})_* \circ \mathrm{pr}_{k'/k}^*$ is multiplication by $[k' : k]$.*

Proof. See [BK18, Proposition 6.1]. $\square$

2.4. **Zero cycles with modulus.** Let k be any field. Given an integral normal curve C over k and an effective divisor $E \subset C$, we say that a rational function f on C has modulus E if $f \in \mathrm{Ker}(\mathcal{O}_{C,E}^{\times} \to \mathcal{O}_E^{\times})$. Here, $\mathcal{O}_{C,E}$ is the semi-local ring of C at the union of E and the generic point of C. In particular, $\mathrm{Ker}(\mathcal{O}_{C,E}^{\times} \to \mathcal{O}_E^{\times})$ is just $k(C)^{\times}$ if $|E| = \varnothing$. Let $G(C, E)$ denote the group of such rational functions.

Let X be a reduced quasi-projective scheme over k and let D be an effective Cartier divisor on X (we allow D to be empty). Let $Z_0(X, D)$ be the free abelian group on the set of closed points of $X \setminus D$. Let C be an integral normal curve over k and let $\varphi_C \colon C \to X$ be a finite morphism such that $\varphi_C(C) \not\subset D$. The push forward of cycles along φ_C gives a well defined group homomorphism

$$(\varphi_C)_* \circ \mathrm{div} \colon G(C, \varphi_C^*(D)) \to Z_0(X, D).$$

Definition 2.8 (Kerz-Saito)**.** We define the Chow group $\mathrm{CH}_0(X|D)$ of 0-cycles of X with modulus D as the cokernel of the homomorphism

$$(2.3) \qquad \bigoplus_{\varphi_C : C \to X} G(C, \varphi_C^*(D)) \to Z_0(X, D),$$

where the sum is taken over the set of finite morphisms $\varphi_C : C \to X$ from an integral normal curve such that $\varphi_C(C) \notin D$.

A general theory of higher Chow groups with modulus $\mathrm{CH}_p(X|D, q)$ was introduced in [BS19], where it was shown that $\mathrm{CH}_0(X|D, 0) \cong \mathrm{CH}_0(X|D)$. It is clear from the definition that the inclusion $Z_0(X|D) \subset Z_0(X)$ defines a canonical 'forget modulus' map $\mathrm{CH}_0(X|D) \to \mathrm{CH}_0^F(X)$ (the latter being Fulton's Chow group of zero cycles). It is known that the Chow group of 0-cycles with modulus is covariantly functorial for the proper maps: if $f : X' \to X$ is a proper map, D and D' are effective Cartier divisors on X and X' such that $f^*(D) = D'$, then there is a push-forward map $f_* : \mathrm{CH}_0(X'|D') \to \mathrm{CH}_0(X|D)$. If f is flat (but not necessarily proper) of relative dimension zero, then there is a pull-back map $f^* : \mathrm{CH}_0(X|D) \to \mathrm{CH}_0(X'|D')$ (see [BS19, Lemma 2.7] or [KP17, Propositions 2.10, 2.12]).

Analogous to the case of the lci Chow group, the Chow group of 0-cycles with modulus has the following nice behavior with respect to the change of the base field.

Proposition 2.9. *Let $k \hookrightarrow k'$ be a separable algebraic (possibly infinite) extension of fields. Let X be a non-singular quasi-projective scheme over k with an effective Cartier divisor D. Let $X' = X_{k'}$ and $D' = D_{k'}$ denote the base change of X and D, respectively. Let $\mathrm{pr}_{k'/k} : X' \to X$ be the projection map. Then the following hold.*

(1) *There exists a pull-back $\mathrm{pr}_{k'/k}^* : \mathrm{CH}_0(X|D) \to \mathrm{CH}_0(X'|D')$.*

(2) *If there exists a sequence of separable algebraic field extensions $k = k_0 \subset k_1 \subset \cdots \subset k'$ with $k' = \cup_i k_i$, then we have $\varinjlim_{i} \mathrm{CH}_0(X_{k_i}|D_{k_i}) \xrightarrow{\simeq} \mathrm{CH}_0(X'|D')$.*

(3) *If $k \hookrightarrow k'$ is finite, then there exists a push-forward $\mathrm{pr}_{k'/k\ *} : \mathrm{CH}_0(X'|D') \to \mathrm{CH}_0(X|D)$ such that $(\mathrm{pr}_{k'/k})_* \circ \mathrm{pr}_{k'/k}^*$ is multiplication by $[k' : k]$.*

Proof. See [BK18, Proposition 6.2]. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\square$

Remark 2.10. The reader should be aware of the fact that in the literature there exists another notion of Chow group of zero cycles with modulus, introduced by H. Russell [Rus13] in the study of a higher dimensional analogue of the generalized Jacobian of Rosenlicht-Serre. Russell used his generalized Albanese with modulus to rephrase Lang's class field theory of function fields of varieties over finite fields in explicit terms. Note however that the modulus condition in *loc.cit.* is different from the one used here (and in [KS16]), and our work is not directly related to his. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\square$

2.5. **The double construction.** Let X be a non-singular quasi-projective scheme of dimension d over k and let $D \subset X$ be an effective Cartier divisor. Recall from [BK18, § 2.1] that the double of X along D is a quasi-projective scheme $S(X, D) = X \amalg_D X$ so that

$$(2.4) \qquad \begin{array}{ccc} D & \xrightarrow{\ \iota\ } & X \\ {\scriptstyle\iota}\downarrow & & \downarrow{\scriptstyle\iota_+} \\ X & \xrightarrow{\ \iota_-\ } & S(X, D) \end{array}$$

is a co-Cartesian square in $\mathbf{Sch}_k$. In particular, the identity map of X induces a finite map $\nabla : S(X, D) \to X$ such that $\nabla \circ \iota_\pm = \mathrm{id}_X$ and $\pi = \iota_+ \amalg \iota_- : X \amalg X \to S(X, D)$ is the normalization map. We let $X_\pm = \iota_\pm(X) \subset S(X, D)$ denote the two irreducible components of $S(X, D)$. We

use S_X as a shorthand for $S(X, D)$ when the divisor D is understood. S_X is a reduced quasi-projective scheme whose singular locus is $D_{\mathrm{red}} \subset S_X$. It is projective (resp. affine) whenever X is so. It follows from [Kri18, Lemma 2.2] that (2.4) is also a Cartesian square.

It is clear that the map $\mathcal{Z}_0(S_X, D) \xrightarrow{(\iota_+^*, \iota_-^*)} \mathcal{Z}_0(X_+, D) \oplus \mathcal{Z}_0(X_-, D)$ is an isomorphism, and there are push-forward inclusion maps $p_{\pm *}: \mathcal{Z}_0(X, D) \to \mathcal{Z}_0(S_X, D)$ such that $\iota_+^* \circ p_{+*} = \mathrm{id}$ and $\iota_+^* \circ p_{-*} = 0$.

The fundamental result that connects the 0-cycles with modulus on X and 0-cycles on S_X is the following.

Theorem 2.11. *Let k be any field. Let X be a smooth quasi-projective surface over k and let $D \subset X$ be an effective Cartier divisor. Then there is a split short exact sequence*

$$0 \to \mathrm{CH}_0(X|D) \xrightarrow{p_{+*}} \mathrm{CH}_0(S_X) \xrightarrow{\iota_-^*} \mathrm{CH}_0(X) \to 0.$$

Proof. This is the restatement of the dimension two case of [BK18, Theorem 1.9] if we assume k is perfect. However, the only place in the proof of this theorem where the perfectness is used is the construction of the map $\tau_X^*: \mathrm{CH}_0^{\mathrm{LW}}(S_X) \to \mathrm{CH}_0(X|D)$ in [BK18, § 5] when k is infinite. The only reason for this assumption was to be able to apply Bertini theorems for reduced schemes. However, the relevant Bertini theorem was known only for schemes which are geometrically reduced (the two notions coincide over a perfect field) when [BK18] was written. But we can now remove this extra assumption in view of the new Bertini theorem [GK20a, Corollary 3.10] for reduced schemes over any infinite field. $\square$

3. THE CYCLE CLASS AND BLOCH-KATO MAPS

In this section, we shall recall the connection between various Chow groups defined in § 2 and the algebraic K-groups. We shall construct the Bloch-Kato map which will be used in the proof of the Bloch-Kato formula. We fix a field k. For a k-scheme X, we let $K_0(X)$ denote $\pi_0(K(X))$, where $K(X)$ is the Bass-Thomason-Trobaugh K-theory spectrum of X. If X is quasi-projective over k, then $K_0(X)$ coincides with the Grothendieck group of locally free sheaves on X. For a closed immersion of k-schemes $D \hookrightarrow X$, let $K(X, D)$ denote the homotopy fiber of the restriction map of Bass-Thomason-Trobaugh K-theory spectra $K(X) \to K(D)$. We let $K_i(X, D)$ denote the i-th stable homotopy group of $K(X, D)$.

3.1. **The cycle class maps.** Let us assume that X is a reduced quasi-projective scheme over k and let $Y \subset X$ be any closed subscheme containing X_{sing} (but not necessarily nowhere dense). If $x \in X \smallsetminus Y$ is a closed point, then it lies in the regular locus of X. In particular, the inclusion map $\mathrm{Spec}\,(k(x)) \hookrightarrow X$ has finite tor-dimension. This yields the maps of spectra $K(\mathrm{Spec}\,(k(x))) \xrightarrow{(\iota_x)_*} K(X) \xrightarrow{\iota_Y^*} K(Y)$. Since the composite map is canonically null-homotopic, $(\iota_x)_*$ has a canonical factorization $(\iota_x)_*: K(\mathrm{Spec}\,(k(x))) \to K(X, D)$. In particular, we have a map $(\iota_x)_*: \mathbb{Z} = K_0(\mathrm{Spec}\,(k(x))) \to K_0(X, D)$. Letting $cyc_{(X,Y)}([x]) = (\iota_x)_*(1)$, and extending it linearly on $\mathcal{Z}_0(X, D)$, we get a cycle class map

(3.1) $$cyc_{(X,Y)}: \mathcal{Z}_0(X, Y) \to K_0(X, Y).$$

Composing $cyc_{(X,Y)}$ with the canonical map $K_0(X, Y) \to K_0(X)$, we obtain the cycle class map $cyc_{(X,Y)}: \mathcal{Z}_0(X, Y) \to K_0(X)$. It is shown in [BK18, Lemma 3.13] that if Y is nowhere dense in X, this map kills $\mathcal{R}_0(X, Y)$ and hence defines a cycle class map

(3.2) $$cyc_{(X,Y)}: \mathrm{CH}_0(X, Y) \to K_0(X).$$

We shall write $cyc_{(X, X_{\mathrm{sing}})}$ in short as cyc_X. We shall denote the image of cyc_X by $F^2 K_0(X)$. Lemma 2.6 implies the following.

Corollary 3.1. *Assume that k is infinite, $\dim(X) = 2$ and Y is nowhere dense in X. Then the cycle class map $\mathcal{Z}_0(X,Y) \to K_0(X)$ induces the commutative diagram*

$$\text{(3.3)} \qquad \mathrm{CH}_0^L(X,Y) \overset{\phi_{(X,Y)}}{\twoheadrightarrow} \underbrace{\mathrm{CH}_0(X,Y) \overset{cyc_{(X,Y)}}{\longrightarrow} K_0(X),}_{cyc_{(X,Y)}^L}$$

where the map $cyc_{(X,Y)}^L$ is the cycle class map for the Levine Chow group defined in [LW85] *(see* [BK18, *Section 3]).*

We next recall the cycle class map for the 0-cycle group with modulus. Let X be a non-singular quasi-projective surface over k and let $D \subset X$ be an effective Cartier divisor. By (3.1), there is a cycle class map $cyc_{(X,D)} \colon \mathcal{Z}_0(X,D) \to K_0(X,D)$. It follows from [Kri15, Theorem 1.2] that this map factors through $\mathrm{CH}_0(X|D)$ so that there is a cycle class map

$$\text{(3.4)} \qquad\qquad cyc_{X|D} \colon \mathrm{CH}_0(X|D) \to K_0(X,D).$$

When X is non-singular, it is clear from the above construction that the composition $\mathrm{CH}_0(X|D)$ $\xrightarrow{cyc_{X|D}} K_0(X,D) \to K_0(X)$ coincides with the composition $\mathrm{CH}_0(X|D) \to \mathrm{CH}_0^F(X) \xrightarrow{cyc_X} K_0(X)$, where cyc_X is as in (3.2).

3.2. The Cohomology of K-theory sheaves.

We shall now establish a key step in the proof of Bloch-Kato formula for the Chow group of 0-cycles. We first recall the definition of relative Milnor K-theory.

Let A be a Noetherian local ring and let $I \subset A$ be an ideal. We let $K_n^M(A)$ denote the Milnor K-group of A as defined by Kato-Saito [KS86, § 1.3]. We let $K_n^M(A, I) = \mathrm{Ker}(K_n^M(A) \twoheadrightarrow K_n^M(A/I))$. The Milnor K-theory of local rings was also defined by Kerz in [Ker09]. The two definitions agree for $n \le 2$. However, this may not be the case when $n \ge 3$ and A has finite residue field. Since we are interested only in $n \le 2$ case in this paper, we shall ignore this subtlety.

We let $\widehat{K}_n^M(A)$ denote the improved Milnor K-group of A as defined in [Ker10]. Then the canonical map from the Milnor to the Quillen K-theory of A has a factorization $K_n^M(A) \to \widehat{K}_n^M(A) \to K_n(A)$. The first map is surjective for all $n \ge 0$ by [Ker10, Theorem 13]. It is an isomorphism when A is a field. The second map is an isomorphism when $n \le 2$ by [Ker10, Propositions 2, 10]. Since we shall only use $\widehat{K}_n^M(A)$ for $n \le 2$, we shall make no difference between $\widehat{K}_n^M(A)$ and $K_n(A)$ and denote the common group by the latter notation. When the residue field of A is infinite, then the map $K_n^M(A) \to K_n(A)$ is an isomorphism for $n \le 2$ by [Ker10, Proposition 10]. It follows that the map $K_n^M(A, I) \to \overline{K}_n(A, I)$ is also an isomorphism in this case, where the latter group is defined to be the kernel of the restriction map $K_n(A) \to K_n(A/I)$.

We now fix a field k. Let X be a reduced quasi-projective surface over k and let $Y \subset X$ be a nowhere dense closed subscheme containing X_{sing}. Let $\mathcal{I}_Y$ denote the sheaf of ideals on X which defines Y. We let $\mathcal{I}_{Y,x}$ denote the stalk of $\mathcal{I}_Y$ at a point $x \in X$. For any integer $n \ge 0$, we let $\mathcal{K}_{n,(X,Y)}^M$ denote the Zariski (resp. Nisnevich) sheaf on X whose stalk at a point $x \in X$ is the Milnor K-group $K_n^M(\mathcal{O}_{X,x}, \mathcal{I}_{Y,x})$ (resp. $K_n^M(\mathcal{O}_{X,x}^h, \mathcal{I}_{Y,x}^h)$). The sheaf $\mathcal{K}_{n,(X,Y)}$ is defined similarly. We write $\mathcal{K}_{n,(X,\varnothing)}^M$ as $\mathcal{K}_{n,X}^M$. We use a similar notation for $\mathcal{K}_{n,(X,\varnothing)}$.

Let τ denote Zariski or Nisnevich topology and assume $n \le 2$. Then it follows from the above that the canonical map of τ-sheaves $\mathcal{K}_{n,X}^M \to \mathcal{K}_{n,X}$ is surjective and generically an isomorphism. It is clear that the map $\mathcal{K}_{n,(X,Y)}^M \to \overline{\mathcal{K}}_{n,(X,Y)}$ is generically an isomorphism. It is shown in [DS73, Theorem 2.1] (see the proof of [KS86, Proposition 9.9]) that this map is surjective. If

k is infinite, then both maps are isomorphisms. Since the surjection $\mathcal{K}_{n,(X,Y)} \twoheadrightarrow \overline{\mathcal{K}}_{n,(X,Y)}$ is an isomorphism away from Y, we therefore get the following.

Lemma 3.2. *Suppose $n \leq 2$. Then the following hold.*

(1) $H^2_\tau(X, \mathcal{K}_{n,(X,Y)}) \to H^2_\tau(X, \overline{\mathcal{K}}_{n,(X,Y)})$ *is an isomorphism.*

(2) *The map $\mathcal{K}^M_{n,X} \to \mathcal{K}_{n,X}$ induces an isomorphism*

$$H^2_\tau(X, \mathcal{K}^M_{n,X}) \xrightarrow{\cong} H^2_\tau(X, \mathcal{K}_{n,X}).$$

(3) *The map $\mathcal{K}^M_{n,(X,Y)} \twoheadrightarrow \overline{\mathcal{K}}_{n,(X,Y)}$ induces an isomorphism*

$$H^2_\tau(X, \mathcal{K}^M_{n,(X,Y)}) \xrightarrow{\cong} H^2_\tau(X, \mathcal{K}_{n,(X,Y)}).$$

We next show the following.

Lemma 3.3. *The change of topology maps*

(3.5) $$\lambda_X : H^2_{\mathrm{zar}}(X, \mathcal{K}^M_{2,X}) \to H^2_{\mathrm{nis}}(X, \mathcal{K}^M_{2,X});$$

(3.6) $$\lambda_X : H^2_{\mathrm{zar}}(X, \mathcal{K}_{2,X}) \to H^2_{\mathrm{nis}}(X, \mathcal{K}_{2,X})$$

are injective.

Proof. In view of Lemma 3.2, both statements are equivalent. So we shall show the injectivity of the second map.

The Thomason-Trobaugh descent spectral sequence and its compatibility with respect to change of topologies yield a commutative diagram with exact rows

(3.7)
$$\begin{array}{ccccccc}
K_1(X) & \to & H^0_{\mathrm{zar}}(X, \mathcal{K}_{1,X}) & \xrightarrow{d^{0,1}} & H^2_{\mathrm{zar}}(X, \mathcal{K}_{2,X}) & \xrightarrow{\gamma_X} & K_0(X) \\
\| & & \downarrow & & \downarrow & & \| \\
K_1(X) & \to & H^0_{\mathrm{nis}}(X, \mathcal{K}_{1,X}) & \xrightarrow{d^{0,1}} & H^2_{\mathrm{nis}}(X, \mathcal{K}_{2,X}) & \xrightarrow{\gamma_X} & K_0(X),
\end{array}$$

where the vertical arrows are the change of topology maps and the maps γ_X are the edge homomorphisms for the Thomason-Trobaugh spectral sequence.

It is shown in [Kri15, Lemma 2.1] (see also [KS02, p. 162]) that the map $K_1(X) \to H^0_\tau(X, \mathcal{K}_{1,X})$ is naturally split surjective when τ is either Zariski or Nisnevich topology. Hence, we get injective maps

(3.8) $$H^2_{\mathrm{zar}}(X, \mathcal{K}_{2,X}) \hookrightarrow H^2_{\mathrm{nis}}(X, \mathcal{K}_{2,X}) \hookrightarrow K_0(X).$$

This finishes the proof. $\square$

3.3. **The Bloch-Kato map for singular surfaces.** We now assume again that X is a reduced quasi-projective surface over an arbitrary field k and Y is any closed subscheme of X containing X_{sing}. Let $x \in X \setminus Y$ be a closed point. The excision theorem for the cohomology with support shows that the restriction map $H^2_{\{x\}}(X, \mathcal{K}^M_{2,X}) \to H^2_{\{x\}}(X_{\mathrm{reg}}, \mathcal{K}^M_{2,X_{\mathrm{reg}}})$ is an isomorphism. On the other hand, we have a commutative diagram of Zariski cohomology groups

(3.9)
$$\begin{array}{ccc}
H^2_{\{x\}}(X_{\mathrm{reg}}, \mathcal{K}^M_{2,X_{\mathrm{reg}}}) & \longrightarrow & H^2(X, \mathcal{K}^M_{2,X}) \\
\downarrow & \overset{\alpha_x}{\nearrow} & \downarrow \scriptstyle{\cong} \\
H^2_{\{x\}}(X_{\mathrm{reg}}, \mathcal{K}_{2,X_{\mathrm{reg}}}) & \longrightarrow & H^2(X, \mathcal{K}_{2,X}),
\end{array}$$

where the horizontal arrows are the 'forget support' maps and the vertical arrows are induced by the canonical Milnor to Quillen K-theory map. The right vertical arrow is an isomorphism by Lemma 3.2. It follows that there is a unique arrow α_x as indicated such that the inner triangles and the outer square in (3.9) commute. We remark that the left vertical arrow in this diagram is also an isomorphism if X_{reg} is smooth over k (see [Kat86, Theorem 2]), but we do not need it.

The Gersten resolution of $\mathcal{K}_{2,X_{\mathrm{reg}}}$ [Qui73, Theorem 5.11] shows that $H^2_{\{x\}}(X_{\mathrm{reg}}, \mathcal{K}_{2,X_{\mathrm{reg}}}) \cong K_0^{\{x\}}(X) \cong K_0(k(x)) \cong \mathbb{Z}$. Hence, x defines a unique cycle class $[x] \in H^2_{\{x\}}(X_{\mathrm{reg}}, \mathcal{K}_{2,X_{\mathrm{reg}}})$. Taking its image under α_x, we get a well-defined cycle class $\rho_{(X,Y)}([x]) \in H^2_{\mathrm{zar}}(X, \mathcal{K}^M_{2,X})$. Extending this assignment linearly on $\mathcal{Z}_0(X,Y)$, we get a map $\rho_{(X,Y)} \colon \mathcal{Z}_0(X,Y) \to H^2_{\mathrm{zar}}(X, \mathcal{K}^M_{2,X})$. Composing this with the change of topology map $\lambda_X \colon H^2_{\mathrm{zar}}(X, \mathcal{K}^M_{2,X}) \to H^2_{\mathrm{nis}}(X, \mathcal{K}^M_{2,X})$, we obtain maps

$$(3.10) \qquad \mathcal{Z}_0(X,Y) \xrightarrow{\rho_{(X,Y)}} H^2_{\mathrm{zar}}(X, \mathcal{K}^M_{2,X}) \xrightarrow{\lambda_X} H^2_{\mathrm{nis}}(X, \mathcal{K}^M_{2,X}).$$

We also have the cycle class map $cyc_{(X,Y)} \colon \mathcal{Z}_0(X,Y) \to K_0(X,Y) \to K_0(X)$ from (3.2). These maps are related by the the following.

Lemma 3.4. *There is a commutative diagram*

$$(3.11)$$

$$\begin{array}{ccc} \mathcal{Z}_0(X,Y) \xrightarrow{\rho_{(X,Y)}} H^2_{\mathrm{zar}}(X, \mathcal{K}^M_{2,X}) & \xrightarrow{\lambda_X} & H^2_{\mathrm{nis}}(X, \mathcal{K}^M_{2,X}) \\ & {}_{cyc_{(X,Y)}} \searrow & \downarrow{\gamma_X} \\ & & K_0(X). \end{array}$$

In particular, $\rho_{(X,Y)}$ factors through its quotient $\mathrm{CH}_0(X,Y)$ if Y is nowhere dense in X.

Proof. The commutativity of the diagram is [GK20b, Lemma 3.2]. Note that the proof of the cited result uses no assumption on the nature of the field k. The injectivity of λ_X and γ_X is (3.8). The last assertion follows from the injectivity of $\gamma_X \circ \lambda_X$ and (3.2). $\qquad \square$

Lemma 3.5. *The map $\mathcal{Z}_0(X,Y) \xrightarrow{\rho_{(X,Y)}} H^2_{\mathrm{zar}}(X, \mathcal{K}^M_{2,X})$ is surjective if Y is nowhere dense in X. The map λ_X is an isomorphism if $X \smallsetminus Y$ is furthermore smooth over k (e.g., if k is perfect).*

Proof. Suppose first that k is infinite. Let $\alpha \in \mathcal{Z}_0(X, X_{\mathrm{sing}})$ be a 0-cycle. Since k is infinite, we can use [Lev87, Lemma 1.3] to find a complete intersection reduced curve $C \subset X$ such that $C \cap Y$ is finite and $C \cap X_{\mathrm{reg}}$ is regular. Moreover, the latter contains the support of α. This yields a push-forward map (e.g., see [ESV99, Lemma 1.8]) $\mathrm{Pic}(C) \to \mathrm{CH}_0(X, X_{\mathrm{sing}})$. Since Y is nowhere dense in X, it is well known that $\mathrm{Pic}(C)$ is generated by closed points away from $C \cap Y$ (note that $C_{\mathrm{sing}} \subset Y$ because $X_{\mathrm{sing}} \subset Y$ and $C \cap X_{\mathrm{reg}}$ is regular), it follows that the canonical map $\mathcal{Z}_0(X,Y) \to \mathrm{CH}_0^L(X, X_{\mathrm{sing}}) = \mathrm{CH}_0^L(X)$ is surjective. On the other hand, the composite map $\mathrm{CH}_0^L(X) \twoheadrightarrow \mathrm{CH}_0(X) \xrightarrow{\rho_X} H^2_{\mathrm{zar}}(X, \mathcal{K}^M_{2,X})$ is surjective by [Lev85a, p. 169] and Lemma 3.2. We have thus shown that the map $\rho_{(X,Y)}$, which factorizes as $\mathcal{Z}_0(X,Y) \twoheadrightarrow \mathrm{CH}_0^L(X) \twoheadrightarrow \mathrm{CH}_0(X) \xrightarrow{\rho_X} H^2_{\mathrm{zar}}(X, \mathcal{K}^M_{2,X})$, is surjective.

If k is finite, then $X \smallsetminus Y$ is smooth over k and dense open in X. So it suffices now to prove the lemma when $X \smallsetminus Y$ is smooth over k and dense open in X. In the latter case, $\lambda_X \circ \rho_{(X,Y)}$ is surjective by [KS86, Theorem 2.5]. In particular, λ_X is surjective. We are therefore done by Lemma 3.3. $\qquad \square$

We can summarize the above results in the form of the following.

Theorem 3.6. *Let X be a reduced quasi-projective surface over a field k. Let $Y \subset X$ be a nowhere dense closed subscheme containing X_{sing}. Then there are canonical maps*

$$(3.12) \qquad \text{CH}_0(X,Y) \xrightarrow{\rho_{(X,Y)}} H^2_{\text{zar}}(X, \mathcal{K}^M_{2,X}) \overset{\lambda_X}{\cong} H^2_{\text{nis}}(X, \mathcal{K}^M_{2,X}) \overset{\gamma_X}{\cong} K_0(X)$$

such that the composite $\gamma_X \circ \lambda_X \circ \rho_{(X,Y)}$ is the cycle class map $\text{cyc}_{(X,Y)}$ of (3.2). The map $\rho_{(X,Y)}$ is surjective. The map λ_X is an isomorphism if $X \smallsetminus Y$ is smooth over k (e.g., if k is perfect).

When X has only one singular closed point and k is infinite, then $\rho_{(X,Y)}$ was shown to be an isomorphism by Collino [Col81]. When X is a normal surface, $\rho_{(X,Y)}$ was shown to be an isomorphism by Pedrini and Weibel [PW86]. Note here that Collino and Pedrini-Weibel prove their results only for $H^2_{\text{zar}}(X, \mathcal{K}_{2,X})$ (i.e., they use Quillen K-theory sheaf). When k is algebraically closed, $\rho_{(X,Y)}$ was shown to be an isomorphism by Levine [Lev85a]. We shall extend Levine's result to all infinite fields.

3.4. **The Bloch-Kato map for cycles with modulus.** We shall now prove the modulus version of Theorem 3.6. In order to do this, we need to generalize the construction of § 3.3 little bit. We fix an arbitrary field k.

Let X be a reduced quasi-projective surface over k and let $Y \subset X$ be a closed subscheme such that $X^o \coloneqq X \smallsetminus Y$ is smooth over k. Notice that we did not impose this extra condition in § 3.3.

Let $x \in X^o$ be a closed point. The excision theorem for the cohomology with support shows that the restriction map $H^2_{\{x\}}(X, \mathcal{K}^M_{2,(X,Y)}) \to H^2_{\{x\}}(X^o, \mathcal{K}^M_{2,X^o})$ is an isomorphism (in the Zariski topology). Since X^o is smooth over k, Kato showed that the Gersten complex for $\mathcal{K}^M_{2,X^o}$ is exact except at its left-most end. This implies that $H^2_{\{x\}}(X^o, \mathcal{K}^M_{2,X^o}) \cong K_0^{\{x\}}(X) \cong K_0(k(x)) \cong \mathbb{Z}$ (see [Kat86, Theorem 2]). Hence, x defines a unique cycle class $[x] \in H^2_{\{x\}}(X, \mathcal{K}^M_{2,(X,Y)})$. The image of this cycle class under the 'forget support' map $H^2_{\{x\}}(X, \mathcal{K}^M_{2,(X,Y)}) \to H^2(X, \mathcal{K}^M_{2,(X,Y)})$ yields a cycle class $[x] \in H^2_{\text{zar}}(X, \mathcal{K}^M_{2,(X,Y)})$. Extending this assignment linearly on $\mathcal{Z}_0(X,Y)$, we get a map $\rho_{(X,Y)} \colon \mathcal{Z}_0(X,Y) \to H^2_{\text{zar}}(X, \mathcal{K}^M_{2,(X,Y)})$. Composing this with the change of topology map $\lambda_{(X,Y)} \colon H^2_{\text{zar}}(X, \mathcal{K}^M_{2,(X,Y)}) \to H^2_{\text{nis}}(X, \mathcal{K}^M_{2,(X,Y)})$, we obtain maps

$$(3.13) \qquad \mathcal{Z}_0(X,Y) \xrightarrow{\rho_{(X,Y)}} H^2_{\text{zar}}(X, \mathcal{K}^M_{2,(X,Y)}) \xrightarrow{\lambda_{(X,Y)}} H^2_{\text{nis}}(X, \mathcal{K}^M_{2,(X,Y)}).$$

We now let k be a as above and let X be a smooth quasi-projective surface over k. Let $D \subset X$ be an effective Cartier divisor. Recall that the Thomason-Trobaugh descent spectral sequence yields the Nisnevich descent map $H^2_{\text{nis}}(X, \mathcal{K}_{2,(X,D)}) \to K_0(X,D)$. We let $\gamma_{(X,D)}$ denote its composition with the canonical map $H^2_{\text{nis}}(X, \mathcal{K}^M_{2,(X,D)}) \to H^2_{\text{nis}}(X, \mathcal{K}_{2,(X,D)})$.

Theorem 3.7. *Let X be a smooth quasi-projective surface over k. Then there are canonical maps*

$$(3.14) \qquad \text{CH}_0(X|D) \xrightarrow{\rho_{X|D}} H^2_{\text{zar}}(X, \mathcal{K}^M_{2,(X,D)}) \overset{\lambda_{(X,D)}}{\underset{\cong}{\to}} H^2_{\text{nis}}(X, \mathcal{K}^M_{2,(X,D)}) \xrightarrow{\gamma_{(X,D)}} K_0(X,D)$$

such that the composite $\gamma_{(X,D)} \circ \lambda_{(X,D)} \circ \rho_{X|D}$ is the cycle class map $\text{cyc}_{X|D}$ of (3.4). The map $\rho_{X|D}$ is surjective.

Proof. We first show that $cyc_{X|D} = \gamma_{(X,D)} \circ \lambda_{(X,D)} \circ \rho_{(X,D)}$ on $\mathcal{Z}_0(X,D)$, where $\rho_{(X,D)}$ is as in (3.13). It is enough to check this for every closed point $x \in X \smallsetminus D$. We let $\rho^{\mathrm{nis}}_{(X,D)} = \lambda_{(X,D)} \circ \rho_{(X,D)}$.

We let S_X be the double of X along D. We identify X as $X_+ \subset S_X$. It is then immediate from the construction of the cycle class map on $\mathcal{Z}_0(X,D) = \mathcal{Z}_0(X_+,D)$ in § 3.1 that the composite map

(3.15) $\qquad \mathcal{Z}_0(X_+,D) = \mathcal{Z}_0(S_X,X_-) \xrightarrow{cyc_{(S_X,X_-)}} K_0(S_X,X_-) \xrightarrow{\iota_+^*} K_0(X,D)$

coincides with $cyc_{X|D}$. We now consider the diagram

(3.16)
$$\begin{array}{ccccc}
\mathcal{Z}_0(S_X,X_-) & \xrightarrow{\rho^{\mathrm{nis}}_{(S_X,X_-)}} & H^2_{\mathrm{nis}}(S_X,\mathcal{K}^M_{2,(S_X,X_-)}) & \xrightarrow{\gamma_{(S_X,X_-)}} & K_0(S_X,X_-) \\
\big\uparrow & & \Big\downarrow{\iota_+^*} & & \Big\downarrow{\iota_+^*} \\
\mathcal{Z}_0(X,D) = \mathcal{Z}_0(X_+,D) & \xrightarrow{\rho^{\mathrm{nis}}_{(X,D)}} & H^2_{\mathrm{nis}}(X,\mathcal{K}^M_{2,(X,D)}) & \xrightarrow{\gamma_{(X,D)}} & K_0(X,D).
\end{array}$$

Here, γ denotes again the edge homomorphism for the Thomason-Trobaugh spectral sequence. The map $\rho^{\mathrm{nis}}_{(S_X,X_-)}$ is the composition of the two maps in (3.13). Like $cyc_{X|D}$, it follows immediately from the constructions of $\rho^{\mathrm{nis}}_{(S_X,X_-)}$ and $\rho^{\mathrm{nis}}_{(X,D)}$ that the left square in (3.16) commutes. The right square in (3.16) commutes by the functoriality of the Thomason-Trobaugh spectral sequence with respect to the inclusion $(X_+,D) \hookrightarrow (S_X,X_-)$. Using (3.15), it suffices therefore to show that composite map on the top row of (3.16) is $cyc_{(S_X,X_-)}$.

To prove this last assertion, we consider the commutative diagram

(3.17)
$$\begin{array}{ccccccccc}
0 & \longrightarrow & \mathcal{Z}_0(S_X,X_-) & \xrightarrow{p_{+*}} & \mathcal{Z}_0(S_X,D) & \xrightarrow{\iota_-^*} & \mathcal{Z}_0(X \smallsetminus D) & \longrightarrow & 0 \\
& & \downarrow{\rho^{\mathrm{nis}}_{(S_X,X_-)}} & & \downarrow{\rho^{\mathrm{nis}}_{(S_X,D)}} & & \downarrow{\rho^{\mathrm{nis}}_X} & & \\
0 & \to & H^2_{\mathrm{nis}}(S_X,\mathcal{K}^M_{2,(S_X,X_-)}) & \xrightarrow{p_{+*}} & H^2_{\mathrm{nis}}(S_X,\mathcal{K}^M_{2,S_X}) & \xrightarrow{\iota_-^*} & H^2_{\mathrm{nis}}(X,\mathcal{K}^M_{2,X}) & \to & 0 \\
& & \downarrow{\gamma_{(S_X,X_-)}} & & \downarrow{\gamma_{S_X}} & & \downarrow{\gamma_X} & & \\
0 & \longrightarrow & K_0(S_X,X_-) & \xrightarrow{p_{+*}} & K_0(S_X) & \xrightarrow{\iota_-^*} & K_0(X) & \longrightarrow & 0.
\end{array}$$

The three rows are split exact. It follows from Lemma 3.4 that

(3.18) $\qquad p_{+*} \circ cyc_{(S_X,X_-)} = cyc_{(S_X,D)} \circ p_{+*} = \gamma_{S_X} \circ \rho^{\mathrm{nis}}_{(S_X,D)} \circ p_{+*}.$

Since (3.17) commutes, we also have $p_{+*} \circ \gamma_{(S_X,X_-)} \circ \rho^{\mathrm{nis}}_{(S_X,X_-)} = \gamma_{S_X} \circ \rho^{\mathrm{nis}}_{(S_X,D)} \circ p_{+*}$. Combining this with (3.18) and using that p_{+*} is injective, we get $cyc_{(S_X,X_-)} = \gamma_{(S_X,X_-)} \circ \rho^{\mathrm{nis}}_{(S_X,X_-)}$. We have therefore shown that $\gamma_{(X,D)} \circ \lambda_{(X,D)} \circ \rho_{(X,D)}$ in (3.14) is $cyc_{X|D}$, which we wanted to show.

We now show that $\rho_{(X,D)}$ factors through the rational equivalence (this is due to Rülling-Saito [RS18] if D_{red} is a SNCD on X). For this, we note that (3.16) and (3.17) remain valid if we replace the Nisnevich cohomology by the Zariski cohomology. Suppose now that $\alpha \in \mathcal{R}_0(X|D)$. Then $p_{+*}(\alpha) \in \mathcal{R}_0(S_X,D)$ by Theorem 2.11. It follows from Theorem 3.6 (applied to S_X) that $\rho_{(S_X,D)} \circ p_{+*}(\alpha) = 0$. This implies by (3.17) that $\rho_{(S_X,X_-)}(\alpha) = 0$ in $H^2_{\mathrm{zar}}(S_X,\mathcal{K}^M_{2,(S_X,X_-)})$. Since the left square in (3.16) is commutative, we deduce that $\rho_{(X,D)}(\alpha) = 0$ in $H^2_{\mathrm{zar}}(X,\mathcal{K}^M_{2,(X,D)})$. We have thus shown that $\rho_{(X,D)}$ factors through its quotient $\mathrm{CH}_0(X|D)$. We let $\rho_{X|D}: \mathrm{CH}_0(X|D) \to H^2_{\mathrm{zar}}(X,\mathcal{K}^M_{2,(X,D)})$ be the induced map.

The map $\lambda_{(X,D)}$ is an isomorphism by a combination of Lemma 3.2 and [KS86, Proposition 9.8] (or [Kri15, Lemma 2.1]). It remains to show that $\rho_{X|D}$ is surjective. For this, it suffices to show that the map $\mathcal{Z}_0(X,D) \xrightarrow{\rho_{(X,D)}} H^2_{\mathrm{zar}}(X, \mathcal{K}^M_{2,(X,D)})$ is surjective. Now, we note in (3.17) that all rows are compatibly split, where all splittings are given by the pull-back via the flat projection $\pi: S_X \to X$. It follows therefore from Theorem 3.6 that $\rho_{(S_X,D)}$ and ρ_X are surjective. But this implies that $\rho_{(S_X,X_-)}$ is also surjective. Using the left square of (3.16), it suffices now to show that the restriction map $\iota_+^*: H^2_{\mathrm{zar}}(S_X, \mathcal{K}^M_{2,(S_X,X_-)}) \to H^2_{\mathrm{zar}}(X, \mathcal{K}^M_{2,(X,D)})$ is surjective. But this is clear because the map of Zariski sheaves $\mathcal{K}^M_{2,(S_X,X_-)} \xrightarrow{\iota_+^*} \mathcal{K}^M_{2,(X,D)}$ is an isomorphism away from D and $\dim(S_X) = 2$. $\qquad\square$

3.5. Some functoriality properties.
We shall use the following functoriality properties of the Chow group of 0-cycles and their cycle classes. For any $Y \in \mathbf{Sch}_k$, recall our notation that $\mathrm{CH}^F_*(Y)$ is the direct sum of all Chow groups of Y in the sense of [Ful75, Chapter 1].

Proposition 3.8. *Let k be any field and let X be a reduced quasi-projective surface over k. Let $j: U \hookrightarrow X$ be an open immersion such that $X_{\mathrm{sing}} \subset U$. Then we have an exact sequence of abelian groups*

$$\mathrm{CH}^F_0(X \smallsetminus U) \xrightarrow{i_*} \mathrm{CH}^L_0(X) \xrightarrow{j^*} \mathrm{CH}^L_0(U) \to 0.$$

The same holds for the lci Chow group.

Proof. The proofs of the proposition for the lci and the Levine-Weibel Chow groups are completely identical. We give the proof for the latter case. We have to first explain the maps i_* and j^*. It is clear that the restriction of cycles with respect to the open immersion j indeed gives the map j^*. It is clearly surjective because $X_{\mathrm{sing}} \subset U$.

The map i_* is the push-forward map with respect to the closed immersion $i: Z := X \smallsetminus U \hookrightarrow X$. It is clearly well-defined at the level of cycles. If $C \subset Z \times \mathbb{P}^1_k$ is an irreducible 1-cycle dominant over $\mathbb{P}^1_k$, then it is also clear (note that $X_{\mathrm{sing}} \subset U$) that $i_*(C)$ is a Cartier curve on $\mathbb{P}^1_X$ which defines the rational equivalence between $i_*(C_0 - C_\infty)$. Hence, the maps in the above sequence are defined. We only need to see that this sequence is exact in the middle.

So suppose that $\alpha = \sum_{i=1}^r n_i[x_i]$ is a 0-cycle on X such that $j^*(\alpha) = 0$. We can assume without loss of generality that $x_i \in U$ for each i. We let $C^1, \ldots, C^s$ be Cartier curves on $\mathbb{P}^1_U$ such that $j^*(\alpha) = \sum_{i=1}^s (C^i_0 - C^i_\infty)$. We let $\overline{C}^i$ be the scheme-theoretic closure of C^i is $\mathbb{P}^1_X$. Since $X_{\mathrm{sing}} \subset U$, it follows that $\overline{C}^i$ is a Cartier curve on $\mathbb{P}^1_X$ and $\overline{C}^i_0 - \overline{C}^i_\infty = (C^i_0 - C^i_\infty) + \beta_i$, where β_i is a 0-cycle supported on Z. Letting $\beta = \sum_{i=1}^s \beta_i$, we get $\alpha - \beta = \sum_{i=1}^s (\overline{C}^i_0 - \overline{C}^i_\infty)$. This finishes the proof. $\qquad\square$

Proposition 3.9. *Let k be an infinite field and let X be a reduced quasi-projective surface over k. Let $\pi: \widetilde{X} \to X$ be a resolution of singularities of X. Then we have a commutative diagram*

(3.19)
$$\begin{array}{ccc} \mathrm{CH}^L_0(X) & \xrightarrow{cyc^L_X} & K_0(X) \\ {\scriptstyle \pi^*}\downarrow & & \downarrow{\scriptstyle \pi^*} \\ \mathrm{CH}^F_0(\widetilde{X}) & \xrightarrow{cyc_{\widetilde{X}}} & K_0(\widetilde{X}). \end{array}$$

The same holds for the lci Chow group over any field.

Proof. Using Corollary 3.1, it suffices to prove the proposition for the lci Chow group. We need to first define the vertical arrow on the left. All other maps are defined. Since the map $\pi^{-1}(X_{\mathrm{reg}}) \to X_{\mathrm{reg}}$ is an isomorphism, the pull-back $\pi^*: \mathcal{Z}_0(X, X_{\mathrm{sing}}) \to \mathcal{Z}_0(\widetilde{X})$ is simply the inclusion map. It is now an easy exercise using the definitions of rational equivalences to check that this map preserves the subgroups of rational equivalences. We shall nonetheless provide a proof which simultaneously shows that (3.19) is commutative.

We now use Lemma 3.4 to get a diagram

$$(3.20) \qquad \begin{array}{ccc}
\mathcal{Z}_0(X, X_{\mathrm{sing}}) \xrightarrow{\rho_X} H^2_{\mathrm{zar}}(X, \mathcal{K}^M_{2,X}) \xrightarrow{\lambda_X} K_0(X) \\
\pi^* \downarrow \qquad\qquad \pi^* \downarrow \qquad\qquad \pi^* \downarrow \\
\mathrm{CH}_0^F(\widetilde{X}) \xrightarrow{\rho_{\widetilde{X}}} H^2_{\mathrm{zar}}(\widetilde{X}, \mathcal{K}^M_{2,\widetilde{X}}) \xrightarrow{\lambda_{\widetilde{X}}} K_0(\widetilde{X}).
\end{array}$$

It is clear from the various definitions above that this diagram is commutative. The classical Bloch's formula for non-singular surfaces and Lemma 3.2 together imply that $\rho_{\widetilde{X}}$ is an isomorphism. We now conclude from Lemma 3.4 that the left vertical arrow kills $\mathcal{R}_0(X, X_{\mathrm{sing}})$ and 3.19 commutes. $\qquad \square$

Lemma 3.10. *Let k be any field and let X be a reduced quasi-projective surface over k. Let $k \hookrightarrow k'$ be a separable algebraic extension (possibly infinite) of fields. Let $X' = X_{k'}$ and let $\mathrm{pr}^*_{k'/k}: X' \to X$ be the projection map. Then the diagram*

$$(3.21) \qquad \begin{array}{ccc}
\mathrm{CH}_0(X) & \xrightarrow{cyc_X} & K_0(X) \\
\mathrm{pr}^*_{k'/k} \downarrow & & \downarrow \mathrm{pr}^*_{k'/k} \\
\mathrm{CH}_0(X') & \xrightarrow{cyc_{X'}} & K_0(X')
\end{array}$$

is commutative.

Proof. Let $x \in X_{\mathrm{reg}}$ be a closed point and let $Y = \mathrm{Spec}\,(k(x)) \times_X X'$. Since k' is separable and algebraic over k, it follows that Y is a 0-dimensional reduced closed subscheme of X'. We let $[Y] = \mathrm{pr}^*_{k'/k}([x]) \in \mathcal{Z}_0(X', X'_{\mathrm{sing}})$. We let $\iota_x: \mathrm{Spec}\,(k(x)) \hookrightarrow X$ and $\iota_Y: Y \hookrightarrow X'$ be the inclusion maps. Then the diagram

$$(3.22) \qquad \begin{array}{ccc}
K_0(\mathrm{Spec}\,(k(x))) & \xrightarrow{(\iota_x)_*} & K_0(X) \\
\mathrm{pr}^*_{k'/k} \downarrow & & \downarrow \mathrm{pr}^*_{k'/k} \\
K_0(Y) & \xrightarrow{(\iota_Y)_*} & K_0(X')
\end{array}$$

is commutative by [TT90, Proposition 3.18]. Since $(\iota_Y)_*(1) = cyc_{X'}([Y])$, the lemma follows. $\qquad \square$

Corollary 3.11. *Let k be any field and let X be a smooth quasi-projective surface over k. Let $D \subset X$ be an effective Cartier divisor. Let $k \hookrightarrow k'$ be a separable algebraic extension (possibly infinite) of fields. Let $X' = X_{k'}, D' = D_{k'}$ and let $\mathrm{pr}_{k'/k}: X' \to X$ be the projection map. Then the diagram*

$$(3.23) \qquad \begin{array}{ccc}
\mathrm{CH}_0(X|D) & \xrightarrow{cyc_{X|D}} & K_0(X, D) \\
\mathrm{pr}^*_{k'/k} \downarrow & & \downarrow \mathrm{pr}^*_{k'/k} \\
\mathrm{CH}_0(X'|D') & \xrightarrow{cyc_{X'|D'}} & K_0(X', D')
\end{array}$$

is commutative.

Proof. We consider the diagram

$$(3.24) \qquad \begin{array}{ccccc} \mathcal{Z}_0(X, D) & \xrightarrow{cyc(S_X, X_-)} & \bar{K}_0(S_X, X_-) & \xrightarrow{\iota_+^*} & K_0(X, D) \\ {\scriptstyle \mathrm{pr}_{k'/k}^*} \downarrow & & \downarrow {\scriptstyle \mathrm{pr}_{k'/k}^*} & & \downarrow {\scriptstyle \mathrm{pr}_{k'/k}^*} \\ \mathcal{Z}_0(X', D') & \xrightarrow{cyc(S_{X'}, X'_-)} & \bar{K}_0(S_{X'}, X'_-) & \xrightarrow{\iota_+^*} & K_0(X', D'). \end{array}$$

We saw in (3.15) that the composite horizontal arrows in this diagram are the cycle class maps $cyc_{X|D}$ and $cyc_{X'|D'}$. Hence, it suffices to show that the two squares in (3.24) commute. The right square clearly commutes. The left square commutes by applying Lemma 3.10 to S_X and X and then using (3.17). $\qquad \square$

4. The family of good cycles

Our next goal is to show that the cycle class map $\mathrm{CH}_0^L(X, Y) \to K_0(X)$ is injective if X is a reduced quasi-projective surface over an infinite field. As we stated in § 1, this was proven by Levine in his yet unpublished manuscript [Lev85b] when the base field is algebraically closed. A published account of Levine's proof is available in [BS98]. In the next few sections, our goal is to revisit Levine's proof and show that it can be carried over to all infinite fields with the help of new arguments at every step of the proof.

In this section, we shall construct families $\phi \colon \Gamma_U \to U$ of cycles on X. We shall state and prove an upgraded version of Kleiman's transversality theorem [Kle74, Theorem 2] and its generalization by Levine [Lev85b, Lemma 1.2] (see also [Blo86, Lemma 1.1]) which works over an arbitrary infinite field. This theorem will be used to ensure that every member of a family ϕ as above gives rise to a well defined 0-cycle on X. The proof closely follows the one in [Lev85b, Lemma 1.2]. Unless stated otherwise, k will always denote an infinite ground field in this section.

4.1. **Some recollection.** We begin with some standard algebraic geometry results that we shall use repeatedly. We collect them here for reader's convenience.

Lemma 4.1. *Let $f \colon X \to Y$ be a continuous map of topological spaces such that the following hold.*

    (1) *Y is irreducible.*
    (2) *f is open.*
    (3) *There exists a dense set of points $y \in Y$ such that $f^{-1}(y)$ is irreducible.*

Then X is irreducible.

Proof. See [Spa21, Tag 004Z]. $\qquad \square$

Lemma 4.2. *Let $f \colon X \to Y$ be an open morphism in $\mathbf{Sch}_k$. Assume that X is equidimensional, Y is irreducible and every irreducible component of X is dominant over Y. Then f has equidimensional fibers. That is,*

$$\dim(f^{-1}(y)) = \dim(X) - \dim(Y) \quad \text{for all } y \in Y.$$

Proof. See [GW10, Theorem 14.114]. $\qquad \square$

Corollary 4.3. *Let $f \colon X \to Y$ be an open as well as closed morphism in $\mathbf{Sch}_k$. Assume that X is equidimensional and Y is irreducible. Then f has equidimensional fibers.*

Proof. Under our assumption, every irreducible component of X must be dominant over Y and therefore Lemma 4.2 applies. $\qquad \square$

Lemma 4.4. *Let* $X, Y \in \mathbf{Sch}_k$ *be equidimensional schemes. Then* $X \times Y$ *is equidimensional.*

Proof. Let $\overline{k}$ be an algebraic closure of k. Since $\dim(X \times Y) = \dim((X \times Y)_{\overline{k}})$ (e.g., see [GW10, Proposition 5.38]) and $(X \times Y)_{\overline{k}} \cong X_{\overline{k}} \times_{\overline{k}} Y_{\overline{k}}$, we can assume k is algebraically closed. In this case, the lemma follows from our assumption because $X' \times Y'$ is irreducible of dimension equal to $\dim(X') + \dim(Y')$ if $X', Y' \in \mathbf{Sch}_k$ are irreducible. $\square$

4.2. **Morphisms to homogeneous spaces.** We recall homogeneous spaces and describe some properties of morphisms from schemes to these spaces. The set-up of this section will be used throughout the next few sections of this paper.

Let G be a connected reductive algebraic group over k and let $H = G/P$ be a projective homogeneous space for G. Recall that a smooth closed subgroup $P' \subset G$ is parabolic if and only if the variety G/P' is complete (equivalently, projective). It follows that P is a parabolic subgroup of G. In particular, it is connected (e.g., see [Bor91, Theorem 11.16]). We shall let $d_P = \dim(P)$.

Let $\pi \colon G \to H$ be the resulting P-torsor. Note that since P is affine, π is an fppf locally trivial P-torsor, it follows that π is fppf locally an affine morphism. But this implies that π is an affine morphism. Since G is reductive, it is smooth over k and an fppf descent argument shows that H must also be smooth over k. Furthermore, as G is connected, it must be geometrically connected (any $X \in \mathbf{Sch}_k$ which is connected and $X(k) \neq \varnothing$ is geometrically connected). It follows that G is geometrically integral. In particular, $H = G/P$ is also geometrically integral.

Another property we shall use frequently is that, being reductive, G is a uni-rational variety over k (i.e., admits a dominant k-morphism from a dense open subset of an affine space over k) by [Bor91, Theorem 18.2]. It follows that H is also uni-rational. Since k is infinite, this implies that for any dense open $U \subset H$ (or in G), the set of k-rational points $U(k)$ is Zariski dense in H (or in G).

Let G act on a reduced quasi-projective scheme X over k and let $\mu \colon G \times X \to X$ be the action map. Let $\Phi = (\mu, \mathrm{id}_X) \colon G \times X \to X \times X$ denote the map $\Phi(g, x) = (\mu(g, x), x)$. Let $\widetilde{\mu} = (\mathrm{id}_G, \mu) \colon G \times X \to G \times X$ denote the map $\Phi(g, x) = (g, \mu(g, x))$. Since $\widetilde{\mu}$ is an isomorphism and G is smooth, it follows that μ is a smooth surjective morphism.

For a closed point $g \in G$, the composite morphism $\lambda_g \colon \mathrm{Spec}\,(k(g)) \times X = X_{k(g)} \hookrightarrow G \times X \xrightarrow{\mu} X$ can be easily seen to be closed. In particular, $gY \coloneqq \lambda_g(Y)$ is closed in X for any closed subscheme $Y \subset X$. If $g \in G(k)$, we can identify $X_{k(g)}$ with X and then λ_g defines an automorphism of X. It is easy to see that $\{\lambda_g\}_{g \in G(k)}$ define a group homomorphism $\lambda \colon G(k) \to \mathrm{Aut}_k(X)$.

4.3. **The parameter space of good cycles.** We now let $H = G/P$ be a homogeneous space for G as above. Then G acts transitively on H. This action gives rise to the maps $\mu, \widetilde{\mu}$ and Φ be as above. Let $Y \subset H$ be an equidimensional closed subscheme. Let X be an equidimensional reduced quasi-projective k-scheme and let $f \colon X \to H$ be a k-morphism.

We consider the commutative diagram

(4.1)
$$
\begin{array}{ccc}
\Gamma & \xrightarrow{\delta'} & Y \times X \\
{\scriptstyle \phi} \Big/ \Big\downarrow {\scriptstyle f'} & & \Big\downarrow {\scriptstyle \iota_Y \times f} \\
G \xleftarrow{p} G \times H & \xrightarrow{\delta} & H \times H,
\end{array}
$$

where $\iota \colon G \to G$ is the inverse morphism ($\iota(g) = g^{-1}$), δ is the composite

$$
G \times H \xrightarrow{\iota \times \mathrm{id}_H} G \times H \xrightarrow{\Phi} H \times H
$$

and p is the projection. The scheme Γ is defined so that the right square is Cartesian. The map ϕ is defined so that the left triangle is commutative. For $U \subset G$, we let $\Gamma_U := U \times_G \Gamma$. We study some properties of Γ.

Since $\Phi = (\mu, \mathrm{id}_H)$ and μ is affine, it follows that Φ is affine. Hence, δ is affine. It is also clear from the definition of Φ that for every point $z \in H \times H$, the scheme-theoretic fiber $\Phi^{-1}(z)$ is isomorphic to $P_{k(z)}$. In particular, Φ is equidimensional of relative dimension d_P. Since the source and the target of Φ are both regular, it follows from [Har77, Exc. III.10.9] that Φ is flat. Since P is smooth over k, it follows that Φ is a smooth morphism. Since $\iota \times \mathrm{id}_H$ is an isomorphism, we conclude that δ is a smooth surjective affine morphism of relative dimension d_P. This implies that δ' is also a smooth surjective affine morphism of relative dimension d_P.

Since X and Y are equidimensional, it follows from Lemma 4.4 that $Y \times X$ is also equidimensional. As P is geometrically integral, we see that over every irreducible component V of $Y \times X$, the map δ' is smooth surjective with irreducible fibers. It follows therefore from Lemma 4.1 that $\delta'^{-1}(V)$ is irreducible of dimension $\dim(V) + d_P$. We conclude that Γ is equidimensional and

$$(4.2) \qquad \dim(\Gamma) = \dim(X) + \dim(Y) + d_P.$$

Lemma 4.5. *There exists a dense open subscheme $U \subset G$ such that $\phi : \Gamma_U \to U$ is flat, and for all $g \in U(k)$, the scheme $f^{-1}(gY) = \phi^{-1}(g)$ is equidimensional. Furthermore,*

$$\dim(f^{-1}(gY)) = \dim(X) - (\dim(H) - \dim(Y)).$$

Equivalently, $\mathrm{codim}_X(f^{-1}(gY)) = \mathrm{codim}_H(Y)$ for all $g \in U(k)$.

Proof. Since ϕ is surjective, at least one irreducible component of Γ is dominant over G. Let us write $\Gamma_{\mathrm{red}} = \Gamma' \cup \Gamma''$ as a union of closed subschemes such that Γ' is the union of all irreducible components of Γ which are dominant over G and Γ'' is the union of those irreducible components which are not dominant over G. We can choose a dense open subscheme $U' \subset G$ such that $\phi^{-1}(U') \cap \Gamma'' = \emptyset$. In particular, $\phi^{-1}(U') = \phi^{-1}(U') \cap \Gamma'$.

We let $\Gamma_{U'} = \phi^{-1}(U')$. Then $\phi : \Gamma_{U'} \to U'$ is surjective morphism with irreducible base. Hence, it follows from the generic flatness theorem (see EGA IV$_2$ 6.9.1) that there is a dense open subscheme $U \subset U'$ such that $\phi : \Gamma_U \to U$ is flat and surjective. Since $\Gamma_U = \Gamma'_U$ and Γ is equidimensional, it follows that we have a morphism $\phi : \Gamma_U \to U$ which is flat, whose source is equidimensional and each irreducible component of the source is dominant over the base. We can therefore apply Lemma 4.2 to conclude that $\phi : \Gamma_U \to U$ has equidimensional fibers.

In particular, using the identification $f^{-1}(gY) = \phi^{-1}(g)$, induced by the commutative diagram (4.1), we have for any $g \in U(k)$,

$$(4.3) \qquad \begin{array}{rcl} \dim(f^{-1}(gY)) & = & \dim(\phi^{-1}(g)) = \dim(\Gamma') - \dim(G) \\ & = & \dim(\Gamma) - \dim(G) \\ & = & \dim(X) + \dim(Y) + d_P - \dim(G) \\ & = & \dim(X) - (\dim(H) - \dim(Y)). \end{array}$$

Since $U \subset G$ is dense open, the proof of the lemma is complete. $\qquad\square$

We let $\alpha : \Gamma \to G \times X$ be the map induced by the projections $\phi : \Gamma \to G$ and $\Gamma \xrightarrow{\delta'} Y \times X \to X$. We let $\beta : G \times X \to H$ be the composite $p_1 \circ \delta \circ (\mathrm{id}_G, f)$, where $p_1 : H \times H \to H$ is the first projection.

If we let G act on $G \times X$ by right multiplication on itself and trivially on X (i.e., $g_1 \cdot (g, x) = (gg_1^{-1}, x)$), then $\beta : G \times X \to H$ is G-equivariant. Since β is surjective and H is reduced, it follows from the generic flatness theorem that β is flat over a dense open subscheme of H.

Since G acts transitively on H and β is G-equivariant, it follows that it must actually be flat everywhere.

We now consider the diagram

(4.4)

$$
\begin{array}{c}
\Gamma' \\
\end{array}
$$

where ι_Y is the inclusion, and $\Gamma' := (G \times X) \times_H Y$ with respect to maps β and ι_Y. It is easy to check from the definition of Γ in (4.1) that $\beta \circ \alpha = \iota_Y \circ p_1 \circ \delta'$. It follows that there is a unique morphism $\gamma : \Gamma \to \Gamma'$ such that $\iota'_Y \circ \gamma = \alpha$ and $\beta' \circ \gamma = p_1 \circ \delta'$. Furthermore, it is easy to check that γ is an isomorphism. In particular, β' is flat (since p_1 and δ' are) and α is a closed immersion.

We let $U \subset G$ be the dense open as in Lemma 4.5 and let $g \in U(k)$. We consider the (equivalent) Cartesian diagrams

(4.5)

$$
\begin{array}{ccc}
f^{-1}(gY) \xrightarrow{\lambda_{g^{-1}} \circ f} Y & \qquad & f^{-1}(gY) \xrightarrow{f} gY \\
\downarrow \qquad \downarrow {\iota_Y} & & \downarrow \qquad \downarrow {\iota_{gY}} \\
X \xrightarrow{\lambda_{g^{-1}} \circ f} H, & & X \xrightarrow{f} H.
\end{array}
$$

Lemma 4.6. *The squares in* (4.5) *are Tor-independent.*

Proof. We consider another commutative diagram

(4.6)

$$
\begin{array}{ccccc}
f^{-1}(gY) & \to & \mathrm{Spec}\,(k(g)) \times X & \to & \mathrm{Spec}\,(k(g)) \\
\downarrow & & \downarrow {\iota_g} & & \downarrow \\
\Gamma_U & \xrightarrow{\alpha} & U \times X & \xrightarrow{p_1} & U,
\end{array}
$$

where the composition of the horizontal arrows on the bottom is ϕ. Since ϕ is flat, it follows that the big outer square is Tor-independent. Since p_1 is flat and the vertical arrows are closed immersions, it follows by an elementary verification that the left square is also Tor-independent.

Let us now consider a resolution $\mathcal{E}^\bullet \to \mathcal{O}_Y \to 0$ of $\mathcal{O}_Y$ by locally free $\mathcal{O}_H$ modules of finite ranks. Since β is flat, it follows that $\beta^*(\mathcal{E}^\bullet)$ is a locally free resolution of $\mathcal{O}_\Gamma$ under the closed immersion $\alpha : \Gamma \hookrightarrow G \times X$. In particular, $\mathcal{E}_U^\bullet := \beta^*(\mathcal{E}^\bullet)|_{U \times X}$ is a locally free resolution of $\mathcal{O}_{\Gamma_U}$. Since the left square in (4.6) is Tor-independent, it follows that $\iota_g^* \mathcal{E}_U^\bullet \to \iota_g^* \mathcal{O}_{\Gamma_U} \to 0$ is a locally free resolution on X. Equivalently, $\iota_g^* \mathcal{E}_U^\bullet \to \mathcal{O}_{f^{-1}(gY)} \to 0$ is a locally free resolution on X. That is, $\mathrm{Tor}_i^{(\beta \circ \iota_g)^{-1} \mathcal{O}_H}((\beta \circ \iota_g)^{-1}\mathcal{O}_Y, \mathcal{O}_X) = 0$ for all $i > 0$. Since $\beta \circ \iota_g = \lambda_{g^{-1}} \circ f$ and since λ_g is an isomorphism, this implies that $\mathrm{Tor}_i^{f^{-1}\mathcal{O}_H}(f^{-1}\mathcal{O}_{gY}, \mathcal{O}_X) = 0$ for all $i > 0$. Equivalently, (4.5) is Tor-independent. $\qquad \square$

Corollary 4.7. *Let $V \subset H$ be an open subscheme such that $Y \cap V$ is a local complete intersection in V. Then $f^{-1}(gY \cap gV)$ is a local complete intersection in $f^{-1}(gV)$ for every $g \in U(k)$.*

Proof. Follows directly from Lemma 4.6 and [Mat97, 16.4]. ☐

4.4. **Kleiman-Levine transversality theorem.** We shall now prove the following transversality result over k. We let G and H be as above. We shall follow the notations of § 4.2 and § 4.3.

Theorem 4.8. *Let X be an equidimensional reduced quasi-projective k-scheme and let $f \colon X \to H$ be a k-morphism. Let $Y \subset H$ be an equidimensional reduced closed subscheme. Then there exists an open dense subscheme $U(f, Y) \subset G$ such that for every $g \in U(f, Y)(k)$, the following hold.*

 (1) *The scheme $f^{-1}(gY)$ is either empty or is equidimensional of dimension $\dim(X) + \dim(Y) - \dim(H)$.*

 (2) $\operatorname{Tor}_i^{f^{-1}\mathcal{O}_H}(f^{-1}\mathcal{O}_{gY}, \mathcal{O}_X) = 0$ *for all $i > 0$.*

 (3) *The inclusion $f^{-1}(gY) \hookrightarrow X$ is a local complete intersection at every generic point of $f^{-1}(gY) \cap X_{\mathrm{sing}}$.*

Proof. Let $U \subset G$ be the open subscheme obtained in Lemma 4.5. The item (1) then follows directly from (4.3) and (2) follows from Lemma 4.6.

We now prove (3). First of all, we can apply (4.3) to every irreducible component of X_{sing} to see that after shrinking U if necessary, every $g \in U(k)$ has the property that either $f^{-1}(gY) \cap X_{\mathrm{sing}}$ is empty or $\dim(f^{-1}(gY) \cap X_{\mathrm{sing}}) = \dim(X_{\mathrm{sing}}) + \dim(Y) - \dim(H)$. Combining this with (4.3), we get the inequality

$$(4.7) \qquad \dim(f^{-1}(gY) \cap X_{\mathrm{sing}}) \leq \dim(X_{\mathrm{sing}}) + \dim(f^{-1}(gY)) - \dim(X),$$

where the equality holds if $f^{-1}(gY) \cap X_{\mathrm{sing}} \neq \varnothing$. In other words, $f^{-1}(gY)$ and X_{sing} intersect properly in X. The same token shows that by possibly shrinking U further, we have that $f^{-1}(gY_{\mathrm{sing}})$ and X_{sing} intersect properly in X for all $g \in U(k)$. This means, in particular, that every generic point of $f^{-1}(gY) \cap X_{\mathrm{sing}}$ is contained in $f^{-1}(gY_{\mathrm{reg}})$.

We now observe that $Y_{\mathrm{reg}} \hookrightarrow H$ is a local complete intersection morphism because H is regular. The same is true for $gY_{\mathrm{reg}} \hookrightarrow H$, since $\lambda_{g^{-1}} \in \operatorname{Aut}_k(H)$. We can therefore apply Corollary 4.7 to conclude (3). We take $U(f, Y)$ to be the above U to finish the proof. ☐

Let us write $\mathbb{P}^1_k$ as the homogeneous space $\operatorname{PGL}_{2,k}/B$, where B is the image of the upper-triangular matrices under the quotient map $\operatorname{GL}_{2,k} \twoheadrightarrow \operatorname{PGL}_{2,k}$. Let G and H be as above. Then $H \times \mathbb{P}^1_k$ becomes a homogeneous space for $\widetilde{G} := G \times \operatorname{PGL}_{2,k}$ via the coordinate-wise action. We shall now apply the previous Theorem in this setting to get the following result on generic translates in $H \times \mathbb{P}^1_k$.

Let X be an equidimensional reduced quasi-projective scheme over k, and let $f \colon X \to H$ be a k-morphism as before. Let $W \subset H \times \mathbb{P}^1_k$ be an equidimensional reduced closed subscheme. Write gW for the pullback of W along $\lambda_{g^{-1}} \times \mathrm{id} \colon H \times \mathbb{P}^1_k \to H \times \mathbb{P}^1_k$. Suppose that the composition $W \to \mathbb{P}^1_k$ of the inclusion of W in $H \times \mathbb{P}^1_k$ followed by the second projection $H \times \mathbb{P}^1_k \to \mathbb{P}^1_k$ is flat over a neighborhood of $\{0, \infty\}$. Let $(f \times \mathrm{id}_{\mathbb{P}^1_k})^{-1}(gW)$ denote the fiber product $gW \times_{(H \times \mathbb{P}^1_k)} (X \times \mathbb{P}^1_k)$. We write $f \times \mathrm{id}_{\mathbb{P}^1_k}$ as $\widetilde{f}$. Let G (resp. $\operatorname{PGL}_{2,k}$) act on $H \times \mathbb{P}^1_k$ by acting trivially on $\mathbb{P}^1_k$ (resp. on H). These actions of G and $\operatorname{PGL}_{2,k}$ on $H \times \mathbb{P}^1_k$ commute with each other.

Proposition 4.9. *There exists an open dense subscheme $U = U(\widetilde{f}, W)$ of G such that for every k-point $g \in U(k)$, the following hold.*

 (1) $\widetilde{f}^{-1}(gW)$ *is either empty or equidimensional of dimension*

$$\dim(\widetilde{f}^{-1}(gW)) = \dim(X) + \dim(W) - \dim(H).$$

(2) $\widetilde{f}^{-1}(gW)$ *intersects* $\mathbb{P}^1_{X_{\mathrm{sing}}}$ *and* $X_{\mathrm{sing}} \times \{0, \infty\}$ *properly.*

(3) *The composition* $\widetilde{f}^{-1}(gW) \hookrightarrow \mathbb{P}^1_X \to \mathbb{P}^1_k$ *is flat over a neighborhood of* $\{0, \infty\}$.

(4) *The inclusion* $\widetilde{f}(gW) \hookrightarrow \mathbb{P}^1_X$ *is a local complete intersection at each generic point of* $\widetilde{f}(gW) \cap \mathbb{P}^1_{X_{\mathrm{sing}}}$.

Proof. We shall use the structure of the homogeneous space on $\mathbb{P}^1_H$ for the group $\widetilde{G}$ as above. We then observe that for every $t \in \mathrm{PGL}_{2,k}(k)$, the map $\lambda_t \colon \mathbb{P}^1_X \to \mathbb{P}^1_X$ is an isomorphism which keeps every fiber of the projection $\mathbb{P}^1_X \to X$ invariant. It follows that for any $(g, t) \in \widetilde{G}(k)$, the assertions (1), (2) and (4) of the proposition will hold for $\widetilde{f}^{-1}(gW)$ if and only if they hold for $t(\widetilde{f}^{-1}(gW))$.

On the other hand, Theorem 4.8 says that there is a dense open $\widetilde{U} \subset \widetilde{G}$ such that for every $(g, t) \in \widetilde{U}(k)$, the scheme $\widetilde{f}^{-1}((g, t)W)$ satisfies (1), (2) and (4). Since

$$(4.8) \qquad \widetilde{f}^{-1}((g, t)W) = t(\widetilde{f}^{-1}(gW)),$$

we conclude that for every $(g, t) \in \widetilde{U}(k)$, the scheme $t(\widetilde{f}^{-1}(gW))$ satisfies (1), (2) and (4). Letting U be the image of $\widetilde{U}$ under the projection $p_1 \colon \widetilde{G} \to G$, we see that $U \subset G$ is dense open. Furthermore, for any $g \in G(k)$, the fiber $p_1^{-1}(g) \cong \mathrm{PGL}_{2,k}$ has a dense set of k-rational points. For any such point t, the scheme $t(\widetilde{f}^{-1}(gW))$ satisfies (1), (2) and (4). But we have seen in the beginning of the proof that this is equivalent to saying that $\widetilde{f}^{-1}(gW)$ satisfies (1), (2) and (4).

For (3), it is enough to show that the map $\widetilde{f}^{-1}(gW) \to \mathbb{P}^1_k$ is flat in a neighborhood of each point $\epsilon \in \{0, \infty\}$. However, we know that W satisfies this property. Hence, gW too satisfies this property for every $g \in G(k)$ as G acts trivially on $\mathbb{P}^1_k$. Now, if we replace $\mathbb{P}^1_k$ by $S = \mathrm{Spec}\,(\mathcal{O}_{\mathbb{P}^1_k, \epsilon})$ and correspondingly replace all schemes over $\mathbb{P}^1_k$ by their base change to S, then $\widetilde{f}^{-1}(gW)$ will not be flat over S if and only if it is supported on the closed point $\{\epsilon\} \subset S$. The latter condition is easily seen to imply that

$$(4.9) \qquad \mathrm{Tor}_1^{\mathcal{O}_S}(k(\epsilon), \mathcal{O}_{gW \times_{H_S} X_S}) \neq 0.$$

Since H_S and gW are both flat over S, an elementary homological algebra shows that (4.9) implies that $\mathrm{Tor}_1^{\mathcal{O}_{H_\epsilon}}(\mathcal{O}_{gW_\epsilon}, \mathcal{O}_{X_\epsilon}) \neq 0$. But this contradicts Lemma 4.6 if we choose $g \in U(k)$ (after possibly shrinking U). It follows that (3) holds if $g \in U(k)$. Letting $U(\widetilde{f}, W) = U$, we conclude the proof of the proposition. $\qquad\square$

5. The family of rational equivalences

In § 4, we constructed families of good cycles on a reduced quasi-projective scheme over an infinite field k which are obtained by pulling back cycles from homogeneous spaces. In this section, we shall construct a family which will parameterize rational equivalences between the members of a given family of good cycles. We shall prove some properties of this family that will be used in the following sections. We fix an infinite field k throughout this section.

5.1. $\mathbb{A}^1$-path connectivity of reductive groups.

We write $(\mathbb{P}^1_k)^m$ as $\overline{\square}^m_k$ for any integer $m \geq 1$. For any integer $n \geq 1$, we fix a compactification $\mathrm{GL}_{n,k} \subset \mathbb{A}^{n^2}_k \hookrightarrow \overline{\square}^{n^2}_k$, where the first inclusion is the inverse image of $\mathbb{G}_{m,k}$ under the determinant map $\det \colon \mathbb{A}^{n^2}_k \cong \mathbb{M}_{n,k} \to \mathbb{A}^1_k$ and the second inclusion is the product of the canonical inclusion $\mathbb{A}^1_k \subset \overline{\square}^1_k$. We remark here that there are many other choices of a smooth compactification of $\mathrm{GL}_{n,k}$. But we choose the above one with some purpose. This will be evident in Lemma 5.1.

We let $\mathrm{GL}_{n,k}^{\times} \coloneqq \mathrm{GL}_{n,k} \cap \mathbb{G}_{m,k}^{n^2} \subset \mathbb{A}_k^{n^2}$. It is clear that $\mathrm{GL}_{n,k}^{\times}$ is a dense open subscheme of $\mathrm{GL}_{n,k}$ whose complement is its intersection with the union of the coordinate axes of $\mathbb{A}_k^{n^2}$. Let $\mu\colon \mathbb{A}_k^n \times \mathbb{A}_k^1 \to \mathbb{A}_k^n$ denote the multiplication operation induced by the k-algebra homomorphism $k[t_1,\ldots,t_n] \to k[t,t_1,\ldots,t_n]$ which sends t_i to tt_i. Note that μ is flat everywhere of relative dimension one and is smooth over $\mathbb{A}_k^n \smallsetminus \{0\}$. We consider the morphisms

$$(5.1) \qquad \mathbb{A}_k^n \times \mathbb{A}_k^n \times \mathbb{A}_k^1 \xrightarrow{\tau'} \mathbb{A}_k^n \times \mathbb{A}_k^n \times \mathbb{A}_k^1 \times \mathbb{A}_k^1 \xrightarrow{\tau} (\mathbb{A}_k^n \times \mathbb{A}_k^1) \times (\mathbb{A}_k^n \times \mathbb{A}_k^1)$$

$$\Phi_n' \downarrow \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \downarrow \mu \times \mu$$

$$\mathbb{A}_k^n \xleftarrow{\qquad\qquad \mu^+ \qquad\qquad} \mathbb{A}_k^n \times \mathbb{A}_k^n,$$

where $\tau'(x,x',y) = (x,x',y,1-y)$, $\tau(x,x',y,y') = (x,y,x',y')$ and μ^+ is the group operation map for the structure of the additive group $\mathbb{G}_{a,k}^n$ on $\mathbb{A}_k^n$.

It is clear that τ is an isomorphism, μ^+ is a smooth morphism and μ is smooth over $\mathbb{A}_k^n \smallsetminus \{0\}$. One can easily check that Φ_n' (defined as the composition) is a surjective morphism between regular schemes. Furthermore, its restriction on $W_n \coloneqq \mathbb{G}_{m,k}^n \times \mathbb{G}_{m,k}^n \times (\mathbb{A}_k^1 \smallsetminus \{0,1\})$ is surjective with each fiber isomorphic to the base change of $\mathbb{G}_{m,k}^n \smallsetminus (\mathbb{A}_k^1 \smallsetminus \{0,1\})$ by some field extension of k. It follows that Φ_n' is a smooth surjective morphism on W_n of relative dimension $m+1$.

Note that each of the arrows in (5.1) is defined over k. In particular, the composition of all arrows Φ_n' is a k-morphism. On the rational points, it is given by $\Phi_n'(x,x',t) = tx + (1-t)x'$. It is also easy to see that Φ_n' defines a rational map $\Phi_n'\colon \mathbb{A}_k^n \times \mathbb{A}_k^n \times \mathbb{P}_k^1 \dashrightarrow \overline{\square}_k^n$ whose base locus is $Z_n \times \{\infty\}$, where Z_n is the union of coordinate axes on $\mathbb{A}_k^n \times \mathbb{A}_k^n$. In particular, if we let $m = n^2$, we get a k-morphism

$$(5.2) \qquad \Phi_m'\colon \mathrm{GL}_{n,k}^{\times} \times \mathrm{GL}_{n,k}^{\times} \times \mathbb{P}_k^1 \to \overline{\square}_k^m.$$

More generally, if $n_1,\ldots,n_r$ are positive integers and we let $m = \sum_{i=1}^r n_i^2$, then Φ_m' defines a rational map $\Phi_m'\colon (\prod_{i=1}^r \mathbb{A}_k^{n_i^2}) \times (\prod_{i=1}^r \mathbb{A}_k^{n_i^2}) \times \mathbb{P}_k^1 \dashrightarrow \overline{\square}_k^m$ which is a morphism on the open subscheme $(\prod_{i=1}^r \mathbb{G}_{m,k}^{n_i^2}) \times (\prod_{i=1}^r \mathbb{G}_{m,k}^{n_i^2}) \times \mathbb{P}_k^1$. We let $G = \prod_{i=1}^r \mathrm{GL}_{n_i,k}$ and $G^{\times} = \prod_{i=1}^r \mathrm{GL}_{n_i,k}^{\times}$. We then get a k-morphism

$$(5.3) \qquad \Phi_m'\colon G^{\times} \times G^{\times} \times \mathbb{P}_k^1 \to \overline{\square}_k^m.$$

We let $B = G^{\times} \times G^{\times}$ and let Φ_m be the composite $B \times \mathbb{P}_k^1 \xrightarrow{\mathrm{id}_B \times \eta} B \times \mathbb{P}_k^1 \xrightarrow{\Phi_m'} \overline{\square}_k^m$, where $\eta(t) = t/(t-1)$. We fix the open embedding $j\colon G \hookrightarrow \overline{\square}_k^m$ via the composition of open embeddings $G = \prod_{i=1}^r \mathrm{GL}_{n_i,k} \hookrightarrow \prod_{i=1}^r \mathbb{M}_{n_i,k} \cong \mathbb{A}_k^m \hookrightarrow \overline{\square}_k^m$. We also have open embedding $G^{\times} \hookrightarrow G$. Let $\underline{\infty} \in \overline{\square}_k^m$ be the closed point whose every coordinate is ∞. The following lemma says, along with other things, that $G^{\times}$ is 'path-connected' in the sense of $\mathbb{A}^1$-homotopy theory.

Lemma 5.1. *The k-morphism*

$$\Phi_m\colon B \times \mathbb{P}_k^1 \to \overline{\square}_k^m$$

has following properties.

(1) $\Phi_m(x,x',t) = \frac{t}{t-1}x + (1 - \frac{t}{t-1})x'$.
(2) $\Phi_m(B \times \{0,\infty\}) \subset G^{\times}$.
(3) $\Phi_m(B \times \{1\}) = \underline{\infty}$.
(4) *For every pair of points $g_1,g_2 \in G^{\times}(k)$, we have $\Phi_m(g,0) = g_1$ and $\Phi_m(g,\infty) = g_2$, where $g = (g_2,g_1) \in B(k)$.*

(5) $G^\times \subset \Phi_m(B \times \mathbb{P}^1_k)$. In particular, Φ_m is dominant.
(6) Φ_m is flat of relative dimension $m + 1$ over $G^\times$.
(7) Φ_m is smooth on $B \times (\mathbb{P}^1_k \smallsetminus \{0, 1, \infty\})$.
(8) The projection map $\Phi_m^{-1}(G^\times) \to \mathbb{P}^1_k$ is flat.

Proof. All properties (except possibly (6) and (8)) are clear from our explicit construction of Φ_m. The property (6) follows because (5) shows that Φ_m is a surjective morphism between regular schemes over $G^\times$ and (1) shows that all fibers of Φ_m over $G^\times$ have relative dimension $m + 1$. Hence, it must be flat (see [Har77, Exc. III.10.9]). The property (8) follows easily from (2) since the image of $\Phi_m^{-1}(G^\times)$ in $\mathbb{P}^1_k$ is anyway open.   □

Remark 5.2. Recall that every k-rational variety X (e.g., the variety G above) is separably uni-ruled. That is, there exists a separable dominant rational map $\Phi: X' \times \mathbb{P}^1_k \dashrightarrow X$, where $\dim(X') = \dim(X) - 1$. However, the purpose of Lemma 5.1 is to show that the reductive group G satisfies many other properties which can not be directly deduced from uni-ruledness. We shall need Lemma 5.1 to construct our parameter space of rational equivalences between good cycles in a family. Remark also that not every uni-rational variety in positive characteristic in separably uni-ruled.

5.2. **The partial parameter space Σ^o.** Let G be as in § 5.1. Let $H = G/P$ be a homogeneous space for G as in § 4.2. Define a morphism $\psi: \Phi_m^{-1}(G) \times G \times H \to H$ as the composition

$$\Phi_m^{-1}(G) \times G \times H \xrightarrow{(\Phi_m, \mathrm{id}_{G \times H})} G \times G \times H \xrightarrow{(\iota, \iota)} G \times G \times H \xrightarrow{(m, \mathrm{id}_H)} G \times H \xrightarrow{\mu_H} H,$$

where $\iota: G \to G$ is the inverse operation, $m: G \times G \to G$ is the multiplication in G and μ_H is the G-action on H (we denoted this action by μ in § 5.1). Let $Y \subset H$ be an integral closed subscheme and let Σ^o be the pullback

(5.4)
$$\begin{array}{ccc}
\Sigma^o & \hookrightarrow & \Phi_m^{-1}(G) \times G \times H \\
{\scriptstyle \psi'} \downarrow & & \downarrow {\scriptstyle \psi} \\
Y & \hookrightarrow & H.
\end{array}$$

We note that the set of k-points of Σ^o is given by

$$\Sigma^o(k) = \{(g_1, g_2, t, g, h) \,|\, \Phi_m(g_1, g_2, t) \in G \text{ and } (\Phi_m(g_1, g_2, t))^{-1} g^{-1} h \in Y\}.$$

There is an action of G on Σ^o induced by the diagonal action of G on $G \times H$. Explicitly,

(5.5)
$$g' \cdot (g_1, g_2, t, g, h) = (g_1, g_2, t, g'g, g'h).$$

Or, in other words, Σ^o is stable for the G action on $\Phi_m^{-1}(G) \times G \times H$ given by the trivial action on the first component and by the canonical action on the second and the third components.

We let $\Gamma := Y \times_H (G \times H)$ with respect to the composite map $\mu_H \circ (\iota \times \mathrm{id}_H): G \times H \to H$. It is easy to check that the diagram

(5.6)
$$\begin{array}{ccc}
\Gamma & \xrightarrow{\iota'_Y} & G \times H \\
{\scriptstyle \widetilde{\mu}'_H} \downarrow & & \downarrow {\scriptstyle \mu'_H} \\
G \times Y & \xrightarrow{\mathrm{id} \times \iota_Y} & G \times H
\end{array}$$

is Cartesian, where $\iota_Y: Y \hookrightarrow H$ is the inclusion, ι'_Y is the second projection of Γ and μ'_H is the automorphism of $G \times H$ given by $(g, h) \mapsto (g, g^{-1}h)$ (see § 4.2). We thus have the following simple expression for the scheme Γ.

Lemma 5.3. *The action map* μ'_H *induces an isomorphism of schemes*

$$\widetilde{\mu}'_H \colon \Gamma \xrightarrow{\cong} G \times Y.$$

In particular, Γ *is integral.*

Proof. We only need to prove the second part. But this follows from the known fact that the product two schemes, one of which is integral and the other is geometrically integral, is integral. □

Let ϕ be the composite map $\Phi_m^{-1}(G) \times G \xrightarrow{(\Phi_m, \mathrm{id}_G)} G \times G \xrightarrow{(\iota, \iota)} G \times G \xrightarrow{m} G$ so that $\psi = \mu_H \circ (\phi, \mathrm{id}_H)$. It is then easy to see that (5.4) being Cartesian is equivalent to saying that the square

(5.7)
$$
\begin{array}{ccc}
\Sigma^o & \xrightarrow{\phi'} & \Gamma \\
{\scriptstyle \theta'} \downarrow & & \downarrow {\scriptstyle \theta} \\
\Phi_m^{-1}(G) \times G & \xrightarrow{\phi} & G
\end{array}
$$

is Cartesian, where the vertical arrows are the projections.

Since the projection $G \times H \to G$ is projective (because H is projective), and Γ is closed in $G \times H$ (because Y is closed in H), it follows that θ is a projective morphism. Hence, θ' is also a projective morphism.

Let G act on itself by left multiplication and diagonally on $G \times H$. Then $\Gamma \subset G \times H$ is G-invariant. In particular, Γ is equipped with G-action such that θ is G-equivariant. Moreover, θ is surjective and equidimensional of relative dimension equal to $\dim(Y)$. As G is integral, the generic flatness theorem says that θ is flat over a dense open subscheme of G. Since θ is G-equivariant, $G(k)$ is Zariski dense in G whose action is transitive on the base of θ, it follows that the latter is flat everywhere on G. It follows in particular that θ' is flat and surjective.

Since Γ and G are integral, it follows that the generic fiber of θ is integral. Moreover, the fiber of θ over every k-rational point is isomorphic to Y. In particular, every k-rational fiber of θ is integral. Hence, the same holds for θ' as well. Since $\Phi_m^{-1}(G) \times G$ is irreducible, θ' is flat (and hence open), $(\Phi_m^{-1}(G) \times G)(k)$ is Zariski dense in $\Phi_m^{-1}(G) \times G$ (as the latter is a rational variety) and every k-rational fiber of θ' is irreducible, it follows from Lemma 4.1 that Σ^o is irreducible.

It follows from Lemma 5.1 (7) that the map $\phi \colon \mathrm{Spec}\,(k(\Phi_m^{-1}(G) \times G)) \to \mathrm{Spec}\,(k(G))$ is smooth. Since θ is generically integral, it follows that θ' is generically reduced. Since it is also surjective, it follows that Σ^o is generically reduced. Hence, it is generically integral. We have thus shown that Σ^o is irreducible and generically integral. Although this will suffice for our main proofs, the following lemma says more.

Lemma 5.4. Σ^o *is an integral scheme.*

Proof. We have seen above that Σ^o is irreducible. So we only need to show that it is reduced. One knows that a Noetherian scheme is reduced if and only if all its local rings satisfy Serre's R_0 and S_1 conditions. Since we have shown above that Σ^o is generically integral, all its local rings clearly satisfy the R_0 condition (i.e., the localization at every minimal prime is regular). It remains to show that the local rings of Σ^o satisfy the S_1 condition. Equivalently, we have to show that if $x \in \Sigma^o$ is a point of codimension at least one, then the local ring $\mathcal{O}_{\Sigma^o, x}$ contains a non-zero divisor.

We let $Z = \overline{\{x\}} \subset \Sigma^o$ with its reduced induced closed subscheme structure. Since θ' is projective and Z is integral, its scheme-theoretic image $\theta'(Z)$ is an integral closed subscheme

of $\Phi_m^{-1}(G) \times G$. We let $U = \mathrm{Spec}(A) \subset \Phi_m^{-1}(G) \times G$ be an affine neighborhood of $\theta'(x)$ and let $V = \mathrm{Spec}(A') \subset \Sigma^o$ be an affine neighborhood of x such that $\theta'(V) \subset U$. Let $\mathfrak{p}$ be the prime ideal of A such that $\theta'(Z) \cap U = V(\mathfrak{p})$ and let $\mathfrak{p}'$ be the prime ideal of A' such that $Z \cap V = V(\mathfrak{p}')$. Since θ' is dominant whose base is integral, it follows that it induces a k-algebra monomorphism $A \hookrightarrow A'$ such that $\mathfrak{p}' \cap A = \mathfrak{p}$.

We have to consider two cases. Suppose first that $\mathfrak{p} = \{0\}$. Then the inclusion $A \hookrightarrow A'$ induces an inclusion $k(A) \hookrightarrow A'_{\mathfrak{p}'}$, where $k(A)$ is the quotient field of A. Since we have shown above that the generic fiber of θ' is integral, it follows that $A'_{\mathfrak{p}'}$ is integral. Since the codimension of Z is at least one in Σ^o, it follows that $A'_{\mathfrak{p}'}$ has dimension at least one. In particular, it contains non-zero divisors.

In the second remaining case, we can assume that $\mathfrak{p} \neq \{0\}$. Let $a \in \mathfrak{p}$ be any non-zero element. Then $a \in \mathfrak{p}'$. Since A is an integral domain, a is a non-zero divisor in A. Since we have shown above that θ' is flat, it follows that the map $A \to A'_{\mathfrak{p}'}$ is also flat. This implies that $a \in A'_{\mathfrak{p}'} = \mathcal{O}_{\Sigma^o, x}$ must be a non-zero divisor. This finishes the proof. $\square$

Let p_{BG} denote the projection map $B \times \mathbb{P}_k^1 \times G \to B \times G$ and let $p_{BG}^o = p_{BG}|_{\Phi_m^{-1}(G) \times G}$. If $p_B \colon B \times \mathbb{P}_k^1 \to B$ is the projection map, then note that $p_{BG} = p_B \times \mathrm{id}_G$. Let $\pi^o \colon \Sigma^o \to B \times G$ denote the composition $p_{BG}^o \circ \theta'$.

Lemma 5.5. π^o *is a flat and surjective morphism whose generic and rational fibers are integral.*

Proof. Since θ' is flat and surjective and p_{BG} is smooth, it follows that π^o is flat. Since p_B is surjective, by Lemma 5.1 (2), it follows that so are p_{BG}^o and π^o (note that p_{BG} is trivially surjective). Since Σ^o (see Lemma 5.4) and $B \times G$ are integral, it follows that the generic fiber of π^o is integral.

We now fix a point $w \in (B \times G)(k)$. We can write this point uniquely as $w = (g_1, g_2, g) \in G^\times(k) \times G^\times(k) \times G(k)$. Then $p_{BG}^{-1}(w) = \mathrm{Spec}(k(w)) \times \mathbb{P}_k^1 \cong \mathbb{P}_k^1$ and $U_w^o := (p_{BG}^o)^{-1}(w) \subset \mathbb{P}_k^1$ is open. Since the projection $p_B \colon \Phi_m^{-1}(G) \to B$ is surjective by Lemma 5.1 (2), we see that U_w^o is dense open in $\mathbb{P}_k^1$. Let $\Sigma_w^o = (\pi^o)^{-1}(w)$.

Let ϕ_w denote the restriction of ϕ to the closed subscheme U_w^o. It follows from (5.7) that $\Sigma_w^o = U_w^o \times_G \Gamma$ via the map ϕ_w. It follows from this that every rational fiber of the projection $p_w^o \colon \Sigma_w^o \to U_w^o$ is isomorphic to Y. In particular, it is integral. Since θ' is flat and surjective, so is p_w^o. In particular, it is open. Since $U_w^o(k)$ is Zariski dense in U_w^o, and the latter is irreducible, we conclude from Lemma 4.1 that Σ_w^o is irreducible.

We shall show that Σ_w^o is reduced by following the same argument that we used for proving this property for Σ^o. We first need to show that the generic fiber of p_w^o is integral.

Let $S = \mathrm{Spec}(k(U_w^o))$ denote the generic point of U_w^o. We need to show that $S \times_G \Gamma$ is integral via the maps $\phi_w \colon S \to G$ and $\theta \colon \Gamma \to G$. For this, we consider the diagram

$$(5.8) \qquad \begin{array}{ccc}
\Gamma & \xrightarrow{\iota_Y'} & G \times H \\
{\scriptstyle \widetilde{\mu}_H'} \downarrow & {\scriptstyle \widetilde{\mu}_H'} \downarrow & \searrow {\scriptstyle p_1} \\
G \times Y & \xrightarrow{\mathrm{id} \times \iota_Y} G \times H \xrightarrow{p_1} & G \\
\uparrow & \uparrow & \uparrow {\scriptstyle \phi_w} \\
S \times Y & \xrightarrow{\mathrm{id} \times \iota_Y} S \times H \xrightarrow{p_1} & S,
\end{array}$$

where p_1 is the projection to the first factor of its source and other notations are as in (5.6).

It is clear that all squares and the lone triangle in (5.8) are commutative. Moreover, all squares are Cartesian and $\theta = p_1 \circ \iota_Y'$. In particular, we get $\theta = p_1 \circ (\mathrm{id} \times \iota_Y) \circ \widetilde{\mu}_H'$. Since $\widetilde{\mu}_H'$

is an isomorphism, the integrality of $S \times_G \Gamma$ is equivalent to showing that $S \times_G (G \times Y)$ is integral via the maps ϕ_w and $p_1 \circ (\mathrm{id} \times \iota_Y)$. Since all lower squares are Cartesian, the desired integrality is finally equivalent to saying that $S \times Y$ is integral. But this is clear because Y is integral and $k(U_w^o)$ is purely transcendental over k.

We have now shown that Σ_w^o is irreducible, the map $p_w^o : \Sigma_w^o \to U_w^o$ is flat and surjective whose generic fiber is integral. Since $\Sigma_w^o = U_w^o \times_G \Gamma$ and $\theta : \Gamma \to G$ is projective, it follows that p_w^o is also projective. We can now repeat the argument of the proof of Lemma 5.4 to conclude that Σ_w^o is reduced. Hence, it is integral. This finishes the proof of the lemma. $\square$

5.3. **The parameter space of rational equivalences.** Let Σ denote the scheme-theoretic closure of Σ^o in $B \times \mathbb{P}_k^1 \times G \times H$. Let θ' be the projection map $\Sigma \to B \times \mathbb{P}_k^1 \times G$ and let π be the composition $\Sigma \xrightarrow{\theta'} B \times \mathbb{P}_k^1 \times G \xrightarrow{p_{BG}} B \times G$.

Lemma 5.6. *In the above notations, Σ is a G-invariant, geometrically integral closed subscheme of $B \times \mathbb{P}_k^1 \times G \times H$, for the G-action induced by the trivial action on $B \times \mathbb{P}_k^1$ and the diagonal action on $G \times H$. Moreover, the projection map π is G-equivariant.*

Proof. Since all factors of $B \times \mathbb{P}_k^1 \times G \times H$ are geometrically integral, we know that it is (geometrically) integral. Since Σ^o is an integral closed subscheme of an open subscheme of $B \times \mathbb{P}_k^1 \times G \times H$, it follows that Σ is an integral closed subscheme of $B \times \mathbb{P}_k^1 \times G \times H$. Since θ' is projective on Σ (since H projective) and it is dominant on Σ^o, it follows that the map $\theta' : \Sigma \to B \times \mathbb{P}_k^1 \times G$ is surjective. Furthermore,

$$(5.9) \qquad \Sigma^o = \Sigma \cap (\Phi_m^{-1}(G) \times G \times H) = \Sigma \cap \theta'^{-1}(\Phi_m^{-1}(G) \times G).$$

Let $\nu : G \times (B \times \mathbb{P}_k^1 \times G \times H) \to B \times \mathbb{P}_k^1 \times G \times H$ denote the action map of the lemma. We have seen in (5.5) that Σ^o is G-invariant with respect to this action. Since G is geometrically integral and Σ is integral, it follows that $G \times \Sigma$ is an integral closed subscheme of $G \times (B \times \mathbb{P}_k^1 \times G \times H)$. Since Σ^o is dense open in Σ, it follows that $G \times \Sigma^o$ is dense open in $G \times \Sigma$. Since Σ^o is G-invariant, it follows that $G \times \Sigma^o \subset \nu^{-1}(\Sigma^o) \subset \nu^{-1}(\Sigma)$. As Σ is closed in $B \times \mathbb{P}_k^1 \times G \times H$, it follows that $G \times \Sigma = \overline{G \times \Sigma^o} \subset \nu^{-1}(\Sigma)$. But this means that Σ is a G-invariant closed subscheme of $B \times \mathbb{P}_k^1 \times G \times H$. Moreover, the projections $\Sigma \xrightarrow{\theta'} B \times \mathbb{P}_k^1 \times G \xrightarrow{p_{BG}} B \times G$ are G-equivariant. $\square$

Let $\pi : \Sigma \to B \times G$ denote the composite projection map. For $w \in B \times G$, we let Σ_w denote the scheme-theoretic fiber $\pi^{-1}(w)$.

Lemma 5.7. *There exists an open dense subscheme V_Y of B such that the projection map $\Sigma_Y := \pi^{-1}(V_Y \times G) \xrightarrow{\pi} V_Y \times G$ has following properties.*

1. *π is flat and surjective.*
2. *The generic fiber of π is integral.*
3. *For every $w \in (V_Y \times G)(k)$, there exists an open neighborhood (depending on w) U_w^o of $w \times \{0, \infty\}$ in $\mathbb{P}_{k(w)}^1 = p_{BG}^{-1}(w)$ such that $\pi^{-1}(w) \cap \theta'^{-1}(U_w^o)$ is an integral scheme which is flat over U_w^o under the projection map $\theta' : \pi^{-1}(w) \to \mathbb{P}_{k(w)}^1$.*
4. *If Y is geometrically reduced (e.g., if k is perfect), then we can choose V_Y so that $\pi^{-1}(w)$ is geometrically reduced for every $w \in V_Y \times G$.*
5. *If Y is geometrically irreducible (e.g., if k is separably closed), then we can choose V_Y so that $\pi^{-1}(w)$ is geometrically irreducible for every $w \in V_Y \times G$.*

Proof. Since θ' and p_{BG} are surjective (see the proof of Lemma 5.6), we see that π is a surjective morphism whose base is integral. Hence, the generic flatness theorem says that there is a dense open subscheme $V' \subset B \times G$ over which π is flat. We let V_Y be the image of

V' under the projection $B \times G \to B$. Since this projection is open, V_Y is open dense in B. Since $B \times G$ (resp. G) is rational, every open subset in $B \times G$ (resp. G) has the property that the set of its k-rational points is Zariski dense. Since G acts transitively on itself, and π is G-equivariant by Lemma 5.6, we deduce that π must be flat over $V_Y \times G$. Since Σ and $V_Y \times G$ are integral, it follows that the generic fiber of π is integral.

We now fix a point $w \in (V_Y \times G)(k)$. We can write this point uniquely as $w = (g_1, g_2, g) \in G^{\times}(k) \times G^{\times}(k) \times G(k)$. Then $p_{BG}^{-1}(w) = \mathrm{Spec}\,(k(w)) \times \mathbb{P}_k^1 \cong \mathbb{P}_k^1$ and $U_w^o := (p_{BG}^o)^{-1}(w) \subset \mathbb{P}_k^1$ is open. Here, p_{BG}^o is the projection $(\Phi_m^{-1}(G) \cap (V_Y \times \mathbb{P}_k^1)) \times G \to V_Y \times G$. Since the projection $p_B : \Phi_m^{-1}(G) \cap (V_Y \times \mathbb{P}_k^1) \to V_Y$ is surjective, by Lemma 5.1 (2), we see that U_w^o is dense open in $\mathbb{P}_k^1$. It also follows from Lemma 5.1 (2) that $\{0, \infty\} \subset U_w$. Let $\Sigma_{Y,w}^o = (\pi^o)^{-1}(w)$ and $\Sigma_{Y,w} = \pi^{-1}(w)$.

Under the above situation, it follows from (5.9) that we have a Cartesian square

(5.10)
$$\begin{array}{ccc} \Sigma_{Y,w}^o & \xrightarrow{j'} & \Sigma_{Y,w} \\ {\scriptstyle p_w^o}\downarrow & & \downarrow{\scriptstyle p_w} \\ U_w^o & \xrightarrow{j} & \mathbb{P}_k^1. \end{array}$$

We saw in the proof of Lemma 5.5 that p_w^o is surjective. Hence, it is flat, since U_w^o is a regular scheme of dimension 1. It also follows from the same lemma that $\Sigma_{Y,w}^o$ is integral. Since p_w is proper, it follows that the map $\overline{\Sigma_{Y,w}^o} \to \mathbb{P}_k^1$ is flat and surjective, where $\overline{\Sigma_{Y,w}^o}$ is the scheme-theoretic closure of $\Sigma_{Y,w}^o$ in $\Sigma_{Y,w}$. We have thus shown $(1) \sim (3)$.

Suppose now that Y is geometrically reduced. Recall that a scheme W over a field k' is geometrically reduced if and only if $W_{k''}$ is reduced for every field extension $k' \subset k''$. Thus we see using (5.6) and Lemma 5.3 that the generic fiber of the map $\theta : \Gamma \to G$ is geometrically reduced. Since ϕ is dominant, the same token shows that the generic fiber of θ' is geometrically reduced.

Let η (resp. η') denote the generic point of $B \times G$ (resp. $\Phi_m^{-1}(G) \times G$). Then (2) shows that $\Sigma_{Y,\eta} = \Sigma_{Y,\eta}^o = \pi^{-1}(\eta) = \theta'^{-1}(\mathbb{P}_\eta^1)$ is integral. Clearly, η' is the generic point of $\mathbb{P}_\eta^1$. Since $\theta'^{-1}(\eta')$ is geometrically reduced, as shown above, it follows (e.g., see [GW10, Proposition 5.49]) that the function field of $\Sigma_{Y,\eta}$ is separable over $k(\eta')$. Since $k(\eta')$ is purely transcendental over $k(\eta)$, it follows that the function field of $\Sigma_{Y,\eta}$ is separable over $k(\eta)$. We thus see that $\Sigma_{Y,\eta} \in \mathbf{Sch}_{k(\eta)}$ is integral whose function field is separable over $k(\eta)$. We conclude from EGA IV$_2$ 4.6.3 (see also [GW10, Proposition 5.49]) that $\Sigma_{Y,\eta}$ is geometrically reduced.

Since π is a flat and projective over $V_Y \times G$ whose generic fiber is geometrically reduced, it follows from EGA IV$_3$ 12.2.1 that there is a dense open subscheme $V' \subset V_Y \times G$ such that $\pi^{-1}(w)$ is geometrically reduced for every $w \in V'$. If we let $V_Y' \subset V_Y$ be the image of V' in V_Y, then using again that G acts transitively on itself, and π is G-equivariant, we deduce that $\pi^{-1}(w)$ is geometrically reduced for every $w \in V_Y' \times G$. We have thus shown that by shrinking V_Y if necessary, we can achieve (4).

Suppose now that Y is geometrically irreducible. Recall that a scheme W over a field k' is geometrically irreducible if and only if $W_{k''}$ is geometrically irreducible for every field extension $k' \subset k''$. It follows therefore from (5.6) and Lemma 5.3 that the generic fiber of the map $\theta : \Gamma \to G$ is geometrically irreducible. Since ϕ is dominant, the same holds for the generic fiber of θ'. In particular, $k(\eta')$ is separably closed in the function field of $\Sigma_{Y,\eta} = \Sigma_{Y,\eta}^o$ by [GW10, Proposition 5.50].

Since $k(\eta')$ is purely transcendental over $k(\eta)$, it follows that the latter is separably closed in the function field of $\Sigma_{Y,\eta}$. We thus see that $\Sigma_{Y,\eta} \in \mathbf{Sch}_{k(\eta)}$ is integral such that $k(\eta)$ is

separably closed in the function field of $\Sigma_{Y,\eta}$. We conclude from [GW10, Proposition 5.50] that $\Sigma_{Y,\eta}$ is geometrically irreducible. Since $V_Y \times G$ is integral, we can apply [Spa21, Tag 0559] to find a dense open subscheme $V' \subset V_Y \times G$ such that $\pi^{-1}(w)$ is geometrically irreducible for every $w \in V'$. We can now argue as before to show that after shrinking V_Y, we can achieve (5). This finishes the proof. $\qquad\square$

Lemma 5.8. *Let $V_Y \subset B$ be the open subscheme of Lemma 5.7. Assume that $Y \subset H$ has codimension i. Let $w = (v,g) \in (V_Y \times G)(k)$ and let $p_w : \Sigma_{Y,w} \to \mathbb{P}^1_k$ be the composite projection $\Sigma \to B \times \mathbb{P}^1_k \times G \to \mathbb{P}^1_k$. Then the following hold.*

(1) $\Sigma_{Y,w}$ *is a closed subscheme of $H \times \mathbb{P}^1_k$ of pure codimension i.*
(2) p_w *is flat over an open neighborhood of $\{0, \infty\}$.*
(3) *If we write $v = (g_1, g_2) \in B(k) = G^\times(k) \times G^\times(k)$, then*

$$p_w^{-1}(0) = g_1 g \cdot Y \quad and \quad p_w^{-1}(\infty) = g_2 g \cdot Y.$$
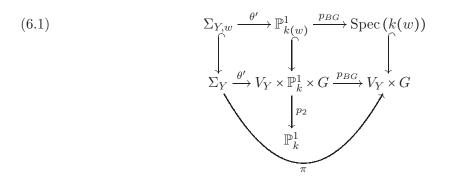
Proof. The property (1) follows because $\pi : \Sigma_Y \to V_Y \times G$ is a flat surjective morphism between two integral schemes and hence has equidimensional fibers. The property (2) follows from Lemma 5.7 (3) because the composition $\mathbb{P}^1_{k(w)} \hookrightarrow B \times \mathbb{P}^1_k \times G \to \mathbb{P}^1_k$ is an isomorphism. The last property follows directly Lemma 5.1 (2) and the definition of ϕ in (5.7). $\qquad\square$

6. Pull-backs of cycles from homogeneous spaces

We shall now use the results of the previous sections to construct the pull-back maps from the Chow groups of codimension two cycles on suitable homogeneous spaces to the Levine-Weibel Chow group of a singular surface. Note that for smooth surfaces, this is an easy consequence of the Chow moving lemma or Fulton's deformation to normal cone techniques. We can not use the latter trick in the non-$\mathbb{A}^1$-invariant world. So we shall use some sort of moving lemma tricks to achieve our goal. Over algebraically closed fields, this construction is due to Levine [Lev85b]. We shall follow Levine's outline to carry this out for surfaces over any infinite field.

We fix an infinite field k. We consider the linear algebraic group $G = \prod_{i=1}^r \mathrm{GL}_{n_i,k}$ and let $H = G/P$ be a projective homogeneous space for G as in § 4.2. We let $B = G^\times \times G^\times \subset G \times G$ be the open subscheme as in § 5.1. For an integral closed subscheme $Y \subset H$, we let $V_Y \subset B$ be the open and $\Sigma_Y \subset B \times \mathbb{P}^1_k \times G \times H$ the closed subschemes of Lemma 5.7. We shall continue to follow the notations of the previous sections.

We recall the maps used in Lemma 5.7 in the following diagram as we shall need to use them in this section.

(6.1)

Here, p_2 is the projection map. The two squares on the top are Cartesian and the composite vertical arrow in the middle is an isomorphism if $w \in (V_Y \times G)(k)$. We shall let p_w denote the composite map $p_w \colon \Sigma_{Y,w} \hookrightarrow \Sigma_Y \xrightarrow{\ p_2 \circ \theta'\ } \mathbb{P}^1_k$.

We fix an equidimensional reduced quasi-projective surface X over k. Note that since X is reduced, it is regular in codimension zero. That is, $\dim(X_{\mathrm{sing}}) \le 1$. Since the associated and minimal primes of a reduced Noetherian commutative ring coincide (e.g., see [GK20b, Lemma 3.3]), we note furthermore that X is Cohen-Macaulay in codimension one. Since the Cohen-Macaulay locus X_{CM} of X is open (e.g., see EGA IV$_2$ 6.11.2), it follows that its complement X_{nCM} is a finite closed subscheme (with reduced induced structure) of X contained in X_{sing}. We fix a morphism $f \colon X \to H$.

6.1. Construction of the map $f^* \colon \mathcal{Z}^2(H) \to \mathrm{CH}_0^L(X)$.
Let $\mathcal{Z}^2(H)$ be the free abelian group of integral cycles on H of codimension two. We shall first define a map

$$f^* \colon \mathcal{Z}^2(H) \to \mathrm{CH}_0^L(X).$$

Let $Y \subset H$ be an integral closed subscheme of codimension two. By applying Theorem 4.8 to X and X_{sing} simultaneously, we find a dense open subscheme $U(f,Y) \subset G$ such that for every $g \in U(f,Y)(k)$, the pull-back $f^{-1}(gY) := gY \times_H X$ has pure codimension two and $f^{-1}(gY) \cap X_{\mathrm{sing}} = \varnothing$. We can then define

$$(6.2) \qquad f^*[Y] = [f^{-1}(gY)] \in \mathcal{Z}_0^L(X),$$

where $[f^{-1}(gY)]$ is the fundamental class of the closed subscheme $f^{-1}(gY)$ in the sense of [Ful75, § 1.5]. Our goal now is to show that (after possibly shrinking $U(f,Y)$) this class does not depend on the choice of $g \in U(f,Y)(k)$ up to rational equivalence in $\mathcal{Z}_0^L(X, X_{\mathrm{sing}})$. We shall prove this using the following lemmas.

Until we have proven the desired independence, we fix $Y \subset H$ as above and simplify our notations. We write $U(f,Y)$ simply as U. Recall also that if $\Sigma \subset B \times \mathbb{P}^1_k \times G \times H$ is as in § 5.3, then $\Sigma_Y = \pi^{-1}(V_Y \times G)$ for the projection map $\pi \colon \Sigma \to B \times G$. Hence, we have $\Sigma_w = \Sigma_{Y,w}$ for all $w \in V_Y \times G$. We shall therefore use the common notation Σ_w. We recall our notation $\widetilde{f}$ for the map $f \times \mathrm{id} \colon X \times \mathbb{P}^1_k \to H \times \mathbb{P}^1_k$ from § 4.4. For every $w \in (V_Y \times G)(k)$, we know that Σ_w is a closed subscheme of $\mathbb{P}^1_H$. We shall write $\widetilde{f}^{-1}(\Sigma_w) = \Sigma_w \times_{\mathbb{P}^1_H} \mathbb{P}^1_X$ as Γ_w.

We consider the composite map

$$(6.3) \qquad \tau \colon V_Y \times G \hookrightarrow B \times G \hookrightarrow G \times G \times G \to G \times G;$$

$$(g_1, g_2, g) \mapsto (g \cdot \Phi_m(g_1, g_2, 0), g \cdot \Phi_m(g_1, g_2, \infty)) = (gg_1, gg_2).$$

Note that this definition makes sense by Lemma 5.1 (2). The equality on the right follows from Lemma 5.1 (4). It is easy to check that the last map in (6.3) is a surjective morphism between regular schemes with smooth fibers, namely, G. Hence, this map is flat by [Har77, Exc.III.10.9], and therefore smooth. Since the first and second maps in (6.3) are open immersions, it follows that τ is a smooth morphism. If $\tau(w) = (g_1, g_2) \in (G \times G)(k)$, then Lemma 5.8 says that

$$(6.4) \qquad \Sigma_w \times_{\mathbb{P}^1_k} \{0\} = g_1 Y \quad \text{and} \quad \Sigma_w \times_{\mathbb{P}^1_k} \{\infty\} = g_2 Y.$$

We let $\widetilde{U} := \tau^{-1}(U \times U)$ and consider the subset $W \subset \widetilde{U} \subset V_Y \times G$ consisting of points w having the following properties.

  a) Γ_w is of pure dimension 1.
  b) $\Gamma_w \cap \mathbb{P}^1_{X_{\mathrm{nCM}}} = \varnothing$.
  c) $(\Sigma_w)_{\mathrm{nsm}} \cap f(X_{\mathrm{sing}}) \times \mathbb{P}^1_k = \varnothing$.

d) $\Gamma_w \cap \mathbb{P}^1_{X_{\mathrm{sing}}}$ is a finite set.

Our goal is to show that W is a constructible subset of $V_Y \times G$ containing the generic point of the latter. In particular, it is non-empty. Note that $\widetilde{U}$ is dense open in $V_Y \times G$ because τ is smooth and $G \times G$ is irreducible.

Lemma 6.1. *W is a constructible subset of $V_Y \times G$.*

Proof. Note that W is constructible in $V_Y \times G$ if and only if it is so in $\widetilde{U}$. We consider the maps $f \times \mathrm{id} \colon V_Y \times \mathbb{P}^1_k \times G \times X \to V_Y \times \mathbb{P}^1_k \times G \times H$. We let $\Sigma' = (f \times \mathrm{id})^{-1}(\Sigma_Y)$ and $f' = (f \times \mathrm{id})|_{\Sigma'}$. This gives us maps $\Sigma' \xrightarrow{f'} \Sigma \xrightarrow{\pi} V_Y \times G$. Note that for every $w \in V_Y \times G$, the induced map on the fibers $f'_w \colon \Sigma'_w \to \Sigma_w$ is same as the restriction of the map $\widetilde{f} \colon (\mathbb{P}^1_X)_{k(w)} \to (\mathbb{P}^1_H)_{k(w)}$ to $\widetilde{f}^{-1}(\Sigma_w)$. We can therefore replace $\Gamma_w = \widetilde{f}^{-1}(\Sigma_w)$ in (a) ~ (d) above by Σ'_w. We let π' be the restriction of the map $\pi \circ f' \colon \Sigma' \to V_Y \times G$ to $\Sigma'_U := (\pi \circ f')^{-1}(\widetilde{U})$.

We let $W_a \subset \widetilde{U}$ be the set of points which satisfy condition (a) above. We define W_b, W_c and W_d similarly. It suffices to show separately that each of the sets W_a, W_b, W_c and W_d is constructible.

Now, the constructibility of W_a follows by applying [GW10, E.1 (5)] to the morphism $\pi' \colon \Sigma'_U \to \widetilde{U}$. The constructibility of W_b and W_d follows by applying [GW10, E.1 (4)] to the compositions $\Sigma'_U \times_X X_{\mathrm{nCM}} \hookrightarrow \Sigma'_U \xrightarrow{\pi'} \widetilde{U}$ and $\Sigma'_U \times_X X_{\mathrm{sing}} \hookrightarrow \Sigma'_U \xrightarrow{\pi'} \widetilde{U}$, respectively. For W_c, we first observe that since $\Sigma \xrightarrow{\pi} V_Y \times G$ is flat by Lemma 5.7(1), one checks that for any point $y \in \Sigma$, the morphism π is smooth at y if and only if $\Sigma_{\pi(y)}$ is smooth over $\mathrm{Spec}\,(k(\pi(y)))$ at y (e.g., see [Spa21, Tag 07BT]). Hence, we have the equality $(\Sigma_w)_{\mathrm{nsm}} = (\Sigma_{\mathrm{nsm}})_w$. We can now deduce the constructibility of W_c by replacing π by its composition with the inclusion $\Sigma_{\mathrm{nsm}} \cap (V_Y \times \mathbb{P}^1_k \times G \times f(X_{\mathrm{sing}})) \cap \Sigma'_U \hookrightarrow \Sigma'_U$ and applying [GW10, E.1 (4)]. $\square$

Lemma 6.2. *For every point $w \in (V_Y \times G)(k)$ lying in W, the following hold.*

(1) $\Gamma_w \cap \mathbb{P}^1_{X_{\mathrm{sing}}} \subset \mathbb{P}^1_{X_{\mathrm{CM}}}$.

(2) *For $x \in \Gamma_w \cap \mathbb{P}^1_{X_{\mathrm{sing}}}$, one has that $\widetilde{f}(x) \in (\Sigma_w)_{\mathrm{sm}}$.*

(3) *For every $x \in \Gamma_w \cap \mathbb{P}^1_{X_{\mathrm{sing}}}$, the ideal of Σ_w in the local ring $\mathcal{O}_{\mathbb{P}^1_H, \widetilde{f}(x)}$ is a complete intersection of height two.*

Proof. The item (1) follows from the item (b) before Lemma 6.1 and (2) follows from the item (c) before Lemma 6.1. By Lemma 5.8, Σ_w is of pure codimension two in $\mathbb{P}^1_H$. Since Σ_w is smooth over $\mathrm{Spec}\,(k(w))$ at $\widetilde{f}(x)$ by (2), it is regular at $\widetilde{f}(x)$. In particular, $\Sigma_w \subset \mathbb{P}^1_H$ is a closed immersion of schemes both of which are regular at $\widetilde{f}(x)$. Hence, (3) follows. $\square$

Lemma 6.3. *For every point $w \in (V_Y \times G)(k)$ lying in W, the following hold.*

(1) Γ_w *is purely 1-dimensional.*

(2) $\Gamma_w \cap \mathbb{P}^1_{X_{\mathrm{sing}}}$ *is a finite set.*

(3) $\Gamma_w \cap (X_{\mathrm{sing}} \times \{0, \infty\}) = \varnothing$.

(4) *The inclusion $\Gamma_w \subset \mathbb{P}^1_X$ is a local complete intersection morphism at every point of $\Gamma_w \cap \mathbb{P}^1_{X_{\mathrm{sing}}}$.*

(5) *The projection map $\Gamma_w \to \mathbb{P}^1_k$ is flat over an open neighborhood of $\{0, \infty\}$.*

Proof. The items (1), (2) and (3) are respectively same as the items (a), (c) and (d) before Lemma 6.1. If $x \in \Gamma_w \cap \mathbb{P}^1_{X_{\mathrm{sing}}}$, then it must be a closed point of $\mathbb{P}^1_X$ by (2). Let $J \subset \mathcal{O}_{\mathbb{P}^1_X, x}$ be the ideal of Γ_w in $\mathcal{O}_{\mathbb{P}^1_X, x}$. Now, item (3) of Lemma 6.3 shows that the ideal I of Σ_w in the local ring $\mathcal{O}_{\mathbb{P}^1_H, \widetilde{f}(x)}$ is a complete intersection of height two. In particular, we can write

$I = (a_1, a_2)$. We have the morphism of local rings $\mathcal{O}_{\mathbb{P}^1_H, \widetilde{f}(x)} \to \mathcal{O}_{\mathbb{P}^1_X, x}$ and $J = I\mathcal{O}_{\mathbb{P}^1_X, x}$. This implies that $J = (a_1, a_2)$. On the other hand, $\mathcal{O}_{\mathbb{P}^1_X, x}$ is Cohen-Macaulay by Lemma 6.2 (1). Since $\mathrm{ht}(J) = 2$, it must be a complete intersection (e.g., see [Mat97, Theorem 17.4]). This proves (4).

For (5), we first note that the projection map $\Sigma_w \to \mathbb{P}^1_k$ is flat over an open neighborhood of $\{0, \infty\}$ by Lemma 5.8. Hence, if we repeat the proof of Proposition 4.9 (3), the only thing we need to know is that $\mathrm{Tor}_1^{\mathcal{O}_{H_\epsilon}}(\mathcal{O}_{(\Sigma_w)_\epsilon}, \mathcal{O}_{X_\epsilon}) = 0$ for $\epsilon = 0, \infty$. But for $w = (v, g)$ with $v \in V_Y(k)$ and $g \in G(k)$ as above, we have

$$(6.5) \qquad \Gamma_w \times_{\mathbb{P}^1} \{0\} = f^{-1}(g\Phi_m(v, 0) \cdot Y) = f^{-1}(g_1 Y) \quad \text{and}$$

$$\Gamma_w \times_{\mathbb{P}^1} \{\infty\} = f^{-1}(g\Phi_m(v, \infty) \cdot Y) = f^{-1}(g_2 Y),$$

where $\tau(v, g) = (g_1, g_2) \in U \times U$ (see (6.4)). Hence, the desired Tor-vanishing follows from our choice of U and Lemma 4.6. $\square$

We shall need the following elementary lemma to show that W contains a dense open subset of $V_Y \times G$.

Lemma 6.4. *Let $f \colon X_1 \to X_2$ be a morphism in $\mathbf{Sch}_k$ such that $X_1(k)$ is Zariski dense in X_1. Let $Z \subset X_1$ be a closed subset which has the property that $f^{-1}(x) \cap Z$ is dense in $f^{-1}(x)$ for every $x \in X_2(k)$. Then $Z = X_1$.*

Proof. Suppose that $Z \neq X_1$. Our assumption then implies that $(X_1 \smallsetminus Z)(k) \neq \varnothing$. Choose a point in $x' \in (X_1 \smallsetminus Z)(k)$ and let $x = f(x')$. Then $x \in X_2(k)$ and $f^{-1}(x) \cap Z$ is a proper closed subset of $f^{-1}(x)$. Hence, it can not be dense in $f^{-1}(x)$, a contradiction. $\square$

Lemma 6.5. *W contains a dense open subset of $V_Y \times G$.*

Proof. Since $V_Y \times G$ is irreducible, it suffices to show that W contains the generic point η of $V_Y \times G$. Suppose to the contrary that $\eta \notin W$. As W is constructible and $V_Y \times G$ irreducible, this means that $\eta \notin \overline{W}$, where the latter is the closure of W in $V_Y \times G$. Since $(V_Y \times G)(k)$ is Zariski dense in $V_Y \times G$, it follows from Lemma 6.4 that there exists $v_0 \in V_Y(k)$ such that $p_1^{-1}(v_0) \cap \overline{W}$ is not dense in $p_1^{-1}(v_0)$.

Since $p_1^{-1}(v_0) \cong G$ has a Zariski dense set of k-rational points, we can choose $w_0 = (v_0, g_0) \in (V_Y \times G)(k)$. Let $\Sigma_{w_0} = \pi^{-1}(w_0) \subset \mathbb{P}^1_H$. By applying Proposition 4.9 to Σ_{w_0}, we get a dense open subscheme $U_{w_0} \subset G$ such that for every $g \in U_{w_0}(k)$, the pull-back $F_g := \widetilde{f}^{-1}(g\Sigma_{w_0})$ satisfies the following.

(1) F_g has pure dimension 1.
(2) $F_g \cap \mathbb{P}^1_{X_{\mathrm{nCM}}} = \varnothing$.
(3) $F_g \cap \mathbb{P}^1_{X_{\mathrm{sing}}}$ is finite and $F_g \cap (X_{\mathrm{sing}} \times \{0, \infty\}) = \varnothing$.
(4) The inclusion $F_g \subset \mathbb{P}^1_X$ is a local complete intersection at each point of $\mathbb{P}^1_{X_{\mathrm{sing}}}$.
(5) The projection $F_g \to \mathbb{P}^1_k$ is flat over a neighborhood of $\{0, \infty\}$.

Since G acts on $\Sigma \subset B \times \mathbb{P}^1_k \times G \times H$ by Lemma 5.6 where its action is trivial on the first two factors and diagonal on the product of the other two factors (see § 5.2), it follows that every $g \in G(k)$ acts as an automorphism of $B \times \mathbb{P}^1_k \times G \times H$ which takes $g\Sigma_{w_0}$ onto $\Sigma_{(v_0, gg_0)}$. Hence, we see that for every $g \in U_{w_0}(k)$, the pull-back $\widetilde{f}^{-1}(\Sigma_{(v_0, gg_0)})$ satisfies (1) ~ (5) above.

We let $U'_{w_0} = (U_{w_0})g_0 \subset G$. Then the right multiplication by g_0 defines an automorphism $G \xrightarrow{\cong} G$ which takes U_{w_0} onto U'_{w_0}. It follows that U'_{w_0} is dense open in G and $\widetilde{f}^{-1}(\Sigma_{(v_0, g)})$ satisfies (1) ~ (5) above for every $g \in U'_{w_0}(k)$. In particular, $\{v_0\} \times U'_{w_0} \subset W$. But $\{v_0\} \times U'_{w_0}$

is clearly dense in $p_1^{-1}(v_0)$. This implies that $p_1^{-1}(v_0) \cap W$ is dense in $p_1^{-1}(v_0)$. This leads to a contradiction. We have thus proven the lemma. $\qquad\square$

Lemma 6.6. *There exists a dense open subscheme $V(f, Y) \subset U(f, Y)$ of G, depending only on f and Y, such that for every $g_1, g_2 \in V(f, Y)(k)$, the classes $[f^{-1}(g_1 Y)]$ and $[f^{-1}(g_2 Y)]$ define the same element of $\mathrm{CH}_0^L(X)$. In particular, the assignment (6.2) gives a group homomorphism*

$$f^* : \mathcal{Z}^2(H) \to \mathrm{CH}_0^L(X).$$

Proof. Let $W' \subset W$ be a subset which is dense open in $V_Y \times G$. Such a subset exists by Lemma 6.5. Since τ is smooth (this is shown just below (6.3)), its image $\tau(W')$ is dense open in $G \times G$. If we take the projection of $\tau(W')$ to the first factor of $G \times G$, it is also dense open in G. Hence, it contains a k-rational point g_0. Then $p_1^{-1}(g_0) \cap \tau(W')$ is dense open in $p_1^{-1}(g_0)$. Since the composite map $p_1^{-1}(g_0) \hookrightarrow G \times G \xrightarrow{p_2} G$ is an isomorphism, it takes $p_1^{-1}(g_0) \cap \tau(W')$ to a dense open subscheme of G. Let $V(f, Y)$ be such a dense open subscheme. Note that $V(f, Y) \subset U(f, Y)$ since $W \subset \tau^{-1}(U \times U)$.

For every $g \in V(f, Y)(k)$, we have that $(g_0, g) \in \tau(W)$. Let $w \in W$ be such that $\tau(w) = (g_0, g)$. It is clear from (6.3) that $w \in (V_Y \times G)(k)$. It follows from Lemma 6.3 and Definition 2.1 that Γ_w is a Cartier curve on $\mathbb{P}_X^1$. Furthermore, it follows from (6.5) that Γ_w defines a rational equivalence between $[f^{-1}(g_0 Y)]$ and $[f^{-1}(g Y)]$. It follows that for any pair of points $g_1, g_2 \in V(f, Y)(k)$, the cycles $[f^{-1}(g_1 Y)]$ and $[f^{-1}(g_2 Y)]$ are both rationally equivalent to $[f^{-1}(g_0 Y)]$. Hence, they are rationally equivalent each other. This finishes the proof. $\qquad\square$

6.2. The pull-back map on the Chow groups.

We let G, H, X and $f : X \to H$ be as we described in the beginning of § 6. We have shown in § 6.1 that f induces a pull-back map $f^* : \mathcal{Z}^2(H) \to \mathrm{CH}_0^L(X)$. We shall now show that this map factors through the rational equivalence in the classical Chow group of H.

Proposition 6.7. *The morphism $f^* : \mathcal{Z}^2(H) \to \mathrm{CH}_0^L(X)$ descends to a homomorphism*

$$f^* : \mathrm{CH}^2(H) \to \mathrm{CH}_0^L(X)$$

satisfying the following property: for every $Y \subset H$ integral subvariety of codimension 2, the equality $f^*[Y] = [f^{-1}(gY)]$ holds in $\mathrm{CH}_0^L(X)$ for every k-point $g \in V(Y, f)$, where $V(Y, g)$ is the open dense subset of G of Lemma 6.6.

Proof. We only need to show that f^* respect the rational equivalence, since it is already clear by construction that on generators the pullback can be computed by first translating Y by a rational point of $V(Y, f)$ followed by the naive pullback.

It is well-known that the rational equivalence in $\mathrm{CH}^2(H)$ can be presented in terms of subvarieties of H of codimension 1 with rational functions on them, or in terms of subvarieties of codimension 2 of $\mathbb{P}_H^1$. The latter presentation is easier to handle in our context. Following Levine [Lev85b], let $W \subset \mathbb{P}_H^1$ be an integral subscheme of codimension 2, flat over $\mathbb{P}_k^1$, and let $p_1' : \mathbb{P}_H^1 \to H$ be the projection.

Let $W_0 = p_1'(W \cdot (H \times \{0\}))$ and $W_\infty = p_1'(W \cdot (H \times \{\infty\}))$. Note that the intersection product is well defined (this is simply the pull-back to W of the Cartier divisors 0 and ∞ of $\mathbb{P}_k^1$, which is well defined since the map is flat). We need to show that the difference $f^*[W_0] - f^*[W_\infty]$ is an element of $\mathcal{R}_0^L(X, X_{\mathrm{sing}})$. Let $V_0 = V(f, W_0)$ and let $V_\infty = V(f, W_\infty)$. If we take $g \in (V_0 \cap V_\infty)(k)$, we have that $f^*[W_0] = [f^{-1}(gW_0)]$ and $f^*[W_\infty] = [f^{-1}(gW_\infty)]$ (for the same k-point g of G).

By Proposition 4.9, there exists an open dense subset $U = U(f \times \mathrm{id}_{\mathbb{P}^1}, W)$ of G such that for every $g \in U(k)$, the inverse image $(f \times \mathrm{id}_{\mathbb{P}^1})^{-1}(gW)$ defines a purely 1-dimensional closed subscheme Γ of $\mathbb{P}^1_X$ satisfying the following properties.

(1) The inclusion $\Gamma \subset \mathbb{P}^1_X$ is a local complete intersection morphism at every point of $\Gamma \cap \mathbb{P}^1_{X_{\mathrm{sing}}}$.
(2) $\Gamma \cap \mathbb{P}^1_{X_{\mathrm{sing}}}$ is a finite set and $\Gamma \cap (X_{\mathrm{sing}} \times \{0, \infty\}) = \varnothing$.
(3) The induced map $\Gamma \to \mathbb{P}^1_k$ is flat over an open neighborhood of $\{0, \infty\}$.

In particular, $(p_1)_*(\Gamma \cdot (X \times \{0\})) - (p_1)_*(\Gamma \cdot (X \times \{\infty\})) \in \mathcal{R}^L_0(X, X_{\mathrm{sing}})$. If we choose $g \in (U \cap V_0 \cap V_\infty)(k)$, we conclude that

$$
\begin{aligned}
f^*[W_0] = [f^{-1}(gW_0)] &= (p_1)_*(\Gamma \cdot (X \times \{0\})) \sim (p_1)_*(\Gamma \cdot (X \times \{\infty\})) \\
&= [f^{-1}(gW_\infty)] = f^*[W_\infty],
\end{aligned}
$$

as required. $\square$

6.3. **Functoriality of f^* in G and H.** We now show that the pull-back on the Chow groups behaves well with respect to the change in the group G and the homogeneous space H. Let (G, H) and (G', H') be two pairs of groups and their homogeneous spaces of the types described in the beginning of § 6. A morphism $\Psi: (G, H) \to (G', H')$ is a pair of maps $\rho: G \to G'$ and $h: H \to H'$ such that ρ is a group homomorphism and h is G-equivariant, where the latter acts on H' via ρ.

Let $\Psi: (G, H) \to (G'H')$ be as above. Let X be an equidimensional reduced quasi-projective surface and let there be a commutative diagram

(6.6)
$$
\begin{array}{ccc}
X & \xrightarrow{\ f\ } & H \\
& {\scriptstyle f'}\searrow & \big\downarrow{\scriptstyle h} \\
& & H'.
\end{array}
$$

We now prove the following functoriality property with respect to the change of homogeneous spaces.

Lemma 6.8. *There is a commutative diagram*

(6.7)
$$
\begin{array}{ccc}
\mathrm{CH}^2(H') & \xrightarrow{\ h^*\ } & \mathrm{CH}^2(H) \\
& {\scriptstyle f'^*}\searrow & \big\downarrow{\scriptstyle f^*} \\
& & \mathrm{CH}^L_0(X).
\end{array}
$$

Proof. Let Y be an integral subscheme of H' of codimension 2. Using Lemma 6.6 (applied simultaneously to h and f'), we let $U' \subset G'$ be a dense open such that $\alpha^*([Y]) = [\alpha^{-1}(gY)]$ and $[\alpha^{-1}(gY)] \sim [\alpha^{-1}(g'Y)]$ for $g, g' \in U'(k)$ and $\alpha \in \{h, f'\}$. We fix $g_0 \in U'(k)$ and let $Y_0 = g_0 Y$ and $U'_0 = (U')g_0^{-1}$. Then U'_0 is a dense open subset such that $\alpha^*([Y_0]) = [\alpha^{-1}(gY_0)]$ and $[\alpha^{-1}(gY_0)] \sim [\alpha^{-1}(g'Y_0)]$ for $g, g' \in U'_0(k)$ and $\alpha \in \{h, f'\}$.

Using Lemma 6.6 (applied to f), we now let $U_1 \subset G$ be a dense open subset such that

(6.8)
$$
f^*(h^*([Y])) = f^*([h^{-1}(Y_0)]) = [f^{-1}(gh^{-1}(Y_0))] \quad \text{and}
$$

$$
[f^{-1}(gh^{-1}(Y_0))] \sim [f^{-1}(g'h^{-1}(Y_0))] \quad \text{for any} \quad g, g' \in U_1(k).
$$

Since U'_0 contains the identity element of G' by its construction, it follows that $U'_0 \cap \rho(G) \neq \varnothing$. In particular, $\rho^{-1}(U'_0)$ is dense open in G. Hence, $U_1 \cap \rho^{-1}(U'_0)$ is dense open in G. We can therefore replace U_1 by $U := U_1 \cap \rho^{-1}(U'_0)$ and (6.8) continues to hold for $g, g' \in U(k)$.

We now choose any $g \in U(k)$ so that $g' := \rho(g) \in U_0'$. Then we get

$$\begin{aligned}
f^* h^*[Y] = f^*([h^{-1}(Y_0)]) \quad &= \quad [f^{-1}(gh^{-1}(Y_0))] \\
&=^1 \quad [f^{-1}(h^{-1}(g'Y_0))] \\
&= \quad [f'^{-1}(g'Y_0)] = [f^{-1}(g'g_0 Y)] \\
&=^2 \quad f'^*([Y]),
\end{aligned}$$

where $=^1$ follows from the G-equivariance of h and $=^2$ follows from the choice U' because $g' g_0 \in U'(k)$. This proves the lemma. $\qquad\square$

We shall also need the following obvious functoriality property of the pull-back map on the Chow groups.

Lemma 6.9. *Let* $\pi \colon \widetilde{X} \to X$ *be a resolution of singularities of* X *and let* $\widetilde{f} \colon \widetilde{X} \to X \to H$ *be the composite map. Then* $\widetilde{f}^* = \pi^* \circ f^*$ *as maps between the Chow groups* $\mathrm{CH}^2(H)$ *and* $\mathrm{CH}^2(\widetilde{X})$ *of smooth schemes.*

Proof. Let $Y \subset H$ be an integral closed subscheme of codimension two. Let $V(f, Y) \subset G$ be as in Lemma 6.6 so that $f^*([Y]) = [f^{-1}(gY)]$ for every $g \in V(f, Y)(k)$. Since $f^{-1}(gY)$ is a 0-cycle supported on X_{reg}, it follows from the definition of π^* in Proposition 3.9 that $\pi^*([f^{-1}(gY)]) = [f^{-1}(gY)] = [\widetilde{f}^{-1}(gY)] = \widetilde{f}^*([gY])$. Since G is rationally connected, we know on the other hand that $[Y] \sim [gY]$ in $\mathscr{Z}^2(H)$. Hence, we get $\widetilde{f}^*([gY]) = \widetilde{f}^*([Y])$ in $\mathrm{CH}^2(\widetilde{X})$. We have thus shown that $\pi^*(f^*([Y])) = \widetilde{f}^*([Y])$. $\qquad\square$

6.4. $\mathbb{P}^1$-homotopy between two maps.

We shall now prove a suitable $\mathbb{P}^1$-invariance of maps between the Chow groups. Let $H = G/P$ be a projective homogeneous space as before. For any $t \in \mathbb{P}^1_k(k)$, let $\iota_t \colon \{t\} \hookrightarrow \mathbb{P}^1_k$ be the inclusion.

Lemma 6.10. *Let* $F \colon X \times \mathbb{P}^1_k \to H$ *be a morphism. Let* $f_t = F \circ \iota_t \colon X \to H$. *Then* $f_0^* = f_\infty^* \colon \mathrm{CH}^2(H) \to \mathrm{CH}_0^L(X)$.

Proof. Let $Y \subset H$ be an integral cycle of codimension two. Let $U \subset G$ be the intersection of the dense open subsets obtained by applying Lemma 6.6 to f_0 and f_∞. Let $\widetilde{F} \colon X \times \mathbb{P}^1_k \to H \times \mathbb{P}^1_k$ be the morphism $\widetilde{F} = (F, p_2)$, where $p_2 \colon X \times \mathbb{P}^1_k \to \mathbb{P}^1_k$ is the projection.

By applying Theorem 4.8 to $(\widetilde{G} := G \times \mathrm{PGL}_{2,k}, H \times \mathbb{P}^1_k)$, we get an open dense subset $\widetilde{U} \subset \widetilde{G}$ such that for $\widetilde{g} = (g, \sigma) \in \widetilde{U}(k)$, the subscheme $\widetilde{F}^{-1}(gY \times \mathbb{P}^1_k) = \widetilde{F}^{-1}(\widetilde{g} \cdot \mathbb{P}^1_Y)$ satisfies (1) ~ (3) of Theorem 4.8. Since $(U \times \mathrm{PGL}_{2,k}) \cap \widetilde{U}$ is dense open in $\widetilde{G}$, we can find a k-rational point $\widetilde{g} = (g, \sigma)$ in this intersection.

We let $\Gamma = \widetilde{F}^{-1}(gY \times \mathbb{P}^1_k)$. Then for any point $\widetilde{g}$ as above, we see from the choice of U that the fibers of the map $\Gamma \to \mathbb{P}^1_k$ over $\{0, \infty\}$ are supported on X_{reg} and $\mathrm{Tor}_i^{\mathcal{O}_H}(\mathcal{O}_{gY}, \mathcal{O}_X) = 0$ for $i \geq 1$ with respect to the maps f_0 and f_∞. In particular, the argument in the proof of Proposition 4.9 (3) shows that this map is flat over an open neighborhood of $\{0, \infty\}$. Hence, Γ determines a Cartier curve on $\mathbb{P}^1_X$ such that $\Gamma_0 = [f_0^{-1}(gY)] = f_0^*([Y])$ and $\Gamma_\infty = [f_\infty^{-1}(gY)] = f_\infty^*([Y])$. This finishes the proof. $\qquad\square$

7. The Chern classes for singular surfaces

We fix an infinite field k and a connected equidimensional reduced quasi-projective surface X over k. In this section, we shall review the ring structure on $\mathrm{CH}_0^L(X)$ due to Levine [Lev85b] and define Chern classes of vector bundles in this ring.

7.1. **Intersection product of Cartier divisors.** For a closed subscheme $D \subset X$, we let $|D|$ denote the support of D. We let $\mathcal{K}_X$ denote the Zariski sheaf of total quotient rings on X. We let $\mathcal{Z}^1(X, X_{\mathrm{sing}})$ denote the subgroup of Cartier divisors $D \in H^0_{\mathrm{zar}}(X, \mathcal{K}_X^{\times}/\mathcal{O}_X^{\times})$ such that $|D| \cap X_{\mathrm{sing}}$ is finite. We let $\mathcal{R}^1(X, X_{\mathrm{sing}})$ be the subgroup of $\mathcal{Z}^1(X, X_{\mathrm{sing}})$ generated by principal Cartier divisors $\mathrm{div}(f)$, where $f \in H^0_{\mathrm{zar}}(X, \mathcal{K}_X^{\times})$ is invertible at all generic points of X_{sing}. We let $\mathrm{CH}^1(X) = \mathcal{Z}^1(X, X_{\mathrm{sing}})/\mathcal{R}^1(X, X_{\mathrm{sing}})$. It is easy to check that there is a canonical injective map $\mathrm{CH}^1(X) \hookrightarrow \mathrm{Pic}(X)$ which takes a Cartier divisor D to the associated line bundle $\mathcal{O}_X(D)$. We let $\mathrm{CH}^0(X)$ denote the free abelian group on the cycle $[X]$ so that there is a canonical isomorphism $\mathbb{Z} \xrightarrow{\cong} \mathrm{CH}^0(X)$. We let $\mathrm{CH}^*(X) = \mathrm{CH}^0(X) \oplus \mathrm{CH}^1(X) \oplus \mathrm{CH}^L_0(X)$. Then a ring structure on $\mathrm{CH}^*(X)$ is completely determined by defining the intersection product

$$(7.1) \qquad \mathrm{CH}^1(X) \otimes \mathrm{CH}^1(X) \to \mathrm{CH}^L_0(X).$$

This construction is identical to the case of smooth surfaces. We recall it here. Let D, E be two effective Cartier divisors on X with no common components. Let $x \in X_{\mathrm{reg}}$ be a closed point. Recall that the intersection multiplicity of D and E at x is defined as

$$i_x(D, E) = \ell(\mathcal{O}_{X,x}/\mathcal{O}_X(-D - E)_x).$$

The number $i_x(D, E)$ is non-zero only if $x \in \mathrm{Supp}(D) \cap \mathrm{Supp}(E)$, and satisfies the standard properties of symmetry and additivity. In particular, if $|D| \cap |E| \subset X_{\mathrm{reg}}$, the product $(D \cdot E) = \sum_{x \in |D| \cap |E|} i_x(D, E)[x]$ is a well-defined element of the Chow group $\mathrm{CH}^L_0(X)$.

Note that since X is quasi-projective, the map $\mathrm{CH}^1(X) \hookrightarrow \mathrm{Pic}(X)$ is in fact an isomorphism. In fact, we can write up to linear equivalence $D = D_1 - D_2$ as difference of effective and very ample Cartier divisors without common components (e.g., see [Har77, II Ex.7.5]). Since k is infinite, we can use the classical Bertini theorem (e.g., see [Jou83, 6.3 b)]) to ensure that $|D_1| \cap |D_2| \cap X_{\mathrm{sing}} = \varnothing$ and $|D_i| \cap X_{\mathrm{sing}}$ finite for $i = 1, 2$. We do the same for E, which we write (up to linear equivalence) as $E_1 - E_2$ with the additional property that $D_i \cap E_j$ is finite and disjoint from X_{sing}. At this point, we can define

$$(D \cdot E) = (D_1 - D_2) \cdot (E_1 - E_2) = D_1 \cdot E_1 - D_1 \cdot E_2 - D_2 \cdot E_1 + D_2 \cdot E_2.$$

It is straightforward to check that the product is well-defined (i.e., it does not depend on the presentation of D and E). We leave out the details.

7.2. **Compatibility of product with pull-backs.** Let G and H be as in § 6. We let $\mathrm{CH}^*(H) = \oplus_{i \geq 0} \mathrm{CH}^i(H)$ denote the classical Chow ring of H. We assume that $\dim(H) \geq 2$. Let $f: X \to H$ be a morphism. We have a well-defined pull-back map $f^*: \mathrm{Pic}(H) \to \mathrm{Pic}(X)$. Equivalently, a pull-back map $f^*: \mathrm{CH}^1(H) \to \mathrm{CH}^1(X)$. To give a more explicit description of this, note that for any integral divisor $D \subset H$, we can apply Theorem 4.8 (to X and X_{sing}) to find a dense open $U \subset G$ such that for every $g \in U(k)$, the scheme-theoretic pull-back $f^*(gD)$ satisfies conditions (1) ∼ (3) of Theorem 4.8. But this precisely means that $f^*(gD)$ is an effective Cartier divisor on X whose no irreducible component is contained in X_{sing}. It is then clear from the definition of $\mathrm{CH}^1(X)$ in the beginning of § 7.1 that $f^*([D]) = [f^*(gD)] \in \mathrm{CH}^1(X)$.

We have also defined the pull-back $f^*: \mathrm{CH}^2(H) \to \mathrm{CH}^L_0(X)$. Since $f^*: \mathrm{CH}^0(H) \to \mathrm{CH}^0(X)$ is identity as X is connected, we have a well-defined codimension preserving (we define f^* to be zero on $\mathrm{CH}^{\geq 3}(H)$) group homomorphism $f^*: \mathrm{CH}^*(H) \to \mathrm{CH}^*(X)$.

Proposition 7.1. *The map $f^*: \mathrm{CH}^*(H) \to \mathrm{CH}^*(X)$ is a ring homomorphism. If $\Psi = (\rho, h): (G, H) \to (G', H')$ is a morphism as in § 6.3, then $(h \circ f)^* = f^* \circ h^*$ as ring homomorphisms.*

Proof. We only need to show this for the product of cycles lying in $\mathrm{CH}^1(H)$. For this, we note that $\mathrm{CH}^1(H) \cong \mathrm{Pic}(H)$ and we know that every line bundle on H is a difference of two very ample line bundles. Furthermore, if D is a Cartier divisor on H such that $\mathcal{O}_H(D)$ is very ample, then the classical Bertini smoothness theorem over infinite fields tells us that $D \sim D'$, where D' is a smooth effective Cartier divisor on H. It follows therefore that $\mathrm{CH}^1(H)$ is generated by smooth very ample divisors.

Now, given two smooth very ample divisors D_1 and D_2 on H, we can use the Bertini theorem of Jouanolou (see also [KA79, Theorem 1]) to find elements D_1' and D_2' in the linear systems $|H^0(H, \mathcal{O}_H(D_1))|$ and $|H^0(H, \mathcal{O}_H(D_2))|$, respectively such that D_1' and D_2' are geometrically integral smooth schemes which intersect transversely in a smooth and (geometrically) integral codimension two subscheme $Y \subset H$. We remind here that H is smooth and geometrically integral. We therefore need to show that if D_1 and D_2 are two smooth connected very ample effective Cartier divisors on H whose scheme-theoretic intersection Y is smooth and integral codimension two cycle on H, then $[f^*(D_1)] \cdot [f^*(D_2)] = [f^*(Y)]$ in $\mathrm{CH}_0^L(X)$.

We now apply Theorem 4.8 again. We can then find an open dense subset U_i of G, for $i = 1, 2$ such that, for every $g \in U_i(k)$, the scheme-theoretic pull-back $f^{-1}(gD_i)$ is an effective Cartier divisor on X having finite intersection with X_{sing}. By replacing U_1 and U_2 by their intersection, we can assume that $U_1 = U_2 = U$ and that, the same g works for both D_1 and D_2.

Let $V(f, Y)$ be the open subset of Lemma 6.6. Then for every $g \in V(f, Y)(k)$, we have $f^*(D_1 \cdot D_2) = f^*[Y] = [f^{-1}(gY)]$, with $f^{-1}(gY) \cap X_{\mathrm{sing}} = \varnothing$. Choosing $g \in (V(f, Y) \cap U)(k)$, we obtain

$$\begin{aligned}
f^*([D_1] \cdot [D_2]) = [f^{-1}(gY)] &= [f^{-1}([gD_1] \cdot [gD_2])] \\
&= [f^{-1}(gD_1)] \cdot [f^{-1}(gD_2)] \\
&= f^*[D_1] \cdot f^*[D_2],
\end{aligned}$$

as required (the second equality follows directly from the local definition of the intersection product in § 7.1). The second assertion of the proposition follows directly from Lemma 6.8. $\square$

7.3. The Chern classes.

Let X be as in the beginning of § 7. We now review the construction of Chern classes in $\mathrm{CH}^*(X)$ of vector bundles on X. This construction depends solely on the results of the previous sections of this manuscript and does not use any further information about the nature of the field k. Hence, all the proofs in the construction of the Chern classes given in [Lev85b] and [BS98] verbatim remain valid and we have nothing extra to add. We shall therefore only recall it very briefly and refer to [BS98, § 5] for details.

For any vector bundle $\mathcal{E}$ of rank n on X, we let $c_0(\mathcal{E}) = 1$ and $c_1(\mathcal{E}) = c_1(\wedge^n(\mathcal{E})) = [\wedge^n(\mathcal{E})] \in \mathrm{Pic}(X) = \mathrm{CH}^1(X)$. It is easy to check that $c_1(\mathcal{E}) = c_1(\mathcal{E}') + c_1(\mathcal{E}'')$ if

$$0 \to \mathcal{E}' \to \mathcal{E} \to \mathcal{E}'' \to 0$$

is an exact sequence of vector bundles. These definitions are identical to the classical case.

We now recall the construction of $c_2(\mathcal{E})$. We first recall that for integers $0 \leq n \leq r$, the Grassmanian variety $\mathrm{Gr}_k(n, r)$ is a representable functor on $\mathbf{Sch}_k$ which to any $Y \in \mathbf{Sch}_k$ associates the set of quotients $\mathcal{O}_Y^r \twoheadrightarrow \mathcal{F}$, where $\mathcal{F}$ is locally free of rank n on Y. Note that this construction holds over any base scheme S. It is classically known that $\mathrm{GL}_{r,k}$ acts transitively on $\mathrm{Gr}_k(n, r)$ and the latter is projective. Hence, we have that $\mathrm{Gr}_k(n, r) \cong \mathrm{GL}_{r,k}/P$ with P parabolic.

Let us now assume that $\mathcal{E}$ is globally generated by the sections $s_1, \ldots, s_r \in H^0(X, \mathcal{E})$. Then it follows from the above definition of $\mathrm{Gr}_k(n, r)$ that there is a unique k-morphism $f : X \to \mathrm{Gr}_k(n, r)$ such that $\mathcal{E} \cong f^*(\mathcal{Q}_{n,r})$, where $\mathcal{Q}_{n,r}$ is the universal quotient vector bundle on $\mathrm{Gr}_k(n, r)$. We let $c_2(\mathcal{E}) := f^*(c_2(\mathcal{Q}_{n,r}))$. Note that this makes sense since we have a

well-defined theory of Chern classes of vector bundles on smooth schemes. The fact that this definition does not depend on the choice of the chosen sections follows from the following.

Lemma 7.2. *Let $\{t_1, \ldots, t_r\}$ and $\{g_1, \ldots, g_s\}$ be two sets of global sections generating $\mathcal{E}$. Let $f_1 \colon X \to \mathrm{Gr}_k(n, r)$ and $f_2 \colon X \to \mathrm{Gr}_k(n, s)$ be the classifying morphisms. Then $f_1^*(c_2(\mathcal{Q}_{n,r})) = f_2^*(c_2(\mathcal{Q}_{n,s}))$ in $\mathrm{CH}_0^L(X)$.*

Proof. This is straightforward using Lemmas 6.8 and 6.10 (e.g., see [BS98, Lemma 10]). $\square$

Lemma 7.3. *Suppose that $\mathcal{E}$ is globally generated by $\{t_1, \ldots, t_r\}$ and let $f \colon X \to \mathrm{Gr}_k(n, r)$ be the associated classifying morphism. Let $\mathcal{L}$ be a line bundle on X globally generated by $\{g_1, \ldots, g_s\}$. Let $h \colon X \to \mathrm{Gr}_k(n, rs)$ be the classifying morphism for the vector bundle $\mathcal{E} \otimes_{\mathcal{O}_X} \mathcal{L}$ and sections $\{t_i \otimes g_j \mid 1 \le i \le s, 1 \le j \le s\}$. Then*

$$(7.2) \qquad h^*(c_2(\mathcal{Q}_{n,rs})) = f^*(c_2(\mathcal{Q}_{n,r})) + (n-1)c_1(\mathcal{E}) \cdot c_1(\mathcal{L}) + \binom{n}{2} c_1(\mathcal{L})^2.$$

Proof. This is straightforward using Lemma 6.8 and the theory of Chern classes on smooth schemes (e.g., see [BS98, Lemma 11]). $\square$

Let $\mathcal{E}$ be a rank n vector bundle on X. Let $\mathcal{L}$ be a globally generated line bundle on X such that $\mathcal{E} \otimes_{\mathcal{O}_X} \mathcal{L}$ is generated by global sections $t_1, \ldots, t_r$. Let $f \colon X \to \mathrm{Gr}_k(n, r)$ be the classifying morphism given by the t_i's. In view of Lemmas 7.2 and 7.3, the following definition is meaningful.

Definition 7.4. We let

$$c_2(\mathcal{E}) = f^*(c_2(\mathcal{Q}_{n,r})) - (n-1)c_1(\mathcal{E}) \cdot c_1(\mathcal{L}) - \binom{n}{2} c_1(\mathcal{L})^2.$$

We define the total Chern class of $\mathcal{E}$ to be

$$c(\mathcal{E}) = 1 + c_1(\mathcal{E}) + c_2(\mathcal{E}).$$

Note that the above definitions coincide with the classical definitions of Chern classes if X happened to be non-singular.

Lemma 7.5. *The following hold for vector bundles on X and their Chern classes.*

(1) *Suppose $H = G/P$ is as in § 5.1 and $f \colon X \to H$ is a morphism. If $\mathcal{E}$ is a G-equivariant vector bundle on H, then $f^*(c(\mathcal{E})) = c(f^*(\mathcal{E}))$.*

(2) *If $\mathcal{E}$ is a vector bundle on X, then there exists $H = G/P$ as in § 5.1 and a G-equivariant vector bundle $\mathcal{E}'$ on H such that $\mathcal{E} \cong f^*(\mathcal{E}')$.*

(3) *If*

$$0 \to \mathcal{E}' \to \mathcal{E} \to \mathcal{E}'' \to 0$$

is an exact sequence of vector bundles on X, then $c(\mathcal{E}) = c(\mathcal{E}') \cdot c(\mathcal{E}'')$.

(4) *If $\pi \colon \widetilde{X} \to X$ is a resolution of singularities of X, then $c(\pi^*(\mathcal{E})) = \pi^*(c(\mathcal{E}))$.*

Proof. We first prove (1). It suffices to prove the lemma for $c_2(\mathcal{E})$ as it is obvious for other Chern classes by their definitions. We now note that since G and H are smooth and connected, it follows from [Tho87, Lemma 2.10] that H admits G-equivariant very ample line bundles. Hence, we can find a very ample G-equivariant line bundle $\mathcal{L}$ generated by its global sections such that $\mathcal{E}' := \mathcal{E} \otimes_{\mathcal{O}_H} \mathcal{L}$ is also generated by its global sections.

Let $\{s_1, \ldots, s_r\}$ be a k-basis of $V := H^0(H, \mathcal{E}')$. Then V becomes a rational representation of G, giving a group homomorphism $\rho \colon G \to \mathrm{GL}_k(V)$. Furthermore, the classifying morphism $h \colon H \to \mathrm{Gr}_k(n, V)$ given by the above basis of V is G-equivariant. It follows that $h \circ f \colon X \to \mathrm{Gr}_k(n, V)$ is the classifying morphism for the vector bundle $f^*(\mathcal{E}')$ with sections

$\{f^*(s_1), \ldots, f^*(s_r)\}$. The item (1) of the lemma is clear for $h \circ f$ by the construction of the Chern classes on X. Since $\mathcal{E}' = h^*(\mathcal{Q}_{n,r})$, it follows from the theory of Chern classes on smooth schemes that $c_2(\mathcal{E}') = h^*(c_2(\mathcal{Q}_{n,r}))$. We now conclude the proof of (1) by using Lemmas 6.8, 7.2 and 7.3.

For (2), we choose a globally generated line bundle $\mathcal{L}$ such that $\mathcal{E} \otimes_{\mathcal{O}_X} \mathcal{L}$ is also globally generated. This gives rise to the classifying morphisms $f_1 \colon X \to \mathrm{Gr}_k(n, r)$ and $f_2 \colon X \to \mathbb{P}_k^s$ for some $r, s \geq 1$, where $n = \mathrm{rank}(\mathcal{E})$. We thus get a map $f \colon X \to \mathrm{Gr}_k(n, r) \times \mathbb{P}_k^s$, and it is clear from various universal properties that $\mathcal{E} \cong f^*(\mathcal{Q}_{n,r} \boxtimes \mathcal{O}_{\mathbb{P}_k^s}(-1))$.

The item (3) is obtained exactly as (2). Let $\mathrm{Fl}_k(n, n'', r)$ be the (partial) flag variety of quotients $k^r \twoheadrightarrow V \twoheadrightarrow W$, where V and W are k-vector spaces of ranks $n = \mathrm{rank}(\mathcal{E})$ and $n'' = \mathrm{rank}(\mathcal{E}'')$, respectively. Then $\mathrm{Fl}_k(n, n'', r)$ is the projective homogeneous space for $\mathrm{GL}_{r,k}$. This is a closed subvariety of $\mathrm{Gr}_k(n, r) \times \mathrm{Gr}_k(n'', r)$ and the projections to these Grassmanian varieties define a universal exact sequence of $\mathrm{GL}_{r,k}$-equivariant vector bundles

$$0 \to \mathcal{S} \to p_1^*(\mathcal{Q}_{n,r}) \to p_2^*(\mathcal{Q}_{n'',r}) \to 0.$$

We now choose a globally generated line bundle $\mathcal{L}$ such that $\mathcal{E} \otimes_{\mathcal{O}_X} \mathcal{L}$ is also globally generated. Then $\mathcal{E}'' \otimes \mathcal{L}$ is globally generated too. As in (2), this gives rise to the classifying morphism $f \colon \mathrm{Fl}_k(n, n'', r) \times \mathbb{P}_k^s$ such that

$$0 \to \mathcal{S} \boxtimes \mathcal{O}_{\mathbb{P}_k^s}(-1) \to p_1^*(\mathcal{Q}_{n,r}) \boxtimes \mathcal{O}_{\mathbb{P}_k^s}(-1) \to p_2^*(\mathcal{Q}_{n'',r}) \boxtimes \mathcal{O}_{\mathbb{P}_k^s}(-1) \to 0$$

pulls back to the exact sequence of (3) on X via f. We are now done by (1) and Proposition 7.1 because (3) is well-known for exact sequences of vector bundles on smooth schemes.

The item (4) is again clear for c_0 and c_1. For c_2, we let $f \colon X \to \mathrm{Gr}_k(n, r)$ be the morphism obtained just above Definition 7.4. We let $\widetilde{f} \colon \widetilde{X} \xrightarrow{\pi} X \to \mathrm{Gr}_k(n, r)$ be the composite map. We then get

$$\begin{aligned} c_2 \circ \pi^*(\mathcal{E}) = c_2 \circ \pi^* \circ f^*(\mathcal{Q}_{n,r}) \quad &= \quad c_2 \circ \widetilde{f}^*(\mathcal{Q}_{n,r}) \;=^1\; \widetilde{f}^* \circ c_2(\mathcal{Q}_{n,r}) \\ &=^2 \quad \pi^* \circ f^* \circ c_2(\mathcal{Q}_{n,r}) \;=^3\; \pi^* \circ c_2 \circ f^*(\mathcal{Q}_{n,r}) \\ &= \quad \pi^* \circ c_2(\mathcal{E}). \end{aligned}$$

In the above, the equality $=^1$ follows from the known functoriality of Chern classes on smooth schemes, $=^2$ follows from Lemma 6.9 and $=^3$ follows from the item (1) of the lemma. This finishes the proof of (4). $\qquad\square$

7.4. The Chern classes on $K_0(X)$.

Let X be as in the beginning of § 7. We let $\mathrm{CH}^*(X)^\times$ denote the multiplicative group of units in $\mathrm{CH}^*(X)$ whose codimension zero part is equal to 1 (see [Ful75, § 15.3]). An immediate consequence of Lemma 7.5 (3) is that the total Chern class defines a group homomorphism

$$(7.3) \qquad\qquad c_X \colon K_0(X) \to \mathrm{CH}^*(X)^\times,$$

where the left hand side uses the additive group structure of the ring $K_0(X)$.

Let $\widetilde{K}_0(X)$ be the kernel of the the rank map $\mathrm{rk} \colon K_0(X) \twoheadrightarrow \mathbb{Z}$. There is a canonical map $\det \colon \widetilde{K}_0(X) \to \mathrm{Pic}(X)$, induced by taking a vector bundle to its determinant. This map is split by the natural map $\mathrm{Pic}(X) \to \widetilde{K}_0(X)$ that sends a line bundle $\mathcal{L}$ to $[\mathcal{L}] - [\mathcal{O}_X]$. Let $SK_0(X)$ denote the kernel of $\det$. It follows that there is a natural decomposition

$$(7.4) \qquad\qquad K_0(X) = \mathbb{Z} \oplus \mathrm{Pic}(X) \oplus SK_0(X).$$

We let $x \in X_{\mathrm{reg}}$ be a closed point and let $U = X \smallsetminus \{x\}$. Let $j \colon U \hookrightarrow X$ be the inclusion and let $j^* \colon K_0(X) \to K_0(U)$ be the induced map. It is then clear that the rank map of $K_0(X)$ factors through $K_0(U)$. Furthermore, as x is a regular closed point of X and $\dim(X) \geq 2$, one knows that the map $\mathrm{Pic}(X) \to \mathrm{Pic}(U)$ is an isomorphism. Since $j^*([\mathcal{O}_{\{x\}}]) = 0$, it follows

that the first and the second components of $[\mathcal{O}_{\{x\}}] \in K_0(X)$ under the decomposition (7.4) are zero. We conclude that the cycle class map of (3.3) canonically factors through

$$(7.5) \qquad\qquad cyc_X^L \colon \mathrm{CH}_0^L(X) \to SK_0(X).$$

We shall denote the image of the cycle class map cyc_X^L by $F^2 K_0(X)$. It also follows from the above discussion and the definitions of Chern classes that $c_i(cyc_X^L([x])) = 0$ for $i = 0, 1$. In particular, we get $c_X \circ cyc_X^L([x]) = c_2 \circ cyc_X^L([x])$. This also implies that $c_2 \colon F^2 K_0(X) \to \mathrm{CH}_0^L(X)$ is a group homomorphism.

Lemma 7.6. *The composite map* $\mathrm{CH}_0^L(X) \xrightarrow{cyc_X^L} F^2 K_0(X) \xrightarrow{-c_2} \mathrm{CH}_0^L(X)$ *is identity.*

Proof. This is proven in [Lev85b] and [BS98, Proposition 2]. We reproduce the latter proof for sake of completeness. It is enough to check that $c_2 \circ cyc_X^L([x]) + [x] = 0$ if $x \in X_{\mathrm{reg}}$ is a closed point. Fix such a point and let $j \colon U = X \smallsetminus \{x\} \hookrightarrow X$ be as above. Let $j^* \colon \mathrm{CH}_0^L(X) \to \mathrm{CH}_0^L(U)$ be the restriction map of Proposition 3.8. Then it is clear from the construction of the Chern classes that $j^* \circ c_X = c_U \circ j^*$. It follows therefore from the above argument that $j^* \circ c_2 \circ cyc_X^L([x]) = 0$. We conclude from Proposition 3.8 that $c_2 \circ cyc_X^L([x]) = m[x]$ for some $m \in \mathbb{Z}$. It remains to show that $m = -1$.

We let $\pi \colon \widetilde{X} \to X$ denote a resolution of singularities of X. This exists over all base fields since $\dim(X) = 2$. Since $x \in X_{\mathrm{reg}} \subset \widetilde{X}$, we see that $\pi^*([x]) = [x]$ under the pull-back map $\pi^* \colon \mathrm{CH}_0^L(X) \to \mathrm{CH}^2(\widetilde{X})$. Lemma 7.5 (4) easily implies that $\pi^* \circ c_2 \circ cyc_X^L([x]) = c_2 \circ \pi^* \circ cyc_X^L([x])$. This yields

$$
\begin{aligned}
c_2 \circ cyc_{\widetilde{X}}([x]) &= c_2 \circ cyc_{\widetilde{X}} \circ \pi^*([x]) =^{\dagger} c_2 \circ \pi^* \circ cyc_X^L([x]) \\
&= \pi^* \circ c_2 \circ cyc_X^L([x]) = \pi^*(m[x]) \\
&= m\pi^*([x]) = m[x],
\end{aligned}
$$

where $=^{\dagger}$ follows from Proposition 3.9. We thus get $c_2 \circ cyc_{\widetilde{X}}([x]) = m[x]$ on $\widetilde{X}$. Since the lemma is well known for non-singular surfaces, this forces $m = -1$. $\qquad\square$

The above lemma leads us to the following final result of this section which generalizes Levine's result [Lev85a] to all infinite fields.

Theorem 7.7. *Let X be an equidimensional reduced quasi-projective surface over an infinite field. Then the cycle class map*

$$cyc_X^L \colon \mathrm{CH}_0^L(X) \to F^2 K_0(X)$$

is an isomorphism.

Proof. The map cyc_X^L is surjective by definition and is injective by Lemma 7.6. $\qquad\square$

Combining Theorem 7.7, Corollary 3.1 and Lemmas 3.2, 3.4 and 3.5, we obtain the following.

Corollary 7.8. *Let X be an equidimensional reduced quasi-projective surface over an infinite field. Then the cycle class map induces the isomorphisms*

$$\mathrm{CH}_0^L(X) \xrightarrow{\cong} H^2_{\mathrm{zar}}(X, \mathcal{K}^M_{2,X}) \xrightarrow{\cong} H^2_{\mathrm{zar}}(X, \mathcal{K}_{2,X}).$$

8. The main results

In this section, we shall prove the main results of this paper. We shall first prove the Bloch-Kato formula for the lci Chow group of singular surfaces and then use it prove Theorems 1.1 and 1.2.

8.1. **The Bloch-Kato formula for singular surface.** Let k be any field. Let X be an equidimensional reduced quasi-projective surface over k. In § 3.3, we constructed the maps

$$(8.1) \qquad \mathrm{CH}_0(X) \xrightarrow{\rho_X} H^2_{\mathrm{zar}}(X, \mathcal{K}^M_{2,X}) \xrightarrow{\lambda_X} H^2_{\mathrm{nis}}(X, \mathcal{K}^M_{2,X}) \xrightarrow{\gamma_X} K_0(X).$$

The following result extends Theorem 7.7 and Corollary 7.8 to all fields if we use the lci Chow group.

Theorem 8.1. *The cycle class map*

$$cyc_X \colon \mathrm{CH}_0(X) \to F^2 K_0(X)$$

is an isomorphism. In particular, the map

$$\rho_X \colon \mathrm{CH}_0(X) \to H^2_{\mathrm{zar}}(X, \mathcal{K}^M_{2,X})$$

is an isomorphism. If X_{reg} is smooth over k (e.g., if k is perfect), then $\lambda_X \circ \rho_X$ is also an isomorphism.

Proof. In view of Theorem 3.6, we only need to show that cyc_X is injective to prove the theorem. If k is finite, we can use Proposition 2.7, Lemma 3.10 and the standard pro-ℓ extension trick to reduce our problem to the case of infinite fields. In the latter case, the desired injectivity follows directly from Corollary 3.1 and Theorem 7.7. $\square$

8.2. **The Bloch-Kato formula for 0-cycles with modulus.** We again assume k to be an arbitrary field. Let X be a smooth quasi-projective surface over k and let $D \subset X$ be an effective Cartier divisor. Let $cyc_{X|D}$ and $\rho_{X|D}$ be the cycle class and the Bloch-Kato maps constructed in § 3.1 and § 3.4. We let $F^2 K_0(X, D)$ denote the image of the map $cyc_{X|D}$. Consider the following diagram (see § 3.3 and § 3.4).

$$(8.2) \qquad \begin{array}{ccc} \mathrm{CH}_0(X|D) \xrightarrow{\rho_{X|D}} H^2_{\mathrm{zar}}(X, \mathcal{K}^M_{2,(X,D)}) \xrightarrow{\lambda_{(X,D)}} H^2_{\mathrm{nis}}(X, \mathcal{K}^M_{2,(X,D)}) \\ \downarrow \qquad\qquad\qquad \downarrow \qquad\qquad \searrow^{\gamma_{(X,d)}} \\ H^2_{\mathrm{zar}}(X, \mathcal{K}_{2,(X,D)}) \xrightarrow{\lambda_{(X,D)}} H^2_{\mathrm{nis}}(X, \mathcal{K}_{2,(X,D)}) \xrightarrow{\gamma_{(X,d)}} F^2 K_0(X, D), \end{array}$$

where the vertical arrows are the canonical maps from the cohomologies of Milnor to Quillen K-theory sheaves. The following result proves Theorem 1.1.

Theorem 8.2. *All of the maps in* (8.2) *are isomorphisms.*

Proof. In view of Lemma 3.2 and Theorem 3.7, the proof of the theorem reduces to showing that $cyc_{X|D}$ is injective. Using Proposition 2.9, Corollary 3.11 and the standard pro-ℓ extension trick, we can reduce our problem to showing the injectivity of $cyc_{X|D}$ when k is infinite.

We now assume k is infinite and consider the diagram

$$(8.3) \qquad \begin{array}{ccccccccc} 0 & \longrightarrow & \mathrm{CH}_0(X|D) & \xrightarrow{p_{+*}} & \mathrm{CH}_0(S_X) & \xrightarrow{\iota^*_-} & \mathrm{CH}_0(X) & \longrightarrow & 0 \\ & & \downarrow^{cyc_{(S_X,X_-)}} & & \downarrow^{cyc_{S_X}} & & \downarrow^{cyc_X} & & \\ 0 & \longrightarrow & K_0(S_X, X_-) & \xrightarrow{p_{+*}} & K_0(S_X) & \xrightarrow{\iota^*_-} & K_0(X) & \longrightarrow & 0 \\ & & \downarrow^{\iota^*_+} & & \downarrow^{\iota^*_+} & & \downarrow^{\iota^*_D} & & \\ & & K_0(X, D) & \xrightarrow{p_{+*}} & K_0(X) & \xrightarrow{\iota^*_D} & K_0(D). & & \end{array}$$

The top row is exact by Theorem 2.11. The middle row is split exact and the bottom row is exact. It follows by applying Theorem 8.1 to S_X that cyc_{S_X} is injective. It follows that $cyc_{(S_X, X_-)}$ is also injective. The bottom left vertical arrow is an isomorphism by [BK18, Proposition 13.2] because k is infinite. Since $cyc_{X|D} = \iota_+^* \circ cyc_{(S_X, X_-)}$ by (3.15), we conclude that $cyc_{X|D}$ is injective. $\qquad\square$

For a connected projective variety X over a field and an effective Cartier divisor $D \subset X$, let $\mathrm{CH}_0(X|D)^0$ denote the kernel of the degree map $\deg: \mathrm{CH}_0(X|D) \to \mathbb{Z}$. Theorem 8.2 has the following important consequence on the finiteness of $\mathrm{CH}_0(X|D)^0$. We shall generalize this to higher dimensions in Corollary 8.7.

Corollary 8.3. *Let X be a smooth and connected projective surface over a finite field and let $D \subset X$ be an effective Cartier divisor. Then $\mathrm{CH}_0(X|D)^0$ is a finite abelian group.*

Proof. Let $H^2_{\mathrm{nis}}(X, \mathcal{K}^M_{2,(X,D)})^0$ be the image of $\mathrm{CH}_0(X|D)^0$ under the Bloch-Kato cycle class map $\rho^{\mathrm{nis}}_{X|D} = \lambda_{(X,D)} \circ \rho_{X|D}: \mathrm{CH}_0(X|D) \to H^2_{\mathrm{nis}}(X, \mathcal{K}^M_{2,(X,D)})$. The group $H^2_{\mathrm{nis}}(X, \mathcal{K}^M_{2,(X,D)})^0$ is finite by the Kato-Saito class field theory (see [KS86, Theorem 9.1] or [GK20c, Theorem 12.8]). The corollary now follows by Theorem 8.2. $\qquad\square$

8.3. **The theorem of Kerz-Saito.** Let k be a finite field of characteristic $p > 0$. Let U be a smooth and connected quasi-projective scheme over k of dimension $d \geq 1$. Choose a compactification $U \subset X$ with X normal and proper over k such that $C = (X \setminus U)_{\mathrm{red}}$ is the support of an effective Cartier divisor on X. We let

$$(8.4) \qquad\qquad C(U) = \varprojlim_{D} \mathrm{CH}_0(X|D),$$

where the limit is taken over all effective Cartier divisors on X supported on C. We endow each $\mathrm{CH}_0(X|D)$ with the discrete topology and $C(U)$ with the inverse limit topology. It is known (e.g., see [KS16, Lemma 3.1]) that $C(U)$ is independent of the choice of the compactification X. We have the degree map $\deg: C(U) \to \mathbb{Z}$ which takes any 0-cycle to its degree. We let $C(U)^0 = \mathrm{Ker}(\deg)$.

Let $\pi_1^{\mathrm{ab}}(U)$ denote the abelianized étale fundamental group of U. We let $\pi_1^{\mathrm{ab}}(U)^0 = \ker(\pi_1^{\mathrm{ab}}(U) \to \mathrm{Gal}(\overline{k}/k))$. If $x \in U$ is a closed point, then the inclusion $\iota_x: \mathrm{Spec}\,(k(x)) \hookrightarrow U$ defines the natural map $(\iota_x)_*: \pi_1(\mathrm{Spec}\,(k(x))) \to \pi_1^{\mathrm{ab}}(U)$. Letting $\rho_U(x) = (\iota_x)_*(F_x)$ (with F_x being the Frobenius element of the Galois group of $k(x)$) and extending linearly, we get a group homomorphism $\rho_U: \mathcal{Z}_0(U) \to \pi_1^{\mathrm{ab}}(U)$. It follows from [KS16, Proposition 3.2] that this map induces a reciprocity map $\rho_U: C(U) \to \pi_1^{\mathrm{ab}}(U)$.

If $D \subset X$ is an effective Cartier divisor supported on C, we let

$$(8.5) \qquad\qquad \pi_1^{\mathrm{ab}}(X, D) = \mathrm{Hom}_{\mathrm{cont}}(\mathrm{fil}_D H^1(U, \mathbb{Q}/\mathbb{Z}), \mathbb{Q}/\mathbb{Z}),$$

where $\mathrm{fil}_D H^1(U, \mathbb{Q}/\mathbb{Z})$ is the group of continuous characters $\chi: \pi_1^{\mathrm{ab}}(U) \to \mathbb{Q}/\mathbb{Z}$ such that for any integral curve $Z \subset U$, the restriction $\chi|_Z: \pi_1^{\mathrm{ab}}(Z) \to \mathbb{Q}/\mathbb{Z}$ satisfies the following inequality of Cartier divisors on $\overline{Z}^N$:

$$\sum_{y \in \psi_Z^{-1}(C)} \mathrm{art}_y(\chi|_Z)[y] \leq \psi_Z^*(D).$$

Here, $\psi_Z: \overline{Z}^N \to X$ is the projection map from the normalization of the closure of Z in X and $\mathrm{art}_y(\chi|_Z)$ is the Artin conductor of the restriction of χ to $\mathrm{Gal}(k(Z)_y)$, where $k(Z)_y$ is the completion of $k(Z)$ at y (see [Ser68]).

It is easy to check that there is an exact sequence

$$(8.6) \qquad \bigoplus_{Z \subset U} \Big(\bigoplus_{y \in |\psi_Z^{-1}(D)|} G_{k(Z)_y}^{n_y} \Big) \to \pi_1^{\mathrm{ab}}(U) \to \pi_1^{\mathrm{ab}}(X, D) \to 0,$$

where the sum on the left runs over all integral curves $Z \subset U$. Here, $\psi_Z^*(D) = \sum_{y \in |\psi_Z^{-1}(D)|} n_y[y]$ and $G_{k(Z)_y}^{n_y}$ is the higher ramification subgroup (for the upper numbering) of $\mathrm{Gal}(k(Z)_y)$. The first map in (8.6) is the composite

$$G_{k(Z)_y}^{n_y} \to \mathrm{Gal}(k(Z^N)) \to \pi_1(Z^N) \xrightarrow{(\psi_Z)_*} \pi_1^{\mathrm{ab}}(U).$$

It follows from (8.5) that $\pi_1^{\mathrm{ab}}(X, D)$ is the unique quotient of $\pi_1^{\mathrm{ab}}(U)$ which classifies all finite abelian Galois covers of U whose ramification away from U is bounded by the divisor D. It follows [EK12, Proposition 3.9] (see also [KS16, Proposition 2.10]) that the limit of these quotient maps induces an isomorphism

$$(8.7) \qquad \pi_1^{\mathrm{ab}}(U) \xrightarrow{\cong} \varprojlim_D \pi_1^{\mathrm{ab}}(X, D).$$

We let $\rho_{(X,D)}$ denote the composition $\mathcal{Z}_0(U) \xrightarrow{\rho_U} \pi_1^{\mathrm{ab}}(U) \twoheadrightarrow \pi_1^{\mathrm{ab}}(X, D)$. It follows from the classical ramified class field theory for curves that this descends to a map $\rho_{(X,D)} \colon \mathrm{CH}_0(X|D) \to \pi_1^{\mathrm{ab}}(X, D)$, compatible with ρ_U. Furthermore, $\rho_U = \varprojlim_D \rho_{(X,D)}$.

We let $\pi_1^{\mathrm{ab}}(X, D)^0$ denote the kernel of the map $\pi_1^{\mathrm{ab}}(X, D) \to \mathrm{Gal}(\overline{k}/k)) \cong \widehat{\mathbb{Z}}$. It follows from [KS14, Corollary 1.2] that $\pi_1^{\mathrm{ab}}(X, D)^0$ is a finite abelian group. The reciprocity map $\rho_{(X,D)}$ restricts to a continuous homomorphism of discrete abelian groups $\rho_{(X,D)}^0 \colon \mathrm{CH}_0(X|D)^0 \to \pi_1^{\mathrm{ab}}(X, D)^0$. Moreover, $\rho_U^0 = \varprojlim_D \rho_{(X,D)}^0$.

Using (8.7) and the finiteness of $\pi_1^{\mathrm{ab}}(X, D)^0$, we get a commutative diagram of exact sequences of topological abelian groups and continuous group homomorphisms

$$(8.8) \qquad \begin{array}{ccccc} 0 \longrightarrow & C(U)^0 & \longrightarrow & C(U) & \xrightarrow{\deg} \mathbb{Z} \\ & {\scriptstyle \rho_U^0}\Big\downarrow & & {\scriptstyle \rho_U}\Big\downarrow & \Big\downarrow \\ 0 \longrightarrow & \pi_1^{\mathrm{ab}}(U)^0 & \longrightarrow & \pi_1^{\mathrm{ab}}(U) & \longrightarrow \widehat{\mathbb{Z}}, \end{array}$$

where the groups on the bottom have their canonical pro-finite topology and the right-most vertical arrow is the pro-finite completion morphism. The horizontal arrows on the right are surjective if U is geometrically connected over k.

An easy consequence of (8.4), (8.6) and (8.7) is the following (see [KS16, Corollary 3.4]).

Lemma 8.4. *The map ρ_U^0 is an isomorphism of topological abelian groups if and only if $\rho_{(X,D)}^0 \colon \mathrm{CH}_0(X|D)^0 \to \pi_1^{\mathrm{ab}}(X, D)^0$ is an isomorphism of finite abelian groups for all effective Cartier divisors $D \subset X$ which are supported on C.*

The following result is due to Kerz and Saito [KS16] when $p \neq 2$. We shall prove this using Theorem 8.2 and the Kato-Saito class field theory.

Theorem 8.5. *ρ_U^0 is an isomorphism of topological abelian groups.*

Proof. By using Wiesend's trick (see [KS16, Lemma 3.6]), we can replace the chosen compactification X by any of its alterations in the sense of de Jong. We can therefore assume that X is smooth projective and $C = X \smallsetminus U$ is a simple normal crossing divisor on X.

Suppose now that $d \geq 3$. By Lemma 8.4, it suffices to show that $\rho^0_{(X,D)}$ is an isomorphism for all D. Since $\pi^{\mathrm{ab}}_1(X,D)^0$ is finite by [KS14, Corollary 1.2], it follows immediately from the Chebotarev-Lang density theorem (e.g., see [Ser68, Theorem 7] or [Sza09, Theorem 5.8.16]) that $\rho^0_{(X,D)}$ is surjective. The heart of the proof therefore is to show that $\rho_{(X,D)} \colon \mathrm{CH}_0(X|D) \to \pi^{\mathrm{ab}}_1(X,D)$ is injective.

Let $\alpha \in \mathrm{CH}_0(X|D)$ be a 0-cycle such that $\rho_{(X,D)}(\alpha) = 0$. By a generalized version of Poonen's Bertini theorem over finite fields (see [GK20a, Corollary 5.4]), we can find a very ample line bundle $\mathcal{L}$ on X and a section $s \in H^0(X,\mathcal{L})$ such that its zero locus $Y = Z(s) \subset X$ is smooth, $Y \times_X C$ is a simple normal crossing divisor on Y and $|\alpha| \subset Y$. Let $\iota \colon Y \hookrightarrow X$ be the inclusion and let $E = Y \times_X D$. By the choice of Y, there exists $\alpha' \in \mathrm{CH}_0(Y|E)$ such that $\alpha = \iota_*(\alpha')$.

We now have the diagram:

$$(8.9) \qquad \begin{array}{ccc} \mathrm{CH}_0(Y|E) & \xrightarrow{\rho_{(Y,E)}} & \pi^{\mathrm{ab}}_1(Y,E) \\ \iota_* \downarrow & & \downarrow \iota_* \\ \mathrm{CH}_0(X|D) & \xrightarrow{\rho_{(X,D)}} & \pi^{\mathrm{ab}}_1(X,D), \end{array}$$

whose commutativity is immediate from the definition of the reciprocity maps.

By [KS14, Theorem 1.1], we can choose $\mathcal{L}$ ample enough (depending on D) so that the right vertical arrow in (8.9) is an isomorphism. It follows that $\rho_{(Y,E)}(\alpha') = 0$. We have therefore inductively reduced the proof of the theorem to the case when $d \leq 2$.

We shall now show that ρ^0_U is an isomorphism when $d \leq 2$. Since we have already seen that the map $\rho^0_{(X,D)} \colon \mathrm{CH}_0(X|D)^0 \to \pi^{\mathrm{ab}}_1(X,D)^0$ is surjective for all D supported on C and since $\mathrm{CH}_0(X|D)^0$ is finite by Corollary 8.3 (note that the $d = 1$ case of this finiteness is classical), it follows that the map $\rho^0_U \colon C(U)^0 \to \pi^{\mathrm{ab}}_1(U)^0$ is surjective. It suffices therefore to show that the map $\rho_U \colon C(U) \to \pi^{\mathrm{ab}}_1(U)$ is injective.

Let $\rho^{\mathrm{nis}}_{X|D} = \lambda_{(X,D)} \circ \rho_{X|D} \colon \mathrm{CH}_0(X|D) \to H^d_{\mathrm{nis}}(X, \mathcal{K}^M_{d,(X,D)})$ be the cycle class map from (8.2) for any D. Let $H^d_{\mathrm{nis}}(X, \mathcal{K}^M_{d,(X,D)})^0$ be the image of $\mathrm{CH}_0(X|D)^0$ under this map (see the proof of Corollary 8.3). It is clear from the definition of $\rho^{\mathrm{nis}}_{X|D}$ (see § 3.4) that it is compatible with the inclusions of effective Cartier divisors $D \subset D'$. We let $\widetilde{\rho}_U := \varprojlim_D \rho^{\mathrm{nis}}_{X|D}$.

We now consider our key diagram

$$(8.10) \qquad \begin{array}{ccc} C(U) & \xrightarrow{\widetilde{\rho}_U} & \varprojlim_D H^d_{\mathrm{nis}}(X, \mathcal{K}^M_{d,(X,D)}) \\ & \rho_U \searrow & \downarrow \widehat{\rho}_U \\ & & \pi^{\mathrm{ab}}_1(U), \end{array}$$

where $\widehat{\rho}_U$ is the reciprocity map of Kato-Saito [KS86, § 3]. The map $\widetilde{\rho}_U$ is the inverse limit of the cycle class maps $\rho^{\mathrm{nis}}_{X|D}$ from (8.2), taken over effective Cartier divisors $D \subset X$ supported on C. It is immediate from the construction of the three maps in (8.10) and by [KS86, Proposition 3.8] that this diagram is commutative.

Since $H^d_{\mathrm{nis}}(X, \mathcal{K}^M_{d,(X,D)})^0$ is finite for all effective Cartier divisors $D \subset X$ supported on C, it follows that the canonical map $\varprojlim_D H^d_{\mathrm{nis}}(X, \mathcal{K}^M_{d,(X,D)}) \to \varprojlim_{D,m} H^d_{\mathrm{nis}}(X, \mathcal{K}^M_{d,(X,D)}) \otimes \mathbb{Z}/m$ is

injective. On the other hand, the reciprocity map $\widehat{\rho}_U$ has a factorization

$$\varprojlim_{D} H^d_{\mathrm{nis}}(X, \mathcal{K}^M_{d,(X,D)}) \hookrightarrow \varprojlim_{D,m} H^d_{\mathrm{nis}}(X, \mathcal{K}^M_{d,(X,D)}) \otimes \mathbb{Z}/m \xrightarrow{\widehat{\rho}_U} \pi_1^{\mathrm{ab}}(U).$$

The latter arrow is an isomorphism by [KS86, Theorem 9.1 (3)]. It follows that the vertical arrow on the right in (8.10) is injective. Since the horizontal arrow on the top in (8.10) is an isomorphism by Theorem 8.2 (see [Kri15, Lemma 3.1] when $d = 1$) and a limit argument, we conclude that ρ_U is injective. This finishes the proof. $\qquad\square$

Corollary 8.6. *The reciprocity maps* $\{\rho_{(X,D)}\}_{|D|\subset C}$ *induce an isomorphism of pro-finite topological groups*

$$\rho_U: \varprojlim_{D,m} \mathrm{CH}_0(X|D) \otimes \mathbb{Z}/m \xrightarrow{\cong} \pi_1^{\mathrm{ab}}(U).$$

The following result provides an independent and a K-theoretic proof of a finiteness result of Deligne (see [EK12, Theorem 8.1]). This was also obtained independently by Kerz-Saito [KS16] in characteristic $\neq 2$.

Corollary 8.7. *Let X be a normal and connected projective variety over a finite field. Let $D \subset X$ be an effective Cartier divisor such that $X \smallsetminus D$ is regular. Then $\mathrm{CH}_0(X|D)^0$ is a finite abelian group.*

Proof. Combine Lemma 8.4, Theorem 8.5 and [KS14, Corollary 1.2]. $\qquad\square$

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