



ϵ -expansion of multivariable hypergeometric functions appearing in Feynman integral calculus

Souvik Bera

Centre for High Energy Physics, Indian Institute of Science, Bangalore-560012, Karnataka, India

Received 14 November 2022; received in revised form 16 February 2023; accepted 7 March 2023

Available online 13 March 2023

Editor: Stephan Stieberger

Abstract

We present a new methodology, suitable for implementation on computer, to perform the ϵ -expansion of hypergeometric functions with linear ϵ dependent Pochhammer parameters in any number of variables. Our approach allows one to perform Taylor as well as Laurent series expansion of multivariable hypergeometric functions. Each of the coefficients of ϵ in the series expansion is expressed as a linear combination of multivariable hypergeometric functions with the same domain of convergence as that of the original hypergeometric function. We present illustrative examples of hypergeometric functions in one, two and three variables which are typical of Feynman integral calculus.

© 2023 The Author(s). Published by Elsevier B.V. This is an open access article under the CC BY license (<http://creativecommons.org/licenses/by/4.0/>). Funded by SCOAP³.

1. Introduction

In the present work we are concerned with the solution to the problem of finding series expansion in the dimensional regularization parameter ϵ of Feynman integrals originating in the perturbative calculations of quantum field theory. These integrals can be expressed as multivariable hypergeometric functions (MHFs) [1–4]. To give a few examples, two loop massive sunset diagram [5,6], three loop vacuum diagram [7], one loop two, three and four point scalar functions [8–15] are evaluated in terms of MHFs. Functional relations are used to find one loop Feynman integrals (see [16] and references within) in terms of MHFs. Feynman integrals can also be realized as GKZ hypergeometric system [17–21]. In dimensional regularization [22], the

E-mail address: souvikbera@iisc.ac.in.

space-time dimension D (more precisely, integer multiples of $D/2$) of the Feynman integral appears linearly in the Pochhammer parameters of the corresponding MHFs and the ratios of the scales involved in the integral take its place in the arguments of MHFs. Usually the result of Feynman integral is expressed as a series in the parameter $\epsilon = (4 - D)/2$. Thus, it is important to find efficient algorithms to determine the coefficients of the ϵ -expansion of the MHFs analytically as well as numerically that appears in the Feynman integral calculus. There are computer programs [23,24], based on sector decomposition method, that can perform the series expansion of Feynman integrals numerically.

The ϵ -expansion of the one variable Gauss hypergeometric function ${}_2F_1$ [25] and ${}_pF_{p-1}$ [26–28] are well studied. There are publicly available packages that can perform the expansion of ${}_pF_{p-1}$ around integer and half-integer Pochhammer parameters analytically [29–31] and numerically [32]. The ϵ -expansion of double variable Appell and Kampé de Fériet functions are discussed in [33–37]. In [38–40], differential equation approach is used to find the series expansion of MHFs related to Feynman integrals (see also [41,42]). One can find a comprehensive review of multivariable hypergeometric functions and their ϵ -expansion related to the Feynman integral calculus in [40]. Despite all this work, it may be recognized that, a general procedure to find the series expansion of any MHF in ϵ does not exist in the literature.

In view of the above consideration, we present an efficient algorithm, that can be implemented on the computer, to evaluate analytically the series expansion of hypergeometric functions with any number of variables in the parameter ϵ that appears linearly in the Pochhammer parameters. There is no restriction on other parameters of the function and these could be any complex numbers. In contrast to the available methods, each of the coefficients of the ϵ expansion is expressed in terms of MHFs. The domain of convergence of the hypergeometric functions appearing in the coefficients of the ϵ -expansion is the same as that of the starting hypergeometric function. Thus, in the context of Feynman integrals, the hypergeometric structure remains intact throughout this method.

In general the expansion coefficients are expressed in terms of Harmonic polylogarithms (HPLs) [43] and nested sums [44–46]. It is observed that not all MHFs can be expanded in terms of HPLs and nested sums, which leads to further generalization, functions with elliptic integrals. On the other hand, MHFs provide a function space for expressing not only Feynman integrals but also the coefficients of its ϵ -expansion. The summation indices of MHFs, in principle, range over all non-negative integers, but for practical calculation, any large finite number (N) can be taken as the upper limit. The accuracy of the result will depend on N . One has to choose a suitable upper limit N to make sure that the summation is converged and the desired accuracy is reached. Thus, the numerical evaluation of MHFs is not a difficult task. Inside the associated domain of convergence, these analytic expressions can be used to find numerical values. We have applied our methodology to one, two and three variable hypergeometric functions and results are found to be consistent.

The article is organized as follows. In Section 2 we explain each of the steps of our methodology in details. A number of illustrative examples of hypergeometric functions are presented in Section 3 which includes one variable Gauss ${}_2F_1$, ${}_2F_2$, double variable Appell F_1 and F_2 functions and three variable Lauricella function $F_C^{(3)}$. The provided examples are of Appell F_1 and Lauricella $F_C^{(3)}$, which are motivated by one loop three point function and two loop sunset integral respectively. The example section is followed by the summary.

We note here that the ancillary notebook contains the first few terms, including those presented in the article, of the ϵ -expansions of the hypergeometric functions that are considered in Section 3.

2. The methodology

The presented approach of finding the series expansion is applicable when the parameter ϵ appears linearly in the Pochhammer parameters of a MHF, which is usually the case when one expresses a Feynman integral in terms of MHF.

Our method employs two well-established techniques from the literature. The first technique deals with the derivative of a MHF with respect to its Pochhammer parameters. We follow a similar approach introduced by Anacarani-Gasaneo in [47–49] for one variable hypergeometric functions and its generalizations for more than one variables in [50,51], which expresses the result of the Pochhammer derivative of any MHF as MHFs with one extra summation fold, but having the same set of variables. The second technique concerns about the differential operators that generate the contiguous relations of a MHF [52]. We apply these techniques to perform the task of the series expansion of a MHF in a systematic way.

This method can be divided into five steps.

- *Step 1:* Determine whether the expansion of the given MHF (say, $F(\epsilon)$) is of Taylor or Laurent type simply by observing the lower Pochhammer parameters (i.e., the Pochhammer parameters that appear in the denominator of the series representation of the MHF)
- *Step 2:* If the series expansion of the given MHF is of Taylor type, then find the series expansion from the definition of the Taylor expansion

$$F(\epsilon) = \sum_{i=0}^{\infty} \frac{\epsilon^i}{i!} \frac{d^i}{d\epsilon^i} F(\epsilon) \Big|_{\epsilon=0} \tag{1}$$

- *Step 3:* If the series expansion of the given MHF (primary function) is of Laurent type, then find a secondary MHF (say, $G(\epsilon)$) and a differential operator (say, H) such that, the secondary MHF can be expanded in Taylor series and the differential operator can be used to relate the secondary MHF with the primary MHF

$$F(\epsilon) = H \bullet G(\epsilon) \tag{2}$$

Calculate the Taylor expansion of the secondary MHF following *step 2*

$$G(\epsilon) = \sum_{j=0}^{\infty} \epsilon^j G_j \tag{3}$$

- *Step 4:* Find the corresponding differential operator (i.e., H)
- *Step 5:* Expand the differential operator in series and apply it on the Taylor expansion of the secondary MHF and collect the coefficients of different powers of ϵ

$$F(\epsilon) = H \bullet G(\epsilon) = \left[\sum_{i=-n}^{\infty} \epsilon^i H_i \right] \bullet \left[\sum_{j=0}^{\infty} \epsilon^j G_j \right]$$

We now elaborate each of the steps in details.

2.1. Step 1: Determination of the type of the series expansion

The expansion of a MHF around $\epsilon = 0$ can either be of Taylor type or Laurent type. If the lower Pochhammer parameters (i.e., Pochhammer parameters in the denominator) of a MHF has the form

$$(B_0 + B_1\epsilon)_* \tag{4}$$

where B_0 is zero or negative integer, then it is called *singular*. The “*” denotes linear combination of summation indices which does not play any role in what follows. If one or more lower Pochhammer parameters of MHF are singular, then ϵ -expansion of that function may be of Laurent type. If the lower Pochhammer parameters are non-singular, the ϵ -expansion of the corresponding function is of Taylor type. We discuss each of the situations below.

2.2. Step 2: Taylor expansion of MHFs

The Taylor series expansion around $\epsilon = 0$ of a MHF, which we denote by $F(\epsilon)$, suppressing dependencies of all the other parameters but ϵ , can be found by taking successive derivatives with respect to ϵ

$$F(\epsilon) = \sum_{i=0}^{\infty} \frac{\epsilon^i}{i!} \left. \frac{d^i}{d\epsilon^i} F(\epsilon) \right|_{\epsilon=0} \tag{5}$$

In practice we only need first few terms of the series expansion.

Since the parameter ϵ takes its place in the Pochhammer parameters of a MHF, taking derivative of Pochhammer parameters with respect to its argument plays crucial role in finding the Taylor series expansion. Derivatives of Gauss hypergeometric function ${}_2F_1$ with respect to the Pochhammer parameters are studied in [48]. Similar study for general ${}_pF_q$ function can be found in [28,49,53,54]. Discussions on the Pochhammer derivative of hypergeometric functions of more than one variable can be found in [33,34,50,51,55]. In this section we show how the Pochhammer derivative of a hypergeometric function with n fold summations can be represented as hypergeometric function with $n + 1$ fold summations and the same set of variables. This fact is used to find the Taylor series expansion of MHF in ϵ , whose coefficients themselves are MHFs.

2.2.1. Derivative of Pochhammer parameters

We start by taking derivative of Pochhammer parameter with respect to its argument. In what follows, we assume that m, n, k take non-negative integer values.

$$\frac{d(a)_n}{da} = (a)_n \left(\psi^{(0)}(a+n) - \psi^{(0)}(a) \right) = (a)_n \sum_{k=0}^{n-1} \frac{1}{a+k} = (a)_n \frac{1}{a} \sum_{k=0}^{n-1} \frac{(a)_k}{(a+1)_k} \tag{6}$$

where $\psi^{(0)}$ is polygamma function. If a is ϵ dependent, the chain rule can be used to find the derivative with respect to ϵ ,

$$\frac{d(a)_m}{d\epsilon} = \frac{da}{d\epsilon} \frac{d(a)_m}{da} \tag{7}$$

Derivative of Pochhammer in other circumstances can be obtained by Pochhammer identities (listed in the Appendix A), chain rule and Eq. (6). We list few cases below.

- Derivative of powers of Pochhammer can be found by applying chain rule first and then Eq. (6). For example,

$$\frac{d}{da} [(a)_m]^p = p [(a)_m]^{p-1} \frac{d}{da} (a)_m \tag{8}$$

- To find the derivative of $(a)_{pm}$, where p is a positive integer, Gauss’s multiplication formula can be used to simplify $(a)_{pm}$ and then chain rule and Eq. (6) can be applied. For example,

$$\begin{aligned} \frac{d(a)_{2m}}{da} &= \frac{d}{da} \left[2^{2m} \left(\frac{a}{2} + \frac{1}{2} \right)_m \left(\frac{a}{2} \right)_m \right] \\ &= 2^{2m} \frac{d}{da} \left[\left(\frac{a}{2} + \frac{1}{2} \right)_m \right] \left(\frac{a}{2} \right)_m + 2^{2m} \left(\frac{a}{2} + \frac{1}{2} \right)_m \frac{d}{da} \left[\left(\frac{a}{2} \right)_m \right] \end{aligned} \tag{9}$$

- $(a)_{-m}$ can be simplified using the Pochhammer identities and then the derivative of the result can be found using Eq. (6)

$$\frac{d}{da} (a)_{-m} = \frac{d}{da} \left[\frac{(-1)^m}{(1-a)_m} \right] = (-1)^m \frac{d}{da} \left[\frac{1}{(1-a)_m} \right] \tag{10}$$

- Using the Pochhammer identities, Pochhammer parameter like $(a)_{m+n}$ can be written as products of several Pochhammers parameters, then chain rule and Eq. (6) can be used to find the derivative of $(a)_{m+n}$.

$$\frac{d}{da} (a)_{m+n} = \frac{d}{da} [(a)_m (a+m)_n] = \frac{d}{da} [(a)_m] (a+m)_n + (a)_m \frac{d}{da} [(a+m)_n] \tag{11}$$

Equipped with all the techniques of finding derivative of Pochhammer parameters, we are ready to consider Pochhammer derivative of MHF.

2.2.2. Pochhammer derivative of MHF

We assume the parameter with respect to which the derivative of a MHF is to be taken is a . First the Pochhammer containing a can be simplified using Pochhammer identities. Then the chain rule of derivative can be applied. Essentially, the task of finding the Pochhammer derivative of a MHF boils down to evaluating a number of expressions that look like the following

$$\sum_{n=0}^{\infty} B(n) \frac{x^n}{n!} \frac{d}{da} (a)_n \tag{12}$$

Following [51], the summation of the index n is explicitly shown as the Pochhammer parameter that is being differentiated contains n . Here $B(n)$ collectively denotes the summation of indices other than n of the MHF. Using Eq. (6) we write

$$\sum_{n=0}^{\infty} B(n) \frac{x^n}{n!} \frac{d}{da} (a)_n = \sum_{n=0}^{\infty} B(n) \frac{x^n}{n!} (a)_n \frac{1}{a} \sum_{k=0}^{n-1} \frac{(a)_k}{(a+1)_k} \tag{13}$$

Now shifting $n \rightarrow n + 1$,

$$\sum_{n=0}^{\infty} B(n) \frac{x^n}{n!} \frac{d}{da} (a)_n = \sum_{n=0}^{\infty} B(n+1) \frac{x^{n+1}}{(n+1)!} (a)_{n+1} \frac{1}{a} \sum_{k=0}^n \frac{(a)_k}{(a+1)_k} \tag{14}$$

Reshuffling the summation of indices n, k as,

$$\sum_{n=0}^{\infty} \sum_{k=0}^n A(n, k) \rightarrow \sum_{n,k=0}^{\infty} A(n+k, k) \tag{15}$$

we find

$$\begin{aligned} \sum_{n=0}^{\infty} B(n) \frac{x^n}{n!} \frac{d}{da} (a)_n &= \sum_{n,k=0}^{\infty} B(n+k+1) \frac{x^{n+k+1}}{(n+k+1)!} (a)_{n+k+1} \frac{1}{a} \frac{(a)_k}{(a+1)_k} \\ &= x \sum_{n,k=0}^{\infty} B(n+k+1) \frac{(1)_k (1)_n (a)_k (a+1)_{k+n}}{(a+1)_k (2)_{k+n}} \frac{x^n x^k}{k! n!} \end{aligned} \tag{16}$$

Hence, taking the Pochhammer derivative of MHF introduces a new summation index. Thus the summation fold is increased by one. This formula (Eq. (16)) is used to express each of the coefficients of the Taylor series expansion of a MHF (Eq. (5)) in terms of higher summation fold MHFs.

We observe that derivative of a Pochhammer parameter introduces a polygamma function (Eq. (6)), which is then written as a series with new summation index. A MHF that contains polygamma function inside the summand (i.e. the argument of the polygamma function contains summation indices), has the same domain of convergence as that of MHF with out any polygamma function (see Appendix B of [56]). Hence the domain of convergence of a MHF obtained by taking Pochhammer derivative of a original MHF, is the same as that of the original MHF. This fact in turn implies that, the domain of convergence of MHFs appearing in each of the coefficients of ϵ of the Taylor series expansion of original MHF is same as the domain of convergence of the original MHF.

2.3. Step 3: Construction of the secondary MHF

As discussed above, if one or more lower Pochhammer parameters of the given MHF (primary function) are singular, then ϵ expansion of that function may be of Laurent type. In such cases, one can find the secondary MHF by replacing the singular lower Pochhammer parameter by

$$(B_1 \epsilon + B_0)_* \rightarrow (B_1 \epsilon + 1)_*$$

The hypergeometric function with non-singular lower Pochhammer parameters (i.e. secondary function) can be expanded in ϵ as Taylor series as prescribed in Section 2.2.

2.4. Step 4: The differential operator

There exist contiguous relations that relate MHFs whose Pochhammer parameters differ by integers. An algorithm based on Gröbner bases calculation is proposed by Takayama [52] to find the differential operators that generates such relations. These operators play a crucial role in the differential reduction technique of MHFs [57,58]. For our purpose we only need one kind of operator, called the **step down** operator, to decrease the argument of the lower Pochhammer parameters by unity.

Let us define a MHF as

$$F(\mathbf{a}, \mathbf{b}, \mathbf{x}) = \sum_{\mathbf{m} \in \mathbb{N}_0^r} \frac{\Gamma(\mathbf{a} + \mu \cdot \mathbf{m}) / \Gamma(\mathbf{a})}{\Gamma(\mathbf{b} + \nu \cdot \mathbf{m}) / \Gamma(\mathbf{b})} \frac{\mathbf{x}^{\mathbf{m}}}{\mathbf{m}!} = \sum_{\mathbf{m} \in \mathbb{N}_0^r} \frac{(\mathbf{a})_{\mu \cdot \mathbf{m}}}{(\mathbf{b})_{\nu \cdot \mathbf{m}}} \frac{\mathbf{x}^{\mathbf{m}}}{\mathbf{m}!} = \sum_{\mathbf{m} \in \mathbb{N}_0^r} A(\mathbf{m}) \mathbf{x}^{\mathbf{m}} \tag{17}$$

Here \mathbf{a} , \mathbf{b} and \mathbf{m} are vectors of length p , q and r respectively. μ and ν are matrices of size $p \times r$ and $q \times r$ respectively with integers as their elements. The expression is the definition of the r variable hypergeometric function with parameters \mathbf{a} and \mathbf{b} . \mathbb{N}_0 denotes natural numbers including zero. We have used the vector notation here.

- $\mathbf{a} := \{a_1, a_2, \dots, a_n\}$
- $\mathbf{x}^{\mathbf{m}} := \prod_{i=1}^n x_i^{m_i}$
- $\mathbf{m}! := \prod_{i=1}^n (m_i!)$
- $\Gamma(\mathbf{a}) := \prod_{i=1}^n \Gamma(a_i)$
- $(\mathbf{a})_{\mathbf{m}} := \prod_{i=1}^n (a_i)_{m_i} = \prod_{i=1}^n \Gamma(a_i + m_i) / \Gamma(a_i)$
- $\sum_{\mathbf{m} \in \mathbb{N}_0^n} := \sum_{m_1=0}^{\infty} \dots \sum_{m_n=0}^{\infty}$

These MHFs are known to satisfy partial differential equations [1]. Let,

$$P_i = \frac{A(\mathbf{m} + \mathbf{e}_i)}{A(\mathbf{m})} = \frac{g_i(\mathbf{m})}{h_i(\mathbf{m})}, \quad i = 1, \dots, r \tag{18}$$

where \mathbf{e}_i is unit vector with 1 in its i -th entry. The annihilators L_i of $F(\mathbf{a}, \mathbf{b}, \mathbf{x})$ (Eq. (17)) are given by

$$L_i = \left[h_i(\boldsymbol{\theta}) \frac{1}{x_i} - g_i(\boldsymbol{\theta}) \right] \tag{19}$$

where $\boldsymbol{\theta} = \{\theta_1, \dots, \theta_r\}$ is a vector containing Euler operators $\theta_i = x_i \partial_{x_i}$.

Following [52,57], we define the unit step down operator for the lower Pochhammer parameter

$$F(\mathbf{a}, \mathbf{b}, \mathbf{x}) = \frac{1}{b_i} \left(\sum_{j=1}^r v_{ij} \theta_{x_j} + b_i \right) \bullet F(\mathbf{a}, \mathbf{b} + \mathbf{e}_i, \mathbf{x}) = H(b_i) \bullet F(\mathbf{a}, \mathbf{b} + \mathbf{e}_i, \mathbf{x}) \tag{20}$$

Here the bullet means the action of the operator H on a function. The step down operator, if needed, can be applied multiple times to decrease any lower Pochhammer parameter by suitable integer. In such cases, the product of the unit step down operators modulo the ideal generated by L_i 's can be taken as the step down operator.

$$H = \left[\prod_{i=1}^r H(b_i) \right] / \langle L_1, \dots, L_r \rangle \tag{21}$$

The calculation of Gröbner bases of the annihilators and the reduction of the step down operator with respect to the ideal generated by the annihilators can be found using computer programs [59] in Maple as well as [60,61] in Mathematica. We present an example of step down operator for the one variable Gauss hypergeometric function below.

Example of Gauss hypergeometric function

The unit step down operator for the Gauss hypergeometric function is given by $H(c) = \frac{1}{c}(\theta + c)$, i.e.

$${}_2F_1(a, b; c; x) = H(c) \bullet {}_2F_1(a, b; c + 1; x) \tag{22}$$

Here the Euler operator $\theta = x \partial_x$. The above identity can be proved from the definition of the Gauss hypergeometric function.

$$\begin{aligned}
 c {}_2F_1(a, b; c; x) &= \sum_{m=0}^{\infty} \frac{c(a)_m(b)_m}{(c)_m} \frac{x^m}{m!} \\
 &= \sum_{m=0}^{\infty} \frac{(c+m)(a)_m(b)_m}{(c+1)_m} \frac{x^m}{m!} \\
 &= (\theta + c) \bullet {}_2F_1(a, b; c + 1; x)
 \end{aligned}
 \tag{23}$$

In the second step we have used the Pochhammer identity $\frac{c}{(c)_m} = \frac{c(c+m)\Gamma(c)}{(c+m)\Gamma(c+m)} = \frac{(c+m)}{(c+1)_m}$ and the last equality is obtained using $\theta \bullet x^m = mx^m$.

The unit step down operator can be used repetitively to find a relation between ${}_2F_1(a, b; -n + c\epsilon; x)$ to ${}_2F_1(a, b; c\epsilon + 1; x)$, where n is any positive integer. Note that the later hypergeometric function can be expanded in Taylor series. For example

$${}_2F_1(a, b; c - 1; x) = H(c - 1)H(c) \bullet {}_2F_1(a, b; c + 1; x)
 \tag{24}$$

In the case, the step down operator H is $H(c - 1)H(c)$ modulo L , where L the annihilator of the hypergeometric function ${}_2F_1(a, b; c + 1; x)$.

$$\begin{aligned}
 &L \bullet {}_2F_1(a, b; c + 1; x) \\
 &= \left[-ab + (-x(a + b + 1) + c + 1)\partial_x - (x - 1)x\partial_x^2 \right] \bullet {}_2F_1(a, b; c + 1; x) = 0
 \end{aligned}
 \tag{25}$$

Hence we find the step down operator

$$H = \left(1 - \frac{abx}{(c - 1)c(x - 1)} \right) - \frac{x(a + b + 1) - 2cx + c - 1}{(c - 1)c(x - 1)}\theta
 \tag{26}$$

which matches with the result obtained from the package `pfq.m` [57]. The above differential operator when made to act on ${}_2F_1(a, b; c + 1; x)$ yields

$$\begin{aligned}
 {}_2F_1(a, b; c - 1; x) &= \left(1 - \frac{abx}{(c - 1)c(x - 1)} \right) {}_2F_1(a, b; c + 1; x) \\
 &\quad + \frac{abx(x - (a + b + 1)) + c(2x - 1) + 1}{(c - 1)c(c + 1)(x - 1)} {}_2F_1(a + 1, b + 1; c + 2; x)
 \end{aligned}
 \tag{27}$$

2.5. Step 5: Argument derivative of MHF

Usually, the step down operators contain derivative operators with respect to the arguments of the corresponding MHF (see the examples in Section 3). We call the action of these derivative operators on MHF as ‘argument derivative’ in order to distinguish it from Pochhammer derivative discussed above.

Similar to the discussion in Section 2.2.2, we explicitly write the summation of the index n related to the argument with respect to which the differential is being taken.

$$\frac{d}{dx} \sum_{n=0}^{\infty} B(n) \frac{x^n}{n!}
 \tag{28}$$

Here, as before, $B(n)$ collectively denotes the dependence of all the other summation indices in a MHF.

So we have

$$\frac{d}{dx} \sum_{n=0}^{\infty} B(n) \frac{x^n}{n!} = \sum_{n=1}^{\infty} B(n) \frac{x^{n-1}}{(n-1)!} = \sum_{n=0}^{\infty} B(n+1) \frac{x^n}{n!} \tag{29}$$

We have shifted $n \rightarrow n + 1$ to get the last equality. Hence the argument derivative of a MHF is also another MHF. It is straightforward to realize that the domain of convergence of the resultant MHF remains same as the initial one.

Let us summarize the whole procedure. For a given MHF, we first check if any of its lower Pochhammer parameters is singular. If all the lower Pochhammer parameters are non-singular, then the expansion of that MHF is of Taylor type and can be computed by taking successive derivatives with respect to ϵ as discussed in Section 2.2. The coefficients of each order of ϵ are expressed in terms of MHFs with the same domain of convergence as that of the original MHF. If one or more lower Pochhammer parameters of the given MHF are singular, then the expansion of the given MHF may be of Laurent type. In that case, a suitable step down operator is found that relates the given MHF (primary function) with a secondary MHF, which can be expanded in Taylor series. Once the Taylor series expansion of the secondary MHF and the step down operator are found, the Laurent series expansion of the primary MHF can be found by applying the stepdown operator on the Taylor expansion of the secondary MHF. As in the case of Taylor series expansion, each of the coefficient of the Laurent series expansion are expressed in terms of MHFs and the domain of convergences remains same as the given MHF.

It is worth pointing out that, one has to obtain suitable analytic continuation before proceeding to find the ϵ -expansion. Finding analytic continuations of general MHF is an active field of research. There are many results of analytic continuations of MHFs available in the mathematics literature [1–4,62–66]. Recently the analytic continuations of the double variable Appell F_2 are found in [67] which are the backbone of the numerical package `AppellF2.wl`. One can find the numerical value of the Appell F_2 function for general values of Pochhammer parameters and real values of x and y using this package. Some new analytic continuations of the Appell F_4 [68], Horn H_1, H_5 functions [69] are found. Analytic continuations of three variable Srivastava H_C functions are obtained in [70]. Study of analytic continuations and numerical evaluations of other hypergeometric functions is ongoing [71,72]. Recently, the process of finding analytic continuations of MHFs is automated with Mathematica based `Olsson.wl` package [73]. This package eases the calculation of finding certain analytic continuations of MHFs associated with a Feynman integral. Once a suitable analytic continuation is obtained, the series expansion in ϵ of the resultant MHFs can be performed.

It is to be noted that, the ϵ -expansion of a MHF at its singular points have to treated carefully. For example, the Gauss ${}_2F_1(1, 1; 2 + \epsilon; z)$ can be expanded around $z = 1$ in the series of ϵ as

$${}_2F_1(1, 1; 2 + \epsilon; z) = -\frac{1 + \epsilon}{z} \ln(1 - z) + O(\epsilon^2)$$

This series expansion is valid around $z = 1$ but certainly not at $z = 1$. Starting from the defining series representation of the Gauss ${}_2F_1$ or its analytic continuation around $z = 1$, we can find the same series expansion as above. But at $z = 1$, using the Gauss summation theorem,

$${}_2F_1(a, b, c, 1) = \frac{\Gamma(c)\Gamma(c - a - b)}{\Gamma(c - a)\Gamma(c - b)}, \quad \Re(c) > \Re(a + b)$$

one can find the series expansion as

$${}_2F_1(1, 1; 2 + \epsilon; 1) = \frac{1 + \epsilon}{\epsilon}$$

Thus, a careful analysis is required for the series expansion of a MHF at its singular points, which is beyond the scope of the presented method.

We evaluate the ϵ -expansion of some MHFs to demonstrate the methodology below.

3. Examples

3.1. Gauss ${}_2F_1$ example with half integer Pochhammer

We begin this section by providing the ϵ -expansion of the following Gauss hypergeometric function

$$G := {}_2F_1(3, 2, \epsilon - 3/2; x) = \sum_{m=0}^{\infty} \frac{(2)_m(3)_m}{(\epsilon - \frac{3}{2})_m} \frac{x^m}{m!} \quad \text{where } |x| < 1 \tag{30}$$

There is only one lower Pochhammer parameter and it is not singular. Hence this Gauss function has Taylor expansion around $\epsilon = 0$. So we do not need any step down operators in this example. The coefficients of ϵ^i are found by taking successive derivative with respect to ϵ (Eq. (5)). Using Eq. (16) we find the expansion

$$\begin{aligned} G &= {}_2F_1\left(2, 3; -\frac{3}{2}; x\right) \\ &- \frac{8x}{3}\epsilon F\left[\begin{matrix} \{4, \{1, 1\}\}, \{3, \{1, 1\}\}, \{1, \{1, 0\}\}, \{1, \{0, 1\}\}, \{-\frac{3}{2}, \{0, 1\}\} \\ \{2, \{1, 1\}\}, \{-\frac{1}{2}, \{1, 1\}\}, \{-\frac{1}{2}, \{0, 1\}\} \end{matrix} \middle| \{x, x\}\right] \\ &- \frac{16x}{9}\epsilon^2 F\left[\begin{matrix} \{4, \{1, 1\}\}, \{3, \{1, 1\}\}, \{1, \{1, 0\}\}, \{1, \{0, 1\}\}, \{-\frac{3}{2}, \{0, 1\}\} \\ \{2, \{1, 1\}\}, \{-\frac{1}{2}, \{1, 1\}\}, \{-\frac{1}{2}, \{0, 1\}\} \end{matrix} \middle| \{x, x\}\right] \\ &+ 192x^2\epsilon^2 \\ &\times F\left[\begin{matrix} \{5, \{1, 1, 1\}\}, \{4, \{1, 1, 1\}\}, \{1, \{1, 0, 0\}\}, \{-\frac{1}{2}, \{0, 1, 1\}\}, \{1, \{0, 1, 0\}\}, \{1, \{0, 0, 1\}\}, \{-\frac{1}{2}, \{0, 0, 1\}\} \\ \{3, \{1, 1, 1\}\}, \{\frac{1}{2}, \{1, 1, 1\}\}, \{\frac{1}{2}, \{0, 1, 1\}\}, \{\frac{1}{2}, \{0, 0, 1\}\} \end{matrix} \middle| \{x, x, x\}\right] \\ &+ O(\epsilon^3) \end{aligned} \tag{31}$$

The coefficients of ϵ^3 and ϵ^4 can be found in the ancillary file. We show the Taylor expansion obtained using HypExp [30,31] below for comparison.

$$\begin{aligned} G &= \frac{-315i\sqrt{-\frac{x}{x-1}}(4x+7)x^2 \text{HPL}\left(\{\text{plus}\}, i\sqrt{-\frac{x}{x-1}}\right) + 140x^4 + 4294x^3 + 2784x^2 - 320x + 32}{32(x-1)^6} \\ &+ \frac{\epsilon x}{48(x-1)^6} \left[12i(143x+329)\sqrt{-\frac{x}{x-1}}x \text{HPL}\left(\{\text{plus}\}, i\sqrt{-\frac{x}{x-1}}\right) \right. \\ &+ 945i(4x+7)\sqrt{-\frac{x}{x-1}}x \text{HPL}\left(\{0, \text{plus}\}, i\sqrt{-\frac{x}{x-1}}\right) - 144x^3 + 2822x^2 \\ &+ 6912x - 128 \left. \right] + \frac{\epsilon^2 x}{72(x-1)^6} \left[-11340i\sqrt{-\frac{x}{x-1}}x^2 \text{HPL}\left(\{0, 0, \text{plus}\}, i\sqrt{-\frac{x}{x-1}}\right) \right. \\ &\left. - 18i\sqrt{-\frac{x}{x-1}}(42x+145)x \text{HPL}\left(\{\text{plus}\}, i\sqrt{-\frac{x}{x-1}}\right) \right] \end{aligned}$$

$$\begin{aligned}
 & - 36i \sqrt{-\frac{x}{x-1}} (143x + 329)x \text{HPL} \left(\{0, \text{plus}\}, i \sqrt{-\frac{x}{x-1}} \right) \\
 & - 19845i \sqrt{-\frac{x}{x-1}} x \text{HPL} \left(\{0, 0, \text{plus}\}, i \sqrt{-\frac{x}{x-1}} \right) + 36x^3 + 14474x^2 \\
 & \left. + 20736x - 128 \right] \tag{32}
 \end{aligned}$$

Here HPL's are harmonic polylogarithms [43,74]. As our motivation is to keep the hypergeometric structure intact throughout the calculation, we do not find equivalence of Eq. (31) and Eq. (32) analytically. Although, we find excellent numerical agreement of the two results. The numerical comparison of Eq. (31) and Eq. (32) are shown in Appendix C.1 at a certain $x (< 1)$ and found to be consistent.

3.2. Gauss ${}_2F_1$ example

Let us consider the another example of Gauss hypergeometric function

$$G := {}_2F_1(\epsilon, -\epsilon, \epsilon - 1; x) = \sum_{m=0}^{\infty} \frac{(\epsilon)_m (-\epsilon)_m x^m}{(\epsilon - 1)_m m!} \quad \text{where } |x| < 1 \tag{33}$$

Here the lower Pochhammer parameter is singular so we find the step down operator first. This will allow us to relate the above function with another hypergeometric function which can be expanded in Taylor expansion.

$$G = H \bullet {}_2F_1(\epsilon, -\epsilon, \epsilon + 1; x) \tag{34}$$

We find the step down operator H to be,

$$\begin{aligned}
 H &= \frac{(x(\epsilon(2x - 1) - x + 1))}{(\epsilon - 1)\epsilon(x - 1)} \partial_x + \frac{\epsilon(2x - 1) - x + 1}{(\epsilon - 1)(x - 1)} \\
 &= \frac{1}{\epsilon} x \partial_x + \epsilon^0 \left(1 - \frac{x^2 \partial_x}{x - 1} \right) + \epsilon \left(-\frac{x(x \partial_x + 1)}{x - 1} \right) + \epsilon^2 \left(-\frac{x(x \partial_x + 1)}{x - 1} \right) \\
 &\quad + \epsilon^3 \left(-\frac{x(x \partial_x + 1)}{x - 1} \right) + O(\epsilon^4) \tag{35}
 \end{aligned}$$

In the last step, we expanded the step down operator in series of ϵ .

Next we find the Taylor expansion of ${}_2F_1$ appearing in the right hand side of Eq. (34).

$$\begin{aligned}
 & {}_2F_1(\epsilon, -\epsilon, \epsilon + 1; x) \\
 &= 1 - \epsilon^2 (x {}_3F_2(1, 1, 1; 2, 2; x)) + \epsilon^3 \left(\frac{2}{3} x {}_3F_2(1, 1, 1; 2, 2; x) \right. \\
 &\quad \left. + \frac{x}{3} F \left[\begin{matrix} \{1, \{1, 1\}\}, \{1, \{1, 1\}\}, \{1, \{1, 0\}\}, \{1, \{0, 1\}\}, \{1, \{0, 1\}\} \\ \{2, \{1, 1\}\}, \{2, \{1, 1\}\}, \{2, \{0, 1\}\} \end{matrix} \middle| \{x, x\} \right] \right. \\
 &\quad \left. + \frac{x^2}{12} F \left[\begin{matrix} \{2, \{1, 1\}\}, \{2, \{1, 1\}\}, \{1, \{1, 0\}\}, \{2, \{0, 1\}\}, \{1, \{0, 1\}\} \\ \{3, \{1, 1\}\}, \{3, \{1, 1\}\}, \{3, \{0, 1\}\} \end{matrix} \middle| \{x, x\} \right] \right) + O(\epsilon^4) \tag{36}
 \end{aligned}$$

The coefficient of ϵ^4 can be found in the ancillary file.

Now applying the step down operator (i.e. Eq. (35)) on the Taylor series (Eq. (36)), we find the series expansion of the Gauss ${}_2F_1$ we started with (i.e. Eq. (33)),

$$\begin{aligned}
 G = & 1 + \epsilon \left(-\frac{x}{x-1} - x {}_3F_2(1, 1, 1; 2, 2; x) - \frac{x^2}{4} {}_3F_2(2, 2, 2; 3, 3; x) \right) \\
 & + \epsilon^2 \left[-\frac{x}{x-1} + \frac{x(2x+1)}{3(x-1)} {}_3F_2(1, 1, 1; 2, 2; x) + \frac{x^2(5x-2)}{12(x-1)} {}_3F_2(2, 2, 2; 3, 3; x) \right. \\
 & + \frac{x}{3} F \left[\begin{matrix} \{1, \{1, 1\}\}, \{1, \{1, 1\}\}, \{1, \{1, 0\}\}, \{1, \{0, 1\}\}, \{1, \{0, 1\}\} \\ \{2, \{1, 1\}\}, \{2, \{1, 1\}\}, \{2, \{0, 1\}\} \end{matrix} \middle| \{x, x\} \right] \\
 & + \frac{x^2}{24} F \left[\begin{matrix} \{2, \{1, 1\}\}, \{2, \{1, 1\}\}, \{1, \{1, 0\}\}, \{2, \{0, 1\}\}, \{2, \{0, 1\}\} \\ \{3, \{1, 1\}\}, \{3, \{1, 1\}\}, \{3, \{0, 1\}\} \end{matrix} \middle| \{x, x\} \right] \\
 & + \frac{x^2}{12} F \left[\begin{matrix} \{2, \{1, 1\}\}, \{2, \{1, 1\}\}, \{2, \{1, 0\}\}, \{1, \{0, 1\}\}, \{1, \{0, 1\}\} \\ \{3, \{1, 1\}\}, \{3, \{1, 1\}\}, \{2, \{0, 1\}\} \end{matrix} \middle| \{x, x\} \right] \\
 & + \frac{x^2}{6} F \left[\begin{matrix} \{2, \{1, 1\}\}, \{2, \{1, 1\}\}, \{1, \{1, 0\}\}, \{2, \{0, 1\}\}, \{1, \{0, 1\}\} \\ \{3, \{1, 1\}\}, \{3, \{1, 1\}\}, \{3, \{0, 1\}\} \end{matrix} \middle| \{x, x\} \right] \\
 & + \frac{2x^3}{81} F \left[\begin{matrix} \{3, \{1, 1\}\}, \{3, \{1, 1\}\}, \{1, \{1, 0\}\}, \{3, \{0, 1\}\}, \{2, \{0, 1\}\} \\ \{4, \{1, 1\}\}, \{4, \{1, 1\}\}, \{4, \{0, 1\}\} \end{matrix} \middle| \{x, x\} \right] \\
 & + \frac{x^3}{27} F \left[\begin{matrix} \{3, \{1, 1\}\}, \{3, \{1, 1\}\}, \{2, \{1, 0\}\}, \{2, \{0, 1\}\}, \{1, \{0, 1\}\} \\ \{4, \{1, 1\}\}, \{4, \{1, 1\}\}, \{3, \{0, 1\}\} \end{matrix} \middle| \{x, x\} \right] \Big] + O(\epsilon^3)
 \end{aligned} \tag{37}$$

We present the series expansion obtained from HypExp package [30] [31] for comparison,

$$\begin{aligned}
 G = & 1 + \epsilon \left(\log(1-x) - \frac{x}{x-1} \right) + \epsilon^2 \left(-\frac{x}{x-1} + \frac{1}{2} \log^2(1-x) - \frac{x \log(1-x)}{x-1} \right) \\
 & + O(\epsilon^3)
 \end{aligned} \tag{38}$$

Our result (Eq. (37)) is found to be in agreement with the result from HypExp (Eq. (38)) numerically at given $x (< 1)$ evaluated with sufficient number of terms. It is also consistent with NumExp package [32]. The numerical comparison can be found in Appendix C.2.

3.3. Taylor series of Appell F_1 : an application to one loop massive vertex integral

In the paper [14], one loop one- to four-point functions are obtained as a meromorphic function of the space-time dimension d . In particular, the one loop massive vertex integral is expressed in terms of double variable Appell F_1 functions (Eq. B.2 of the mentioned paper)

$$F_1 \left(\frac{d-2}{2}, 1, \frac{1}{2}, \frac{d}{2}, x_c, y_c \right) \tag{39}$$

The authors also provided a table (Table B.1) for numerical evaluation of the Appell function F_1 . The numerical evaluation of Appell F_1 function for complex values of its argument is the subject of our forthcoming article [72]. In this section, we focus on the ϵ -expansion of the above Appell function. With $d = 4 - 2\epsilon$, the function becomes

$$F_1(1 - \epsilon, 1, \frac{1}{2}, 2 - \epsilon, x, y) \tag{40}$$

The numerical values of the coefficients of the series expansion of the F_1 (Eq. (40)) is given in Table B.2 of [14] at $(x_c, y_c) = (11.1 - 10^{-12}i, 12.1 - 10^{-12}i)$. Since this point does not lie in the defining domain of convergence of the Appell F_1 function, which is $|x| < 1 \wedge |y| < 1$, we find a proper analytic continuation before carrying out the series expansion. One such analytic continuation of F_1 from [72] is written below, skipping the derivation, which is valid at the aforementioned values of (x_c, y_c) .

$$\begin{aligned} F_1(a, b_1, b_2, c, x, y) &= (1-x)^{-a} \left(\frac{1}{1-x}\right)^{-a+b_1+b_2} \frac{\Gamma(c)\Gamma(a-b_1-b_2)}{\Gamma(a)\Gamma(c-b_1-b_2)} \\ &\times F_2\left(b_1+b_2, c-a, b_2, -a+b_1+b_2+1, b_1+b_2, \frac{1}{1-x}, \frac{x-y}{x-1}\right) \\ &+ (1-x)^{-a} \frac{\Gamma(c)\Gamma(-a+b_1+b_2)}{\Gamma(b_1+b_2)\Gamma(c-a)} \\ &\times F_2\left(a, -b_1-b_2+c, b_2, a-b_1-b_2+1, b_1+b_2, \frac{1}{1-x}, \frac{x-y}{x-1}\right) \end{aligned} \tag{41}$$

with the domain of convergence

$$|1-x| > 1 \wedge \left|\frac{x-y}{x-1}\right| < 1 \wedge \left|\frac{x-y}{x-1}\right| + \frac{1}{|1-x|} < 1 \tag{42}$$

where the functions appearing in the RHS of Eq. (41) are Appell F_2 (see Appendix B). Applying the analytic continuation on the F_1 function in Eq. (40) we find

$$\begin{aligned} F_1(1 - \epsilon, 1, \frac{1}{2}, 2 - \epsilon, x, y) &= \frac{\left(\frac{1}{1-x}\right)^{\epsilon+\frac{1}{2}} (1-x)^{\epsilon-1} \Gamma(-\epsilon - \frac{1}{2}) \Gamma(2 - \epsilon)}{\Gamma(\frac{1}{2} - \epsilon) \Gamma(1 - \epsilon)} \\ &\times \sum_{m,n=0}^{\infty} \frac{(1)_m \left(\frac{1}{2}\right)_n \left(\frac{3}{2}\right)_{m+n}}{m!n! \left(\frac{3}{2}\right)_n \left(\epsilon + \frac{3}{2}\right)_m} \left(\frac{1}{1-x}\right)^m \left(\frac{x-y}{x-1}\right)^n \\ &+ \frac{2(1-x)^{\epsilon-1} \Gamma(2 - \epsilon) \Gamma(\epsilon + \frac{1}{2})}{\sqrt{\pi}} \sum_{m,n=0}^{\infty} \frac{\left(\frac{1}{2}\right)_n (1-\epsilon)_{m+n}}{m!n! \left(\frac{3}{2}\right)_n} \left(\frac{1}{1-x}\right)^m \left(\frac{x-y}{x-1}\right)^n \end{aligned} \tag{43}$$

$$= g_1(x, y, \epsilon) \times S_1 + g_2(x, y, \epsilon) \times S_2 \tag{44}$$

Now we are ready to perform the series expansion of the above expression. The prefactors g_1 and g_2 can be easily expanded in series of ϵ , where as the series S_1 and S_2 can be expanded in Taylor series in ϵ following the prescription of Section 2.2. The first few terms in the series expansion of the prefactors g_1, g_2 read,

$$g_1 = -2 \left(\frac{1}{1-x}\right)^{3/2} + \epsilon \left[\frac{2\sqrt{\frac{1}{1-x}} \left(\log\left(\frac{1}{1-x}\right) + \log(1-x) - 3\right)}{x-1} \right] + O(\epsilon^2) \tag{45}$$

$$g_2 = -\frac{2}{x-1} + \epsilon \left[\frac{-2 \log(1-x) + 2 + \log(16)}{x-1} \right] + O(\epsilon^2) \tag{46}$$

and the first few terms of the expansion of the series read

$$S_1 = \sum_{m,n=0}^{\infty} \frac{(1)_m \left(\frac{1}{2}\right)_n \left(\frac{3}{2}\right)_{m+n}}{m!n! \left(\frac{3}{2}\right)_m \left(\frac{3}{2}\right)_n} \left(\frac{1}{1-x}\right)^m \left(\frac{x-y}{x-1}\right)^n + \epsilon \left[\frac{2}{3(x-1)} \sum_{m,n,p=0}^{\infty} \frac{(1)_m \left(\frac{1}{2}\right)_n (1)_p \left(\frac{3}{2}\right)_p \left(\frac{5}{2}\right)_{m+n+p}}{m!n!p! \left(\frac{3}{2}\right)_n \left(\frac{5}{2}\right)_p \left(\frac{5}{2}\right)_{m+p}} \left(\frac{1}{1-x}\right)^{m+p} \left(\frac{x-y}{x-1}\right)^n \right] + O(\epsilon^2) \tag{47}$$

$$S_2 = \sum_{m,n=0}^{\infty} \frac{\left(\frac{1}{2}\right)_n (1)_{m+n}}{m!n! \left(\frac{3}{2}\right)_n} \left(\frac{1}{1-x}\right)^m \left(\frac{x-y}{x-1}\right)^n + \epsilon \left[\frac{(x-y)}{(3-3x)} \sum_{m,n,p=0}^{\infty} \frac{(1)_n ((1)_p)^2 \left(\frac{3}{2}\right)_{n+p} (2)_{m+n+p}}{m!n!p! (2)_p (2)_{n+p} \left(\frac{5}{2}\right)_{n+p}} \left(\frac{1}{1-x}\right)^m \left(\frac{x-y}{x-1}\right)^{n+p} - \frac{1}{(1-x)} \sum_{m,n,p=0}^{\infty} \frac{(1)_m \left(\frac{1}{2}\right)_n (1)_p (1)_{n+p} (2)_{m+n+p}}{m!n!p! \left(\frac{3}{2}\right)_n (2)_{m+p} (2)_{n+p}} \left(\frac{1}{1-x}\right)^{m+p} \left(\frac{x-y}{x-1}\right)^n \right] + O(\epsilon^2) \tag{48}$$

Combining the expansion of the prefactors and the series, we write the expansion of the Appell F_1 function Eq. (40) in the notation described in Appendix B

$$F_1(1-\epsilon, 1, \frac{1}{2}, 2-\epsilon, x, y) = -2 \left(\frac{1}{1-x}\right)^{3/2} F \left[\left\{ \frac{3}{2}, \{1, 1\} \right\}, \{1, \{1, 0\}\}, \left\{ \frac{1}{2}, \{0, 1\} \right\} \mid \left\{ \frac{1}{1-x}, \frac{x-y}{x-1} \right\} \right] - \frac{2}{x-1} F \left[\left\{ 1, \{1, 1\} \right\}, \left\{ \frac{1}{2}, \{0, 1\} \right\} \mid \left\{ \frac{1}{1-x}, \frac{x-y}{x-1} \right\} \right] + \epsilon \left[\frac{2\sqrt{\frac{1}{1-x}} \left(\log\left(\frac{1}{1-x}\right) + \log(1-x) - 3 \right)}{x-1} \times F \left[\left\{ \frac{3}{2}, \{1, 1\} \right\}, \{1, \{1, 0\}\}, \left\{ \frac{1}{2}, \{0, 1\} \right\} \mid \left\{ \frac{1}{1-x}, \frac{x-y}{x-1} \right\} \right] - \frac{4\left(\frac{1}{1-x}\right)^{3/2}}{3(x-1)} F \left[\left\{ \frac{5}{2}, \{1, 1, 1\} \right\}, \{1, \{1, 0, 0\}\}, \left\{ \frac{3}{2}, \{0, 1, 0\} \right\}, \{1, \{0, 0, 1\}\} \mid \left\{ \frac{1}{1-x}, \frac{x-y}{x-1}, \frac{1}{1-x} \right\} \right] - \frac{2}{(x-1)^2} F \left[\{2, \{1, 1, 1\}\}, \{1, \{1, 0, 0\}\}, \{1, \{0, 1, 1\}\}, \left\{ \frac{1}{2}, \{0, 1, 0\} \right\}, \{1, \{0, 0, 1\}\} \mid \left\{ \frac{1}{1-x}, \frac{x-y}{x-1}, \frac{1}{1-x} \right\} \right] + \frac{2(x-y)}{3(x-1)^2} F \left[\{2, \{1, 1, 1\}\}, \left\{ \frac{3}{2}, \{0, 1, 1\} \right\}, \{1, \{0, 1, 0\}\}, \{1, \{0, 0, 1\}\}, \{1, \{0, 0, 1\}\} \mid \left\{ \frac{1}{1-x}, \frac{x-y}{x-1}, \frac{x-y}{x-1} \right\} \right] - \frac{2 \left(\log\left(\frac{1-x}{4}\right) - 1 \right)}{x-1} F \left[\left\{ 1, \{1, 1\} \right\}, \left\{ \frac{1}{2}, \{0, 1\} \right\} \mid \left\{ \frac{1}{1-x}, \frac{x-y}{x-1} \right\} \right] \right] + O(\epsilon^2) \tag{49}$$

The higher order terms of the expansion can be found in the ancillary notebook. A numerical comparison of the expansion coefficients of F_1 (Eq. (49)) found above with the results given in Table B.2 of [14] can be found in Appendix C.3.

3.4. Laurent series of Appell F_1

Let us consider the following double variable Appell F_1 function which we wish to find the ϵ expansion of,

$$F_1(3/2, 2\epsilon + 1, 4 - \epsilon, \epsilon - 2, x, y) = \sum_{m,n=0}^{\infty} \frac{\left(\frac{3}{2}\right)_{m+n} (2\epsilon + 1)_m (4 - \epsilon)_n x^m y^n}{(\epsilon - 2)_{m+n} m! n!}$$

where $|x| < 1 \wedge |y| < 1$ (50)

The corresponding Appell F_1 function that can be expanded in Taylor series is given by

$$F_1(3/2, 2\epsilon + 1, 4 - \epsilon, \epsilon + 1, x, y) = \sum_{m,n=0}^{\infty} \frac{\left(\frac{3}{2}\right)_{m+n} (2\epsilon + 1)_m (4 - \epsilon)_n x^m y^n}{(\epsilon + 1)_{m+n} m! n!}$$

where $|x| < 1 \wedge |y| < 1$ (51)

We find the Taylor expansion of the Appell F_1 function in Eq. (51) and act the step down operator on it to find Laurent series expansion of the Appell F_1 function Eq. (50).

The step down operator that relates the above Appell F_1 function in Eq. (50) with the Appell F_1 function in Eq. (51) is found to be,

$$H = C_0 + C_1 \partial_x + C_2 \partial_y \tag{52}$$

where are rational functions C_i 's are

$$C_0 = \frac{1}{4\epsilon(\epsilon^2 - 3\epsilon + 2)(x - 1)^2(y - 1)^2} \left[4\epsilon^3(x - 1)^2(y - 1)^2 - 12\epsilon^2(x^2(2y^2 - 5y + 2) - 2x(y^2 - 4y + 2) - y^2 - y + 1) + \epsilon(x^2(5y^2 - 160y + 68) + 2x(91y^2 + 10y - 14) - 139y^2 + 44y + 8) + 3(x^2(95y^2 - 54y + 11) - 4x(39y^2 - 14y + 1) + 4y(17y - 4)) \right] \tag{53}$$

$$C_1 = \frac{1}{4(\epsilon - 2)(\epsilon - 1)\epsilon(x - 1)^2(y - 1)^2} \left[x(4\epsilon^2(x^2(y^2 + 1) + x(-4y^2 + y - 1) + 4y^2 - 3y + 1) - 2\epsilon(x^2(22y^2 + 6y + 1) + x(-78y^2 + 39y - 19) + 44y^2 - 21y + 6) + x^2(225y^2 - 178y + 57) - 2x(155y^2 - 66y + 15) + 8(15y^2 - 3y + 1)) \right] \tag{54}$$

$$C_2 = \frac{1}{4(\epsilon - 2)(\epsilon - 1)\epsilon(x - 1)^2(y - 1)^2} \left[y(4\epsilon^2(x^2(y + 1)^2 - 2x(3y + 1)y + 7y^2 - 4y + 1) - 4\epsilon(x^2(11y^2 + 18y - 5) - 6x(9y^2 - 2y + 1) + 35y^2 - 14y + 3) + x^2(225y^2 - 106y + 24) - 2x(191y^2 - 57y + 9) + 171y^2 - 36y + 8) \right] \tag{55}$$

Now expanding all the three coefficients C_i , in ϵ , we club coefficients of different powers of ϵ together,

$$H = A_1 \frac{1}{\epsilon} + A_2 \epsilon^0 + A_3 \epsilon + A_4 \epsilon^2 + O(\epsilon^3) \tag{56}$$

where,

$$A_1 = \frac{x(x^2(225y^2 - 178y + 57) - 2x(155y^2 - 66y + 15) + 8(15y^2 - 3y + 1)) \partial_x}{8(x-1)^2(y-1)^2} + \frac{y(x^2(225y^2 - 106y + 24) - 2x(191y^2 - 57y + 9) + 171y^2 - 36y + 8) \partial_y}{8(x-1)^2(y-1)^2} + \frac{3(x^2(95y^2 - 54y + 11) - 4x(39y^2 - 14y + 1) + 4y(17y - 4))}{8(x-1)^2(y-1)^2} \tag{57}$$

$$A_2 = \frac{x(587x^2y^2 - 558x^2y + 167x^2 - 618xy^2 + 240xy - 14x + 184y^2 + 12y) \partial_x}{16(x-1)^2(y-1)^2} + \frac{y(587x^2y^2 - 462x^2y + 112x^2 - 714xy^2 + 246xy - 6x + 233y^2 + 4y) \partial_y}{16(x-1)^2(y-1)^2} + \frac{865x^2y^2 - 806x^2y + 235x^2 - 1040xy^2 + 544xy - 92x + 334y^2 - 56y + 16}{16(x-1)^2(y-1)^2} \tag{58}$$

$$A_3 = \frac{x(1327x^2y^2 - 1318x^2y + 403x^2 - 1298xy^2 + 472xy + 2x + 376y^2 + 36y) \partial_x}{32(x-1)^2(y-1)^2} + \frac{y(1327x^2y^2 - 1142x^2y + 304x^2 - 1474xy^2 + 478xy + 18x + 469y^2 + 20y) \partial_y}{32(x-1)^2(y-1)^2} + \frac{3(643x^2y^2 - 618x^2y + 181x^2 - 696xy^2 + 304xy - 20x + 214y^2 - 8y)}{32(x-1)^2(y-1)^2} \tag{59}$$

The Taylor Expansion of the Appell F_1 in Eq. (51) is found taking successive derivative with respect to ϵ i.e.,

$$F_1(\epsilon, x, y) = F_1(0, x, y) + \epsilon F_1'(\epsilon, x, y) \Big|_{\epsilon=0} + \frac{1}{2!} \epsilon^2 F_1''(\epsilon, x, y) \Big|_{\epsilon=0} + \frac{1}{3!} \epsilon^3 F_1'''(\epsilon, x, y) \Big|_{\epsilon=0} + O(\epsilon^4) \tag{60}$$

$$= f_0 + \epsilon f_1 + \epsilon^2 f_2 + \epsilon^3 f_3 + O(\epsilon^4) \tag{61}$$

All the Pochhammer parameters of the Appell F_1 in Eq. (51) are collectively denoted as ϵ in the argument of F_1 in Eq. (60) to give emphasis on the fact that the parameters are ϵ dependent. The primes in the right hand side refer to the derivative with respect to ϵ . We write the first few f_i below.

$$f_0 = F_1 \left(\frac{3}{2}, 1, 4, 1, x, y \right) \tag{62}$$

$$f_1 = 3x F \left[\left\{ \frac{5}{2}, \{1, 1, 1\} \right\}, \{1, \{1, 0, 0\}\}, \{1, \{1, 0, 0\}\}, \{1, \{0, 1, 0\}\}, \{4, \{0, 0, 1\}\} \mid \{x, x, y\} \right] - \frac{3x}{2} F \left[\left\{ \frac{5}{2}, \{1, 1, 1\} \right\}, \{1, \{1, 0, 1\}\}, \{1, \{1, 0, 0\}\}, \{1, \{0, 1, 0\}\}, \{4, \{0, 0, 1\}\} \mid \{x, x, y\} \right]$$

$$\begin{aligned}
 & -6y F \left[\left\{ \frac{5}{2}, \{1, 1, 1\} \right\}, \{5, \{1, 0, 1\}\}, \{1, \{1, 0, 0\}\}, \{1, \{1, 0, 0\}\}, \{1, \{0, 1, 0\}\}, \{1, \{0, 0, 1\}\} \right. \\
 & \qquad \qquad \qquad \left. \{2, \{1, 1, 1\}\}, \{2, \{1, 0, 1\}\}, \{2, \{1, 0, 0\}\} \right] \left[\begin{matrix} \{y, x, y\} \end{matrix} \right] \\
 & -\frac{3y}{2} F \left[\left\{ \frac{5}{2}, \{1, 1, 1\} \right\}, \{5, \{1, 0, 1\}\}, \{4, \{1, 0, 0\}\}, \{1, \{1, 0, 0\}\}, \{1, \{0, 1, 0\}\}, \{1, \{0, 0, 1\}\} \right. \\
 & \qquad \qquad \qquad \left. \{2, \{1, 1, 1\}\}, \{2, \{1, 0, 1\}\}, \{5, \{1, 0, 0\}\} \right] \left[\begin{matrix} \{y, x, y\} \end{matrix} \right]
 \end{aligned} \tag{63}$$

We do not show the functions f_2, f_3 here due to its long expression. It is provided in the ancillary notebook.

Now applying the step down operator on the Appell F_1 in Eq. (51),

$$F_1(3/2, 2\epsilon + 1, 4 - \epsilon, \epsilon - 2, x, y) = H \bullet F_1(3/2, 2\epsilon + 1, 4 - \epsilon, \epsilon + 1, x, y) \tag{64}$$

Using Eq. (56) and Eq. (61) on the right hand side of the above equation,

$$\begin{aligned}
 & F_1(3/2, 2\epsilon + 1, 4 - \epsilon, \epsilon - 2, x, y) \\
 & = \left(A_1 \frac{1}{\epsilon} + A_2 \epsilon^0 + A_3 \epsilon + A_4 \epsilon^2 + O(\epsilon^3) \right) \bullet (f_0 + \epsilon f_1 + \epsilon^2 f_2 + \epsilon^3 f_3 + O(\epsilon^4)) \\
 & = \frac{1}{\epsilon} [A_1 \bullet f_0] + [A_1 \bullet f_1 + A_2 \bullet f_0] + \epsilon [A_3 \bullet f_0 + A_2 \bullet f_1 + A_1 \bullet f_2] \\
 & \quad + \epsilon^2 [A_4 \bullet f_0 + A_3 \bullet f_1 + A_2 \bullet f_2 + A_1 \bullet f_3] + O(\epsilon^3)
 \end{aligned} \tag{65}$$

We apply A_1 (Eq. (57)) on f_0 (Eq. (62)) to find the coefficient of ϵ^{-1} of the Laurent expansion,

$$\begin{aligned}
 & A_1 \bullet f_0 \\
 & = \frac{3(x^2(95y^2 - 54y + 11) - 4x(39y^2 - 14y + 1) + 4y(17y - 4))}{8(x - 1)^2(y - 1)^2} F_1\left(\frac{3}{2}, 1, 4, 1, x, y\right) \\
 & + \frac{3x(x^2(225y^2 - 178y + 57) - 2x(155y^2 - 66y + 15) + 8(15y^2 - 3y + 1))}{16(x - 1)^2(y - 1)^2} F_1\left(\frac{5}{2}, 2, 4, 2, x, y\right) \\
 & + \frac{3y(x^2(225y^2 - 106y + 24) - 2x(191y^2 - 57y + 9) + 171y^2 - 36y + 8)}{4(x - 1)^2(y - 1)^2} F_1\left(\frac{5}{2}, 1, 5, 2, x, y\right)
 \end{aligned} \tag{66}$$

Similarly one find the higher order term of the Laurent series. The coefficient of ϵ^0, ϵ^1 in the Laurent series can be found in the ancillary notebook. The coefficients are numerically tested and found to be consistent. The results of the numerical tests can be found in the Appendix C.4.

3.5. Appell F_2 example

It is known that the result of the ϵ -expansion of Appell F_2 function around integer Pochhammer parameters can be expressed in terms of HPLs and S-sums [35]. Algorithms are also provided to carry out such expansion in the mentioned article. The example given in Eq. (81) of [35] deals with the following Appell F_2 function

$$F_2(1, 1, \epsilon, 1 + \epsilon, 1 - \epsilon, x, y) \tag{67}$$

However, we consider the Appell F_2 function with parameters $a_i, i = 1, \dots, 4$ in this example

$$F_2 := F_2(a_1, a_2, \epsilon, a_3 + \epsilon, a_4 - \epsilon, x, y) \tag{68}$$

where it is assumed that $a_1, a_2 \in \mathbb{C}$ and $a_3, a_4 \in \mathbb{C} - \mathbb{Z}_{\leq 0}$ in order to ensure that the function can be expanded in Taylor series. In this section, we find the series expansion of the above F_2 function Eq. (68) and express the result in terms of MHFs.

Following the prescription discussed in Section 2.2, we write the first few terms of the series expansion below.

$$\begin{aligned}
 F_2 = & {}_2F_1(a_1, a_2, a_3, x) \\
 & + \epsilon \left[-\frac{a_1 a_2 x}{a_3^2} \sum_{s_1, s_2=0}^{\infty} \frac{(1)_{s_1} (1)_{s_2} (a_1 + 1)_{s_1+s_2} (a_2 + 1)_{s_1+s_2} (a_3)_{s_2}}{(2)_{s_1+s_2} (a_3 + 1)_{s_2} (a_3 + 1)_{s_1+s_2}} \frac{x^{s_1+s_2}}{s_1! s_2!} \right. \\
 & \left. + \frac{a_1 y}{a_4} \sum_{s_1, s_2=0}^{\infty} \frac{((1)_{s_2})^2 (a_1 + 1)_{s_1+s_2} (a_2)_{s_1}}{(2)_{s_2} (a_3)_{s_1} (a_4 + 1)_{s_2}} \frac{x^{s_1} y^{s_2}}{s_1! s_2!} \right] \\
 & + \epsilon^2 \left[\frac{a_1 a_2 x}{a_3^3} \sum_{s_1, s_2=0}^{\infty} \frac{(1)_{s_1} (1)_{s_2} (a_1 + 1)_{s_1+s_2} (a_2 + 1)_{s_1+s_2} (a_3)_{s_2}}{(2)_{s_1+s_2} (a_3 + 1)_{s_2} (a_3 + 1)_{s_1+s_2}} \frac{x^{s_1+s_2}}{s_1! s_2!} \right. \\
 & + \frac{a_1 (a_1 + 1) a_2 (a_2 + 1) x^2}{2 a_3 (a_3 + 1)^3} \\
 & \times \sum_{s_1, s_2, s_3=0}^{\infty} \frac{(1)_{s_1} (1)_{s_2} (1)_{s_3} (a_1 + 2)_{s_1+s_2+s_3} (a_2 + 2)_{s_1+s_2+s_3} (a_3 + 1)_{s_3} (a_3 + 1)_{s_2+s_3}}{(3)_{s_1+s_2+s_3} (a_3 + 2)_{s_3} (a_3 + 2)_{s_2+s_3} (a_3 + 2)_{s_1+s_2+s_3}} \frac{x^{s_1+s_2+s_3}}{s_1! s_2! s_3!} \\
 & + \frac{a_1 y}{2 a_4^2} \sum_{s_1, s_2=0}^{\infty} \frac{((1)_{s_2})^2 (a_1 + 1)_{s_1+s_2} (a_2)_{s_1}}{(2)_{s_2} (a_3)_{s_1} (a_4 + 1)_{s_2}} \frac{x^{s_1} y^{s_2}}{s_1! s_2!} \\
 & + \frac{a_1 y}{2 a_4^2} \sum_{s_1, s_2, s_3=0}^{\infty} \frac{(1)_{s_2} (1)_{s_3} (1)_{s_2+s_3} (a_1 + 1)_{s_1+s_2+s_3} (a_2)_{s_1} (a_4)_{s_3}}{(2)_{s_2+s_3} (a_3)_{s_1} (a_4 + 1)_{s_3} (a_4 + 1)_{s_2+s_3}} \frac{x^{s_1} y^{s_2+s_3}}{s_1! s_2! s_3!} \\
 & - \frac{a_1 (a_1 + 1) a_2 x y}{a_3^2 a_4} \\
 & \times \sum_{s_1, s_2, s_3=0}^{\infty} \frac{(1)_{s_1} ((1)_{s_2})^2 (1)_{s_3} (a_1 + 2)_{s_1+s_2+s_3} (a_2 + 1)_{s_1+s_3} (a_3)_{s_3}}{(2)_{s_2} (2)_{s_1+s_3} (a_3 + 1)_{s_3} (a_3 + 1)_{s_1+s_3} (a_4 + 1)_{s_2}} \frac{x^{s_1+s_3} y^{s_2}}{s_1! s_2! s_3!} \\
 & + \frac{a_1 (a_1 + 1) y^2}{4 a_4 (a_4 + 1)^2} \sum_{s_1, s_2, s_3=0}^{\infty} \frac{(1)_{s_2} (1)_{s_3} (2)_{s_2+s_3} (a_1 + 2)_{s_1+s_2+s_3} (a_2)_{s_1} (a_4 + 1)_{s_3}}{(3)_{s_2+s_3} (a_3)_{s_1} (a_4 + 2)_{s_3} (a_4 + 2)_{s_2+s_3}} \frac{x^{s_1} y^{s_2+s_3}}{s_1! s_2! s_3!} \\
 & \left. + \frac{a_1 y^2 (a_1 + 1)}{a_4 (2 a_4 + 2)} \sum_{s_1, s_2, s_3=0}^{\infty} \frac{(1)_{s_2} ((1)_{s_3})^2 (2)_{s_2+s_3} (a_1 + 2)_{s_1+s_2+s_3} (a_2)_{s_1}}{(2)_{s_3} (3)_{s_2+s_3} (a_3)_{s_1} (a_4 + 2)_{s_2+s_3}} \frac{x^{s_1} y^{s_2+s_3}}{s_1! s_2! s_3!} \right] \\
 & + O(\epsilon^3) \tag{69}
 \end{aligned}$$

The coefficient of ϵ^3 can be found in the ancillary notebook. The result obtained above is crosschecked numerically with Eq. (81) of [35] after putting a_i 's as 1 in Eq. (69). For the purpose of numerical evaluation `PolyLogTools` [75] is used. The results are found to be in good agreement. A table containing the numerical result is provided in Appendix C.5.

Furthermore, we take different sets of values (integers, half-integers, rationals) for the a_i 's in Appendix C.6 and show that, the convergences of the series coefficients are independent of the values of the Pochhammer parameters, as noted in [50].

3.6. The 2-loop sunset integral in $D = 2$ dimension

It is well known that, the 2-loop sunset integral with different internal masses is convergent in $D = 2$ dimension. In this section we apply our methodology to find the analytic expression of the 2-loop sunset integral at $D = 2$ dimension. We use the expression in Eq. (16) of [5],

$$T_{123}(p^2, m_1^2, m_2^2, m_3^2) = -m_3^2 \left(\frac{m_3^2}{4\pi\mu^2} \right)^{-2(\epsilon+1)} \mathcal{S} \tag{70}$$

with,

$$\begin{aligned} \mathcal{S} &= z_1^{-\epsilon} z_2^{-\epsilon} \Gamma(\epsilon)^2 F_C^{(3)}(1, 1 - \epsilon, 1 - \epsilon, 1 - \epsilon, 1 - \epsilon, z_1, z_2, z_3) \\ &\quad - z_1^{-\epsilon} \Gamma(\epsilon)^2 F_C^{(3)}(1, \epsilon + 1, 1 - \epsilon, \epsilon + 1, 1 - \epsilon, z_1, z_2, z_3) \\ &\quad - z_2^{-\epsilon} \Gamma(\epsilon)^2 F_C^{(3)}(1, \epsilon + 1, \epsilon + 1, 1 - \epsilon, 1 - \epsilon, z_1, z_2, z_3) \\ &\quad - \Gamma(-\epsilon)\Gamma(\epsilon)\Gamma(2\epsilon + 1) F_C^{(3)}(2\epsilon + 1, \epsilon + 1, \epsilon + 1, \epsilon + 1, 1 - \epsilon, z_1, z_2, z_3) \\ &= \sum_{i=1}^4 g_i S_i \end{aligned} \tag{71}$$

where we have put the dimension $D = 2 - 2\epsilon$, thus $\nu = -\epsilon$. The g_i 's denote the prefactors and the Lauricella functions $F_C^{(3)}$ are denoted by S_i 's.

The series expansion of the prefactors g_i 's in ϵ can be easily found and the expansion of the $F_C^{(3)}$ functions (i.e., S_i 's) can be obtained using the presented method. Since \mathcal{S} in Eq. (71) is finite and the prefactors g_i 's can have at most second order poles, the Lauricella functions have to be expanded upto $O(\epsilon^2)$. Thus, we have,

$$\mathcal{S} = \sum_{i=1}^4 g_i S_i = \sum_{i=1}^4 \left(\sum_{j=-2}^0 \epsilon^j g_i^{(j)} \right) \left(\sum_{k=0}^2 \epsilon^k S_i^{(k)} \right) \tag{72}$$

Now we collect the coefficients of ϵ up to $O(\epsilon^0)$,

$$\begin{aligned} \mathcal{S} &= \frac{1}{\epsilon^2} \mathcal{S}^{(-2)} + \frac{1}{\epsilon} \mathcal{S}^{(-1)} + \mathcal{S}^{(0)} \\ &= \frac{1}{\epsilon^2} \left[\sum_{i=1}^4 g_i^{(-2)} S_i^{(0)} \right] + \frac{1}{\epsilon} \left[\sum_{i=1}^4 \left(g_i^{(-1)} S_i^{(0)} + g_i^{(-2)} S_i^{(1)} \right) \right] \\ &\quad + \epsilon^0 \left[\sum_{i=1}^4 \left(g_i^{(0)} S_i^{(0)} + g_i^{(-1)} S_i^{(1)} + g_i^{(-2)} S_i^{(2)} \right) \right] \end{aligned} \tag{73}$$

At this point, we expect that the coefficients of ϵ^{-2} and ϵ^{-1} in Eq. (73) to be zero as the \mathcal{S} is finite. Indeed, after expanding the Lauricella functions and the prefactors in series in ϵ and collecting the $O(\epsilon^{-2})$ coefficient we found

$$\mathcal{S}^{(-2)} = \sum_{i=1}^4 g_i^{(-2)} S_i^{(0)} = 0 \tag{74}$$

However, the coefficient of ϵ^{-1} is not identically zero, but

$$\begin{aligned}
 \mathcal{S}^{(-1)} &= \sum_{i=1}^4 \left(g_i^{(-1)} \mathcal{S}_i^{(0)} + g_i^{(-2)} \mathcal{S}_i^{(1)} \right) \\
 &= -z_1 \sum_{n_1, \dots, n_4=0}^{\infty} \frac{(1)_{n_1} (1)_{n_4} (1)_{n_3+n_4} (2)_{n_1+n_2+n_3+n_4} (2)_{n_1+n_2+n_3+n_4}}{(1)_{n_2} (1)_{n_3} (2)_{n_1+n_4} (2)_{n_1+n_4} (2)_{n_3+n_4}} \frac{z_1^{n_1+n_4} z_2^{n_2} z_3^{n_3}}{n_1! n_2! n_3! n_4!} \\
 &\quad + z_2 \sum_{n_1, \dots, n_4=0}^{\infty} \frac{(1)_{n_2} (1)_{n_4} (1)_{n_3+n_4} (2)_{n_1+n_2+n_3+n_4} (2)_{n_1+n_2+n_3+n_4}}{(1)_{n_1} (1)_{n_3} (2)_{n_2+n_4} (2)_{n_2+n_4} (2)_{n_3+n_4}} \frac{z_1^{n_1} z_2^{n_2+n_4} z_3^{n_3}}{n_1! n_2! n_3! n_4!} \\
 &\quad - z_2 \sum_{n_1, \dots, n_4=0}^{\infty} \frac{(1)_{n_2} (1)_{n_4} (1)_{n_1+n_3+n_4} (2)_{n_1+n_2+n_3+n_4} (2)_{n_1+n_2+n_3+n_4}}{(1)_{n_1} (1)_{n_3} (2)_{n_2+n_4} (2)_{n_2+n_4} (2)_{n_1+n_3+n_4}} \frac{z_1^{n_1} z_2^{n_2+n_4} z_3^{n_3}}{n_1! n_2! n_3! n_4!} \\
 &\quad + z_1 \sum_{n_1, \dots, n_4=0}^{\infty} \frac{(1)_{n_1} (1)_{n_4} (1)_{n_2+n_3+n_4} (2)_{n_1+n_2+n_3+n_4} (2)_{n_1+n_2+n_3+n_4}}{(1)_{n_2} (1)_{n_3} (2)_{n_1+n_4} (2)_{n_1+n_4} (2)_{n_2+n_3+n_4}} \frac{z_1^{n_1+n_4} z_2^{n_2} z_3^{n_3}}{n_1! n_2! n_3! n_4!}
 \end{aligned} \tag{75}$$

It turns out to be zero numerically (see Table 13).

We find the coefficient of ϵ^0 (i.e., $\mathcal{S}^{(0)}$) in a similar way but do not write it explicitly to lighten the paper. It is provided in the ancillary notebook. In Appendix C.7 we compare our expressions of $\mathcal{S}^{(-2)}$, $\mathcal{S}^{(-1)}$ and $\mathcal{S}^{(0)}$ against the numerical package FIESTA5 [24] and the x space presentation of the sunset integral in $D = 2$ dimension [5,76], which reads

$$\begin{aligned}
 T_{123}(p^2, m_1^2, m_2^2, m_3^2) &= -\frac{2^4}{(2\pi\mu^2)^{-2}} \mathcal{I} \\
 &= -\frac{2^4}{(2\pi\mu^2)^{-2}} \int_0^\infty dx \, x \, J_0(\sqrt{-p^2x}) K_0(m_1x) K_0(m_2x) K_0(m_3x)
 \end{aligned} \tag{76}$$

where J_0 and K_0 are Bessel’s function of first and second kind respectively. We find that, our expression of $\mathcal{S}^{(0)}$ is in agreement with the results of FIESTA5 and the x -space representation numerically.

4. Summary

We have presented a new method to obtain the ϵ -expansion of MHFs. In our approach, when the expansion is of Taylor type, successive derivative with respect to ϵ has to be evaluated. We have demonstrated the process of Pochhammer derivative of a MHFs which plays important role in finding the Taylor expansion. When lower Pochhammer parameters of a hypergeometric function are singular, the series expansion may be of Laurent type. In this situation, an appropriate step down operator can be found, which relates the starting hypergeometric function to another one with shifted lower Pochhammer parameters. This hypergeometric function with non-singular lower Pochhammer parameters can be expanded in Taylor series expansion. Finally the obtained step down operator can be made to act on the Taylor series of the hypergeometric function with non-singular lower Pochhammer parameters to find the ϵ -expansion of the starting function. In our method, each of the coefficient of ϵ in the series expansion of a MHF is linear combination of hypergeometric functions with same domain of convergence as the original one. These analytic

expressions can be used to find the numerical value at given point in its argument space within the domain of convergence. The summation of such multi-fold series has to be performed with sufficient number of terms for the required precision.

This methodology allows one to find series expansion of any MHF related to the Feynman integrals. To illustrate the approach we have presented examples containing one variable Gauss ${}_2F_1$, two variable Appell F_1 , F_2 and three variable Lauricella function, which is linked to two loop sunset Feynman integral with different masses. In the present work many of the steps of the calculations required for the examples provided in this article are performed manually using *Mathematica*. We plan to build a automated package that performs the ϵ -expansion of MHFs in near future.

In this approach, higher order terms in the series expansion contain MHFs where all the variables are not independent. In such cases, it may be possible to reduce the fold of the hypergeometric functions. Some reduction formulae of Kampé de Fériet series are discussed in [4]. The reduction formulae for a general MHFs are unknown and beyond the scope of the paper.

CRediT authorship contribution statement

Souvik Bera: Conceptualization, Methodology, Writing – original draft, Writing – review & editing.

Declaration of competing interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

Data availability

No data was used for the research described in the article.

Acknowledgements

The author would like to thank B. Ananthanarayan and Tanay Pathak for the discussions and comments on the draft. We would also like to thank the Reviewer for valuable comments and suggestions which helped us in improving the quality of the manuscript. This is a part of author's doctoral work at CHEP, IISc.

Appendix A. Pochhammer identities

Pochhammer parameter is ratio of Euler's gamma functions,

$$(a)_m = \frac{\Gamma(a+m)}{\Gamma(a)} \quad (77)$$

Some useful identities of the Pochhammer parameters are listed below.

- $(a)_{m_1+\dots+m_n} = (a)_{m_1}(a+m_1)_{m_2}(a+m_1+m_2)_{m_3} \dots (a+m_1+\dots+m_{n-1})_{m_n}$
- $(a)_{-n} \rightarrow \frac{(-1)^n}{(1-a)_n}$, $a \notin \mathbb{Z}$ and $n \in \mathbb{N}$

- Gauss’s Multiplication theorem:

$$(a)_{km} = k^{km} \left(\frac{a}{k}\right)_m \left(\frac{a+1}{k}\right)_m \dots \left(\frac{a+k-1}{k}\right)_m \quad \text{where } k \in \mathbb{N}^*$$

Appendix B. Notation used for MHFs

In this article we have used a common notation of hypergeometric functions for more than one variable. We denote a hypergeometric function with r fold summations with the set of variables \mathbf{x} in vector notation as,

$$F \left[\begin{matrix} \{\mathbf{a}, \mathbf{m}\} \\ \{\mathbf{b}, \mathbf{n}\} \end{matrix} \middle| \{\mathbf{x}\} \right] = \sum_{s_1=0}^{\infty} \dots \sum_{s_r=0}^{\infty} \frac{(\mathbf{a})_{\mathbf{m}\cdot\mathbf{s}} \mathbf{x}^{\mathbf{s}}}{(\mathbf{b})_{\mathbf{n}\cdot\mathbf{s}} \mathbf{s}!} \tag{78}$$

where \mathbf{m}, \mathbf{n} are vectors of length p and q respectively. Each of the elements of \mathbf{m} and \mathbf{n} themselves are vectors of length r with integers in their elements.

$$\begin{aligned} \mathbf{s} &= \{s_1, \dots, s_r\} & \{\mathbf{x}\} &= \{x_1, \dots, x_r\} \\ \{\mathbf{a}, \mathbf{m}\} &= \{a_1, \mathbf{m}_1\}, \dots, \{a_p, \mathbf{m}_p\} & \{\mathbf{b}, \mathbf{n}\} &= \{b_1, \mathbf{n}_1\}, \dots, \{b_q, \mathbf{n}_q\} \\ (\mathbf{a})_{\mathbf{m}\cdot\mathbf{s}} &= (a_1)_{\mathbf{m}_1\cdot\mathbf{s}} \dots (a_p)_{\mathbf{m}_p\cdot\mathbf{s}} & (\mathbf{b})_{\mathbf{n}\cdot\mathbf{s}} &= (b_1)_{\mathbf{n}_1\cdot\mathbf{s}} \dots (b_q)_{\mathbf{n}_q\cdot\mathbf{s}} \\ \mathbf{m}_j \cdot \mathbf{s} &= \sum_{i=0}^r (m_j)_i s_i \quad j = 1, \dots, p & \mathbf{n}_j \cdot \mathbf{s} &= \sum_{i=0}^r (n_j)_i s_i \quad j = 1, \dots, q \\ \mathbf{x}^{\mathbf{s}} &= \prod_{i=1}^r x_i^{s_i} & \mathbf{x}! &= \prod_{i=1}^r (x_i!) \end{aligned}$$

In this notation the Appell F_1 can be written as

$$\begin{aligned} F_1(a, b_1, b_2, c, x, y) &= \sum_{m,n=0}^{\infty} \frac{(a)_{m+n} (b_1)_m (b_2)_n x^m y^n}{(c)_{m+n} m! n!} \\ &= F \left[\begin{matrix} \{a, \{1, 1\}\}, \{b_1, \{1, 0\}\}, \{b_2, \{0, 1\}\} \\ \{c, \{1, 1\}\} \end{matrix} \middle| \{x, y\} \right] \end{aligned}$$

The Appell F_2 function is defined as

$$F_2(a, b_1, b_2; c_1, c_2; x, y) = \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \frac{(a)_{m+n} (b_1)_m (b_2)_n x^m y^n}{(c_1)_m (c_2)_n m! n!}$$

Appendix C. Numerical tests

In this Appendix we test the results obtained different examples in Section 3 numerically.

C.1. Gauss ${}_2F_1$ example with half integer Pochhammer

We compare our result from Section 3.1, (i.e. Eq. (31)) numerically with `HyperExp` result (Eq. (32)) and the result obtained from `NumExp` below. The coefficients of the series expansion of ϵ are evaluated at $x = 0.3$ and tabulated in Table 1. We observe that our results are numerically consistent with others.

Table 1

The numerical value of coefficients of ϵ at $x = 0.3$ from Eq. (31), HypExp and NumExp are compared. The NumExp results are obtained with $e_h = 10^{-4}$.

| | ϵ^0 | ϵ^1 | ϵ^2 | ϵ^3 |
|--------------------|--------------|--------------|--------------|--------------|
| Eq. (31) | 127.514 | -38.7907 | 610.309 | -189.456 |
| result from HypExp | 127.514 | -38.7907 | 610.309 | -189.456 |
| result from NumExp | 127.514 | -38.7907 | 610.309 | -189.528 |

Table 2

The numerical value of coefficients of ϵ at $x = 0.1$ from Eq. (37), HypExp and NumExp are compared. The NumExp results are obtained with $e_h = 10^{-4}$.

| | ϵ^0 | ϵ^1 | ϵ^2 | ϵ^3 |
|--------------------|--------------|--------------|--------------|--------------|
| Eq. (37) | 1 | 0.0057506 | 0.104955 | 0.0998262 |
| result from HypExp | 1 | 0.0057506 | 0.104955 | 0.0998262 |
| result from NumExp | 1 | 0.0057506 | 0.104955 | 0.099735 |

Table 3

We compare the numerical results of the coefficients of the series expansion of the Appell function (Eq. (49)) upto ϵ^4 with the results provided in Table B.2 of [14]. The values used for the numerical evaluation is $(x_c, y_c) = (11.1 - 10^{-12}i, 12.1 - 10^{-12}i)$. The upper limit of the sum is taken as $N = 40$ for each of the summation indices while evaluating the coefficients of ϵ^0, ϵ^1 and ϵ^2 . For the coefficients of ϵ^3 and ϵ^4 , N is taken as 20 and 10 respectively. To obtain the numerical value with more accuracy, one may take larger values of N , but the time taken for numerical evaluation of higher order terms increases rapidly due to presence of higher fold series.

| Numerical value of expansion coefficients of Eq. (49) | Results from Table B.2 of [14] |
|--|---|
| $(-0.17504424807351878429 - 0.05422812947330403050 i)\epsilon^0$ | $(-0.17504424807358806571 - 0.05422812947328981004i)\epsilon^0$ |
| $(-0.00861885859121239314 - 0.39051763820460791405 i)\epsilon^1$ | $(-0.00861885859131501092 - 0.39051763820462137566i)\epsilon^1$ |
| $(0.37518853545322500566 - 0.34047477405513330909 i)\epsilon^2$ | $(0.37518853545319785781 - 0.34047477405516524129i)\epsilon^2$ |
| $(0.49765461883465610385 - 0.00717399489426463939 i)\epsilon^3$ | $(0.49765461883470790694 - 0.00717399489427550385i)\epsilon^3$ |
| $(0.32835724868568101641 + 0.23005850008096157264 i)\epsilon^4$ | $(0.32835724868237320395 + 0.23005850008124251183i)\epsilon^4$ |

C.2. Gauss ${}_2F_1$ example

We compare our result from Section 3.2, (Eq. (37)) numerically with HypExp result (Eq. (38)) and the result obtained from NumExp below. The coefficients of the series expansion of ϵ are evaluated at $x = 0.1$ and tabulated in Table 2. We observe that our results are numerically consistent with others.

C.3. Taylor series of Appell F_1

In this section we numerically evaluate the ϵ -expansion coefficients of the Appell F_1 function (Eq. (49)) at $(x_c, y_c) = (11.1 - 10^{-12}i, 12.1 - 10^{-12}i)$. The obtained numerical values are compared with the result given in [14] in Table 3 above.

C.4. Laurent series of Appell F_1

To test our result numerically, we multiply ϵ to the both sides of the Laurent expansion of the Appell function considered in Section 3.4. The right hand side becomes a series with non-negative powers of ϵ .

Table 4

The values of $C_i \epsilon^i$, for $i = 0, 1, 2$ at $x = 0.1$ and $y = 0.2$ with $\epsilon = 10^{-3}$ are shown.

| C_0 | $C_1 \epsilon$ | $C_2 \epsilon^2$ |
|----------------------|------------------------|---------------------------|
| 8.511583950834586989 | 0.00180071895404555128 | 0.00001320237872566561435 |

Table 5

Comparison of the values of both sides of the series of $\epsilon F_1(3/2, 2\epsilon + 1, 4 - \epsilon, \epsilon - 2, x, y)$ upto ϵ^2 terms at $(x, y) = (0.1, 0.2)$ with $\epsilon = 10^{-3}$. The value of the series expansion upto ϵ^2 terms is in agreement upto 8 decimal places with value of $\epsilon F_1(\dots, x, y)$.

| $\epsilon F_1(3/2, 2\epsilon + 1, 4 - \epsilon, \epsilon - 2, x, y)$ | $C_0 + C_1 \epsilon + C_2 \epsilon^2$ |
|--|---------------------------------------|
| 8.513397873773039908 | 8.51339787216735821 |

$$\epsilon F_1(3/2, 2\epsilon + 1, 4 - \epsilon, \epsilon - 2, x, y) = C_0 + C_1 \epsilon + C_2 \epsilon^2 + O(\epsilon^3)$$

Each of the coefficients with appropriate powers of ϵ is evaluated at $(x, y) = (0.1, 0.2)$ with $\epsilon = 10^{-3}$ with sufficient numbers of terms and tabulated in Table 4

The numerical value of $\epsilon F_1(3/2, 2\epsilon + 1, 4 - \epsilon, \epsilon - 2, x, y)$ is calculated with same x, y and ϵ and compared with sum the first three terms of its series expansion in Table 5.

C.5. Appell F_2 example

In this section we numerically compute the ϵ -expansion coefficients of the Appell F_2 function (Eq. (69)) at $(x, y) = (0.6, 0.1)$ with the a_i 's equal to 1. The obtained numerical values are compared with the numerical value of the Eq. (81) of [35] in Table 6 below.

C.6. Appell F_2 example with integers, half-integers and rational parameters

In this section we continue the same example of Appell F_2 (i.e., Eq. (69)) and perform numerical tests where a_i 's take values from integers, half-integers and rational numbers. Each of the series coefficients is evaluated with a set of upper limit (N) to show the dependencies of the results on N.

First, we consider the case where a_i 's take integer values. Writing the series expansion of F_2 with the following values of a_i 's : $a_1 = 1, a_2 = 2, a_3 = 3$ and $a_4 = 4$

$$F_2(1, 2, \epsilon, 3 + \epsilon, 4 - \epsilon, x, y) = f_0 + \epsilon f_1 + \epsilon^2 f_2 + \epsilon^3 f_3 \tag{79}$$

We find the numerical values of each of the series coefficients of Eq. (79) with the upper limit $N = 15, 30, 45$ and 60 at $(x, y) = (0.2, 0.3)$ in Table 7 below.

Now we compare both sides of Eq. (79) with $\epsilon = 10^{-5}$ in Table 8.

Next, we take half-integer values of a_i 's: $a_1 = 1/2, a_2 = 3/2, a_3 = 5/2$ and $a_4 = 7/2$

$$F_2\left(\frac{1}{2}, \frac{3}{2}, \epsilon, \epsilon + \frac{5}{2}, \frac{7}{2} - \epsilon, x, y\right) = g_0 + \epsilon g_1 + \epsilon^2 g_2 + \epsilon^3 g_3 \tag{80}$$

Similar to the previous case, we compute the series coefficients of Eq. (80) at $(x, y) = (0.2, 0.3)$ with the upper limit $N = 15, 30, 45$ and 60 in Table 9 below.

We compare both sides of (80) with $\epsilon = 10^{-5}$ below in Table 10.

Table 6

We compare the numerical results of the coefficients of the series expansion of the Appell function (Eq. (69)) with all the $a_i = 1$ up to ϵ^3 with the results provided in Eq. (81) of [35]. The values used for the numerical evaluation is $(x, y) = (0.6, 0.1)$. Here, as always, N is the upper limit of the summation for each of the indices. We evaluate the series coefficients with a set to values of N, to show how the values change with the upper limit of the summation. The numerical results are found to be in good agreement with each other. To obtain the numerical value with more accuracy, one may take larger values of N, but the time taken for numerical evaluation of higher order terms increases rapidly due to presence of higher fold series.

| Order | N | Numerical value of expansion coefficients of Eq. (69) with $a_i = 1$ | Numerical value of expansion coefficients of Eq. (81) of [35] |
|--------------|-----|--|---|
| ϵ^0 | 25 | 2.4999957354567953718 | 2.4999999999999998612 |
| | 50 | 2.4999999999878756421 | |
| | 75 | 2.499999999999998268 | |
| | 100 | 2.499999999999998612 | |
| ϵ^1 | 25 | -1.571552060080885826 | -1.571521648555935118 |
| | 50 | -1.571521649284227887 | |
| | 75 | -1.571521648555935385 | |
| | 100 | -1.571521648555935119 | |
| ϵ^2 | 25 | 2.608431603015495386 | 2.608484299442508259 |
| | 50 | 2.608484297358779548 | |
| | 75 | 2.608484299442436849 | |
| | 100 | 2.608484299442508257 | |
| ϵ^3 | 25 | -2.154453137514624822 | -2.154366134071260766 |
| | 50 | -2.154366138151998583 | |
| | 75 | -2.154366134071422536 | |

Table 7

We compute the first four series coefficients of expansion of $F_2(1, 2, \epsilon, 3 + \epsilon, 4 - \epsilon, x, y)$ at $(x, y) = (0.2, 0.3)$. The summations are performed with upper limit N for each of the summation indices.

| N | f_0 | f_1 | f_2 | f_3 |
|----|------------------------|------------------------|------------------------|------------------------|
| 15 | 1.15717756570958926382 | 0.04946518465296974939 | 0.04760586647909068011 | 0.00446404842334520315 |
| 30 | 1.15717756571048779862 | 0.04946518465764881786 | 0.04760586648282076076 | 0.00446404842749894999 |
| 45 | 1.15717756571048779862 | 0.04946518465764881791 | 0.04760586648282076074 | 0.00446404842749895002 |
| 60 | 1.15717756571048779862 | 0.04946518465764881791 | 0.04760586648282076074 | 0.00446404842749895002 |

Table 8

We compare both sides of the series expansion of $F_2(1, 2, \epsilon, 3 + \epsilon, 4 - \epsilon, x, y)$ (i.e., (79)) at $(x, y) = (0.2, 0.3)$. The summations are performed with the upper limit N = 60 for each of the indices. Both the sides of Eq. (79) are found to be in good agreement.

| $F_2(1, 2, \epsilon, 3 + \epsilon, 4 - \epsilon, x, y)$ | |
|---|------------------------|
| f_0 | 1.15717756571048779862 |
| $f_0 + \epsilon f_1$ | 1.15717806036233437511 |
| $f_0 + \epsilon f_1 + \epsilon^2 f_2$ | 1.15717806036709496176 |
| $f_0 + \epsilon f_1 + \epsilon^2 f_2 + \epsilon^3 f_3$ | 1.15717806036709496622 |
| | 1.15717806036709496622 |

Lastly, we choose rational values for a_i 's, $a_1 = 1/7, a_2 = 2/7, a_3 = 3/7$ and $a_4 = 4/7$

$$F_2\left(\frac{1}{7}, \frac{2}{7}, \epsilon, \epsilon + \frac{3}{7}, \frac{4}{7} - \epsilon, x, y\right) = h_0 + \epsilon h_1 + \epsilon^2 h_2 + \epsilon^3 h_3 \tag{81}$$

Table 9

We compute the first four series coefficients of expansion of $F_2(1/2, 3/2, \epsilon, 5/2 + \epsilon, 7/2 - \epsilon, x, y)$ at $(x, y) = (0.2, 0.3)$ with upper limit N for each of the summation indices.

| N | g_0 | g_1 | g_2 | g_3 |
|----|------------------------|------------------------|------------------------|------------------------|
| 15 | 1.06740285248337825566 | 0.02629391741076840821 | 0.02793237719286772142 | 0.00201506616058023520 |
| 30 | 1.06740285248347457546 | 0.02629391741162192975 | 0.02793237719337559810 | 0.00201506616133570611 |
| 45 | 1.06740285248347457546 | 0.02629391741162192976 | 0.02793237719337559810 | 0.00201506616133570612 |
| 60 | 1.06740285248347457546 | 0.02629391741162192976 | 0.02793237719337559810 | 0.00201506616133570612 |

Table 10

We compare both sides of the series expansion of $F_2(1/2, 3/2, \epsilon, 5/2 + \epsilon, 7/2 - \epsilon, x, y)$ (i.e., (80)) at $(x, y) = (0.2, 0.3)$. The summations are performed with the upper limit $N = 60$ for each of the indices. Both the sides of Eq. (80) are found to be in good agreement.

| $F_2(1/2, 3/2, \epsilon, 5/2 + \epsilon, 7/2 - \epsilon, x, y)$ | |
|---|------------------------|
| g_0 | 1.06740285248347457546 |
| $g_0 + \epsilon g_1$ | 1.06740311542264869168 |
| $g_0 + \epsilon g_1 + \epsilon^2 g_2$ | 1.06740311542544192940 |
| $g_0 + \epsilon g_1 + \epsilon^2 g_2 + \epsilon^3 g_3$ | 1.06740311542544193141 |

Table 11

We compute the first four series coefficients of expansion of $F_2(1/7, 2/7, \epsilon, 3/7 + \epsilon, 4/7 - \epsilon, x, y)$ at $(x, y) = (0.2, 0.3)$ with upper limit N for each of the summation indices.

| N | h_0 | h_1 | h_2 | h_3 |
|----|-----------------------|-------------------------|------------------------|------------------------|
| 15 | 1.0213175368701332097 | 0.052073040471508997983 | 0.28243204145121175747 | 0.10874010806468834360 |
| 30 | 1.0213175368701836808 | 0.052073040691456832163 | 0.28243204129337127545 | 0.10874010942130990672 |
| 45 | 1.0213175368701836808 | 0.052073040691456863666 | 0.28243204129337121899 | 0.10874010942131012250 |
| 60 | 1.0213175368701836808 | 0.052073040691456863666 | 0.28243204129337121899 | 0.10874010942131012250 |

Table 12

We compare both sides of the series expansion of $F_2(1/7, 2/7, \epsilon, 3/7 + \epsilon, 4/7 - \epsilon, x, y)$ (i.e., (81)) at $(x, y) = (0.2, 0.3)$. The summations are performed with the upper limit $N = 60$ for each of the indices. Both the sides of Eq. (81) are found to be in good agreement.

| $F_2(1/7, 2/7, \epsilon, 3/7 + \epsilon, 4/7 - \epsilon, x, y)$ | |
|---|-----------------------|
| h_0 | 1.0213175368701836808 |
| $h_0 + \epsilon h_1$ | 1.0213180576005905954 |
| $h_0 + \epsilon h_1 + \epsilon^2 h_2$ | 1.0213180576288337995 |
| $h_0 + \epsilon h_1 + \epsilon^2 h_2 + \epsilon^3 h_3$ | 1.0213180576288339083 |

We now compute the series coefficients of Eq. (81) at $(x, y) = (0.2, 0.3)$ with the upper limit $N = 15, 30, 45$ and 60 in Table 11.

Now we compare both sides of (81) with $\epsilon = 10^{-5}$ in Table 12.

We observe that the coefficients of the series expansion of F_2 in Eq. (69) convergence in a same rate, independent of the values of the parameters. In all the cases, at $(x, y) = (0.2, 0.3)$, the upper limit of the summation $N = 60$ of the series coefficients is enough to obtain a match in numerical results with the LHS up to 20 digits.

Table 13

We calculate $\mathcal{S}^{(-2)}$, $\mathcal{S}^{(-1)}$ and $\mathcal{S}^{(0)}$ for three different kinematic points and compare the results with the results obtained from FIESTA5 and by numerically performing the integration in (76) using *Mathematica*.

| kinematic points | $\mathcal{S}^{(-2)}$ | $\mathcal{S}^{(-1)}$ | $\mathcal{S}^{(0)}$ | | result from FIESTA5 | result from the integration ($4m_3^2 \mathcal{I}$) |
|------------------|----------------------|--------------------------------|---------------------|------------------|---------------------|--|
| | | | N | N | | |
| $p^2 = 3$ | 0 | $-1.09434200450461 * 10^{-9}$ | 10 | 13.0801528975881 | 10 | 13.08015 |
| $m_1^2 = 10$ | | $-3.46337421335846 * 10^{-13}$ | 15 | 13.0801528947215 | 15 | ± 0.00007 |
| $m_2^2 = 2$ | | $-1.18293156085287 * 10^{-16}$ | 20 | 13.0801528947195 | 20 | |
| $m_3^2 = 100$ | | $-4.24195869926179 * 10^{-20}$ | 25 | | | 13.0801528947195 |
| $p^2 = 3$ | 0 | $4.60503575832032 * 10^{-18}$ | 10 | 21.8600302323743 | 10 | 21.8600 |
| $m_1^2 = 1$ | | $2.64103442427579 * 10^{-25}$ | 15 | 21.8600302323744 | 15 | ± 0.0001 |
| $m_2^2 = 2$ | | $1.60726507054367 * 10^{-32}$ | 20 | | | |
| $m_3^2 = 100$ | | $1.02402152695267 * 10^{-39}$ | 25 | | | 21.8600302323744 |
| $p^2 = 3$ | 0 | $3.62652142904041 * 10^{-31}$ | 10 | 46.3800201955786 | 10 | 46.3800 |
| $m_1^2 = 1$ | | $3.13801126225243 * 10^{-44}$ | 15 | | | ± 0.0003 |
| $m_2^2 = 2$ | | $2.41027454208700 * 10^{-57}$ | 20 | | | |
| $m_3^2 = 1000$ | | | | | | 46.3800201955786 |

C.7. The sunset example

In this section we numerically compare our results of $\mathcal{S}^{(-2)}$, $\mathcal{S}^{(-1)}$ and $\mathcal{S}^{(0)}$ with the x -space representation of the sunset integral and the result obtained by the numerical package FIESTA5 [24]. We recall that, \mathcal{S} , given by Eq. (71), is expanded in Laurent series in $\epsilon (= (2 - D)/2)$

$$\mathcal{S} = \frac{1}{\epsilon^2} \mathcal{S}^{(-2)} + \frac{1}{\epsilon} \mathcal{S}^{(-1)} + \mathcal{S}^{(0)}$$

Since the sunset integral is finite in $D = 2$ dimension, we expect $\mathcal{S}^{(-2)}$ and $\mathcal{S}^{(-1)}$ to be zero. Thus we compare $\mathcal{S}^{(0)}$ with the x -space representation, given by Eq. (76). Fixing the overall prefactors, we expect the two quantities below to be equal.

$$\mathcal{S}^{(0)} \stackrel{?}{=} 4m_3^2 \mathcal{I} \tag{82}$$

where the integral \mathcal{I} , given by Eq. (76), is calculated in *Mathematica* using the NIntegrate command.

The coefficients of the series expansion of \mathcal{S} (i.e., $\mathcal{S}^{(-2)}$, $\mathcal{S}^{(-1)}$ and $\mathcal{S}^{(0)}$) are calculated for three different kinematic points (within the domain of convergence of the Lauricella series) specified by a set of values of p^2 , m_1^2 , m_2^2 and m_3^2 and tabulated in the Table 13. For each set, the coefficient of ϵ^{-2} (i.e., $\mathcal{S}^{(-2)}$) is identically equal to zero. The coefficients of ϵ^{-1} and ϵ^0 (i.e., $\mathcal{S}^{(-1)}$ and $\mathcal{S}^{(0)}$) are calculated with different values of the upper limit N for each of the summation indices to show the dependence of the result on N. For example, in the first kinematic point, given by $(p^2, m_1^2, m_2^2, m_3^2) = (3, 10, 2, 100)$, $\mathcal{S}^{(-1)}$ is calculated with $N = 10, 15, 20, 25$. As the number of terms in the summation is increased, the value of $\mathcal{S}^{(-1)}$ approaches to zero. Similar trend follows for the second and third kinematic points. $\mathcal{S}^{(0)}$ is calculated with $N = 10, 15, 20$ for the first kinematic point. The result with the upper limit of the summation $N = 20$ matches with the result of the x -space representation (i.e., $4m_3^2 \mathcal{I}$). The numerical results of the sunset integral from FIESTA5 are also provided for comparison.

Appendix D. The ${}_2F_2$ example

In this example we apply the methodology to confluent hypergeometric function ${}_2F_2$. Consider the following function

$$G := {}_2F_2(1, 2; \epsilon, \epsilon - 1, x) = \sum_{m=0}^{\infty} \frac{(1)_m(2)_m}{(\epsilon)_m(\epsilon - 1)_m} \frac{x^m}{m!} \quad \text{where } |x| < \infty \tag{83}$$

Note that the lower Pochhammer parameters are ϵ and $\epsilon - 1$. So, the function G may have Laurent series expansion. On the other hand, the function ${}_2F_2(1, 2; \epsilon + 1, \epsilon + 1, x)$ has Taylor series expansion. We find the step down operator that relates to the former i.e. Eq. (83) with the later

$$G = H \bullet {}_2F_2(1, 2; \epsilon + 1, \epsilon + 1, x) \tag{84}$$

The step down operator is found to be

$$H = \frac{\epsilon^3 - \epsilon^2 + 2x}{(\epsilon - 1)\epsilon^2} + \frac{x(2\epsilon^2 - \epsilon + 4x - 1)}{(\epsilon - 1)\epsilon^2} \partial_x + \frac{x^2(\epsilon + x - 1)}{(\epsilon - 1)\epsilon^2} \partial_x^2 \tag{85}$$

Expanding the operator H in series, and collecting the coefficients of ϵ we find

$$H = \left[-(x - 1)x^2\partial_x^2 + x(1 - 4x)\partial_x - 2x \right] \frac{1}{\epsilon^2} + \left[-x^3\partial_x^2 + (2 - 4x)x\partial_x - 2x \right] \frac{1}{\epsilon} + \left[-x^3\partial_x^2 - 4x^2\partial_x - 2x + 1 \right] + \left[-x^3\partial_x^2 - 4x^2\partial_x - 2x \right] \epsilon + O(\epsilon^2) \tag{86}$$

The Taylor expansion of the ${}_2F_2$ function on the right and side of Eq. (84) is found to be

$$\begin{aligned} &{}_2F_2(1, 2; \epsilon + 1, \epsilon + 1, x) \\ &= {}_1F_1(2; 1, x) - 4\epsilon x F \left[\begin{matrix} \{3, \{1, 1\}\}, \{1, \{1, 0\}\}, \{1, \{0, 1\}\}, \{1, \{0, 1\}\} \\ \{2, \{1, 1\}\}, \{2, \{1, 1\}\}, \{2, \{0, 1\}\} \end{matrix} \middle| \{x, x\} \right] \\ &+ 6x\epsilon^2 F \left[\begin{matrix} \{3, \{1, 1\}\}, \{1, \{1, 0\}\}, \{1, \{0, 1\}\}, \{1, \{0, 1\}\} \\ \{2, \{1, 1\}\}, \{2, \{1, 1\}\}, \{2, \{0, 1\}\} \end{matrix} \middle| \{x, x\} \right] \\ &+ \frac{3x^2}{4}\epsilon^2 F \left[\begin{matrix} \{4, \{1, 1, 1\}\}, \{1, \{1, 0, 0\}\}, \{2, \{0, 1, 1\}\}, \{1, \{0, 1, 0\}\}, \{1, \{0, 1, 0\}\}, \{1, \{0, 0, 1\}\} \\ \{3, \{1, 1, 1\}\}, \{3, \{1, 1, 1\}\}, \{3, \{0, 1, 1\}\}, \{2, \{0, 1, 0\}\} \end{matrix} \middle| \{x, x, x\} \right] \\ &+ \frac{9x^2}{8}\epsilon^2 F \left[\begin{matrix} \{4, \{1, 1, 1\}\}, \{1, \{1, 0, 0\}\}, \{2, \{0, 1, 1\}\}, \{1, \{0, 1, 0\}\}, \{2, \{0, 0, 1\}\}, \{1, \{0, 0, 1\}\} \\ \{3, \{1, 1, 1\}\}, \{3, \{1, 1, 1\}\}, \{3, \{0, 1, 1\}\}, \{3, \{0, 0, 1\}\} \end{matrix} \middle| \{x, x, x\} \right] \\ &+ O(\epsilon^3) \end{aligned} \tag{87}$$

To find the Laurent series expansion of Eq. (83), we apply the step down operator Eq. (86) on the Taylor series expansion of ${}_2F_2(1, 2; \epsilon + 1, \epsilon + 1, x)$, i.e. Eq. (87). The Laurent expansion reads,

$$\begin{aligned} G &= \left[-2x {}_1F_1(2; 1; x) + 2(1 - 4x)x {}_1F_1(3; 2; x) - 3(x - 1)x^2 {}_1F_1(4; 3; x) \right] \frac{1}{\epsilon^2} \\ &+ \left[-2x {}_1F_1(2; 1; x) + 2(2 - 4x)x {}_1F_1(3; 2; x) - 3x^3 {}_1F_1(4; 3; x) \right. \\ &+ 4x(6x - 1) F \left[\begin{matrix} \{3, \{1, 1\}\}, \{1, \{1, 0\}\}, \{1, \{0, 1\}\}, \{1, \{0, 1\}\} \\ \{2, \{1, 1\}\}, \{2, \{1, 1\}\}, \{2, \{0, 1\}\} \end{matrix} \middle| \{x, x\} \right] \\ &+ 9x^2(2x - 1) F \left[\begin{matrix} \{4, \{1, 1\}\}, \{2, \{1, 0\}\}, \{1, \{0, 1\}\}, \{1, \{0, 1\}\} \\ \{3, \{1, 1\}\}, \{3, \{1, 1\}\}, \{2, \{0, 1\}\} \end{matrix} \middle| \{x, x\} \right] \\ &+ \frac{9x^2(2x - 1)}{2} F \left[\begin{matrix} \{4, \{1, 1\}\}, \{1, \{1, 0\}\}, \{2, \{0, 1\}\}, \{2, \{0, 1\}\} \\ \{3, \{1, 1\}\}, \{3, \{1, 1\}\}, \{3, \{0, 1\}\} \end{matrix} \middle| \{x, x\} \right] \end{aligned}$$

Table 14
The values of $C_i \epsilon^i$, for $i = 0, 1, 2, 3$ at $x = 0.1$ with $\epsilon = 10^{-4}$ are shown.

| | |
|------------------|--|
| C_0 | -0.07305179768480031690 |
| $C_1 \epsilon$ | -1.795147633233522925*10 ⁻⁵ |
| $C_2 \epsilon^2$ | 7.021511837197325735*10 ⁻⁹ |
| $C_3 \epsilon^3$ | -1.52211510291782210*10 ⁻¹³ |

Table 15
Comparison of the values of both sides of the series of $\epsilon^2 {}_2F_2(1, 2; \epsilon, \epsilon - 1, x)$ upto ϵ^3 terms at $x = 0.1$ with $\epsilon = 10^{-4}$. The result from the series expansion upto ϵ^3 terms is in agreement with the value of $\epsilon^2 {}_2F_2(\dots, x)$.

| | |
|---|--|
| $\epsilon^2 {}_2F_2(1, 2; \epsilon, \epsilon - 1, x)$ | $C_0 + C_1 \epsilon + C_2 \epsilon^2 + C_3 \epsilon^3$ |
| -0.07306974213977305910 | -0.0730697421397730264 |

$$\begin{aligned}
 & + \frac{8(x-1)x^3}{3} F \left[\begin{matrix} \{5, \{1, 1\}\}, \{3, \{1, 0\}\}, \{1, \{0, 1\}\}, \{1, \{0, 1\}\} \\ \{4, \{1, 1\}\}, \{4, \{1, 1\}\}, \{2, \{0, 1\}\} \end{matrix} \middle| \{x, x\} \right] \\
 & + \frac{4(x-1)x^3}{3} F \left[\begin{matrix} \{5, \{1, 1\}\}, \{2, \{1, 0\}\}, \{2, \{0, 1\}\}, \{2, \{0, 1\}\} \\ \{4, \{1, 1\}\}, \{4, \{1, 1\}\}, \{3, \{0, 1\}\} \end{matrix} \middle| \{x, x\} \right] \\
 & + \frac{8(x-1)x^3}{9} F \left[\begin{matrix} \{5, \{1, 1\}\}, \{1, \{1, 0\}\}, \{3, \{0, 1\}\}, \{3, \{0, 1\}\} \\ \{4, \{1, 1\}\}, \{4, \{1, 1\}\}, \{4, \{0, 1\}\} \end{matrix} \middle| \{x, x\} \right] \Bigg] \frac{1}{\epsilon} + O(\epsilon^0)
 \end{aligned} \tag{88}$$

The next two higher order terms are also evaluated and can be found in the ancillary file. A numerical test of the above result can be found in Appendix D.1.

D.1. The ${}_2F_2$ example

To check the result (i.e. Eq. (88)) of the ${}_2F_2$ example, we multiply both sides of Eq. (88) by ϵ^2 . The right hand side becomes a series with non-negative powers of ϵ .

$$\epsilon^2 {}_2F_2(1, 2; \epsilon, \epsilon - 1, x) = C_0 + C_1 \epsilon + C_2 \epsilon^2 + C_3 \epsilon^3 + O(\epsilon^4)$$

Each of the coefficients with appropriate powers of ϵ are evaluated at $x = 0.1$ with $\epsilon = 10^{-4}$ and tabulated in Table 14.

The left hand side i.e. $\epsilon^2 {}_2F_2(1, 2; \epsilon, \epsilon - 1, x)$ is calculated at the same x and ϵ with sufficient number of terms and compared with the sum of $C_0, C_1 \epsilon, C_2 \epsilon^2$ and $C_3 \epsilon^3$ in the Table 15.

References

[1] H. Bateman, Higher Transcendental Functions, 1953.
 [2] L. Slater, Generalized Hypergeometric Functions, 1966.
 [3] H. Exton, Multiple hypergeometric functions and applications, 1976.
 [4] H.M. Srivastava, P.W. Karlsson, Multiple Gaussian hypergeometric series, 1985.
 [5] F.A. Berends, M. Buza, M. Bohm, R. Scharf, Closed expressions for specific massive multiloop selfenergy integrals, Z. Phys. C 63 (1994) 227–234.
 [6] B. Ananthanarayan, S. Friot, S. Ghosh, New series representations for the two-loop massive sunset diagram, Eur. Phys. J. C 80 (7) (2020) 606, arXiv:1911.10096 [hep-ph].

- [7] Z.-H. Gu, H.-B. Zhang, T.-F. Feng, Hypergeometric expression for a three-loop vacuum integral, *Int. J. Mod. Phys. A* 35 (19) (2020) 2050089.
- [8] A.I. Davydychev, General results for massive N point Feynman diagrams with different masses, *J. Math. Phys.* 33 (1992) 358–369.
- [9] J. Fleischer, F. Jegerlehner, O.V. Tarasov, A new hypergeometric representation of one loop scalar integrals in d dimensions, *Nucl. Phys. B* 672 (2003) 303–328, arXiv:hep-ph/0307113.
- [10] E.E. Boos, A.I. Davydychev, A method of evaluating massive Feynman integrals, *Theor. Math. Phys.* 89 (1991) 1052–1063.
- [11] O.V. Tarasov, Application and explicit solution of recurrence relations with respect to space-time dimension, *Nucl. Phys. B, Proc. Suppl.* 89 (2000) 237–245, arXiv:hep-ph/0102271.
- [12] T.-F. Feng, C.-H. Chang, J.-B. Chen, Z.-H. Gu, H.-B. Zhang, Evaluating Feynman integrals by the hypergeometry, *Nucl. Phys. B* 927 (2018) 516–549, arXiv:1706.08201 [hep-ph].
- [13] T.-F. Feng, C.-H. Chang, J.-B. Chen, H.-B. Zhang, The system of partial differential equations for the C_0 function, *Nucl. Phys. B* 940 (2019) 130–189, arXiv:1809.00295 [hep-th].
- [14] K.H. Phan, T. Riemann, Scalar 1-loop Feynman integrals as meromorphic functions in space-time dimension d, *Phys. Lett. B* 791 (2019) 257–264, arXiv:1812.10975 [hep-ph].
- [15] A.I. Davydychev, Geometrical methods in loop calculations and the three-point function, in: J. Blumlein, W. Friebel, T. Naumann, T. Riemann, P. Wegner, D. Perret-Gallix (Eds.), *Nucl. Instrum. Methods A* 559 (2006) 293–297, arXiv:hep-th/0509233.
- [16] O.V. Tarasov, Functional reduction of one-loop Feynman integrals with arbitrary masses, *J. High Energy Phys.* 06 (2022) 155, arXiv:2203.00143 [hep-ph].
- [17] M.Y. Kalmykov, B.A. Kniehl, Mellin-Barnes representations of Feynman diagrams, linear systems of differential equations, and polynomial solutions, *Phys. Lett. B* 714 (2012) 103–109, arXiv:1205.1697 [hep-th].
- [18] L. de la Cruz, Feynman integrals as A-hypergeometric functions, *J. High Energy Phys.* 12 (2019) 123, arXiv:1907.00507 [math-ph].
- [19] R.P. Klausen, Hypergeometric series representations of Feynman integrals by GKZ hypergeometric systems, *J. High Energy Phys.* 04 (2020) 121, arXiv:1910.08651 [hep-th].
- [20] T.-F. Feng, C.-H. Chang, J.-B. Chen, H.-B. Zhang, GKZ-hypergeometric systems for Feynman integrals, *Nucl. Phys. B* 953 (2020) 114952, arXiv:1912.01726 [hep-th].
- [21] B. Ananthanarayan, S. Banik, S. Bera, S. Datta, FeynGKZ: a Mathematica package for solving Feynman integrals using GKZ hypergeometric systems, *Comput. Phys. Commun.* 287 (2023) 108699, arXiv:2211.01285 [hep-th].
- [22] G. 't Hooft, M.J.G. Veltman, Regularization and renormalization of gauge fields, *Nucl. Phys. B* 44 (1972) 189–213.
- [23] S. Borowka, G. Heinrich, S. Jahn, S.P. Jones, M. Kerner, J. Schlenk, T. Zirke, pySecDec: a toolbox for the numerical evaluation of multi-scale integrals, *Comput. Phys. Commun.* 222 (2018) 313–326, arXiv:1703.09692 [hep-ph].
- [24] A.V. Smirnov, N.D. Shapurov, L.I. Vysotsky, FIESTA5: numerical high-performance Feynman integral evaluation, *Comput. Phys. Commun.* 277 (2022) 108386, arXiv:2110.11660 [hep-ph].
- [25] M.Y. Kalmykov, Gauss hypergeometric function: reduction, epsilon-expansion for integer/half-integer parameters and Feynman diagrams, *J. High Energy Phys.* 04 (2006) 056, arXiv:hep-th/0602028.
- [26] M.Y. Kalmykov, B.F.L. Ward, S.A. Yost, On the all-order epsilon-expansion of generalized hypergeometric functions with integer values of parameters, *J. High Energy Phys.* 11 (2007) 009, arXiv:0708.0803 [hep-th].
- [27] M.Y. Kalmykov, B.A. Kniehl, Towards all-order Laurent expansion of generalized hypergeometric functions around rational values of parameters, *Nucl. Phys. B* 809 (2009) 365–405, arXiv:0807.0567 [hep-th].
- [28] D. Greynat, J. Sesma, A new approach to the epsilon expansion of generalized hypergeometric functions, *Comput. Phys. Commun.* 185 (2014) 472–478, arXiv:1302.2423 [math-ph].
- [29] S. Moch, P. Uwer, XSummer: transcendental functions and symbolic summation in form, *Comput. Phys. Commun.* 174 (2006) 759–770, arXiv:math-ph/0508008.
- [30] T. Huber, D. Maitre, HypExp: a Mathematica package for expanding hypergeometric functions around integer-valued parameters, *Comput. Phys. Commun.* 175 (2006) 122–144, arXiv:hep-ph/0507094.
- [31] T. Huber, D. Maitre, HypExp 2, expanding hypergeometric functions about half-integer parameters, *Comput. Phys. Commun.* 178 (2008) 755–776, arXiv:0708.2443 [hep-ph].
- [32] Z.-W. Huang, J. Liu, NumExp: numerical epsilon expansion of hypergeometric functions, *Comput. Phys. Commun.* 184 (2013) 1973–1980, arXiv:1209.3971 [physics.comp-ph].
- [33] D. Greynat, J. Sesma, G. Vulvert, Epsilon expansion of Appell and Kampé de Fériet functions, arXiv:1310.7700 [math-ph], 2013.
- [34] D. Greynat, J. Sesma, G. Vulvert, Derivatives of the Pochhammer and reciprocal Pochhammer symbols and their use in epsilon-expansions of Appell and Kampé de Fériet functions, *J. Math. Phys.* 55 (2014) 043501.

- [35] S. Moch, P. Uwer, S. Weinzierl, Nested sums, expansion of transcendental functions and multiscale multiloop integrals, *J. Math. Phys.* 43 (2002) 3363–3386, arXiv:hep-ph/0110083.
- [36] S. Weinzierl, Expansion around half integer values, binomial sums and inverse binomial sums, *J. Math. Phys.* 45 (2004) 2656–2673, arXiv:hep-ph/0402131.
- [37] S. Weinzierl, Symbolic expansion of transcendental functions, *Comput. Phys. Commun.* 145 (2002) 357–370, arXiv:math-ph/0201011.
- [38] S.A. Yost, V.V. Bytev, M.Y. Kalmykov, B.A. Kniehl, B.F.L. Ward, The epsilon expansion of Feynman diagrams via hypergeometric functions and differential reduction, arXiv:1110.0210 [math-ph], 2011.
- [39] V.V. Bytev, M.Y. Kalmykov, B.A. Kniehl, When epsilon-expansion of hypergeometric functions is expressible in terms of multiple polylogarithms: the two-variables examples, in: T. Riemann (Ed.), *PoS LL2012*, 2012, p. 029, arXiv:1212.4719 [hep-th].
- [40] M. Kalmykov, V. Bytev, B.A. Kniehl, S.-O. Moch, B.F.L. Ward, S.A. Yost, Hypergeometric functions and Feynman diagrams. antidifferentiation and the calculation of Feynman amplitudes, arXiv:2012.14492 [hep-th], 2020.
- [41] J. Blümlein, M. Saragnese, C. Schneider, Hypergeometric structures in Feynman integrals, arXiv:2111.15501 [math-ph], 2021.
- [42] J. Blümlein, M. Saragnese, C. Schneider, Computer algebra and hypergeometric structures for Feynman integrals, arXiv:2207.08524 [math-ph], 2022.
- [43] E. Remiddi, J.A.M. Vermaseren, Harmonic polylogarithms, *Int. J. Mod. Phys. A* 15 (2000) 725–754, arXiv:hep-ph/9905237.
- [44] J.A.M. Vermaseren, Harmonic sums, Mellin transforms and integrals, *Int. J. Mod. Phys. A* 14 (1999) 2037–2076, arXiv:hep-ph/9806280.
- [45] J. Blümlein, Algebraic relations between harmonic sums and associated quantities, *Comput. Phys. Commun.* 159 (2004) 19–54, arXiv:hep-ph/0311046.
- [46] A.I. Davydychev, M.Y. Kalmykov, Massive Feynman diagrams and inverse binomial sums, *Nucl. Phys. B* 699 (2004) 3–64, arXiv:hep-th/0303162.
- [47] L.U. Ancarani, G. Gasaneo, Derivatives of any order of the confluent hypergeometric function ${}_1F_1(a, b, z)$ with respect to the parameter a or b , *J. Math. Phys.* 49 (6) (2008) 063508, <https://doi.org/10.1063/1.2939395>.
- [48] L.U. Ancarani, G. Gasaneo, Derivatives of any order of the Gaussian hypergeometric function ${}_2F_1(a, b, c; z)$ with respect to the parameters a, b and c , *J. Phys. A, Math. Theor.* 42 (39) (2009) 395208.
- [49] L.U. Ancarani, G. Gasaneo, Derivatives of any order of the hypergeometric function ${}_pF_q(a_1, \dots, a_p; b_1, \dots, b_q; z)$ with respect to the parameters a_i and b_i , *J. Phys. A, Math. Theor.* 43 (8) (2010).
- [50] L.U. Ancarani, J.A. Del Punta, G. Gasaneo, Derivatives of Horn hypergeometric functions with respect to their parameters, *J. Math. Phys.* 58 (7) (2017) 073504, <https://doi.org/10.1063/1.4994059>.
- [51] V.V. Bytev, B.A. Kniehl, Derivatives of any Horn-type hypergeometric functions with respect to their parameters, *Nucl. Phys. B* 952 (2020) 114911.
- [52] N. Takayama, Groebner basis and the problem of contiguous relation, *Jpn. J. Appl. Math.* 6 (1989) 147–160.
- [53] H. Kang, C. An, Differentiation formulas of some hypergeometric functions with respect to all parameters, *Appl. Math. Comput.* 258 (2015) 454–464.
- [54] B.X. Fejzullahu, Parameter derivatives of the generalized hypergeometric function, *Integral Transforms Spec. Funct.* 28 (11) (2017) 781–788.
- [55] V. Sahai, A. Verma, Derivatives of Appell functions with respect to parameters, *Journal of Inequalities and Special Functions* 6 (3.2015).
- [56] S. Friot, D. Greynat, On convergent series representations of Mellin-Barnes integrals, *J. Math. Phys.* 53 (2012) 023508, arXiv:1107.0328 [math-ph].
- [57] V.V. Bytev, M.Y. Kalmykov, B.A. Kniehl, HYPERDIRE, HYPERgeometric functions Differential REDuction: MATHEMATICA-based packages for differential reduction of generalized hypergeometric functions ${}_pF_{p-1}, F_1, F_2, F_3, F_4$, *Comput. Phys. Commun.* 184 (2013) 2332–2342, arXiv:1105.3565 [math-ph].
- [58] V.V. Bytev, M.Y. Kalmykov, S.-O. Moch, HYPERgeometric functions Differential REDuction (HYPERDIRE): MATHEMATICA based packages for differential reduction of generalized hypergeometric functions: F_D and F_S Horn-type hypergeometric functions of three variables, *Comput. Phys. Commun.* 185 (2014) 3041–3058, arXiv:1312.5777 [math-ph].
- [59] F. Chyzak, Gröbner bases, symbolic summation and symbolic integration, in: *Gröbner Bases and Applications*, Cambridge University Press, 1998, pp. 32–60.
- [60] C. Koutschan, A fast approach to creative telescoping, *Math. Comput. Sci.* 4 (2) (2010) 259–266.
- [61] C. Koutschan, Advanced applications of the holonomic systems approach, *ACM Commun. Comput. Algebra* 43 (3/4) (2010) 119.

- [62] S.I. Bezrodnykh, Horn's hypergeometric functions with three variables, *Integral Transforms Spec. Funct.* 32 (3) (2021) 207–223, <https://doi.org/10.1080/10652469.2020.1814770>.
- [63] S.I. Bezrodnykh, Analytic continuation of the Horn hypergeometric series with an arbitrary number of variables, *Integral Transforms Spec. Funct.* 31 (10) (2020) 788–803, <https://doi.org/10.1080/10652469.2020.1744590>.
- [64] S.I. Bezrodnykh, Analytic continuation of the Lauricella function $F_D^{(N)}$ with arbitrary number of variables, *Integral Transforms Spec. Funct.* 29 (1) (2018) 21–42, <https://doi.org/10.1080/10652469.2017.1402017>.
- [65] S.I. Bezrodnykh, Analytic continuation of Lauricella's function $F_D^{(N)}$ for variables close to unit near hyperplanes $z_j = z_l$, *Integral Transforms Spec. Funct.* (2021) 1–15, <https://doi.org/10.1080/10652469.2021.1939329>.
- [66] S.I. Bezrodnykh, Analytic continuation of Lauricella's functions $F_A^{(N)}$, $F_B^{(N)}$, and $F_D^{(N)}$, *Integral Transforms Spec. Funct.* 31 (11) (2020) 921–940, <https://doi.org/10.1080/10652469.2020.1762081>.
- [67] B. Ananthanarayan, S. Bera, S. Friot, O. Marichev, T. Pathak, On the evaluation of the Appell F_2 double hypergeometric function, *Comput. Phys. Commun.* 284 (2023) 108589, arXiv:2111.05798 [math.CA].
- [68] B. Ananthanarayan, S. Friot, S. Ghosh, A. Hurier, New analytic continuations for the Appell F_4 series from quadratic transformations of the Gauss ${}_2F_1$ function, arXiv:2005.07170 [hep-th], 2020.
- [69] S. Bera, T. Pathak, Analytic continuations of the Horn H_1 and H_5 functions, arXiv:2210.17370 [math-ph], 2022.
- [70] S. Friot, G. Suchet-Bernard, Linear transformations of Srivastava's H_C triple hypergeometric function, arXiv:2205.06247 [math-ph], 2022.
- [71] B. Ananthanarayan, S. Bera, S. Friot, T. Pathak, in preparation.
- [72] B. Ananthanarayan, S. Bera, S. Friot, in preparation.
- [73] B. Ananthanarayan, S. Bera, S. Friot, T. Pathak, Olsson.wl: a *Mathematica* package for the computation of linear transformations of multivariable hypergeometric functions, arXiv:2201.01189 [cs.MS], 2021.
- [74] D. Maître, HPL, a *Mathematica* implementation of the harmonic polylogarithms, *Comput. Phys. Commun.* 174 (3) (2006) 222–240, arXiv:hep-ph/0507152v2.
- [75] C. Duhr, F. Dulat, PolyLogTools — polylogs for the masses, *J. High Energy Phys.* 08 (2019) 135, arXiv:1904.07279 [hep-th].
- [76] S. Groote, J.G. Korner, A.A. Pivovarov, A numerical test of differential equations for one- and two-loop sunrise diagrams using configuration space techniques, *Eur. Phys. J. C* 72 (2012) 2085, arXiv:1204.0694 [hep-ph].