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Geometric dominating-set and set-cover via local-search

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ABSTRACT

In this paper, we study two classic optimization problems: minimum geometric dominating set and set cover. In the dominating-set problem, for a given set of objects in the plane as input, the objective is to choose a minimum number of input objects such that every input object is dominated by the chosen set of objects. Here, we say that one object is dominated by another if their intersection is nonempty. For the second problem, for a given set of points and objects in the plane, the objective is to choose a minimum number of objects to cover all the points. This is a particular version of the set-cover problem.

Both problems have been well-studied, subject to various restrictions on the input objects. These problems are APX-hard for object sets consisting of axis-parallel rectangles, ellipses, α-fat objects of constant description complexity, and convex polygons. On the other hand, PTASs (polynomial time approximation schemes) are known for object sets consisting of disks or unit squares. Surprisingly, a PTAS was unknown even for arbitrary squares. For both problems obtaining a PTAS remains open for a large class of objects.

For the dominating-set problem, we prove that a popular local-search algorithm leads to a $(1+\varepsilon)$ approximation for a family of homothets of a convex object (which includes arbitrary squares, k-regular polygons, translated and scaled copies of a convex set, etc.) in $n^{O(1/\varepsilon^2)}$ time. On the other hand, the same approach leads to a PTAS for the geometric covering problem when the objects are convex pseudodisks (which include disks, unit height rectangles, homothetic convex objects, etc.). Consequently, we obtain an easy-to-implement approximation algorithm for both problems for a large class of objects, significantly improving the best-known approximation guarantees.

1. Introduction

1.1. Problems studied

We consider two fundamental combinatorial optimization problems in a geometric context, dominating-set and set-cover. Let $\mathcal{P}$ be a subset of the plane $\mathbb{R}^2$, and let $\mathscr{S}$ be a collection of subsets of $\mathcal{P}$, called *objects*. A subset $\mathscr{S}' \subseteq \mathscr{S}$ is a *dominating-set* if every element of $\mathscr{S}$ has a nonempty intersection with at least one element of $\mathscr{S}'$. A subset $\mathscr{S}'' \subseteq \mathscr{S}$ is a *cover* if every point of $\mathcal{P}$ lies within at least one element of $\mathscr{S}''$. The *dominating-set* and *set-cover* problems involve

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computing a minimum cardinality dominating-set and set-cover, respectively. Both problems have a wealth of theoretical results and practical applications. The geometric set-cover problem has many applications in the real world, for example, wireless sensor networks, optimizing the number of stops in an existing transportation network, and job scheduling [2,7,17].

1.2. Local search

It is well known that both of these problems are NP-hard in the most general setting, so researchers have focused on approximation algorithms. In this paper, we analyze an approach based on local search. Local search is a popular heuristic algorithm. It is an iterative algorithm that starts with a feasible solution and improves it after each iteration until a locally optimal solution is reached. One big advantage of local search is its easy implementability and its parallelizability [8]. As mentioned by Cohen-Addad and Mathieu [8], it is interesting to analyze such algorithms even when alternative, theoretically optimal polynomial-time algorithms are known.

1.3. Our results

Our results on the dominating-set problem apply under the assumption that the input consists of homothets of a convex body in the plane, that is, the elements of $\mathscr{S}$ are equal to each other up to translation and positive uniform scaling. This includes a large class of natural object sets, such as collections of squares of arbitrary size, collections of regular k-gons of arbitrary size, and collections of circular disks of arbitrary radii. Given $\varepsilon > 0$, a $(1+\varepsilon)$-*approximation algorithm* for the dominating-set (resp., set-cover) problem returns a dominating-set (resp., set-cover) whose cardinality is larger than the optimum by a factor of at most $(1+\varepsilon)$ and runs in polynomial (in the input size) time. Such algorithms are known in the literature as polynomial time approximation schemes (PTAS).

First, we show that the standard local search algorithm leads to a PTAS for computing a minimum dominating-set of homothetic convex objects. For the analysis, we use a separator-based technique, that was introduced independently by Chan and Har-Peled [4] and Mustafa and Ray [29]. The main part of this proof technique is to show the existence of a planar graph satisfying a *locality condition* (to be defined in Section 2.1). Gibson et al. [16] used the same approach where the objects were arbitrary disks. Inspired by their work, we ask the question of whether we can generalize their framework to more general objects. Our result on the dominating-set problem can be viewed as a non-trivial generalization of their result. To show the planarity, first, we decompose (or trim) a set of homothetic convex objects (which are returned by the optimum algorithm and the local search algorithm) into a set of interior disjoint objects so that each input object has a "trace" in this new set of objects. This decomposition is motivated by the idea of core decomposition introduced by Mustafa et al. [28], and this technique could be of independent interest. Next, we consider the nearest-site Voronoi diagram for this set of disjoint objects with respect to the well-known convex distance function. The decomposition ensures that each site has a nonempty cell in the Voronoi diagram. Finally, we show that the dual of this Voronoi diagram satisfies the locality condition. Note that if homothets of a centrally symmetric convex object are given, one can avoid the disjoint decomposition, and the analysis is much simpler.

Our result on the set-cover problem applies, assuming that the input consists of a collection of convex pseudodisks in the plane. A set of objects is said to be a collection of *pseudodisks* if the boundaries of every pair of them either do not intersect or cross (i.e., intersect non-tangentially) at exactly two points [30]. Note that this generalizes collections of homothets. We use a similar technique as the previous one. First, we show that we can decompose (or trim) a set of pseudodisks (which are returned by the optimum algorithm and the local search algorithm) into a set of interior disjoint objects so that each input point has a "trace" in this new set of objects. We consider a graph $\mathcal{G}$ in which each vertex corresponds to a trimmed object, and two vertices are joined by an edge if the corresponding objects share an edge in their boundary. Since the trimmed objects are interior disjoint with each other, the graph $\mathcal{G}$ is planar. We prove that the graph $\mathcal{G}$ satisfies the locality condition.

Our results are formally given below.

Theorem 1. *Given a set $\mathscr{S}$ of n convex homothets in $\mathbb{R}^2$ and $\varepsilon > 0$, there exists a $(1+\varepsilon)$ approximation algorithm for dominated set based on local search that runs in time $n^{O(1/\varepsilon^2)}$.*

Theorem 2. *Given a set $\mathscr{S}$ of n convex pseudodisks in $\mathbb{R}^2$ and $\varepsilon > 0$, there exists a $(1+\varepsilon)$ approximation algorithm for set-cover based on local search that runs in time $n^{O(1/\varepsilon^2)}$.*

1.4. Related work

Our work is motivated by recent progress on the approximability of various fundamental geometric optimization problems like finding maximum independent sets [1], minimum hitting set of geometric intersection graphs [29], and minimum geometric set covers [28].

Dominating-Set: The minimum dominating-set problem is NP-hard for general graphs [15]. From the result of Raz and Safra [31], it follows that it is NP-hard even to obtain a $(c \log \Delta)$-approximate dominating-set for general graphs, where Δ is the maximum degree of a node in the graph and $c\,(> 0)$ is any constant (see [24]).

Researchers have studied the problem for different graph classes like planar graphs, intersection graphs, bounded arboricity graphs, etc. Recently, Har-Peled and Quanrud [18] proved that local search produces a PTAS for graphs with polynomially bounded expansion. Gibson and Pirwani [16] gave a PTAS for the intersection graphs of arbitrary disks. Unless P = NP [9],[2] it is not possible to compute a $((1-\varepsilon)\ln n)$-approximate dominating-set in polynomial time for n homothetic polygons [13,20,32]. Erlebach and van Leeuwen [11] proved that the problem is APX-hard for the intersection graphs of axis-parallel rectangles, ellipses, α-fat objects of constant description complexity, and convex polygons with r-corners ($r \geq 4$), i.e., there is no PTAS for these unless P = NP.

The effort has been devoted to related problems involving various objects such as squares, regular polygons, etc. Marx [26] proved that the problem is W[1]-hard for unit squares, that implies that no efficient-polynomial-time-approximation-scheme (EPTAS) is possible unless FPT = W[1] [27]. The best-known approximation factor for homothetic $2k$-regular polygons is $O(k)$ due to Erlebach and van Leeuwen [11], where $k > 0$. They also obtained an $O(k^2)$-approximation algorithm for homothetic $(2k+1)$-regular polygons. Even worse, for the homothetic convex polygons where each polygon has k-corners, the best-known result is $O(k^4)$-approximation. Currently, there is no PTAS known even for arbitrary squares. We consider the problem for a set of homothetic convex objects.

Set-Cover: The set-cover problem is known to be NP-complete [21]. The geometric variant has received a great amount of attention due to its wide applications (for example, the recent breakthrough of Bansal and Pruhs [2]). Unfortunately, the geometric version of the problem also remains NP-complete even when the objects are unit disks or unit squares [3,19].

Erlebach and van Leeuwen [12] obtained a PTAS for the geometric set-cover problem when the objects are unit squares. Recently, Chan and Grant [3] showed that the problem is APX-hard when the objects are axis-aligned rectangles. They extended the results to several other classes of objects, including axis-aligned ellipses in $\mathbb{R}^2$, axis-aligned slabs, downward shadows of line segments, unit balls in $\mathbb{R}^3$, axis-aligned cubes in $\mathbb{R}^3$. A QPTAS was developed by Mustafa et al. [28] for the problem when the objects are pseudodisks. The current state of the art lacks a PTAS when the objects are pseudodisks which includes a large class of objects: arbitrary squares, arbitrary regular polygons, and homothetic convex objects.

In the weighted setting, Varadarajan introduced the idea of quasi-uniform sampling to obtain an $O(\log\phi(\mathrm{OPT}))$-approximation guarantees in the weighted setting for a large class of objects for which such guarantees were known in the unweighted case [33]. Here $\phi(\mathrm{OPT})$ is the union complexity of the objects in the optimum set OPT. Recently, Li and Jin proposed a PTAS for the weighted version of the problem when the objects are unit disks [25].

In [17], the authors described a PTAS for the problem of computing a minimum cover of given points by a set of weighted fat objects by allowing them to expand by some δ-fraction. A multi-cover variant of the problem (where each point is covered by at least k sets) under geometric settings was studied in [5].

1.5. Organization

In Section 1.6, we discuss notation and preliminaries. After that, in Section 2, we present a general algorithm based on the local search technique. Then, in Section 2.1, we present a high-level view of the analysis technique of local search, which was introduced by Chan & Har-Peled [4] and Mustafa & Ray [29]. As a warm-up, in Section 3, we give a simple proof of locality condition for the dominating-set problem for a family of homothetic copies of a centrally symmetric convex object. In Section 4, we prove two results for a set of pseudodisks which are common tools for analysing both dominating-set and geometric set-cover problems. Using these tools, thereafter, in Sections 5 and 6, we prove the locality conditions for the dominating-set and set-cover problems when the objects are homothets of a convex polygon and convex pseudodisks, respectively.

1.6. Notation and preliminaries

Throughout the paper, we use capital letters to denote objects and calligraphic font to denote sets of objects. We make the general-position assumption that if two objects of the input set have a nonempty intersection, then their interiors intersect. No three object boundaries intersect at a common point. We denote the set $\{1, 2, \ldots, n\}$ as $[n]$. By a *geometric object* (or object, in short) R, we refer to a simply connected compact region in $\mathbb{R}^2$ with a nonempty interior. In other words, the object R is a closed region bounded by a closed Jordan curve ∂R. The $\mathrm{int}(R)$ is defined as all the points in R which do not appear in the boundary ∂R. Given two objects U and V, we say that U has an *interior overlap* with V if $\mathrm{int}(U) \cap \mathrm{int}(V) \neq \emptyset$, and given a set of objects $\mathcal{V}$, we say that U has an *interior overlap* with $\mathcal{V}$ if U has an interior overlap with any $V \in \mathcal{V}$.

For a set of objects $\mathcal{R}$, we define the *cover-free region* of any object $R_i \in \mathcal{R}$ as $\mathrm{CF}(R_i, \mathcal{R}) = \bigcap_{\substack{R_j \in \mathcal{R} \\ R_j \neq R_i}} R_i \setminus R_j$. That is, $CF(R_i, R)$ is the portion of R_i that is not contained in any other object of R. Note that $\mathrm{CF}(R_i, \mathcal{R}) \cap R_j = \emptyset$ for all R_i, R_j $(i \neq j) \in \mathcal{R}$. When the underlying set of objects $\mathcal{R}$ is obvious, we use the term $\mathrm{CF}(R_i)$ instead of $\mathrm{CF}(R_i, \mathcal{R})$. A collection of geometric objects $\mathcal{R}$ is said to form a family of *pseudodisks* if the boundaries of every pair of them either do not intersect or cross (i.e., intersect non-tangentially) at exactly two points [30].

[2] Originally the assumption was NP $\not\subseteq$ DTIME$(n^{O(\log\log n)})$. This assumption was improved to P $\neq$ NP recently by Dinur and Steurer [9].

Algorithm 1: Local-Search($\mathscr{S}$, b).

Input: A set of n objects $\mathscr{S}$ in $\mathbb{R}^2$ and a parameter b
1 Initialize $\mathcal{A}$ to an arbitrary subset of $\mathscr{S}$ which is a feasible solution;
2 **while** $\exists\, \mathcal{X} \subseteq \mathcal{A}$ *of size at most* b*, and* $\mathcal{X}' \subseteq \mathscr{S}$ *of size at most* $|\mathcal{X}| - 1$ *such that* $(\mathcal{A} \setminus \mathcal{X}) \cup \mathcal{X}'$ *is a feasible solution* **do**
3 | set $\mathcal{A} \leftarrow (\mathcal{A} \setminus \mathcal{X}) \cup \mathcal{X}'$;
4 **Report**: $\mathcal{A}$;

A collection of geometric objects $\mathcal{R}$ is said to be *cover-free* if no object $R \in \mathcal{R}$ is covered by the union of the objects in $\mathcal{R} \setminus R$, in other words, $\mathrm{CF}(R, \mathcal{R}) \neq \emptyset$ for all objects in $\mathcal{R}$. Two objects are *homothetic* to each other if one object can be obtained from the other by scaling and translating.

Consider the *convex distance function* with respect to a convex object C with a fixed interior point as *center* as follows.

Definition 1. Given $p_1, p_2 \in \mathbb{R}^2$, *convex distance function* induced by C, denoted by $\delta_C(p_1, p_2)$, is the smallest $\alpha \geq 0$ such that $p_1, p_2 \in \alpha C$ while the center of C is at p_1.

It was first introduced by Minkowski in 1911 [22,6]. Note that this function satisfies the following properties.

Property 1.

(i) *The function δ_C is symmetric (i.e., $\delta_C(p_1, p_2) = \delta_C(p_2, p_1)$) if and only if C is centrally symmetric.*
(ii) *Let p_1 and p_3 be any two points in $\mathbb{R}^2$ and let p_2 be any point on the line segment $\overline{p_1 p_3}$, then $\delta_C(p_1, p_3) = \delta_C(p_1, p_2) + \delta_C(p_2, p_3)$.*
(iii) *The distance function δ_C follows the triangular inequality, i.e., and $\delta_C(p_1, p_3) \leq \delta_C(p_1, p_2) + \delta_C(p_2, p_3)$, where p_1, p_2 and p_3 are any three points in $\mathbb{R}^2$.*

2. Local-search algorithm

We use a standard local search algorithm [29] as given in Algorithm 1.

A subset of objects $\mathcal{A} \subseteq \mathscr{S}$ is said to be *b-locally optimal* if one cannot obtain a smaller feasible solution by removing a subset $\mathcal{X} \subseteq \mathcal{A}$ of size at most b from $\mathcal{A}$ and replacing it with a subset of size at most $|\mathcal{X}| - 1$ from $\mathscr{S} \setminus \mathcal{A}$. Our algorithm computes a b-locally optimal set of objects for $b = \frac{\alpha}{\varepsilon^2}$, where $\alpha > 0$ is a suitably large constant. Observe that at the end of the while-loop, the set $\mathcal{A}$ is b-locally optimal, and the set $\mathcal{A}$ is cover-free.

Since the size of $\mathcal{A}$ is decreased by at least one after each update in Line 3, the number of iterations of the while-loop is at most n, and each iteration takes $O(n^b)$ time as it needs to check every subset of size at most b. So, this while-loop needs $O(n^{b+1})$ time. Thus, the total time complexity of the above algorithm is $O(n^{b+1})$.

2.1. Analysis of approximation

We will analyze the algorithm's performance for both problems following the framework of [4,29]. When there is a difference, we will indicate the specific context within which the analysis is performed (set-cover or dominating-set). Let $\mathcal{O}$ be the optimal solution, and $\mathcal{A}$ be the solution returned by our local search algorithm. Note that both $\mathcal{O}$ and $\mathcal{A}$ ensure the following.

Claim 1. *For any object $A \in \mathcal{A}$ (resp., $O \in \mathcal{O}$), $\mathrm{CF}(A, \mathcal{A})$ (resp., $\mathrm{CF}(O, \mathcal{O})$) is nonempty. In other words, $\mathcal{A}$ (resp., $\mathcal{O}$) is cover-free.*

We can assume that no object $S \in \mathscr{S}$ is properly contained in any other object of $\mathscr{S}$. We can ensure this by an initial pass over the input objects in which we remove any object of the input that is contained within another object. Thus, we can assume that there is no object $S \in \mathscr{S} \setminus \mathcal{A}$ that completely contains any object of $\mathcal{A}$. Similarly, we can assume that no object in $\mathcal{O}$ is completely contained in any object from $\mathscr{S} \setminus \mathcal{O}$. Let $\mathcal{A}' = \mathcal{A} \setminus \mathcal{O}$, $\mathcal{O}' = \mathcal{O} \setminus \mathcal{A}$.

In the context of the dominating-set problem, let $\mathscr{S}' \subset \mathscr{S}$ be the set containing all objects of $\mathscr{S}$ that is not dominated by any object in $\mathcal{A} \cap \mathcal{O}$. Note that there does not exist an object $O \in \mathcal{O}'$ which covers $\mathrm{CF}(A_1, \mathcal{A}') \cup \mathrm{CF}(A_2, \mathcal{A}')$, $A_1, A_2 \in \mathcal{A}'$, otherwise local search would replace A_1 and A_2 by O. Similarly, there does not exist an object $A \in \mathcal{A}'$ which covers $\mathrm{CF}(O_1, \mathcal{O}') \cup \mathrm{CF}(O_2, \mathcal{O}')$, $O_1, O_2 \in \mathcal{O}'$ otherwise it would contradict the optimality of $\mathcal{O}$.

Now we are going to eliminate the same number of objects from both $\mathcal{A}'$ and $\mathcal{O}'$ to ensure that for any $A \in \mathcal{A}'$, $\mathrm{CF}(A, \mathcal{A}')$ is not properly contained in any object in $\mathcal{O}'$. Let $O \in \mathcal{O}'$ be an object that properly contains $\mathrm{CF}(A, \mathcal{A}')$ for an object $A \in \mathcal{A}'$. Let $\mathscr{S}''$ be the set containing all objects of $\mathscr{S}'$ that is not dominated by O. Note that both the sets $\mathcal{A}' \setminus A$ and $\mathcal{O}' \setminus O$ dominates $\mathscr{S}''$. We reset $\mathscr{S}' \leftarrow \mathscr{S}''$. We remove A and O from $\mathcal{A}'$ and $\mathcal{O}'$, respectively by updating $\mathcal{A}' \leftarrow \mathcal{A}' \setminus A$ and $\mathcal{O}' \leftarrow \mathcal{O}' \setminus O$. We repeat this until there is no object $O \in \mathcal{O}'$ that properly contains an object $A \in \mathcal{A}'$. Note that the removal of objects only makes cover-free regions larger. As a result, once for an object $A \in \mathcal{A}'$ we ensure that $\mathrm{CF}(A, \mathcal{A}')$ is not properly contained in any object in $\mathcal{O}'$, we do not need to reconsider the object A later.

Similarly, if there exists an object $A \in \mathcal{A}'$ that properly contains $\mathrm{CF}(O, \mathcal{O}')$ for an object $O \in \mathcal{O}'$, we update $\mathcal{A}' \leftarrow \mathcal{A}' \setminus A$ and $\mathcal{O}' \leftarrow \mathcal{O}' \setminus O$. Let $\mathscr{S}''$ be the set containing all objects of $\mathscr{S}'$ that is not dominated by A. We reset $\mathscr{S}' \leftarrow \mathscr{S}''$. We repeat this until there does not exist any object $A \in \mathcal{A}'$ that properly contains $\mathrm{CF}(O, \mathcal{O}')$ for an object $O \in \mathcal{O}'$. This ensures the following.

Claim 2. *For any object $A \in \mathcal{A}'$ (resp., $O \in \mathcal{O}'$), $\mathrm{CF}(A, \mathcal{A}')$ (resp., $\mathrm{CF}(O, \mathcal{O}')$) is not properly contained in any object in $\mathcal{O}'$ (resp., $\mathcal{A}'$).*

Observe that $|\mathcal{O} \setminus \mathcal{O}'| = |\mathcal{A} \setminus \mathcal{A}'|$. Finally, we will show that $|\mathcal{A}'| \leq (1+\varepsilon)|\mathcal{O}'|$ which implies that $|\mathcal{A}| \leq (1+\varepsilon)|\mathcal{O}|$.

In the context of geometric covering, we do a similar process as discussed above to ensure Claim 2. Here, let $\mathcal{P}'$ be the set containing all points of $\mathcal{P}$ that are covered by objects in $\mathcal{A}' \cap \mathcal{O}'$.

Henceforth, $\mathcal{A}', \mathcal{O}', \mathcal{P}'$ and $\mathscr{S}'$ will be denoted as $\mathcal{A}, \mathcal{O}, \mathcal{P}$ and $\mathscr{S}$, respectively, satisfying both Claims 1 and 2.

Now, we present *locality conditions* for the dominating-set and set-cover problems in Lemmas 1 and 2, respectively.

Lemma 1 *(Locality condition for dominating-set). There exists a planar graph $\mathcal{G} = (\mathcal{A} \cup \mathcal{O}, \mathcal{E})$ such that for all $S \in \mathscr{S}$, if S is dominated by at least one object of $\mathcal{A}$ and at least one object of $\mathcal{O}$, then there exists $A \in \mathcal{A}$ and $O \in \mathcal{O}$ both of which dominate S and $(A, O) \in \mathcal{E}$.*

Lemma 2 *(Locality condition for set-cover). There exists a planar graph $\mathcal{G} = (\mathcal{A} \cup \mathcal{O}, \mathcal{E})$ such that for all points $p \in \mathcal{P}$, if p is covered by at least one object of $\mathcal{A}$ and at least one object of $\mathcal{O}$, then there exists $A \in \mathcal{A}$ and $O \in \mathcal{O}$ both of which cover p and $(A, O) \in \mathcal{E}$.*

As a warm-up, in Section 3, we first present a simple proof of the locality condition lemma (Lemma 1) when objects are homothets of a centrally symmetric convex object. In Section 5 (resp., Section 6), we prove the locality condition lemma for the dominating-set (resp., set-cover) problems when objects are homothets of a convex object (resp., convex pseudodisks).

Once we have established both of these locality condition lemmas, the analysis of the algorithm is the same as in [29]. For the sake of completeness, we provide the following analysis. As the graph $\mathcal{G}$ is planar, the following planar separator theorem can be used.

Theorem 3 *(Frederickson [14]). For any planar graph $\mathcal{G} = (\mathcal{V}, \mathcal{E})$ with n vertices and a parameter $1 \leq r \leq n$, there is a set $\mathcal{X} \subseteq \mathcal{V}$ of size at most $\frac{c_1 n}{\sqrt{r}}$, such that $\mathcal{V} \setminus \mathcal{X}$ can be partitioned into $\lceil n/r \rceil$ sets $\mathcal{V}_1, \mathcal{V}_2, \ldots \mathcal{V}_{\lceil n/r \rceil}$ satisfying (i) $|\mathcal{V}_i| \leq c_2 r$, (ii) $N(\mathcal{V}_i) \cap \mathcal{V}_j = \emptyset$ for $i \neq j$, and $|N(\mathcal{V}_i) \cap \mathcal{X}| \leq c_3\sqrt{r}$, where $c_1, c_2, c_3 > 0$ are constants, and $N(\mathcal{V}') = \{U \in \mathcal{V} \setminus \mathcal{V}' \mid \exists V \in \mathcal{V}' \text{ with } (U, V) \in \mathcal{E}\}$.*

We apply Theorem 3 to the graphs described in Lemmas 1 and 2, setting $r = b/c_2$, where c_2 is the constant of Theorem 3. Here, $n = |\mathcal{A}| + |\mathcal{O}|$ and $r = c_4/\varepsilon^2$, for some constant c_4. So, $|\mathcal{V}_i| \leq b$. Let $\mathcal{A}_i = \mathcal{A} \cap \mathcal{V}_i$ and $\mathcal{O}_i = \mathcal{O} \cap \mathcal{V}_i$. Note that we must have

$$|\mathcal{A}_i| \leq |\mathcal{O}_i| + |N(\mathcal{V}_i) \cap \mathcal{X}|, \tag{1}$$

otherwise our local search would continue to replace $\mathcal{A}_i$ by $\mathcal{O}_i \cup N(\mathcal{V}_i)$, resulting in a better solution. For a suitable constant c_5, we now have

$$\begin{aligned}
|\mathcal{A}| &\leq |\mathcal{X}| + \sum_i |\mathcal{A}_i| && \text{(Each element of } \mathcal{A} \text{ either belongs to } \mathcal{A}_i \text{ or } \mathcal{X}\text{)}\\
&\leq |\mathcal{X}| + \sum_i |\mathcal{O}_i| + \sum_i |N(\mathcal{V}_i) \cap \mathcal{X}| && \text{(Follows from Equation (1))}\\
&\leq |\mathcal{O}| + |\mathcal{X}| + \sum_i |N(\mathcal{V}_i) \cap \mathcal{X}| && (\mathcal{O}_i \text{ are disjoint subsets of } \mathcal{O})\\
&\leq |\mathcal{O}| + \frac{c_5(|\mathcal{A}| + |\mathcal{O}|)}{\sqrt{b}} && \left(\text{As } \sum_i |N(\mathcal{V}_i) \cap \mathcal{X}| \leq \lceil n/r \rceil (c_3\sqrt{r}) \text{ and } |\mathcal{X}| \leq c_1 \tfrac{|\mathcal{A}|+|\mathcal{O}|}{\sqrt{r}}\right)\\
|\mathcal{A}| &\leq \frac{1 + c_5/\sqrt{b}}{1 - c_5/\sqrt{b}} |\mathcal{O}| && \text{(By rearranging)}\\
|\mathcal{A}| &\leq (1+\varepsilon)|\mathcal{O}| && \left(b \text{ is large enough constant times } \tfrac{1}{\varepsilon^2}\right)
\end{aligned}$$

3. Dominating-set for homothets of a centrally symmetric convex object

In this section, as a warm-up, we give a simple analysis of the local search algorithm for the dominating-set problem when the objects are homothets of a centrally symmetric convex object. Our analysis is a generalization of Gibson et al. [16].

Let C be a centrally symmetric convex object in the plane with the center $c(C)$. Given a set $\mathscr{S}$ of homothets of C, our objective is to show that the local-search algorithm given in Section 2 is a PTAS for the minimum dominating-set for $\mathscr{S}$. Recall that $\mathcal{A}$ is the set of objects returned by the local-search algorithm, and $\mathcal{O}$ is the minimum dominating-set. As a continuation from Section 2, we assume that both Claims 1 and 2 are satisfied, and we only need to prove the locality condition mentioned in Lemma 1.

We consider a nearest-site Voronoi diagram for all objects in $\mathcal{A} \cup \mathcal{O}$ with respect to a distance function δ^*_C. First, we will extend the convex distance function to provide meaning (albeit negative) to the interior of each site. This would allow us to interpret the Voronoi diagram as a Voronoi diagram of additively weighted points, rather than a Voronoi diagram of (unweighted) regions. For each object $S \in \mathscr{S}$, we define the *weight* $w(S)$ to be α such that $S = c(S) + \alpha C$. Now, we define the distance $\delta^*_C(p, S)$ between a point $p \in \mathbb{R}^2$ and an object $S \in \mathscr{S}$ as follows: $\delta^*_C(p, S) = \delta_C(p, c(S)) - w(S)$. The distance function $\delta^*_C(p, S)$ has the following properties:

Property 2.

(i) *The distance function $\delta^*_C(p, S)$ achieves its minimum value when $p = c(S)$.*
(ii) *If p is contained in the object S, then $\delta^*_C(p, S) \leq 0$.*
(iii) *If $\delta^*_C(p, S) > 0$, then p is outside the object S, and a translated copy of C centered at p with scaling factor $\delta^*_C(p, S)$ touches the object S.*

Note that Property 2(iii) is crucial for our analysis, and it follows due to the symmetric property of δ_C. As a result, this approach cannot be applied when objects are not centrally symmetric.

Now, we define the nearest-site Voronoi diagram NVD_{C^*} for all the objects in $\mathcal{A} \cup \mathcal{O}$ with respect to the distance function δ^*_C. We define Voronoi cell of $S_i \in \mathcal{A} \cup \mathcal{O}$ as $\mathrm{Cell}(S_i) = \{p \in \mathbb{R}^2 | \delta^*_C(p, S_i) \leq \delta^*_C(p, S_j) \textit{ for all } j \neq i\}$. The NVD_{C^*} is a partition on the plane imposed by the collection of cells of all the objects in $\mathcal{A} \cup \mathcal{O}$.

Let us consider the graph $\mathcal{G} = (\mathcal{V}, \mathcal{E})$, the *dual* of the Voronoi diagram NVD_{C^*}, whose vertices $\mathcal{V}$ are the elements of $\mathcal{A} \cup \mathcal{O}$ and the edge set $\mathcal{E}$ consists of pairs $U, V \in \mathcal{V}$ whose Voronoi cells share an edge on their boundaries.

We will show that each object in $\mathcal{A} \cup \mathcal{O}$ has a nonempty cell in this Voronoi diagram (see Lemma 3), and each cell is simply connected (see Lemma 4). As a result, the graph $\mathcal{G} = (\mathcal{V}, \mathcal{E})$ that is the dual of this Voronoi diagram is planar. Finally, we will show (in Lemma 5) that this graph satisfies the locality condition mentioned in Lemma 1. This completes the proof.

Lemma 3. *The cell of every object $S \in \mathcal{A} \cup \mathcal{O}$ is non-empty. Moreover, the center $c(S) \subseteq \mathrm{Cell}(S)$.*

Proof. For the sake of contradiction, assume for some object $S \in \mathcal{A} \cup \mathcal{O}$, $c(S) \notin \mathrm{Cell}(S)$ and $c(S) \in \mathrm{Cell}(S')$ where $S'(\neq S) \in \mathcal{A} \cup \mathcal{O}$. So, $\delta^*_C(c(S), S) \geq \delta^*_C(c(S), S')$. Since $\delta^*_C(c(S), S) = -w(S)$, we have $-w(S) \geq \delta_C(c(S), c(S')) - w(S')$. This implies $w(S') \geq \delta_C(c(S), c(S')) + w(S)$ which means that the object S is contained in the object S'. This contradicts Claim 1 and 2. $\square$

Lemma 4. *Each cell $\mathrm{Cell}(S)$ is simply connected.*

Proof. We first claim that for every point $p \in \mathrm{Cell}(S)$, the line segment $\overline{pc(S)} \subseteq \mathrm{Cell}(S)$. To see this, suppose to the contrary that there exists a point $q \in \overline{pc(S)}$ such that $q \in \mathrm{Cell}(S')$ where $S'(\neq S) \in \mathcal{A} \cup \mathcal{O}$. Then by basic properties of convex distance functions (Property 1), we have

$$\delta^*_C(p, S') = \delta_C(p, c(S')) - w(S') \leq \delta_C(p, q) + \delta_C(q, c(S')) - w(S') \leq \delta_C(p, q) + \delta^*_C(q, S')$$

$$< \delta_C(p, q) + \delta^*_C(q, S) = \delta_C(p, q) + \delta_C(q, c(S)) - w(S) = \delta_C(p, c(S)) - w(S) = \delta^*_C(p, S),$$

contradicting the fact that $p \in \mathrm{Cell}(S)$.

To see that $\mathrm{Cell}(S)$ is connected, observe that any two points $p, p' \in \mathrm{Cell}(S)$ can be connected via $c(S)$ as follows. First, connect p to $c(S)$ and then connect p' to $c(S)$. By the above claim and Lemma 3, all of these segments lie within $\mathrm{Cell}(S)$.

To complete the proof that $\mathrm{Cell}(S)$ is simply connected, we use the well-known equivalent characterization [23] that for any simple closed (i.e., Jordan) curve $\Psi \subset \mathrm{Cell}(S)$, the interior of the region bounded by this curve lies entirely within $\mathrm{Cell}(S)$. Consider any x in the interior of the region bounded by Ψ. Either $x = c(S)$ or (by extending the ray from $c(S)$ through x until it hits Ψ) there exists $p \in \mathrm{Cell}(S)$ such that x lies on the line segment $\overline{pc(S)}$. In the former case, $x \in \mathrm{Cell}(S)$, follows from Lemma 3. For the latter case, by the above claim (that $\overline{pc(S)} \subseteq \mathrm{Cell}(S)$), we have $x \in \mathrm{Cell}(S)$. This completes the proof. $\square$

Lemma 5. *For any arbitrary input object $S \in \mathscr{S}$, there is an edge between $(A, O) \in \mathcal{G}$ such that $A \in \mathcal{A}$ and $O \in \mathcal{O}$, and both A and O dominate S.*

Proof. Let us assume that A and O are the closest objects to $c(S)$ (with respect to δ_C^*) in $\mathcal{A}$ and $\mathcal{O}$, respectively. Since both $\mathcal{A}$ and $\mathcal{O}$ are dominating sets, both A and O must dominate S. Note that $c(S)$ belongs to at least one of the Voronoi cells: Cell(A) or Cell(O). If $c(S)$ belongs to both Cell(A) and Cell(O), then $c(S)$ lies on the common boundary of Cell(A) and Cell(O). Therefore, the edge (A, O) belongs to the graph $\mathcal{G}$. Without loss of generality, we may assume that $c(S)$ does not belong to Cell(O). Note that $c(O)$ lies in Cell(O) (due to Lemma 3). Therefore, $c(S)$ and $c(O)$ lie in different Voronoi cells. So the line segment $\overline{c(S)c(O)}$ must intersect an edge of Cell(O) at some point p. Let Cell(R) denote the cell neighboring the Cell(O) along this edge. We have $\delta_C^*(p, R) = \delta_C^*(p, O)$. By basic properties of the convex distance function (see Property 1) and the definition of the distance function δ_C^*, we obtain

$$
\begin{aligned}
\delta_C^*(c(S), R) &= \delta_C(c(S), c(R)) - w(R) && \text{(Follows from the definition of } \delta_C^*\text{)}\\
&\le \delta_C(c(S), p) + \delta_C(p, c(R)) - w(R) && \text{(Due to Property 1(iii) of convex distance function)}\\
&= \delta_C(c(S), p) + \delta_C^*(p, R) && \text{(Follows from the definition of } \delta_C^*\text{)}\\
&= \delta_C(c(S), p) + \delta_C^*(p, O) && \text{(}p\text{ is on the boundary of both Cell}(R)\text{ and Cell}(O)\text{)}\\
&= \delta_C(c(S), p) + \delta_C(p, c(O)) - w(O) && \text{(Follows from the definition of } \delta_C^*\text{)}\\
&= \delta_C(c(S), c(O)) - w(O) && \text{(Due to Property 1(ii) of convex distance function)}\\
&= \delta_C^*(c(S), O). && \text{(Follows from the definition of } \delta_C^*\text{)}
\end{aligned}
$$

By general position, we may assume that $\delta_C(c(S), R) < \delta_C(c(S), O)$. Since O was chosen to be the closest object in $\mathcal{O}$ to $c(S)$, it follows that $R \in \mathcal{A}$. Clearly, the objects R and O both dominate S. Therefore, there is an edge (R, O) in $\mathcal{G}$, as desired. □

4. Tools for constructing disjoint objects

In this section, we present two tools (or lemmata) which are essential for analyzing our main results. An essential step in our analysis (particularly in constructing the planar graph of Section 2.1) involves replacing a collection of overlapping objects that cover a given region with a collection of non-overlapping objects that cover the same region. This leads to the notion of a *decomposition*. The decomposition, we define here, is inspired by the idea of core decomposition introduced by Mustafa et al. [28].

Definition 2. Given a set of convex objects $\mathcal{R} = \{R_1, \ldots, R_n\}$, a set $\widetilde{\mathcal{R}} = \{\widetilde{R}_1, \ldots, \widetilde{R}_n\}$ of convex objects is called a *sub-decomposition* if for each $i \in [n]$, $\widetilde{R}_i \subseteq R_i$. Such a set $\widetilde{\mathcal{R}}$ is called a *decomposition* if the same region is covered, that is, $\bigcup_{i\in[n]} \widetilde{R}_i = \bigcup_{i\in[n]} R_i$. We refer to $\widetilde{R}_i$ as the *trace* of R_i, $i \in [n]$. Further, if the elements of $\widetilde{\mathcal{R}}$ have pairwise disjoint interiors, the decomposition/sub-decomposition is said to be *disjoint*.

First, we prove the following lemma which is reminiscent of [28, Lem. 3.3]. Edelsbrunner [10] introduced a very similar decomposition in the context of Euclidean disks.

Lemma 6. *For a cover-free set of convex pseudodisks $\mathcal{R} = \{R_1, \ldots, R_n\}$, there exists a disjoint decomposition $\widetilde{\mathcal{R}} = \{\widetilde{R}_1, \ldots, \widetilde{R}_n\}$ such that* CF$(R_j, \mathcal{R}) \subseteq \widetilde{R}_j$*, for all $j \in [n]$.*

Proof. The proof is constructive. The algorithm to construct a disjoint decomposition $\widetilde{\mathcal{R}} = \{\widetilde{R}_1, \ldots, \widetilde{R}_n\}$ of $\mathcal{R} = \{R_1, \ldots, R_n\}$ is as follows. This is an n-phase algorithm. After the i^{th} phase, the following invariants are maintained for all $i \in [n]$.

Invariant 1. *The objects in $\widetilde{\mathcal{R}}^i = \{\widetilde{R}^i_1, \ldots, \widetilde{R}^i_n\}$ form a decomposition of $\mathcal{R} = \{R_1, \ldots, R_n\}$ such that (i)* CF$(R_j) \subseteq \widetilde{R}^i_j$ *for all $j \in [n]$, and (ii)* $\mathrm{int}(\widetilde{R}^i_t) \cap \mathrm{int}(\widetilde{R}^i_q) = \emptyset$ *where $t \neq q$ and $1 \le t \le i$, $1 \le q \le n$.*

Invariant 2. *The objects in $\widetilde{\mathcal{R}}^i = \{\widetilde{R}^i_1, \ldots, \widetilde{R}^i_n\}$ form a collection of convex pseudodisks.*

We initialize $\widetilde{\mathcal{R}}^0 = \mathcal{R}$. This satisfies both invariants. At the beginning of the i^{th} phase, we set $X = \widetilde{R}^{i-1}_i$. Let $\mathcal{R}^i_\pi = \{\widetilde{R}^{i-1}_{\pi(1)}, \ldots, \widetilde{R}^{i-1}_{\pi(\ell)}\}$, $0 \le \ell < n$ be the set of objects in $\widetilde{\mathcal{R}}^{i-1}$ that intersect $\mathrm{int}(\widetilde{R}^{i-1}_i)$. In other words, $\mathrm{int}(\widetilde{R}^{i-1}_i) \cap \mathrm{int}(\widetilde{R}^{i-1}_{\pi(j)}) \neq \emptyset$ for any $\pi(j) \in \Pi$, where $\Pi = \{\pi(1), \ldots, \pi(\ell)\}$.

Consider any object $\widetilde{R}^{i-1}_{\pi(j)} \in \mathcal{R}^i_\pi$. As $\widetilde{R}^{i-1}_{\pi(j)}$ and X are pseudodisks, their respective boundaries intersect in two points. Let p_1 and p_2 be these two intersection points. By convexity, the line segment $\overline{p_1p_2}$ is contained in both $\widetilde{R}^{i-1}_{\pi(j)}$ and X. Let $\mathcal{C}_1$ (respectively, $\mathcal{C}_2$) be the part of the boundary of $\widetilde{R}^{i-1}_{\pi(j)}$ (respectively, X) that lie inside X (respectively, $\widetilde{R}^{i-1}_{\pi(j)}$). We replace

Fig. 1. Illustration of Lemma 6.

both $\mathcal{C}_1$ and $\mathcal{C}_2$ by the line segment $\overline{p_1 p_2}$. In this way, we obtain new convex objects $\widetilde{R}^i_{\pi(j)} \subseteq \widetilde{R}^{i-1}_{\pi(j)}$ and $X_j \subseteq X$ that have interiors that are pairwise disjoint with each other, and $\widetilde{R}^i_{\pi(j)} \cup X_j = \widetilde{R}^{i-1}_{\pi(j)} \cup X$. See Fig. 1 for illustration.

For all $\pi(j) \in \Pi$, we construct the corresponding $\widetilde{R}^i_{\pi(j)}$ as above. At the end of this phase, we assign $\widetilde{R}^i_i = \bigcap_{j\in\Pi} X_j$. Note that $\widetilde{R}^i_i$ is also convex as it is an intersection of some convex objects. We set $\widetilde{R}^i_j = \widetilde{R}^{i-1}_j$ for all $j(\neq i) \in [n] \setminus \Pi$. As a result, we obtain a collection of convex objects $\widetilde{\mathcal{R}}^i$.

Observe that, for any point p, that is contained in the union of $\mathcal{R}^i_\pi$, either there exists a j such that this point lies within $\widetilde{R}^i_{\pi(j)}$, and so is covered by this set, or it lies within X_j for all j. Hence, it lies within their common intersection, which is X. So, $\widetilde{\mathcal{R}}^i$ is a decomposition of $\widetilde{\mathcal{R}}^{i-1}$.

Thus, after the i^{th} phase, we obtain a decomposition $\widetilde{\mathcal{R}}^i$ such that $\text{int}(\widetilde{R}^i_i) \cap \text{int}(\widetilde{R}^i_j) = \emptyset$ for all $j(\neq i) \in \{1, \dots, n\}$. On the other hand, we have $\text{int}(\widetilde{R}^{i-1}_t) \cap \text{int}(\widetilde{R}^{i-1}_q) = \emptyset$ where $t \neq q$ and $1 \leq t \leq i-1$, $1 \leq q \leq n$. Combining these, we obtain $\text{int}(\widetilde{R}^i_t) \cap \text{int}(\widetilde{R}^i_q) = \emptyset$ where $t \neq q$ and $1 \leq t \leq i$, $1 \leq q \leq n$.

Since the union of objects in $\widetilde{\mathcal{R}}^i$ is same as the union of the objects in $\widetilde{\mathcal{R}}^{i-1}$, and the objects in $\widetilde{\mathcal{R}}^{i-1}$ are cover-free, so each object $\widetilde{R}^i_j$ has its cover-free region $\text{CF}(R_j)$ which is not covered by others, for all $j \in [n]$. Thus, Invariant 1 is maintained. Now, we prove that Invariant 2 is also maintained. We prove the objects in $\widetilde{\mathcal{R}}^i$ form pseudodisks by showing the following claim.

Claim 3. *$\widetilde{\mathcal{R}}^i$ is a collection of convex pseudodisks.*

Proof. It suffices to show that for any two objects $\widetilde{R}^{i-1}_{\ell_1}$ and $\widetilde{R}^{i-1}_{\ell_2}$ in $R^{\,i-1}$, their boundaries $\partial\widetilde{R}^i_{\ell_1}$ and $\partial\widetilde{R}^i_{\ell_2}$ can cross each other at most twice.

Recall the definition of X from the above construction. For any $R \in \mathcal{R}^i_\pi$, let $I(R)$ be the interval $R \cap \partial X$ on the boundary of X. Due to the Invariant 1, no pseudodisk in $\widetilde{\mathcal{R}}^{i-1}$ is completely contained in another pseudodisk, so the intervals are well defined.

There are three possible cases:

- Case 1: $I(\widetilde{R}^{i-1}_{\ell_1}) \cap I(\widetilde{R}^{i-1}_{\ell_2}) = \emptyset$,
- Case 2: $I(\widetilde{R}^{i-1}_{\ell_1}) \subseteq I(\widetilde{R}^{i-1}_{\ell_2})$,
- Case 3: $I(\widetilde{R}^{i-1}_{\ell_1}) \cap I(\widetilde{R}^{i-1}_{\ell_2}) \neq \emptyset$ and $I(\widetilde{R}^{i-1}_{\ell_1}) \nsubseteq I(\widetilde{R}^{i-1}_{\ell_2})$.

In both Case 1 and Case 2 (see Fig. 2(a) and (b)), $\partial\widetilde{R}^i_{\ell_1}$ and $\partial\widetilde{R}^i_{\ell_2}$ do not have any new crossing which $\partial\widetilde{R}^{i-1}_{\ell_1}$ and $\partial\widetilde{R}^{i-1}_{\ell_2}$ did not have. In fact, they may lose intersections lying in X. As $\partial\widetilde{R}^{i-1}_{\ell_1}$ and $\partial\widetilde{R}^{i-1}_{\ell_2}$ may cross each other at most twice, so does $\partial\widetilde{R}^i_{\ell_1}$ and $\partial\widetilde{R}^i_{\ell_2}$. In Case 3 (see Fig. 2(c)), $\partial\widetilde{R}^{i-1}_{\ell_1}$ and $\partial\widetilde{R}^{i-1}_{\ell_2}$ crosses each other once in X and once outside X. The outside crossing remains the same for $\partial\widetilde{R}^i_{\ell_1}$ and $\partial\widetilde{R}^i_{\ell_2}$. They cross each other once along a new part of their boundaries, i.e., along the boundary of $X_{\ell_1} \cap X_{\ell_2}$. Thus, the claim follows. $\square$

After completion of the n^{th} phase, we assign $\widetilde{\mathcal{R}} = \widetilde{\mathcal{R}}^n$. The proof of the lemma follows from the Invariant 1. $\square$

Now, we prove the following important lemma, which we use as a tool for obtaining disjoint sub-decompositions. The previous lemma obtains disjoint decomposition when the objects are pseudodisks. When the set of objects does not satisfy the pseudodisk property, but they are trimmed from a set of pseudodisks, we apply the following tool to obtain a disjoint sub-decomposition.

Lemma 7. *Given two sets $\mathcal{U}$ and $\mathcal{V}$ of distinct convex objects such that their union forms a collection of pseudodisks, let $\mathcal{U}^0$ and $\mathcal{V}^0$ be any disjoint sub-decompositions of $\mathcal{U}$ and $\mathcal{V}$, respectively. Let U_i and V_j be any two convex pseudodisks from $\mathcal{U}$ and $\mathcal{V}$, respectively, and*

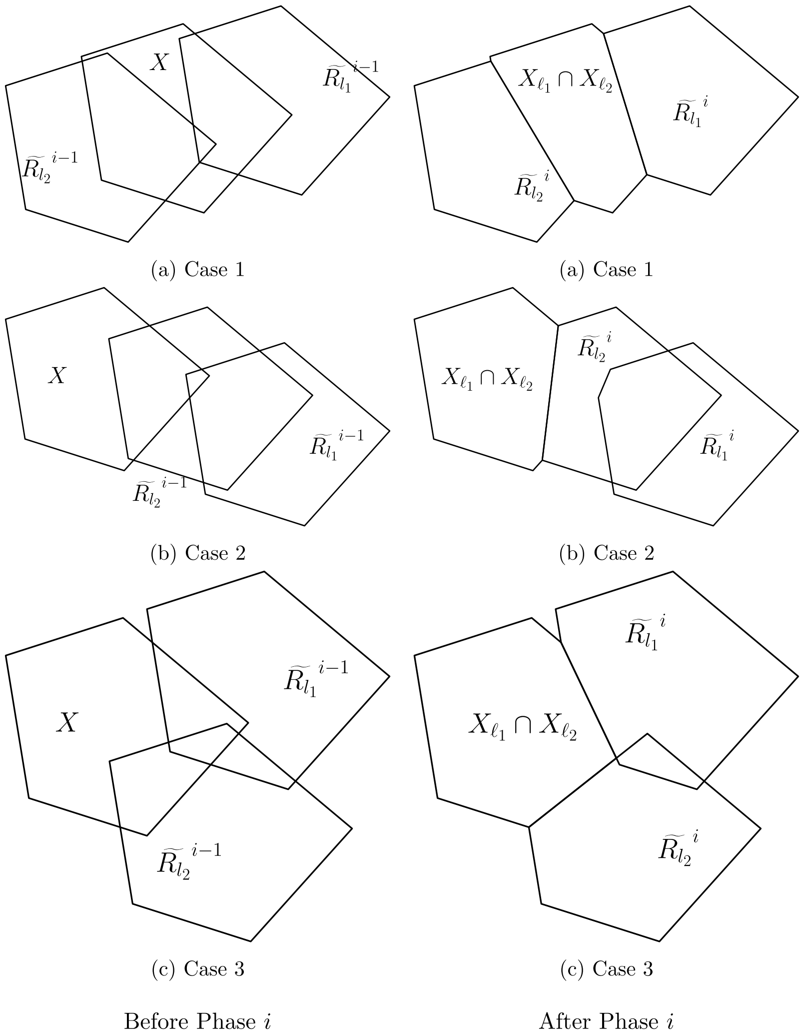


Fig. 2. Illustration of Claim 3.

U_i^0 and V_j^0 be two corresponding convex objects from $\mathcal{U}^0$ and $\mathcal{V}^0$, respectively, such that $\mathrm{CF}(U_i^0, \mathcal{U}^0 \cup \mathcal{V}^0) \neq \emptyset$, $\mathrm{CF}(V_j^0, \mathcal{U}^0 \cup \mathcal{V}^0) \neq \emptyset$ *and* $\mathrm{int}(U_i^0) \cap \mathrm{int}(V_j^0) \neq \emptyset$. *Then we can find $U_{ij}^0 \subseteq U_i^0$ and $V_{ji}^0 \subseteq V_j^0$ such that the following properties are satisfied.*

(i) *U_{ij}^0 and V_{ji}^0 are convex, have nonempty disjoint interiors, and their intersection consists of a separating line segment, which we denote by E_{ij}^0.*
(ii) *$U_i^0 \setminus U_{ij}^0$ is completely contained in V_j.*
(iii) *$V_j^0 \setminus V_{ji}^0$ is completely contained in U_i.*

Proof. Given two convex objects U and V, define a *petal* of U with respect to V to be a connected component of $U \setminus V$. Since U_i^0 and V_j^0 need not be pseudodisks, there may be multiple petals of U_i^0 with respect to V_j^0. Let us assume that there are k such petals, that we denote by $\mathrm{Petal}_t(U_i^0)$, for $1 \leq t \leq k$. Thus, $U_i^0 \setminus V_j^0 = \bigcup_{t=1}^{k} \mathrm{Petal}_t(U_i^0)$. Similarly, we define $\mathrm{Petal}(V_j^0)$ to be the set of petals of V_j^0 with respect to U_i^0, and we let k' denote their number. Observe that each petal is bounded by two boundary arcs, one from ∂U_i^0 and the other from ∂V_j^0 (see Fig. 3). Also, observe that consecutive petals are defined by consecutive intersection points between the boundaries of the two objects.

Since $V_j^0 \subseteq V_j$, we have $U_i^0 \setminus V_j \subseteq U_i^0 \setminus V_j^0$. Define $\mathrm{NCpetal}(U_i^0)$ to be the subset of petals of U_i^0 (with respect to V_j^0) that are not entirely covered by V_j, that is, $\mathrm{NCpetal}(U_i^0) = \{\mathrm{Petal}_t(U_i^0) \in \{U_i^0 \setminus V_j^0\} | \mathrm{Petal}_t(U_i^0) \cap \{U_i^0 \setminus V_j\} \neq \emptyset\}$. Similarly, we define $\mathrm{NCpetal}(V_j^0)$. Because $\mathrm{CF}(U_i^0, \mathcal{U}^0 \cup \mathcal{V}^0) \neq \emptyset$, $\mathrm{NCpetal}(U_i^0)$ contains at least one element, and the same holds for $\mathrm{NCpetal}(V_j^0)$ (see Fig. 3).

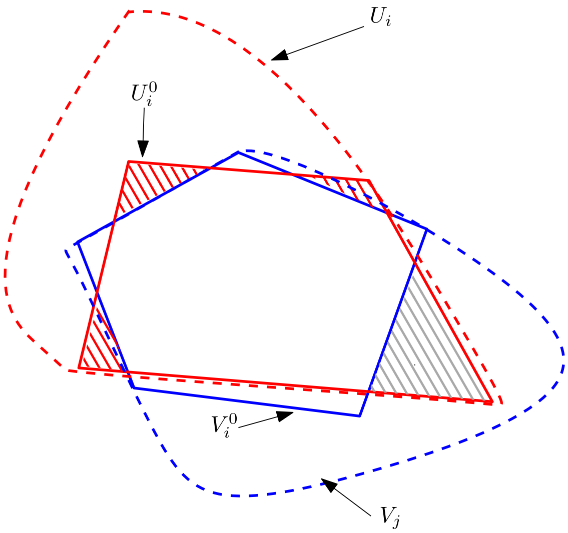


Fig. 3. Petals: tiled regions are Petals of U_i^0; NCpetals are marked with red. (For interpretation of the colors in the figure(s), the reader is referred to the web version of this article.)

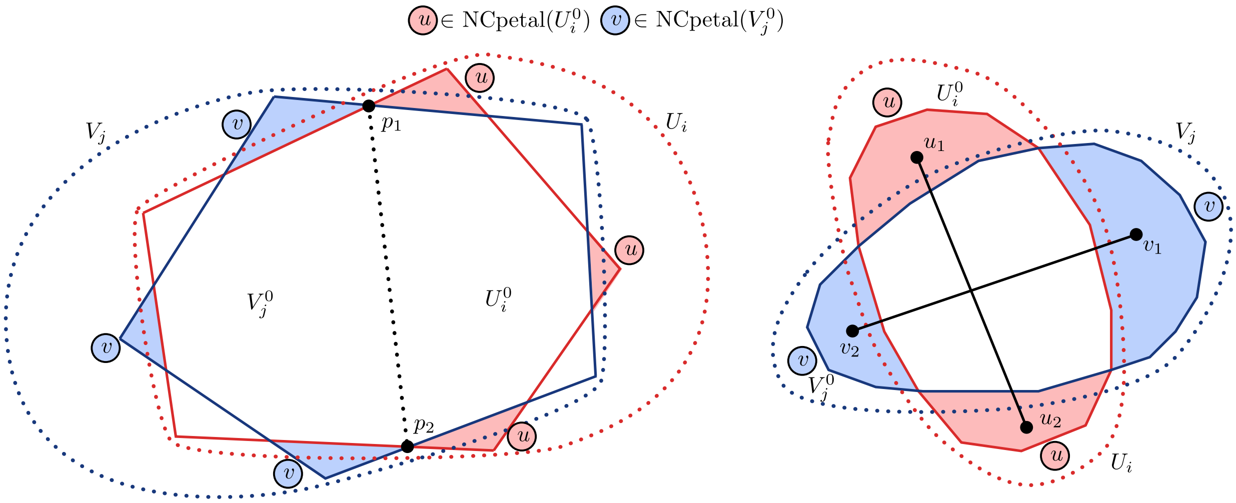


Fig. 4. Illustration of Lemma 7.

Consider only the uncovered petals (that is, $\mathrm{NCpetal}(U_i^0) \cup \mathrm{NCpetal}(V_j^0)$). Let us label the petals of $\mathrm{NCpetal}(U_i^0)$ with the letter "u" and label the petals of $\mathrm{NCpetal}(V_j^0)$ with the letter "v". Let $R_{ij}^0 = U_i^0 \cap V_j^0$. If you consider the cyclic order of these petals around ∂R_{ij}^0, the alternating pattern "u…v…u…v" cannot occur in the cyclic sequence as shown in the following argument (see Fig. 4).

> Suppose to the contrary that the alternating pattern "u…v…u…v" occurs in the cyclic sequence. Then there must exist points u_1, u_2 (from the first and third "u" petals in the sequence) that lie in $U_i^0 \setminus V_j^0$. Similarly, there exist points v_1, v_2 (from the second and fourth "v" petals) that lie in $V_j^0 \setminus U_i^0$. Because of the alternation, the line segments $\overline{u_1u_2}$ and $\overline{v_1v_2}$ intersect in R_{ij}^0. However, the existence of these two line segments violates the hypothesis that U_i and V_j are pseudodisks.

Since the alternation pattern "u…v…u…v" cannot arise in the cyclic sequence, it follows the cyclic order of uncovered petals around ∂R_{ij}^0 consists of a sequence of petals from $\mathrm{NCpetal}(U_i^0)$ followed by a sequence from $\mathrm{NCpetal}(V_j^0)$. As a result, we can find a line segment $\overline{p_1p_2}$ lying in $\mathrm{int}(R_{ij}^0)$ whose two endpoints are on ∂R_{ij}^0 such that all the uncovered petals of U_i^0 (formally $\mathrm{NCpetal}(U_i^0)$) lie on one side of this line segment. The uncovered petals of V_j^0 (formally $\mathrm{NCpetal}(V_j^0)$) lie on the other side. In other words, extension of this line segment $\overline{p_1p_2}$ partitions the plane into two half-spaces $\mathcal{H}_i^0$ and $\mathcal{H}_j^0$ where $\mathcal{H}_i^0$ contains all the petals of NCpetal(U_i^0) and $\mathcal{H}_j^0$ contains all the petals of NCpetal(V_j^0). We define $U_{ij}^0 = \mathcal{H}_i^0 \cap U_i^0$ and $V_{ji}^0 = \mathcal{H}_j^0 \cap V_j^0$. The line segment $\overline{p_1p_2}$ plays the role of the separating line segment E_{ij}^0. Claim (i) follows because p_1 and p_2 lie on the boundary of both U_i^0 and V_j^0. Claim (ii) follows because $U_i^0 \setminus U_{ij}^0$ consists of a portion of R_{ij}^0 (which clearly lies in V_j) together with a subset of petals of U_i^0 that are all covered by V_j. Claim (iii) is symmetrical. Hence U_{ij}^0, V_{ij}^0 satisfy the lemma. □

5. Dominating-set for homothetic convex objects

In this section, we show mainly the existence of a planar graph satisfying the locality condition mentioned in Lemma 1 when objects are homothetic copies of a convex object. Let C be a convex object in the plane. We fix an arbitrary interior point of C as the *center* $c(C)$. We are given a set $\mathscr{S}$ of n homothetic (i.e., translated and uniformly scaled) copies of C, and our objective is to show that the local-search algorithm given in Section 2 produces a PTAS for the minimum dominating-set for $\mathscr{S}$. Recall that $\mathcal{A}$ is the set of objects returned by the local-search algorithm, and $\mathcal{O}$ is a minimum dominating-set. Without loss of generality, we assume that both Claims 1 and 2 are satisfied.

Here is an overview of the proof. First, we find a disjoint sub-decomposition $\widetilde{\mathcal{A}} \cup \widetilde{\mathcal{O}}$ of $\mathcal{A} \cup \mathcal{O}$ (in Lemma 8). Next, we consider a nearest-site Voronoi diagram for the sites in $\widetilde{\mathcal{A}} \cup \widetilde{\mathcal{O}}$ with respect to a distance function. Then we show (in Lemma 12) that the dual of this Voronoi diagram satisfies the locality condition mentioned in Lemma 1.

5.1. Decomposing into interior disjoint convex sites

Using Lemmas 6 and 7 as tools, we now prove the following which is one of the important observations of our work.

Lemma 8. *Let $\mathcal{A}$ be the output of the local-search algorithm for dominating-set on a set $\mathscr{S}$ of homothetic convex objects, and let $\mathcal{O}$ be the optimum dominating-set. Then there exists a disjoint sub-decomposition $\widetilde{\mathcal{A}} \cup \widetilde{\mathcal{O}}$ that satisfies the following: for any input object $S \in \mathscr{S}$ either*

(i) *there exist $\widetilde{A} \in \widetilde{\mathcal{A}}$ and $\widetilde{O} \in \widetilde{\mathcal{O}}$ such that $S \cap \widetilde{A} \neq \emptyset$ and $S \cap \widetilde{O} \neq \emptyset$, or*
(ii) *there exist $A \in \mathcal{A}$ and $O \in \mathcal{O}$ such that $S \cap A \cap O \neq \emptyset$, and their traces $\widetilde{A}$ and $\widetilde{O}$ share an edge on their boundary.*

The remainder of this section is devoted to the proof of this lemma. As a continuation from Section 2.1, we would like to remind the reader that duplicate objects have been pruned from $\mathcal{A}$ and $\mathcal{O}$.

Let $\mathcal{A} = \{A_1, \ldots, A_\ell\}$ and $\mathcal{O} = \{O_1, \ldots, O_t\}$. Our algorithm to obtain a disjoint sub-decomposition $\widetilde{\mathcal{A}} \cup \widetilde{\mathcal{O}} = \{\widetilde{A}_1, \ldots \widetilde{A}_\ell\} \cup \{\widetilde{O}_1, \ldots \widetilde{O}_t\}$ for $\mathcal{A} \cup \mathcal{O}$ satisfying the lemma statement is as follows.

Step 1: Obtaining decompositions individually: Note that the objects in $\mathcal{A}$ (resp., $\mathcal{O}$) are cover-free (follows from Claim 1). So, we apply Lemma 6 on the set $\mathcal{A}$ (resp., $\mathcal{O}$) of objects, to compute the disjoint decomposition of $\mathcal{A}$ (resp., $\mathcal{O}$). Let $\mathcal{A}^0 = \{A_1^0, \ldots, A_\ell^0\}$ (resp., $\mathcal{O}^0 = \{O_1^0, \ldots, O_t^0\}$) be the disjoint decomposition of $\mathcal{A}$ (resp., $\mathcal{O}$). Now, the following claim is obvious.

Claim 4. *Any point $p \in \mathbb{R}^2$ is contained in the interior of at most two objects of $\mathcal{A}^0 \cup \mathcal{O}^0$.*

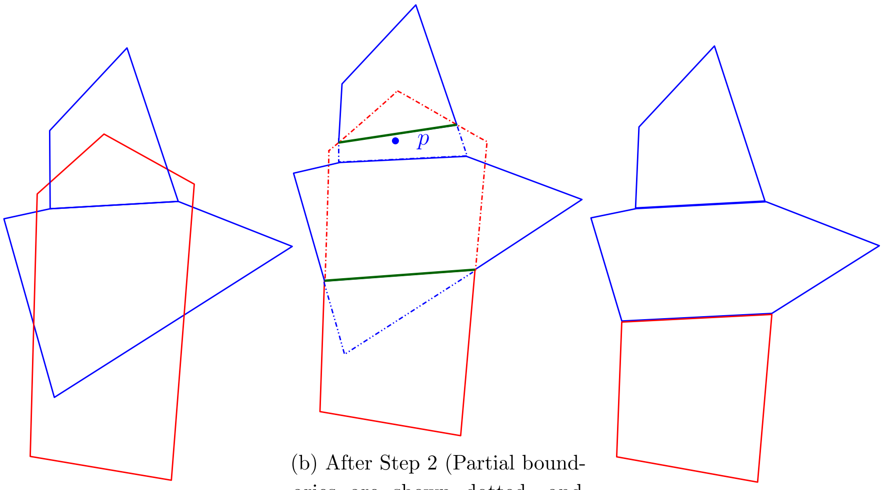
Lemma 6 ensures that $\mathrm{CF}(A_i, \mathcal{A}) \subseteq A_i^0 \neq \emptyset$ and $\mathrm{CF}(O_j, \mathcal{O}) \subseteq O_j^0 \neq \emptyset$ for all $i \in [\ell]$, $j \in [t]$. By Claim 2, no object A_i^0 can be properly contained in any single object from $\mathcal{O}^0$, but it may be completely covered by the union of two or more objects from $\mathcal{O}^0$. We can remedy this as follows.

Replace each object of $\mathcal{A}^0$ and $\mathcal{O}^0$ with an infinitesimally trimmed version of itself. By our general position assumption, the resulting sets of trimmed objects still form dominating-sets. Furthermore, because the elements of $\mathcal{O}^0$ have pairwise disjoint interiors, no single object of $\mathcal{A}^0$ can be contained in the union of two or more of the trimmed objects in $\mathcal{O}^0$. Henceforth, $\mathcal{A}^0$ and $\mathcal{O}^0$ refer to the sets of trimmed objects. Thus we have the following.

Claim 5.

(i) $\mathrm{CF}(A_i^0, \mathcal{A}^0 \cup \mathcal{O}^0) \neq \emptyset$ *for all $i \in [\ell]$,*
(ii) $\mathrm{CF}(O_j^0, \mathcal{A}^0 \cup \mathcal{O}^0) \neq \emptyset$ *for all $j \in [t]$,*
(iii) *For each object $S \in \mathcal{S}$, there exist an object $A_i^0 \in \mathcal{A}^0$ (resp., $O_j^0 \in \mathcal{O}^0$) such that $S \cap A_i^0 \neq \emptyset$ (resp., $S \cap O_j^0 \neq \emptyset$).*

Step 2: Obtaining disjoint sub-decomposition: Now, consider $A_i^0 \in \mathcal{A}^0$ for all $i \in [\ell]$. Lemma 6 ensures that A_i^0 does not have any interior overlap with A_k^0, for any $k \in [\ell] \setminus i$. Similarly, O_j^0 $(j \in [t])$ does not have any interior overlap with O_k^0, for any $k \in [t] \setminus j$. But, A_i^0 may have interior overlap with one or more objects of $\mathcal{O}^0$. Let $L(i)$ be the subset of indices



(a) After Step 1

(b) After Step 2 (Partial boundaries are shown dotted, and the separating line segments are marked in green)

(c) After Step 3

Fig. 5. Illustration of different steps: objects in $\mathcal{A}$ and $\mathcal{O}$ are marked with red and blue, respectively.

$j \in [t]$ such that A_i^0 has an interior overlap with O_j^0. For any $j \in L(i)$, Claim 5 implies that both $\mathrm{CF}(A_i^0, \mathcal{A}^0 \cup \mathcal{O}^0) \neq \emptyset$ and $\mathrm{CF}(O_j^0, \mathcal{A}^0 \cup \mathcal{O}^0) \neq \emptyset$. By applying Lemma 7 to A_i^0 and O_j^0, we obtain two interior-disjoint convex objects $A_{ij}^0 \subseteq A_i^0$ and $O_{ji}^0 \subseteq O_j^0$. Let $A_i^1 = \bigcap_{j \in L(i)} A_{ij}^0$. Similarly, let $M(j)$ be the subset of indices $i \in [l]$ such that O_j^0 has an interior overlap with A_i^0. Let $O_j^1 = \bigcap_{i \in M(j)} O_{ji}^0$ that is a convex object and it contains $\mathrm{CF}(O_j)$. Let $\mathcal{A}^1 = \{A_1^1, \ldots, A_\ell^1\}$ and $\mathcal{O}^1 = \{O_1^1, \ldots, O_t^1\}$. Clearly, $A_i^1 \subseteq A_i^0$ and $O_j^1 \subseteq O_j^0$, and since separating line segments E_{ij}^0 have eliminated all overlaps between the two decompositions, it follows that $\mathcal{A}^1 \cup \mathcal{O}^1$ is a disjoint sub-decomposition of $\mathcal{A} \cup \mathcal{O}$. If we concentrate on the arrangements of all E_{ij}^0 along the boundary of ∂A_i^0, then we observe the following.

Claim 6. *Any two separating line segments E_{ij}^0 and $E_{ij'}^0$ do not intersect each other.*

Proof. If E_{ij}^0 and $E_{ij'}^0$ intersect each other, then assertions (ii) and (iii) of Lemma 7 imply that the corresponding objects O_j^0 and $O_{j'}^0$ also intersect, that is not possible because $\mathcal{O}^0$ is a disjoint decomposition. □

The boundary ∂A_i^1 is actually obtained by replacing zero or more disjoint arcs of ∂A_i^0 with separating line segments. Since each of these separating line segments is part of different disjoint objects in $\mathcal{O}^0$, here we would like to remark that the object A_i^1 is nonempty. Similarly, each object $O_j^1 \in \mathcal{O}^1$ is nonempty. We denote the *partial boundary* ΔA_{ij}^0 (resp., ΔO_{ji}^0) by the portion of the boundary ∂A_i^0 (resp., ∂O_j^0) which is replaced by the edge E_{ij}^0 (see Fig. 5(b) where partial boundary is marked as dotted).

Note the following.

Claim 7. *Let A_i^0 and O_j^0 be any two objects from $\mathcal{A}^0$ and $\mathcal{O}^0$, respectively, such that $\mathrm{int}(A_i^0) \cap \mathrm{int}(O_j^0) \neq \emptyset$ and E_{ij}^0 is not a part of ∂A_i^1. Then, the following properties must be satisfied:*

- *there exists an object $O_{j'}^0$ in $\mathcal{O}^0$ such that $\mathrm{int}(A_i^0) \cap \mathrm{int}(O_{j'}^0) \neq \emptyset$, $E_{ij'}^0$ is a part of ∂A_i^1, and $A_i^0 \setminus A_{ij}^0$ is completely contained in $O_{j'}$.*
- *O_j^0 does not intersect A_i^1.*

Proof. Claim 6 implies that no two separating line segments intersect each other, so the fact that E_{ij}^0 does not contribute to ∂A_i^1 implies that there is another object $O_{j'}^0$ such that the partial-boundary $\Delta A_{ij'}^0$ contains the partial boundary ΔA_{ij}^0. Thus, $A_{ij'}^0 \subseteq A_{ij}^0$ that implies $A_i^0 \setminus A_{ij}^0 \subseteq A_i^0 \setminus A_{ij'}^0$. Since $A_i^0 \setminus A_{ij'}^0$ is completely contained in $O_{j'}$ (by Lemma 7), $A_i^0 \setminus A_{ij}^0$ is also completely contained in $O_{j'}$.

Since O_j^0 and $O_{j'}^0$ are interior disjoint and the partial-boundary $\Delta A_{ij'}^0$ contains the partial boundary ΔA_{ij}^0, O_j^0 cannot intersect A_i^1. Hence, the claim follows. □

By a symmetrical argument, we have the following.

Claim 8. *Let A_i^0 and O_j^0 be any two objects from $\mathcal{A}^0$ and $\mathcal{O}^0$, respectively, such that* $\mathrm{int}(A_i^0) \cap \mathrm{int}(O_j^0) \neq \emptyset$ *and E_{ji}^0 is not a part of ∂O_j^1. Then, the following properties must be satisfied:*

- *there exists an object $A_{i'}^0$ in $\mathcal{A}^0$ such that* $\mathrm{int}(O_j^0) \cap \mathrm{int}(A_{i'}^0) \neq \emptyset$*, $E_{ji'}^0$ is a part of ∂O_j^1, and $O_j^0 \setminus O_{ji}^0$ is completely contained in $A_{i'}$.*
- *A_i^0 does not intersect O_j^1.*

Note that after this step, there might be some point $p \in A_i^0$ but $p \notin A_i^1$ and there does not exist any O_j^1 such that $p \in O_j^1$ (see Fig. 5(a–b)). Hence, the objects of $\mathcal{A}^1 \cup \mathcal{O}^1$ fail to cover the same region as $\mathcal{A}^0 \cup \mathcal{O}^0$, as needed in the decomposition. To remedy this, we expand some of the objects in $\mathcal{A}^1$ and $\mathcal{O}^1$ in the next step.

Step 3: Expansion of objects in $\mathcal{A}^1$ and $\mathcal{O}^1$:

For each $(i, j) \in [\ell] \times [t]$, define $\chi(i, j) = 1$ if E_{ij}^0 is a part of ∂A_i^1 and E_{ji}^0 is also a part of ∂O_j^1, and it is 0 otherwise. Recalling A_{ij}^0 and O_{ji}^0 from Lemma 7, for each $i \in [\ell]$, define $A_i^2 = \bigcap_{\{j \mid \chi(i,j)=1\}} A_{ij}^0$, and for each $j \in [t]$, define $O_j^2 = \bigcap_{\{i \mid \chi(i,j)=1\}} O_{ji}^0$. Let $\mathcal{A}^2 = \{A_1^2, \ldots, A_\ell^2\}$ and $\mathcal{O}^2 = \{O_1^2, \ldots, O_t^2\}$. Note that $\mathcal{A}^2 \cup \mathcal{O}^2$ is a disjoint sub-decomposition of $\mathcal{A} \cup \mathcal{O}$. This construction along with Claims 7 and 8 ensures the following.

Claim 9.

- *For any point $p \in A_i^0 \setminus A_i^2$, there exists some $O_j^2 \in \mathcal{O}^2$ such that A_i^2 and O_j^2 share an edge on their boundary and $p \in O_j$.*
- *For any point $p \in O_j^0 \setminus O_j^2$, there exists some $A_i^2 \in \mathcal{A}^2$ such that A_i^2 and O_j^2 share an edge on their boundary and $p \in A_i$.*

By renaming each set A_i^2 as $\widetilde{A}_i$ for $i \in [\ell]$ and each O_j^2 as $\widetilde{O}_j$ for $j \in [t]$, we obtain the final decomposition $\widetilde{\mathcal{A}} \cup \widetilde{\mathcal{O}} = \mathcal{A}^2 \cup \mathcal{O}^2$. Finally, we claim the following that completes the proof of the lemma statement.

Claim 10. *For any input object $S \in \mathscr{S}$ either (i) there exist $\widetilde{A} \in \widetilde{\mathcal{A}}$ and $\widetilde{O} \in \widetilde{\mathcal{O}}$ such that $S \cap \widetilde{A} \neq \emptyset$ and $S \cap \widetilde{O} \neq \emptyset$, or (ii) there exist $A \in \mathcal{A}$ and $O \in \mathcal{O}$ such that $S \cap A \cap O \neq \emptyset$, and $\widetilde{A}$ and $\widetilde{O}$ share an edge on their boundary.*

Proof. Let S be any input object in $\mathscr{S}$. From Claim 5 (iii), we know that there exist $A_i^0 \in \mathcal{A}^0$ and $O_j^0 \in \mathcal{O}^0$ such that $S \cap A_i^0 \neq \emptyset$ and $S \cap O_j^0 \neq \emptyset$ for some $i \in [\ell]$ and $j \in [t]$. If after Step 3, $S \cap A_i^2 \neq \emptyset$ and $S \cap O_j^2 \neq \emptyset$, then the claim follows. So, without loss of generality, assume that $S \cap A_i^2 = \emptyset$. Consider any point $p \in S \cap A_i^0$. As $p \in A_i^0 \setminus A_i^2$, there exist some $O_j^2 \in \mathcal{O}^2$ such that A_i^2 and O_j^2 share an edge on their boundary and $p \in O_j$ (follows from Claim 9). Thus the claim follows. □

5.2. Nearest-site Voronoi diagram

Recalling the definition of the convex distance function δ_C from Definition 1, we define the distance $\delta_C(p, P)$ from a point p to any object P (that need not be convex and homothetic to C) as follows.

Definition 3. Let p be a point, and P be an object in a plane. The distance $\delta_C(p, P)$ from p to P is defined as $\delta_C(p, P) = \min_{q \in P} \delta_C(p, q)$.

This distance function has the following properties.

Property 3.

(i) *If p is contained in the object P, then $\delta_C(p, P) = 0$.*
(ii) *If $\delta_C(p, P) > 0$, then p is outside the object P, and a translated copy of C centered at p with scaling factor $\delta_C(p, P)$ touches the object P.*

Now, we define a nearest-site Voronoi diagram NVD_C for all the objects in $\widetilde{\mathcal{A}} \cup \widetilde{\mathcal{O}}$ with respect to the distance function δ_C. We define Voronoi cell of $S_i \in \widetilde{\mathcal{A}} \cup \widetilde{\mathcal{O}}$ as $\mathrm{Cell}(S_i) = \{p \in \mathbb{R}^2 \mid \delta_C(p, S_i) \leq \delta_C(p, S_j)\ \textit{for all}\ j \neq i\}$. The NVD_C is a partition on the plane imposed by the collection of cells of all the objects in $\widetilde{\mathcal{A}} \cup \widetilde{\mathcal{O}}$. A point p is in $\mathrm{Cell}(S)$ for some object $S \in \widetilde{\mathcal{A}} \cup \widetilde{\mathcal{O}}$, implies that if we place a homothetic copy of C centered at p with a scaling factor $\delta_C(p, S)$, then C touches S and the interior of C is empty. Now, we have the following two lemmas.

Lemma 9. *The cell of every object $S \in \widetilde{\mathcal{A}} \cup \widetilde{\mathcal{O}}$ is nonempty. Moreover, $S \subseteq \mathrm{Cell}(S)$.*

Proof. This follows from Property 3(i) and the fact that $\widetilde{\mathcal{A}} \cup \widetilde{\mathcal{O}}$ is a set of interior disjoint objects (from Lemma 8(a)). □

Lemma 10. *Each cell $\mathrm{Cell}(S)$ is simply connected.*

Proof. For every $S \in \widetilde{\mathcal{A}} \cup \widetilde{\mathcal{O}}$, let us define the function $\pi_S : \mathbb{R}^2 \to S$, that maps any point to one of its closest points in S. (If $p \in S$, then $\pi_S(p) = p$.)

We first claim that for every point $p \in \mathrm{Cell}(S)$, the line segment $\overline{p\pi_S(p)} \subseteq \mathrm{Cell}(S)$. To see this, suppose to the contrary that there exists a point $q \in \overline{p\pi_S(p)}$ such that $q \in \mathrm{Cell}(S')$ where $S'(\neq S) \in \widetilde{\mathcal{A}} \cup \widetilde{\mathcal{O}}$. Then by basic properties of convex distance functions (Property 1), we have

$$\delta_C(p, S') \leq \delta_C(p, \pi_{S'}(q)) \leq \delta_C(p,q) + \delta_C(q, \pi_{S'}(q)) < \delta_C(p,q) + \delta_C(q, \pi_S(p)) = \delta_C(p, \pi_S(p)),$$

contradicting the fact that $p \in \mathrm{Cell}(S)$.

To see that $\mathrm{Cell}(S)$ is connected, observe that any two points $p, p' \in \mathrm{Cell}(S)$ can be connected as follows. First, connect p to $\pi_S(p)$ and p' to $\pi_S(p')$. Then connect these two points through S. By the above claim and Lemma 9, all of these segments lie within $\mathrm{Cell}(S)$.

To complete the proof that $\mathrm{Cell}(S)$ is simply connected, we use the well-known equivalent characterization [23] that for any simple closed (i.e., Jordan) curve $\Psi \subset \mathrm{Cell}(S)$, the interior of the region bounded by this curve lies entirely within $\mathrm{Cell}(S)$. Consider any x in the interior of the region bounded by Ψ. Either $x \in S$ or (by extending the ray from $\pi_S(x)$ through x until it hits Ψ) there exists $p \in \mathrm{Cell}(S)$ such that x lies on the line segment $\overline{p\pi_S(x)}$. In the former case, $x \in \mathrm{Cell}(S)$, follows from Lemma 9. Now, we are going to argue that $x \in \mathrm{Cell}(S)$ for the latter case as well. To see this, suppose to the contrary that $x \in \mathrm{Cell}(S')$ where $S'(\neq S) \in \widetilde{\mathcal{A}} \cup \widetilde{\mathcal{O}}$. Then by basic properties of convex distance functions (Property 1), we have

$$\delta_C(p, S') \leq \delta_C(p, \pi_{S'}(x)) \leq \delta_C(p,x) + \delta_C(x, \pi_{S'}(q)) < \delta_C(p,x) + \delta_C(x, \pi_S(p)) = \delta_C(p, \pi_S(p)),$$

contradicting the fact that $p \in \mathrm{Cell}(S)$. Therefore $x \in \mathrm{Cell}(S)$, as desired. □

5.3. Locality condition

Let us consider the graph $\mathcal{G} = (\mathcal{V}, \mathcal{E})$, the *dual* of the Voronoi diagram NVD_C, whose vertices $\mathcal{V}$ are the elements of $\mathcal{A} \cup \mathcal{O}$ and the edge set $\mathcal{E}$ consists of pairs $U, V \in \mathcal{V}$ whose Voronoi cells share an edge on their boundaries. From Lemma 9 and Lemma 10, we have the following.

Lemma 11. *The graph $\mathcal{G} = (\mathcal{A} \cup \mathcal{O}, \mathcal{E})$ is a planar graph.*

Now, we prove that the graph $\mathcal{G}$ satisfies the property needed in the locality condition (Lemma 1).

Lemma 12. *For any arbitrary input object $S \in \mathscr{S}$, if S is dominated by at least one object of $\mathcal{A}$ and at least one object of $\mathcal{O}$, then there exists $A \in \mathcal{A}$ and $O \in \mathcal{O}$ both of which dominate S and $(A, O) \in \mathcal{E}$ of $\mathcal{G}$.*

Proof. Let S be any object in $\mathscr{S}$. According to Lemma 8, there exists a disjoint sub-decomposition $\widetilde{\mathcal{A}} \cup \widetilde{\mathcal{O}}$ such that either:

(i) there exist $\widetilde{A} \in \widetilde{\mathcal{A}}$ and $\widetilde{O} \in \widetilde{\mathcal{O}}$ such that $S \cap \widetilde{A}$ and $S \cap \widetilde{O}$ are both nonempty, or
(ii) there exist $A \in \mathcal{A}$ and $O \in \mathcal{O}$ such that $S \cap A \cap O \neq \emptyset$, and their respective traces $\widetilde{A}$ and $\widetilde{O}$ share an edge in common on their boundaries.

For case (ii), clearly, both A and O dominate S. The fact that $\widetilde{A}$ and $\widetilde{O}$ share a common edge on their boundary implies (by Lemma 9) that $\mathrm{Cell}(\widetilde{A})$ and $\mathrm{Cell}(\widetilde{O})$ also share a common edge on their boundaries. Therefore, (A, O) is an edge of $\mathcal{G}$, as desired.

For case (i), the proof is similar to Lemma 5. Let $c = c(S)$ denote the center of S. Let us assume that A and O have been chosen so that $\widetilde{A}$ and $\widetilde{O}$ are the closest objects to c (with respect to δ_C) in $\widetilde{\mathcal{A}}$ and $\widetilde{\mathcal{O}}$, respectively. Without loss of generality, we may assume that c does not lie in $\mathrm{Cell}(\widetilde{O})$. Let $o \in \widetilde{O}$ denote the closest point to c in $\widetilde{O}$. Clearly, c and o lie in different Voronoi cells, so this segment must intersect an edge of $\mathrm{Cell}(\widetilde{O})$ at some point p. Let $\mathrm{Cell}(\widetilde{R})$ denote the cell neighboring the $\mathrm{Cell}(\widetilde{O})$ along this edge. Letting r denote the closest point to p in $\widetilde{R}$, we have $\delta_C(p,r) = \delta_C(p, \widetilde{R}) = \delta_C(p, \widetilde{O}) \leq \delta_C(p, o)$. By basic properties of the convex distance function (see Property 1), we obtain

$$\delta_C(c,r) \leq \delta_C(c,p) + \delta_C(p,r) \leq \delta_C(c,p) + \delta_C(p,o) = \delta_C(c,o).$$

By general position, we may assume that $\delta_C(c, \widetilde{R}) < \delta_C(c, \widetilde{O})$. Since $\widetilde{O}$ was chosen to be the closest object in $\widetilde{O}$ to c, it follows that $\widetilde{R} \in \widetilde{\mathcal{A}}$. Clearly, the associated objects R and O (that contain $\widetilde{R}$ and $\widetilde{O}$, respectively) both dominates S. Therefore, there is an edge (R, O) in $\mathcal{G}$, as desired. □

6. Geometric set-cover for convex pseudodisks

Given a set $\mathscr{S}$ of n convex pseudodisks and a set $\mathcal{P}$ of points in $\mathbb{R}^2$, the objective is to cover all the points in $\mathcal{P}$ using a subset of $\mathscr{S}$ of minimum cardinality. Here, we analyze that the local search algorithm would give a polynomial time approximation scheme, as given in Section 2. The analysis is similar to the previous problem. Recall from Section 2.1 that $\mathcal{O}$ is an optimal covering set for $\mathcal{P}$ and $\mathcal{A}$ is the covering set returned by our local search algorithm satisfying both Claims 1 and 2. Here, we need to show that the locality condition mentioned in Lemma 2 is satisfied.

If we restrict the proof of Lemma 8 up to Claim 9, it is straightforward to obtain the following.

Lemma 13. *Let $\mathcal{A}$ be the output of the local-search algorithm for set-cover on a set $\mathscr{S}$ of convex pseudodisks and a set $\mathcal{P}$ of points in $\mathbb{R}^2$, and let $\mathcal{O}$ be the optimum. Then there exists a disjoint sub-decomposition $\widetilde{\mathcal{A}} \cup \widetilde{\mathcal{O}}$ that satisfies the following: for any input point $p \in \mathcal{P}$ there exists $A \in \mathcal{A}$ and $O \in \mathcal{O}$ such that $p \in A$ and $p \in O$, and their traces $\widetilde{A}$ and $\widetilde{O}$ share an edge on their boundary.*

Proof. Let $\mathcal{A} = \{A_1, \ldots, A_\ell\}$ and $\mathcal{O} = \{O_1, \ldots, O_t\}$. Our algorithm to obtain a disjoint sub-decomposition $\widetilde{\mathcal{A}} \cup \widetilde{\mathcal{O}} = \{\widetilde{A}_1, \ldots \widetilde{A}_\ell\} \cup \{\widetilde{O}_1, \ldots \widetilde{O}_t\}$ for $\mathcal{A} \cup \mathcal{O}$ satisfying the lemma statement is precisely same as the three steps mentioned in Section 5.1 for Lemma 8. The main difference is in the statement of Claim 8. For the set-cover problem, we have the following.

Claim 11.

(i) $\mathrm{CF}(A_i^0, \mathcal{A}^0 \cup \mathcal{O}^0) \neq \emptyset$ *for all* $i \in [\ell]$,
(ii) $\mathrm{CF}(O_j^0, \mathcal{A}^0 \cup \mathcal{O}^0) \neq \emptyset$ *for all* $j \in [t]$,
(iii) *Each point $p \in \mathcal{P}$ is covered by exactly one object from $\mathcal{A}^0$ (resp., $\mathcal{O}^0$).*

Finally, we claim the following statement instead of Claim 10.

Claim 12. *For any input point $p \in \mathcal{P}$, there exist $A \in \mathcal{A}$ and $O \in \mathcal{O}$ such that $p \in A$ and $p \in O$, and $\widetilde{A}$ and $\widetilde{O}$ share an edge on their boundary.*

Proof. Let p be any input point in $\mathcal{P}$. By Claim 11 (iii), there exist $A_i^0 \in \mathcal{A}^0$ and $O_j^0 \in \mathcal{O}^0$ such that $p \in A_i^0$ and $p \in O_j^0$ for some $i \in [\ell]$ and $j \in [t]$. After Step 3, since $\mathcal{A}^2 \cup \mathcal{O}^2$ is a disjoint decomposition of $\mathcal{A} \cup \mathcal{O}$, p cannot be both in A_i^2 and O_j^2. Therefore, either of the following happens: $p \notin A_i^2$, or $p \notin O_j^2$. In both cases, the claim follows from Claim 9. □

Thus the lemma follows. □

Now, consider a graph $\mathcal{G} = (\mathcal{V}, \mathcal{E})$, where each vertex $V \in \mathcal{V}$ corresponds to an object in $\widetilde{\mathcal{A}} \cup \widetilde{\mathcal{O}}$, and we create an edge in between two vertices whenever the corresponding objects in $\widetilde{\mathcal{A}} \cup \widetilde{\mathcal{O}}$ share an edge in their boundary. Since the objects of $\widetilde{\mathcal{A}} \cup \widetilde{\mathcal{O}}$ are convex and have disjoint interiors, this graph is planar. From Lemma 13, it follows that the graph $\mathcal{G}$ satisfies the *locality condition* mentioned in Lemma 2. This completes the proof of Theorem 2.

7. Concluding remarks

In this paper, we have shown that the well-known local search algorithm gives a PTAS for finding the minimum cardinality dominating-set and geometric set-cover when the objects are homothetic convex objects and convex pseudodisks, respectively. Consequently, we obtain an algorithm that is easy to implement along with a guarantee on its approximation error for a broad class of objects which encompasses arbitrary squares, k-regular polygons and translates of convex polygons. A QPTAS is known for the weighted set-cover problem where objects are pseudodisks [28]. But, no QPTAS is known for the weighted dominating-set problem when objects are homothetic convex objects. Note that the separator-based analysis for local search to obtain PTAS has a limitation for handling the weighted version of the problems. Thus, finding a polynomial time approximation scheme for the weighted version of both minimum dominating-set and minimum geometric set-cover problems for homothetic convex objects, pseudodisks, remains open in this context. Especially for the weighted version of the problem, it would be interesting to analyze the approximation guarantees of the local search algorithm.

Declaration of competing interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

Data availability

No data was used for the research described in the article.

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