THE REDUCED BERGMAN KERNEL AND ITS PROPERTIES

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ABSTRACT. In this article, we study some properties of the n-th order weighted reduced Bergman kernels for planar domains, $n \geq 1$. Specifically, we look at Ramadanov type theorems, localization, and boundary behaviour of the weighted reduced Bergman kernel and its higher-order counterparts. We also give a transformation formula for these kernels under biholomorphisms.

1. INTRODUCTION

Recall that the Bergman space associated with a domain $D \subset \mathbb{C}$ consists of square-integrable holomorphic functions on it. An important space that has a close relationship with this space is the collection of all holomorphic functions whose derivatives are square-integrable with respect to the area measure. This space can be associated with a closed subspace of the Bergman space, which we call the reduced Bergman space. The reduced Bergman space is a reproducing kernel Hilbert space with its reproducing kernel called the reduced Bergman kernel; see below for the definitions.

In a previous work (see [3]), we proved a transformation formula for the weighted reduced Bergman kernels under proper holomorphic maps between bounded planar domains. In continuation of our previous effort, this note is a step forward in creating a dictionary between the Bergman kernel and the reduced Bergman kernel. This note consists of two parts.

In the first part, we prove Ramadanov type theorem for the weighted reduced Bergman kernel. We also prove Ramadanov type theorem for the higher-order weighted reduced Bergman kernels (see Definition 1.2).

The second part of this article is about the local analysis of these n-th order weighted reduced Bergman kernels near a boundary point. In particular, we study localization of the n-th order weighted reduced Bergman kernels. Using this localization, we make some observations about the boundary asymptotics of these kernels. Similar ideas have been used to study the boundary behaviour of the weighted Bergman kernels, see [6]. In a related work by the third author, similar boundary behaviour was observed for the class of finitely connected domains bounded by Jordan curves for the span metric, closely connected to the reduced Bergman kernel, see [13].

We also prove a transformation formula for these n-th order reduced Bergman kernels under biholomorphisms, that we will use to get the boundary asymptotes for these kernels; which is interesting in its own right.

For a domain $D \subset \mathbb{C}$, we will be working with admissible weights due to Pasternak-Winiarski. The space of admissible weights on D is denoted by $AW(D)$.

Definition 1.1. (See, [8], [9]) Let $D \subset \mathbb{C}$ be a domain and μ be a positive measurable real-valued function on D. The weight μ is called an admissible weight on D if

(i) the weighted Bergman space $A_\mu^2(D)$ is a closed subspace of $L_\mu^2(D)$ (and hence a Hilbert space), and

(ii) the evaluation functional $A_\mu^2(D) \ni f \mapsto f(z) \in \mathbb{C}$ is continuous for every $z \in D$.

M. Sakai ([11], [12]) defined the n-th order weighted reduced Bergman kernel in the following way:

Definition 1.2. Let $D \subset \mathbb{C}$ be a domain, $\mu \in AW(D)$, $\zeta \in D$, and n be a positive integer. Define

$$AD^\mu(D, \zeta^n) = \left\{ f \in \mathcal{O}(D) : f(\zeta) = f'(\zeta) = \cdots = f^{(n-1)}(\zeta) = 0, \int_D |f'(z)|^2 \mu(z) dA(z) < \infty \right\}.$$

This is a Hilbert space with respect to the inner product

$$\langle f, g \rangle_{AD^\mu(D, \zeta^n)} = \int_D f'(z) \overline{g'(z)} \mu(z) \, dA(z), \quad f, g \in AD^\mu(D, \zeta^n).$$

The linear functional defined by $AD^\mu(D, \zeta^n) \ni f \mapsto f^{(n)}(\zeta) \in \mathbb{C}$, is continuous. By Riesz representation theorem, there exists a unique function $M_{D,\mu,n}(\cdot, \zeta) \in AD^\mu(D, \zeta^n)$ such that $f^{(n)}(\zeta) = \langle f, M_{D,\mu,n}(\cdot, \zeta) \rangle_{AD^\mu(D, \zeta^n)}$ for every $f \in AD^\mu(D, \zeta^n)$. Define

$$\tilde{K}_{D,\mu,n}(z, \zeta) = \frac{\partial}{\partial z} M_{D,\mu,n}(z, \zeta), \quad z, \zeta \in D.$$

The kernel $\tilde{K}_{D,\mu,n}$ is called the n-th order weighted reduced Bergman kernel of D with respect to the weight μ. Putting $n = 1$ gives us the weighted reduced Bergman kernel $\tilde{K}_{D,\mu}$ of D with respect to the weight μ. So,

$$f^{(n)}(\zeta) = \int_D f'(z) \overline{\tilde{K}_{D,\mu,n}(z, \zeta)} \mu(z) \, dA(z) \quad \text{for } f \in AD^\mu(D, \zeta^n).$$

We now compute the n-th order reduced Bergman kernels for the unit disc $\mathbb{D}$.

Example 1. Let $\zeta \in \mathbb{D}$ and $f \in AD(\mathbb{D}, \zeta^n)$ for $n \in \mathbb{Z}^+$, that is $f^{(k)}(\zeta) = 0$ for all $0 \leq k \leq n-1$. For $g \in AD(\mathbb{D}, 0^n)$ and $0 < r < 1$, the Cauchy integral formula gives us

$$\begin{aligned}
g^{(n)}(0) &= \frac{(n-1)!}{2\pi i} \int_{|\xi|=r} \frac{g'(\xi)}{\xi^n} \, d\xi = \frac{(n-1)!}{2\pi i} \int_0^{2\pi} \frac{g'(re^{it})}{(re^{it})^n} re^{it} i \, dt \\
&= \frac{(n-1)!}{2\pi} \int_0^{2\pi} \frac{g'(re^{it})}{r^{n-1}} \overline{(e^{it})^{n-1}} \, dt.
\end{aligned}$$

Multiplying both sides by r^{2n-1} and integrating with respect to parameter r, we get

$$\int_0^1 g^{(n)}(0) r^{2n-1} dr = \frac{(n-1)!}{2\pi} \int_0^1 \int_0^{2\pi} g'(re^{it}) \overline{(re^{it})^{n-1}} \, r \, dr \, dt.$$

By change of variables on the right hand side of the above equation, we get

$$g^{(n)}(0) = \frac{n!}{\pi} \int_{\mathbb{D}} g'(\xi) \overline{(\xi)^{n-1}} \, dA(\xi). \tag{1}$$

Let $\tilde{K}_n(\cdot, \cdot)$ denote the n-th order reduced Bergman kernel of $\mathbb{D}$. For all $\zeta \in \mathbb{D}$, and $f \in AD(\mathbb{D}, \zeta^n)$, we have

$$f^{(n)}(\zeta) = \int_{\mathbb{D}} f'(\xi) \overline{\tilde{K}_n(\xi, \zeta)} \, dA(\xi). \tag{2}$$

Let $\phi_\zeta : \mathbb{D} \to \mathbb{D}$ be the automorphism of unit disc given by $\phi_\zeta(z) = \frac{\zeta - z}{1 - z\bar\zeta}$. Note that $\phi_\zeta(0) = \zeta$, $\phi_\zeta(\zeta) = 0$, and $\phi_\zeta \circ \phi_\zeta(z) = z$ for all $z \in \mathbb{D}$. Consider the holomorphic function $f \circ \phi_\zeta$. Observe that $f \circ \phi_\zeta \in AD(\mathbb{D}, 0^n)$ and $(f \circ \phi_\zeta)^{(n)}(0) = f^{(n)}(\zeta)(\phi'_\zeta(0))^n$. On substituting $g = f \circ \phi_\zeta$ in equation (1), we get

$$
\begin{aligned}
f^{(n)}(\zeta)(\phi'_\zeta(0))^n &= \frac{n!}{\pi} \int_{\mathbb{D}} (f \circ \phi_\zeta)'(\chi) \overline{(\chi)^{n-1}} \, dA(\chi) \\
&= \frac{n!}{\pi} \int_{\mathbb{D}} f'(\phi_\zeta(\chi)) \phi'_\zeta(\chi) \overline{(\chi)^{n-1}} \, dA(\chi).
\end{aligned}
$$

Now by doing change of variables $\xi = \phi_\zeta(\chi)$, we get $\chi = \phi_\zeta(\xi)$, and $\phi'_\zeta(\chi) = \frac{1}{\phi'_\zeta(\xi)}$. Therefore

$$
\begin{aligned}
f^{(n)}(\zeta)(\phi'_\zeta(0))^n &= \frac{n!}{\pi} \int_{\mathbb{D}} f'(\xi)(\phi'_\zeta(\xi))^{-1} \overline{(\phi_\zeta(\xi))^{n-1}} \, |\phi'_\zeta(\xi)|^2 \, dA(\xi) \\
&= \frac{n!}{\pi} \int_{\mathbb{D}} f'(\xi) \overline{(\phi_\zeta(\xi))^{n-1}} \, \overline{\phi'_\zeta(\xi)} \, dA(\xi).
\end{aligned}
$$

Therefore we get

$$
f^{(n)}(\zeta) = \int_{\mathbb{D}} f'(\xi) \frac{n! \overline{(\phi_\zeta(\xi))^{n-1}} \, \overline{\phi'_\zeta(\xi)}}{\pi(\phi'_\zeta(0))^n} \, dA(\xi). \tag{3}
$$

Comparing the equations (2) and (3), we get

$$
\tilde{K}_n(\xi, \zeta) = \frac{n!}{\pi} \frac{(\phi_\zeta(\xi))^{n-1} \, \phi'_\zeta(\xi)}{\overline{(\phi'_\zeta(0))^n}}.
$$

We can check that $\phi'_\zeta(\xi) = \frac{|\zeta|^2 - 1}{(1 - \bar\zeta\xi)^2}$, which gives $\phi'_\zeta(0) = |\zeta|^2 - 1$. Therefore,

$$
\tilde{K}_n(\xi, \zeta) = \frac{n!}{\pi} \frac{(\zeta - \xi)^{n-1}}{(1 - \bar\zeta\xi)^{n-1}} \frac{|\zeta|^2 - 1}{(1 - \bar\zeta\xi)^2} \frac{1}{(|\zeta|^2 - 1)^n}.
$$

Thus, for $n \geq 1$, the n-th order reduced Bergman kernel is given by

$$
\tilde{K}_n(\xi, \zeta) = \frac{n!}{\pi} \frac{(\xi - \zeta)^{n-1}}{(1 - \bar\zeta\xi)^{n+1}(1 - |\zeta|^2)^{n-1}}. \tag{4}
$$

The following is an equivalent definition of the 1-st order weighted reduced Bergman kernel. We shall use both definitions interchangeably.

Definition 1.3. Let D be a domain in $\mathbb{C}$ and $\mu \in \mathrm{AW}(D)$. The weighted reduced Bergman space of D is the space of all the square-integrable holomorphic functions on D whose primitive exists on D, i.e.,

$$
\mathcal{D}^\mu(D) = \left\{ f \in \mathcal{O}(D) : f = g' \text{ for some } g \in \mathcal{O}(D) \text{ and } \int_D |f(z)|^2 \mu(z) dA(z) < \infty \right\}.
$$

This is a Hilbert space with respect to the inner product

$$
\langle f, g \rangle := \int_D f(z) \overline{g(z)} \mu(z) dA(z).
$$

For every $\zeta \in D$, the evaluation functional

$$f \mapsto f(\zeta), \qquad f \in \mathcal{D}^\mu(D)$$

is a bounded linear functional, and therefore $\mathcal{D}^\mu(D)$ is a reproducing kernel Hilbert space. The reproducing kernel of $\mathcal{D}^\mu(D)$, denoted by $\tilde{K}_{D,\mu}(\cdot,\cdot)$, is called the (1-st order) weighted reduced Bergman kernel of D with respect to the weight μ. It satisfies the reproducing property:

$$f(\zeta) = \int_D f(z)\overline{\tilde{K}_{D,\mu}(z,\zeta)}\mu(z)dA(z)$$

for all $f \in \mathcal{D}^\mu(D)$ and $\zeta \in D$.

Remark 1.4. It will easily follow from the above definition that $\tilde{K}_{D,\mu}$ is holomorphic in the first variable, anti-holomorphic in the second variable, and that $\tilde{K}_{D,\mu} \in C^\infty(D \times D)$.

It is known (see [1, p. 26], [2, p. 476]) that for a domain $D \subset \mathbb{C}$, $\mu \in AW(D)$, and $n \geq 2$,

$$\tilde{K}_{D,\mu,n}(z,\zeta) = \frac{(-1)^{n-1}}{J_{n-2}} \det \begin{pmatrix} \tilde{K}_{0,\bar{0}}(z,\zeta) & \dots & \tilde{K}_{0,\overline{n-1}}(z,\zeta) \\ \tilde{K}_{0,\bar{0}} & \dots & \tilde{K}_{0,\overline{n-1}} \\ \tilde{K}_{1,\bar{0}} & \dots & \tilde{K}_{1,\overline{n-1}} \\ \vdots & & \vdots \\ \tilde{K}_{n-2,\bar{0}} & \dots & \tilde{K}_{n-2,\overline{n-1}} \end{pmatrix}, \tag{5}$$

where $J_n = \det\left(\tilde{K}_{j\bar{k}}\right)_{j,k=0}^n$ and

$$\tilde{K}_{j\bar{k}}(z,\zeta) = \frac{\partial^{j+k}}{\partial z^j \partial \bar{\zeta}^k}\tilde{K}_{D,\mu}(z,\zeta), \quad \tilde{K}_{j\bar{k}} \equiv \tilde{K}_{j\bar{k}}(\zeta,\zeta).$$

Here $J_n > 0$ for all $\zeta \notin N_D(\mu) := \{z \in D : \tilde{K}_{D,\mu}(z,z) = 0\}$. Thus, $\tilde{K}_{D,\mu,n} \in C^\infty(D \times (D \setminus N_D(\mu))$. If $\mu \in L^1(D)$, then $N_D(\mu) = \emptyset$, and therefore $\tilde{K}_{D,\mu,n} \in C^\infty(D \times D)$.

Notation: For a non-negative integer p, we denote $\tilde{K}_{D,\mu,n}^{(p)}(z,\zeta) := \frac{\partial^p}{\partial z^p}\tilde{K}_{D,\mu,n}(z,\zeta)$.

Remark 1.5. Let $D \subset \mathbb{C}$ be a bounded domain, $\mu \in L^1(D)$ be an admissible weight on D, $\zeta \in D$ and n be a positive integer.

1. Note that, $(M_{D,\mu,n}(\cdot,\zeta))^{(n)}(\zeta) = \|M_{D,\mu,n}(\cdot,\zeta)\|^2_{AD^\mu(D,\zeta^n)}$. Thus, $(M_{D,\mu,n}(\cdot,\zeta))^{(n)}(\zeta) = 0$ would imply that $M_{D,\mu,n}(\cdot,\zeta) \equiv 0$ and hence $f^{(n)}(\zeta) = 0$ for all $f \in AD^\mu(D,\zeta^n)$. This is not true since D is bounded and $\mu \in L^1(D)$. Hence, $(M_{D,\mu,n}(\cdot,\zeta))^{(n)}(\zeta) > 0$.

2. Consider the extremal problem $\min\{\|f\|_{AD^\mu(D,\zeta^n)} : f \in AD^\mu(D,\zeta^n), f^{(n)}(\zeta) = 1\}$. Then

$$g(z) = \frac{M_{D,\mu,n}(z,\zeta)}{\tilde{K}_{D,\mu,n}^{(n-1)}(\zeta,\zeta)}, \quad z \in D$$

is the unique function solving the extremal problem with $\|g\|_{AD^\mu(D,\zeta^n)} = \sqrt{\tilde{K}_{D,\mu,n}^{(n-1)}(\zeta,\zeta)}$.

3. Consider the extremal problem $\max\{|f^{(n)}(\zeta)| : f \in AD^\mu(D,\zeta^n), \|f\|_{AD^\mu(D,\zeta^n)} = 1\}$. Then

$$h(z) = \frac{M_{D,\mu,n}(z,\zeta)}{\sqrt{\tilde{K}_{D,\mu,n}^{(n-1)}(\zeta,\zeta)}}, \quad z \in D$$

is the unique function solving the extremal problem such that $h^{(n)}(\zeta) > 0$. Here, $h^{(n)}(\zeta) = \sqrt{\tilde{K}_{D,\mu,n}^{(n-1)}(\zeta,\zeta)}$.

Ramadanov [10] showed that for a sequence of domains $D_j \subset \mathbb{C}$ such that $D_j \subset D_{j+1}$ for all $j \in \mathbb{Z}^+$, and $D := \bigcup_{j=1}^{\infty} D_j$, the Bergman kernel $K_j(\cdot, \cdot)$ corresponding to the domain D_j converges uniformly on compacts of $D \times D$ to the Bergman kernel $K(\cdot, \cdot)$ corresponding to the domain D. Over the years, different versions of Ramadanov type theorems have been proved for the Bergman kernels and the weighted Bergman kernels (see, for instance, [7], [14]), for different types of convergence of the sequence of domains. Let us prove some of these Ramadanov type theorems for the n-th order weighted reduced Bergman kernels.

Theorem 1.6. *Let $\{D_i\}_{i=1}^{\infty}$ be a sequence of domains in $\mathbb{C}$ with $\mu_i \in AW(D_i)$.*

(1) *Set $D := \cup_i D_i$ and let $\mu \in AW(D)$. Assume that for any $i \in \mathbb{Z}^+$, there exists a $j = j(i) \in \mathbb{Z}^+$ such that $D_i \subset D_k$ and $\mu_i(z) \le \mu_k(z) \le \mu(z)$ for all $k \ge j(i)$ and $z \in D_i$. If $\mu_i \longrightarrow \mu$ pointwise a.e. on D then*

(A) *we have*

$$\lim_{i \to \infty} \tilde{K}_{D_i, \mu_i} = \tilde{K}_{D, \mu}$$

locally uniformly on $D \times D$.

(B) *for $n > 1$ and $w \in D$,*

$$\lim_{i \to \infty} \tilde{K}_{D_i, \mu_i, n}(\cdot, w) = \tilde{K}_{D, \mu, n}(\cdot, w)$$

locally uniformly on D. In particular, $\tilde{K}_{D_i, \mu_i, n}$ converges to $\tilde{K}_{D, \mu, n}$ pointwise on $D \times D$. Furthermore, all the derivatives of $\tilde{K}_{D_i, \mu_i, n}$ converge locally uniformly to the respective derivatives of $\tilde{K}_{D, \mu, n}$ on $D \times (D \setminus N_D(\mu))$.

(2) *Suppose that $D = \cap_i D_i$ is a domain and $\mu \in AW(D)$ be such that $\mu(z) \le \mu_i(z)$ for all $i \in \mathbb{Z}^+$ and $z \in D$. Assume that $\mu_i \longrightarrow \mu$ pointwise a.e. on D and that for every fixed $t \in D$, we have*

$$\lim_{i \to \infty} \tilde{K}_{D_i, \mu_i}(t, t) = \tilde{K}_{D, \mu}(t, t).$$

Then,

(A) *we have*

$$\lim_{i \to \infty} \tilde{K}_{D_i, \mu_i} = \tilde{K}_{D, \mu}$$

locally uniformly on $D \times D$.

(B) *for $n > 1$, all the derivatives of $\tilde{K}_{D_i, \mu_i, n}$ converge locally uniformly to the respective derivatives of $\tilde{K}_{D, \mu, n}$ on $D \times (D \setminus N_D(\mu))$.*

Remark 1.7. If $\mu \in L^1(D)$, we have uniform convergence on all the compact subsets of $D \times D$.

We have the following corollary about the convergence of the corresponding kernel functions $M_{D_i, \mu_i, n}$ of the domains D_i.

Corollary 1.8. *Let (D_i, μ_i, D, μ) for $i \in \mathbb{Z}^+$ be as in Theorem 1.6 (1), then for $n \ge 1$ and for all $\xi \in D$, the n-th order kernel function $M_{D_i, \mu_i, n}(\cdot, \xi)$ converges locally uniformly on the domain D to the kernel function $M_{D, \mu, n}(\cdot, \xi)$.*

If (D_i, μ_i, D, μ) are as in Theorem 1.6 (2), then

(1) *for all $\xi \in D$, the 1-st order kernel function $M_{D_i, \mu_i}(\cdot, \xi)$ converges locally uniformly on the domain D to the kernel function $M_{D, \mu}(\cdot, \xi)$.*

(2) *for $n > 1$ and for all $\xi \in D \setminus N_D(\mu)$, the n-th order kernel function $M_{D_i, \mu_i, n}(\cdot, \xi)$ converges locally uniformly on the domain D to the kernel function $M_{D, \mu, n}(\cdot, \xi)$.*

Moving ahead to the second part of this note, we study the boundary behaviour of $\tilde{K}_{D,\mu,n}^{(n-1)}(z,z)$ as z approaches ∂D. We will start by studying two main ingredients, namely, the localization of $\tilde{K}_{D,\mu,n}^{(n-1)}(z,z)$ and the transformation formula for $\tilde{K}_{D,\mu,n}^{(n-1)}$ under biholomorphisms.

For a domain $D \subset \mathbb{C}$ and a point $p \in \partial D$, let $\mathcal{A}(D,p)$ denote the collection of all admissible weights μ on D that are in $L^\infty(D)$ and extend continuously to p with $\mu(p) > 0$.

Definition 1.9. Let $D \subset \mathbb{C}$ be a domain. A boundary point $p \in \partial D$ is called a holomorphic local peak point of D if there is a neighborhood W of p in $\mathbb{C}$ and a function h such that

   (i) h is continuous on $\overline{D} \cap W$,
   (ii) h is holomorphic on $D \cap W$,
   (iii) $|h(z)| < 1$ for all $z \in (\overline{D} \cap W) \setminus \{p\}$ and $h(p) = 1$.

Theorem 1.10. *Let D be a bounded domain in $\mathbb{C}$, $p \in \partial D$, $\mu \in \mathcal{A}(D,p)$ and n be a positive integer. There exists a neighborhood U of p in $\mathbb{C}$ such that $D \cap U$ is connected and $\mu \geq c$ a.e. on $D \cap U$ for some constant $c > 0$. Hence, the n-th order weighted reduced Bergman kernel $\tilde{K}_{D \cap U,\mu,n}$ of $D \cap U$ with respect to the weight $\mu|_{D \cap U}$ is well-defined.*

Suppose that p is a holomorphic local peak point, and there exists a neighborhood N_p of p in $\mathbb{C}$ such that $D \cap N_p$ is simply connected. Then,

$$\lim_{\zeta \to p} \frac{\tilde{K}_{D \cap U,\mu,n}^{(n-1)}(\zeta,\zeta)}{\tilde{K}_{D,\mu,n}^{(n-1)}(\zeta,\zeta)} = 1.$$

Let us see an example where we study boundary behaviour of these kernel functions explicitly using the above localization before studying it for general domains.

Example 2. Consider the domain $D = \mathbb{D} \setminus \cup_{i=1}^{N} \overline{D(q_i,r_i)}$ for $N \in \mathbb{Z}^+$, such that $\overline{D(q_i,r_i)} \cap \overline{D(q_j,r_j)} = \emptyset$, if $i \neq j$, where $q_i \in \mathbb{D}$, and $\overline{D(q_i,r_i)} \subset \mathbb{D}$ for all $1 \leq i \leq N$. We want to study the behaviour of the kernel functions on the diagonal near the boundary of D. Let $p \in \partial D$, i.e., $|p| = 1$, and observe that for a sufficiently small neighourhood U of p, we have $D \cap U = \mathbb{D} \cap U$. Therefore, the above localization theorem 1.10 tells us that

$$\lim_{z \to p} \frac{\tilde{K}_{D,n}^{(n-1)}(z,z)}{\tilde{K}_{\mathbb{D},n}^{(n-1)}(z,z)} = 1. \tag{6}$$

Now using the Leibniz rule to differentiate the equation (4), with respect to the variable z upto $(n-1)$ times, we get

$$\tilde{K}_{\mathbb{D},n}^{(n-1)}(z,w) = \frac{1}{\pi(1-|w|^2)^{n-1}} \sum_{k=0}^{n-1} \binom{n-1}{k}^2 (n+k)!\,(n-k-1)!\, \frac{\overline{w}^k(z-w)^k}{(1-\overline{w}z)^{n+k+1}}.$$

Taking $z = w$, all the terms in the above summation vanishes except for $k = 0$, so we get

$$\tilde{K}_{\mathbb{D},n}^{(n-1)}(z,z) = \frac{n!\,(n-1)!}{\pi(1-|z|^2)^{2n}}.$$

Using this in the equation (6), we get

$$\lim_{z \to p} \left((1-|z|^2)^{2n}\, \tilde{K}_{D,n}^{(n-1)}(z,z) \right) = \frac{n!\,(n-1)!}{\pi}. \tag{7}$$

This is true for every finitely connected domain $D \subset \mathbb{D}$ such that $\partial \mathbb{D} \subset \partial D$, and for every $p \in \partial \mathbb{D}$.

Now we would like to study these kernel functions near other boundary components. As we can see $\partial D = \partial \mathbb{D} \cup \left(\cup_{i=1}^{N} \partial D(q_i,r_i) \right)$. Take a point $p_i \in \partial D(q_i,r_i) = \{|z-q_i| = r_i\}$. Consider

the univalent holomorphic map $f : D \to \mathbb{D}$ given by $f(z) = \frac{r_i}{z - q_i}$. One can easily see that $\tilde{D} := f(D) \subset \mathbb{D}$ is an $(N+1)$−connected domain with $\partial D \subset \partial \tilde{D}$ (since $f(\{|z - q_i| = r_i\}) = \{|w| = 1\}$). Therefore equation (7) holds true for the domain $\tilde{D}$. Now using the transformation formula proved in Theorem 1.11, we get

$$\tilde{K}_{D,n}^{(n-1)}(z,z) = |f'(z)|^{2n}\, \tilde{K}_{\tilde{D},n}^{(n-1)}\left(\frac{r_i}{z - q_i}, \frac{r_i}{z - q_i}\right).$$

Note that $f'(z) = \frac{-r_i}{(z - q_i)^2}$. Now multiplying both sides by $\left(1 - |\frac{r_i}{z - q_i}|^2\right)^{2n}$, and taking the limit $z \to p_i$, we get

$$\lim_{z \to p_i} \left(1 - \left|\frac{r_i}{z - q_i}\right|^2\right)^{2n} \tilde{K}_{D,n}^{(n-1)}(z,z)$$

$$= \lim_{z \to p_i} \frac{r^{2n}}{|z - q_i|^{4n}} \left(1 - \left|\frac{r_i}{z - q_i}\right|^2\right)^{2n} \tilde{K}_{\tilde{D},n}^{(n-1)}\left(\frac{r_i}{z - q_i}, \frac{r_i}{z - q_i}\right).$$

Using the change of variables $\tilde{z} = \frac{r_i}{z - q_i}$, with $\tilde{p}_i = \frac{r_i}{p_i - q_i} \in \partial\mathbb{D}$, we get

$$\lim_{z \to p_i} \frac{r_i^{2n}}{|z - q_i|^{4n}} \left(1 - \left|\frac{r_i}{z - q_i}\right|^2\right)^{2n} \tilde{K}_{\tilde{D},n}^{(n-1)}\left(\frac{r_i}{z - q_i}, \frac{r_i}{z - q_i}\right)$$

$$= \frac{1}{r_i^{2n}} \lim_{\tilde{z} \to \tilde{p}_i} |\tilde{z}|^{4n} (1 - |\tilde{z}|^2)^{2n}\, \tilde{K}_{\tilde{D},n}^{(n-1)}(\tilde{z}, \tilde{z}).$$

Now using the fact that $|\tilde{p}_i| = 1$, and equation (7) for the domain $\tilde{D}$ gives us

$$\lim_{z \to p_i} \left(1 - \left|\frac{r_i}{z - q_i}\right|^2\right)^{2n} \tilde{K}_{D,n}^{(n-1)}(z,z) = \frac{n!\,(n-1)!}{\pi r_i^{2n}}. \tag{8}$$

Theorem 1.11. *Let D_1, D_2 be domains in $\mathbb{C}$ and n be a positive integer. Let ν be an admissible weight on D_2 and $f : D_1 \to D_2$ be a biholomorphism. Then,*

$$(\overline{f'(\zeta)})^n\, M_{D_2,\nu,n}(f(z), f(\zeta)) = M_{D_1, \nu \circ f, n}(z, \zeta), \quad z, \zeta \in D_1.$$

Consequently, we have

$$(f'(z))^n\, \tilde{K}_{D_2,\nu,n}^{(n-1)}(f(z), f(\zeta))\, (\overline{f'(\zeta)})^n = \tilde{K}_{D_1, \nu \circ f, n}^{(n-1)}(z, \zeta), \quad z, \zeta \in D_1.$$

Theorem 1.12. *Let $D \subset \mathbb{C}$ be a bounded domain and $p \in \partial D$. Assume that ∂D is C^2-smooth near p. If ψ is a C^2-defining function for D near p with $\frac{\partial \psi}{\partial z}(p) = 1$, then for a positive integer n*

$$\tilde{K}_{D,n}^{(n-1)}(z,z) \sim \frac{n!(n-1)!}{\pi} \frac{1}{(\psi(z))^{2n}} \approx \frac{1}{(\delta(z))^{2n}} \quad as \quad z \to p.$$

Here, $\delta(z)$ denotes the Euclidean distance of z from ∂D. For functions f and g on D, the notation $f \sim \lambda g$ as $z \to p$ means that $f(z)/g(z) \to \lambda$ as $z \to p$, and $f \approx g$ as $z \to p$ means that f/g is bounded above and below by positive constants in some neighborhood of p in $\overline{D}$.

Theorem 1.13. *Let $D \subset \mathbb{C}$ be a bounded domain, $p \in \partial D$ and $\nu \in \mathcal{A}(D, p)$. Assume that ∂D is C^2-smooth near p. Then*

$$\tilde{K}_{D,\nu,n}^{(n-1)}(z,z) \sim \frac{1}{\nu(p)} \tilde{K}_{D,n}^{(n-1)}(z,z)$$

as $z \to p$.

2. Ramadanov Type Theorems

We will use the following characterization for the kernel functions.

Lemma 2.1. *Let $D \subset \mathbb{C}$ be a domain, $\mu \in AW(D)$ and $n \in \mathbb{Z}^+$. For $w \in D$, let $S_{D,\mu,n}^w \subset AD^\mu(D, w^n)$ denote the set of all functions f such that*

$$f^{(n)}(w) \geq 0 \quad and \quad \|f\|_{AD^\mu(D,w^n)} \leq \sqrt{f^{(n)}(w)}.$$

Then the n-th order weighted kernel function $M_{D,\mu,n}(\cdot, w)$ is uniquely characterized by the properties:

   (i) $M_{D,\mu,n}(\cdot, w) \in S_{D,\mu,n}^w$;

   (ii) *if $f \in S_{D,\mu,n}^w$ and $f^{(n)}(w) \geq \tilde{K}_{D,\mu,n}^{(n-1)}(w, w)$, then $f(\cdot) \equiv M_{D,\mu,n}(\cdot, w)$.*

Proof. Let $\varphi_1, \varphi_2 \in AD^\mu(D, w^n)$ satisfy (i) and (ii). Either $\varphi_1^{(n)}(w) \geq \varphi_2^{(n)}(w)$ or $\varphi_2^{(n)}(w) \geq \varphi_1^{(n)}(w)$. In both the cases, we have $\varphi_1 \equiv \varphi_2$. Therefore, the function satisfying both (i) and (ii) is unique. We shall show that $M_{D,\mu,n}(\cdot, w)$ satisfies the two properties. Since

$$(M_{D,\mu,n}(\cdot, w))^{(n)}(w) = \tilde{K}_{D,\mu,n}^{(n-1)}(w, w) \geq 0 \quad and \quad \|M_{D,\mu,n}(\cdot, w)\|_{AD^\mu(D,w^n)} = \sqrt{\tilde{K}_{D,\mu,n}^{(n-1)}(w, w)},$$

property (i) holds. Let $f \in S_{D,\mu,n}^w$ such that $f^{(n)}(w) \geq \tilde{K}_{D,\mu,n}^{(n-1)}(w, w)$. If $f^{(n)}(w) = 0$ then $\tilde{K}_{D,\mu,n}^{(n-1)}(w, w) = 0$ and therefore $\|f\|_{AD^\mu(D,w^n)} = \|M_{D,\mu,n}(\cdot, w)\|_{AD^\mu(D,w^n)} = 0$. Hence, f and $M_{D,\mu,n}(\cdot, w)$ are constant functions on D that vanish at $w \in D$. Thus,

$$f(\cdot) \equiv M_{D,\mu,n}(\cdot, w) \equiv 0.$$

So, assume that $f^{(n)}(w) > 0$. If $\tilde{K}_{D,\mu,n}^{(n-1)}(w, w) = 0$ then by the similar argument as above $M_{D,\mu,n}(\cdot, w) \equiv 0$ and therefore $g^{(n)}(w) = \langle g, M_{D,\mu,n}(\cdot, w)\rangle_{AD^\mu(D,w^n)} = 0$ for all $g \in AD^\mu(D, w^n)$. This is a contradiction as $f^{(n)}(w) \neq 0$. Therefore, $\tilde{K}_{D,\mu,n}^{(n-1)}(w, w) > 0$.

Note that the function $f(\cdot)/f^{(n)}(w)$ belongs to the set $\{h \in AD^\mu(D, w^n) : h^{(n)}(w) = 1\}$. Observe that

$$\left\| \frac{f(\cdot)}{f^{(n)}(w)} \right\|_{AD^\mu(D,w^n)} \leq \frac{\sqrt{f^{(n)}(w)}}{f^{(n)}(w)} = \frac{1}{\sqrt{f^{(n)}(w)}} \leq \frac{1}{\sqrt{\tilde{K}_{D,\mu,n}^{(n-1)}(w, w)}} = \left\| \frac{M_{D,\mu,n}(\cdot, w)}{\tilde{K}_{D,\mu,n}^{(n-1)}(w, w)} \right\|_{AD^\mu(D,w^n)}$$

By the uniqueness of the minimal problem: $\min\{\|h\|_{AD^\mu(D,w^n)} : h \in AD^\mu(D, w^n), h^{(n)}(w) = 1\}$, we therefore have

$$\frac{f(\cdot)}{f^{(n)}(w)} \equiv \frac{M_{D,\mu,n}(\cdot, w)}{\tilde{K}_{D,\mu,n}^{(n-1)}(w, w)} \quad and \quad \frac{1}{\sqrt{f^{(n)}(w)}} = \frac{1}{\sqrt{\tilde{K}_{D,\mu,n}^{(n-1)}(w, w)}}.$$

So, $f(\cdot) \equiv M_{D,\mu,n}(\cdot, w)$. $\qquad\square$

In a similar manner, we can characterize the 1-st order reduced Bergman kernel as follows.

Lemma 2.2. *Let $D \subset \mathbb{C}$ be a domain, and μ be an admissible weight on D. For $w \in D$, let $\tilde{S}_{D,\mu}^w \subset \mathcal{D}^\mu(D)$ denote the set of all functions f such that $f(w) \geq 0$ and $\|f\|_{L_\mu^2(D)} \leq \sqrt{f(w)}$. Then the 1-st order weighted reduced Bergman kernel function $\tilde{K}_{D,\mu}(\cdot, w)$ is uniquely characterized by the properties:*

   (i) $\tilde{K}_{D,\mu}(\cdot, w) \in \tilde{S}_{D,\mu}^w$;

(ii) *if $f \in \tilde{S}^w_{D,\mu}$ and $f(w) \geq \tilde{K}_{D,\mu}(w,w)$, then $f(\cdot) \equiv \tilde{K}_{D,\mu}(\cdot,w)$.*

Proof. Similar to the proof of Lemma 2.1. $\qquad\square$

Proof of Theorem 1.6 (1). We will start by proving the monotonicity property: $\tilde{K}^{(n-1)}_{D_i,\mu_i,n}(z,z) \geq \tilde{K}^{(n-1)}_{D_k,\mu_k,n}(z,z)$ for all $z \in D_i$ and $k \geq j(i)$.

Fix $z \in D_i$ and $k \geq j(i)$. If $\tilde{K}^{(n-1)}_{D_k,\mu_k,n}(z,z) = 0$ then there is nothing to prove. Therefore, assume that $\tilde{K}^{(n-1)}_{D_k,\mu_k,n}(z,z) > 0$. Observe that

$$\int_{D_i} \left| \frac{(M_{D_k,\mu_k,n}(\cdot,z))'(\zeta)}{\tilde{K}^{(n-1)}_{D_k,\mu_k,n}(z,z)} \right|^2 \mu_i(\zeta) dA(\zeta) \leq \int_{D_i} \left| \frac{(M_{D_k,\mu_k,n}(\cdot,z))'(\zeta)}{\tilde{K}^{(n-1)}_{D_k,\mu_k,n}(z,z)} \right|^2 \mu_k(\zeta) dA(\zeta)$$

$$\leq \int_{D_k} \left| \frac{(M_{D_k,\mu_k,n}(\cdot,z))'(\zeta)}{\tilde{K}^{(n-1)}_{D_k,\mu_k,n}(z,z)} \right|^2 \mu_k(\zeta) dA(\zeta) = \frac{1}{\tilde{K}^{(n-1)}_{D_k,\mu_k,n}(z,z)} < \infty.$$

Therefore $M_{D_k,\mu_k,n}(\cdot,z)/\tilde{K}^{(n-1)}_{D_k,\mu_k,n}(z,z)$ belongs to the set $\{h \in AD^{\mu_i}(D_i,z^n) : h^{(n)}(z) = 1\}$. Note that $\tilde{K}^{(n-1)}_{D_i,\mu_i,n}(z,z) > 0$ as we have a function in $AD^{\mu_i}(D_i,z^n)$ whose n-th derivative does not vanish at z.

Thus, by the minimality, we have

$$\frac{1}{\tilde{K}^{(n-1)}_{D_i,\mu_i,n}(z,z)} = \int_{D_i} \left| \frac{(M_{D_i,\mu_i,n}(\cdot,z))'(\zeta)}{\tilde{K}^{(n-1)}_{D_i,\mu_i,n}(z,z)} \right|^2 \mu_i(\zeta)\, dA(\zeta)$$

$$\leq \int_{D_i} \left| \frac{(M_{D_k,\mu_k,n}(\cdot,z))'(\zeta)}{\tilde{K}^{(n-1)}_{D_k,\mu_k,n}(z,z)} \right|^2 \mu_i(\zeta)\, dA(\zeta) \leq \frac{1}{\tilde{K}^{(n-1)}_{D_k,\mu_k,n}(z,z)}.$$

Hence, $\tilde{K}^{(n-1)}_{D_i,\mu_i,n}(z,z) \geq \tilde{K}^{(n-1)}_{D_k,\mu_k,n}(z,z)$. Since $D_i \subset D$ for all $i \in \mathbb{Z}^+$ and $\mu_i(z) \leq \mu(z)$ for all $z \in D_i$, it can be proved in a similar manner that $\tilde{K}^{(n-1)}_{D_i,\mu_i,n}(z,z) \geq \tilde{K}^{(n-1)}_{D,\mu,n}(z,z)$ for all $i \in \mathbb{Z}^+$ and $z \in D_i$.

Now we will show that $\{\tilde{K}_{D_k,\mu_k}\}$ and $\{\tilde{K}_{D_k,\mu_k,n}(\cdot,w)\}$ are normal families for any $w \in D$.

Let $K \subset D$ be a compact subset. There exists an $i \in \mathbb{Z}^+$ such that $K \subset D_i$, and therefore $K \subset D_i \subset D_k$ for all $k \geq j(i)$. Choose $i \in \mathbb{Z}^+$ so that $w \in D_i$. For $z, \zeta \in K$ and $k \geq j(i)$, we have by monotonicity that

$$|\tilde{K}_{D_k,\mu_k}(z,\zeta)| \leq \sqrt{\tilde{K}_{D_k,\mu_k}(z,z)} \sqrt{\tilde{K}_{D_k,\mu_k}(\zeta,\zeta)} \leq \sqrt{\tilde{K}_{D_i,\mu_i}(z,z)} \sqrt{\tilde{K}_{D_i,\mu_i}(\zeta,\zeta)} \leq M$$

and

$$\begin{aligned}
|\tilde{K}_{D_k,\mu_k,n}(z,w)| &= |(M_{D_k,\mu_k,n}(\cdot,w))'(z)| = |\langle (M_{D_k,\mu_k,n}(\cdot,w))', \tilde{K}_{D_k,\mu_k}(\cdot,z) \rangle_{L^2_{\mu_k}(D_k)}| \\
&\leq \|(M_{D_k,\mu_k,n}(\cdot,w))'\|_{L^2_{\mu_k}(D_k)} \|\tilde{K}_{D_k,\mu_k}(\cdot,z)\|_{L^2_{\mu_k}(D_k)} \\
&= \sqrt{\tilde{K}^{(n-1)}_{D_k,\mu_k,n}(w,w)} \sqrt{\tilde{K}_{D_k,\mu_k}(z,z)} \leq \sqrt{\tilde{K}^{(n-1)}_{D_i,\mu_i,n}(w,w)} \sqrt{\tilde{K}_{D_i,\mu_i}(z,z)} \leq C,
\end{aligned}$$

where

$$M = \sup_{t \in K} |\tilde{K}_{D_i,\mu_i}(t,t)| \geq 0 \quad \text{and} \quad C = \sup_{t \in K} \sqrt{\tilde{K}_{D_i,\mu_i}(t,t)} \sqrt{\tilde{K}^{(n-1)}_{D_i,\mu_i,n}(w,w)} \geq 0.$$

By Montel's theorem, $\{\tilde{K}_{D_k,\mu_k}\}_{k=1}^{\infty}$ and $\{\tilde{K}_{D_k,\mu_k,n}(\cdot,w)\}_{k=1}^{\infty}$ are normal families. We will show that for $n \geq 1$ and $w \in D$, every convergent subsequence of $\{\tilde{K}_{D_k,\mu_k,n}(\cdot,w)\}_{k=1}^{\infty}$ converges to $\tilde{K}_{D,\mu,n}(\cdot,w)$. This will imply that

$$\lim_{k\to\infty} \tilde{K}_{D_k,\mu_k} = \tilde{K}_{D,\mu} \quad \text{and} \quad \lim_{k\to\infty} \tilde{K}_{D_k,\mu_k,n}(\cdot,w) = \tilde{K}_{D,\mu,n}(\cdot,w)$$

locally uniformly on $D \times D$ and D respectively.

So, without loss of generality, assume that

$$\lim_{k\to\infty} \tilde{K}_{D_k,\mu_k,n}(\cdot,w) = \tilde{K}$$

locally uniformly on D for some holomorphic function $\tilde{K}$. Since $\tilde{K}_{D_k,\mu_k,n}^{(p)}(w,w) = 0$ for all $0 \leq p \leq n-2$ and large k, we have $\tilde{K}^{(p)}(w) = 0$ for all $0 \leq p \leq n-2$. Let $\gamma \subset\subset D$ be a closed curve. Then $\gamma \subset D_k$ for large enough k. Since the holomorphic functions $\tilde{K}_{D_k,\mu_k,n}(\cdot,w)$ have a primitive on D_k and converge locally uniformly to $\tilde{K}$ on D,

$$\int_{\gamma} \tilde{K}(z)dz = \int_{\gamma} \lim_{k\to\infty} \tilde{K}_{D_k,\mu_k,n}(z,w)dz = \lim_{k\to\infty} \int_{\gamma} \tilde{K}_{D_k,\mu_k,n}(z,w)dz = 0.$$

Thus, there exists a holomorphic function M on D such that $M' = \tilde{K}$. Choose M so that $M(w) = 0$. Therefore,

$$M(w) = M'(w) = \cdots = M^{(n-1)}(w) = 0.$$

Let $K \subset D$ be a compact subset. Then $K \subset D_k$ for large enough k. By Fatou's lemma,

$$
\begin{aligned}
\int_K |M'(z)|^2 \mu(z) dA(z) &= \int_K |\tilde{K}(z)|^2 \mu(z) dA(z) \leq \liminf_{k\to\infty} \int_K |\tilde{K}_{D_k,\mu_k,n}(z,w)|^2 \mu_k(z) dA(z) \\
&\leq \liminf_{k\to\infty} \int_{D_k} |\tilde{K}_{D_k,\mu_k,n}(z,w)|^2 \mu_k(z) dA(z) = \liminf_{k\to\infty} \tilde{K}_{D_k,\mu_k}^{(n-1)}(w,w) \\
&= \tilde{K}^{(n-1)}(w) = M^{(n)}(w).
\end{aligned}
$$

Since compact $K \subset D$ was arbitrary, we have

$$\|M'\|_{L^2_\mu(D)}^2 = \int_D |M'(z)|^2 \mu(z)\, dA(z) \leq M^{(n)}(w) < \infty.$$

Therefore, $M \in S_{D,\mu,n}^w$. Since $\tilde{K}_{D_i,\mu_i,n}^{(n-1)}(w,w) \geq \tilde{K}_{D,\mu,n}^{(n-1)}(w,w)$ for all $i \in \mathbb{Z}^+$, we have that $M^{(n)}(w) = \tilde{K}^{(n-1)}(w) \geq \tilde{K}_{D,\mu,n}^{(n-1)}(w,w)$. Hence, Lemma 2.1 implies that $M(\cdot) \equiv M_{D,\mu,n}(\cdot,w)$ and therefore $\tilde{K} = \tilde{K}_{D,\mu,n}(\cdot,w)$.

It now follows from the equation (5) that all the derivatives of $\tilde{K}_{D_i,\mu_i,n}$ are the fractions where the numerator is a polynomial in the derivatives of $\tilde{K}_{D_i,\mu_i}$ and the denominator is some power of J_{n-2} (corresponding to the domain D_i and weight μ_i) which is non-zero on $D_i \times (D_i \setminus N_{D_i}(\mu_i))$ for all $i \in \mathbb{Z}^+$.

If $K \subset D$ is compact such that $\tilde{K}_{D,\mu}(z,z) \neq 0$ for all $z \in K$, then $\tilde{K}_{D_i,\mu_i}(z,z) \neq 0$ for all large i and $z \in K$. Thus, any compact set in $D \times (D \setminus N_D(\mu))$ is eventually contained in $D_i \times (D_i \setminus N_{D_i}(\mu_i))$.

Since $\tilde{K}_{D_i,\mu_i}$ are holomorphic in the first and anti-holomorphic in the second variable, all of its derivatives converge locally uniformly to the respective derivatives of $\tilde{K}_{D,\mu}$. Therefore, for $n > 1$, all the derivatives of $\tilde{K}_{D_i,\mu_i,n}$ converge locally uniformly to the respective derivatives of $\tilde{K}_{D,\mu,n}$ on $D \times (D \setminus N_D(\mu))$. $\qquad \square$

Proof of Theorem 1.6 (2). We will first show part (A). Let $K \subset D$ be a compact subset. For $z, w \in K$ and $k \in \mathbb{Z}^+$, we have by monotonicity that

$$|\tilde{K}_{D_k,\mu_k}(z,w)| \leq \sqrt{\tilde{K}_{D_k,\mu_k}(z,z)}\sqrt{\tilde{K}_{D_k,\mu_k}(w,w)} \leq \sqrt{\tilde{K}_{D,\mu}(z,z)}\sqrt{\tilde{K}_{D,\mu}(w,w)} \leq M,$$

where $M = \sup_{t \in K}|\tilde{K}_{D,\mu}(t,t)| > 0$. By Montel's theorem, $\{K_{D_k,\mu_k}\}_{k=1}^{\infty}$ is a normal family and therefore has a subsequence that converges locally uniformly on $D \times D$. We will show that every such subsequence must converge to $\tilde{K}_{D,\mu}$. So, without loss of generality, assume that

$$\lim_{k \to \infty} \tilde{K}_{D_k,\mu_k} = \tilde{K}$$

locally uniformly on $D \times D$, for some $\tilde{K}$. Fix $w \in D$. Again, let $K \subset D$ be a compact subset. By Fatou's lemma,

$$
\begin{aligned}
\int_K |\tilde{K}(z,w)|^2 \mu(z)\,dA(z) &\leq \liminf_{k \to \infty} \int_K |\tilde{K}_{D_k,\mu_k}(z,w)|^2 \mu_k(z)\,dA(z) \\
&\leq \liminf_{k \to \infty} \int_{D_k} |\tilde{K}_{D_k,\mu_k}(z,w)|^2 \mu_k(z)\,dA(z) \\
&= \liminf_{k \to \infty} \tilde{K}_{D_k,\mu_k}(w,w) = \tilde{K}_{D,\mu}(w,w) = \tilde{K}(w,w).
\end{aligned}
$$

Since compact $K \subset D$ was arbitrary, we have

$$\|\tilde{K}(\cdot,w)\|^2_{L^2_\mu(D)} = \int_D |\tilde{K}(z,w)|^2 \mu(z)\,dA(z) \leq \tilde{K}_{D,\mu}(w,w) < \infty.$$

Let $\gamma \subset\subset D$ be a closed curve. Then $\gamma \subset D_k$ for large enough k. Since the holomorphic functions $\tilde{K}_{D_k,\mu_k}(\cdot,w)$ have a primitive on D_k and converge locally uniformly to $\tilde{K}(\cdot,w)$ on D,

$$\int_\gamma \tilde{K}(z,w)dz = \int_\gamma \lim_{k \to \infty} \tilde{K}_{D_k,\mu_k}(z,w)dz = \lim_{k \to \infty} \int_\gamma \tilde{K}_{D_k,\mu_k}(z,w)dz = 0.$$

Thus, $\tilde{K}(\cdot,w)$ has a primitive on D. Therefore, $\tilde{K}(\cdot,w) \in \mathcal{D}^\mu(D)$. Thus, the function $\tilde{K}(\cdot,w) \in \tilde{S}^w_{D,\mu}$ and $\tilde{K}(w,w) = \tilde{K}_{D,\mu}(w,w)$. Hence, Lemma 2.2 implies that $\tilde{K}(\cdot,w) \equiv \tilde{K}_{D,\mu}(\cdot,w)$. Since $w \in D$ was arbitrary, we conclude that $\tilde{K} \equiv \tilde{K}_{D,\mu}$.

Part (B) follows from the determinant formula (5) as in the proof of Theorem 1.6 (1). $\square$

3. BOUNDARY BEHAVIOUR

Proof of Theorem 1.10. Since μ extends continuously to p with $\mu(p) > 0$, there exists a neighborhood U of p in $\mathbb{C}$ such that $D \cap U$ is connected and $\mu \geq c$ a.e. on $D \cap U$ for some constant $c > 0$. So, $(\mu|_{D \cap U})^{-1} \in L^1(D \cap U)$. Therefore, $\mu|_{D \cap U}$ is an admissible weight on $D \cap U$ and hence $\tilde{K}_{D \cap U,\mu,n}$ is well-defined.

As $D \cap U \subset D$, the monotonicity of the solution of the extremal problems in Remark 1.5 implies that $\tilde{K}^{(n-1)}_{D \cap U,\mu,n}(\zeta,\zeta) \geq \tilde{K}^{(n-1)}_{D,\mu,n}(\zeta,\zeta)$ for all $\zeta \in D \cap U$. Therefore,

$$1 \leq \liminf_{\zeta \to p} \frac{\tilde{K}^{(n-1)}_{D \cap U,\mu,n}(\zeta,\zeta)}{\tilde{K}^{(n-1)}_{D,\mu,n}(\zeta,\zeta)}.$$

It is now enough to show that

$$\limsup_{\zeta \to p} \frac{\tilde{K}^{(n-1)}_{D \cap U,\mu,n}(\zeta,\zeta)}{\tilde{K}^{(n-1)}_{D,\mu,n}(\zeta,\zeta)} \leq 1. \tag{9}$$

Since p is a holomorphic local peak point, choose h and W as in the Definition 1.9. For any neighborhood $\hat{U} \subset U$ of p, monotonicity gives that

$$\limsup_{\zeta \to p} \frac{\tilde{K}_{D \cap U, \mu, n}^{(n-1)}(\zeta, \zeta)}{\tilde{K}_{D, \mu, n}^{(n-1)}(\zeta, \zeta)} \leq \limsup_{\zeta \to p} \frac{\tilde{K}_{D \cap \hat{U}, \mu, n}^{(n-1)}(\zeta, \zeta)}{\tilde{K}_{D, \mu, n}^{(n-1)}(\zeta, \zeta)}.$$

Therefore, we may assume that $U \subset (N_p \cap W)$ without loss of generality. Choose an open neighborhood $U_0 \subset\subset U$ of p such that $h \neq 0$ on $\bar{D} \cap U_0$. Then there exists a constant $a \in (0, 1)$ such that $|h| \leq a$ on $\overline{(U \setminus U_0) \cap D}$. Choose a cut-off function $\chi : \mathbb{C} \longrightarrow [0, 1]$ such that $\chi \equiv 1$ on some neighborhood $\hat{U}_0$ of $\overline{U}_0$ and $supp\,\chi \subset U$. Fix $\zeta \in D \cap U_0$ and put

$$f = \frac{M_{D \cap U, \mu, n}(\cdot, \zeta)}{\sqrt{(M_{D \cap U, \mu, n}(\cdot, \zeta))^{(n)}(\zeta)}}.$$

Then, $f \in AD^\mu(D \cap U, \zeta^n)$ such that $\|f\|_{AD^\mu(D \cap U, \zeta^n)} = 1$ and $|f^{(n)}(\zeta)|^2 = \tilde{K}_{D \cap U, \mu, n}^{(n-1)}(\zeta, \zeta)$.

Without loss of generality, assume that p lies on the outer boundary curve of D, i.e., the boundary of the unbounded component of $\mathbb{C} \setminus D$. Let $\hat{D}$ be the domain obtained by filling all the bounded components of $\mathbb{C} \setminus D$. Then, $\hat{D}$ is a simply connected domain. For $k \geq 1$, define

$$\alpha_k = \begin{cases} \bar{\partial}(\chi f' h^k) = f' h^k \frac{\partial \chi}{\partial \bar{z}}\, d\bar{z} & \text{, on } D \cap U = \hat{D} \cap U \\ 0 & \text{, on } \hat{D} \setminus U \end{cases}.$$

Then, α_k is a smooth $\bar{\partial}$-closed $(0,1)$-form on $\hat{D}$. Put $\phi(z) = (2n+2) \log|z - \zeta|$ for $z \in \hat{D}$. The function ϕ is a subharmonic function on $\hat{D}$. Also,

$$\begin{aligned}
\int_{\hat{D}} |\alpha_k|^2(z) \exp(-\phi(z))\, dA(z) &= \int_{D \cap U} |f'(z)|^2 |h(z)|^{2k} \left|\frac{\partial \chi}{\partial \bar{z}}(z)\right|^2 |z - \zeta|^{-(2n+2)}\, dA(z) \\
&\leq C_1 \int_{D \cap (U \setminus \hat{U}_0)} |f'(z)|^2 |h(z)|^{2k}\, dA(z) \\
&\leq C_1\, a^{2k} \frac{1}{c} \int_{D \cap U} |f'(z)|^2 \mu(z)\, dA(z) = C_2\, a^{2k} < \infty.
\end{aligned}$$

where $C_1 > 0$ is chosen independent of $\zeta \in D \cap U_0$ and $C_2 = C_1/c$. Therefore, α_k belongs to $L^2_{(0,1)}(\hat{D}, \exp(-\phi))$, the Hilbert space of $(0,1)$-forms that are square-integrable with respect to the weight function $\exp(-\phi)$. By Hormander's theory (see [5]), there exists a smooth function g_k on $\hat{D}$ such that $\bar{\partial} g_k = \alpha_k$ and

$$\int_{\hat{D}} |g_k(z)|^2 (1 + |z|^2)^{-2} \exp(-\phi(z))\, dA(z) \leq C_2\, a^{2k}.$$

Since D is bounded, $\hat{D}$ is also bounded. Therefore, we have

$$\int_{\hat{D}} |g_k(z)|^2 |z - \zeta|^{-(2n+2)}\, dA(z) \leq C_3\, a^{2k} < \infty,$$

where $C_3 = C_2 \sup_{\hat{D}}(1 + |z|^2)^2 > 0$. Since the above integral is finite, we must have that $g_k(\zeta) = g_k'(\zeta) = \cdots = g_k^{(n-1)}(\zeta) = 0$. As $\mu \in L^\infty(D)$, there exists a constant $M > 0$ so that

$$\begin{aligned}
\int_{\hat{D}} |g_k(z)|^2 \mu(z) dA(z) &\leq M \sup_{\hat{D}} |z - \zeta|^{2n+2} \int_{\hat{D}} |g_k(z)|^2 |z - \zeta|^{-(2n+2)} dA(z) \\
&\leq M (diam D)^{2n+2} C_3\, a^{2k} < \infty.
\end{aligned}$$

Therefore, $g_k \in L^2_\mu(\hat{D})$ and $\|g_k\|_{L^2_\mu(\hat{D})} \leq C_0 a^k$, where $C_0 = \sqrt{M(diam D)^{2n+2} C_3} > 0$. Put, $\hat{F}_k = \chi f' h^k - g_k$. Then, $\hat{F}_k \in \mathcal{O}(\hat{D})$. We also have $\hat{F}_k(\zeta) = \hat{F}_k'(\zeta) = (\hat{F}_k)^{(n-2)}(\zeta) = 0$ and

$(\hat{F}_k)^{(n-1)}(\zeta) = f^{(n)}(\zeta)(h(\zeta))^k$. Since $\hat{D}$ is simply connected, $\hat{F}_k$ has a primitive on $\hat{D}$. Choose $F_k \in \mathcal{O}(D)$ such that $F_k(\zeta) = 0$ and $F_k' = \hat{F}_k$. Since $F_k(\zeta) = F_k'(\zeta) = \cdots = F_k^{(n-1)}(\zeta) = 0$ and

$$\|F_k'\|_{L_\mu^2(D)} = \|\hat{F}_k\|_{L_\mu^2(D)} \leq \|f'\|_{L_\mu^2(D \cap U)} + \|g_k\|_{L_\mu^2(D)} \leq 1 + C_0\, a^k,$$

we have $F_k \in AD^\mu(D, \zeta^n)$ and $\|F_k\|_{AD^\mu(D,\zeta^n)} \leq 1 + C_0\, a^k$. Hence,

$$
\begin{aligned}
|h(\zeta)|^{2k}\, \tilde{K}_{D\cap U,\mu,n}^{(n-1)}(\zeta,\zeta) &= |h(\zeta)|^{2k}\,|f^{(n)}(\zeta)|^2 = |F_k^{(n)}(\zeta)|^2 \\
&= \left| \langle F_k, M_{D,\mu,n}(\cdot,\zeta) \rangle_{AD^\mu(D,\zeta^n)} \right|^2 \\
&\leq \|F_k\|_{AD^\mu(D,\zeta^n)}^2 \, \|M_{D,\mu,n}(\cdot,\zeta)\|_{AD^\mu(D,\zeta^n)}^2 \leq (1 + C_0\, a^k)^2\, \tilde{K}_{D,\mu,n}^{(n-1)}(\zeta,\zeta).
\end{aligned}
$$

Therefore,

$$\frac{\tilde{K}_{D\cap U,\mu,n}^{(n-1)}(\zeta,\zeta)}{\tilde{K}_{D,\mu,n}^{(n-1)}(\zeta,\zeta)} \leq \frac{(1 + C_0\, a^k)^2}{|h(\zeta)|^{2k}}.$$

Since C_0 is independent of ζ, we have

$$\limsup_{\zeta \to p} \frac{\tilde{K}_{D\cap U,\mu,n}^{(n-1)}(\zeta,\zeta)}{\tilde{K}_{D,\mu,n}^{(n-1)}(\zeta,\zeta)} \leq (1 + C_0\, a^k)^2.$$

Finally, since $a \in (0,1)$ and C_0 is independent of k, taking the limit $k \to \infty$ gives (9). $\qquad\square$

Proof of Theorem 1.11. Let $\{\varphi_n\}$ be a Cauchy sequence in $A_{\nu \circ f}^2(D_1)$. Then, applying the change of variables formula

$$\int_{D_1} |\varphi_n - \varphi_m|^2(z)\,(\nu \circ f)(z)\, dA(z) = \int_{D_2} |(\varphi_n \circ f^{-1}) - (\varphi_m \circ f^{-1})|^2(z)\, \nu(z)\, |(f^{-1})'(z)|^2\, dA(z)$$

implies that $\{(\varphi_n \circ f^{-1})\,(f^{-1})'\}$ is a Cauchy sequence in $A_\nu^2(D_2)$. Therefore, there exists a function $\varphi \in A_\nu^2(D_2)$ such that $\|(\varphi_n \circ f^{-1})\,(f^{-1})' - \varphi\|_{L_\nu^2(D_2)} \to 0$ as $n \to \infty$. It can now be checked using change of variables formula, as above, that $(\varphi \circ f)\, f' \in A_{\nu \circ f}^2(D_1)$ and $\|\varphi_n - (\varphi \circ f)\, f'\|_{L_{\nu \circ f}^2(D_1)} \to 0$ and $n \to \infty$. Thus, $A_{\nu \circ f}^2(D_1)$ is a closed subspace of $L_{\nu \circ f}^2(D_1)$, and hence a Hilbert space. Now for $z \in D_1$, the evaluation functional $A_\nu^2(D_2) \ni h \mapsto h(f(z))$ is continuous, i.e., there exists a constant $C_z > 0$ such that

$$|h(f(z))| \leq C_z \|h\|_{L_\nu^2(D_2)} \quad \text{for all } h \in A_\nu^2(D_2).$$

For every $g \in A_{\nu \circ f}^2(D_1)$, the function $(g \circ f^{-1})\,(f^{-1})' \in A_\nu^2(D_2)$. Therefore,

$$
\begin{aligned}
|g(z)| &= |f'(z)|\,|g(z)\,(f^{-1})'(f(z))| \leq |f'(z)|\,C_z\,\|(g \circ f^{-1})\,(f^{-1})'\|_{L_\nu^2(D_2)} \\
&= |f'(z)|\,C_z\,\|g\|_{L_{\nu \circ f}^2(D_1)}.
\end{aligned}
$$

Hence, the evaluation functional $A_{\nu \circ f}^2(D_1) \ni g \mapsto g(z)$ is continuous. So, $\nu \circ f$ is an admissible weight on D_1 and therefore $M_{D_1,n,\nu \circ f}$ is well-defined.

Let $\zeta \in D_1$ be arbitrary. Let $\varphi \in AD^\nu(D_2, (f(\zeta))^n)$ be arbitrary. Note that $(\varphi \circ f)(\zeta) = (\varphi \circ f)'(\zeta) = \cdots = (\varphi \circ f)^{(n-1)}(\zeta) = 0$. Also,

$$
\begin{aligned}
\int_{D_1} |(\varphi \circ f)'(z)|^2 (\nu \circ f)(z)\, dA(z) &= \int_{D_2} |\varphi'(w)|^2\,|f'(f^{-1}(w))|^2\,|(f^{-1})'(w)|^2 \nu(w)\, dA(w) \\
&= \int_{D_2} |\varphi'(w)|^2 \nu(w)\, dA(w).
\end{aligned}
$$

Therefore, $\varphi \circ f \in AD^{\nu \circ f}(D_1, \zeta^n)$. So,

$$
\begin{aligned}
\varphi^{(n)}(f(\zeta))(f'(\zeta))^n &= (\varphi \circ f)^{(n)}(\zeta) = \int_{D_1} (\varphi \circ f)'(z) \, \overline{(M_{D_1,\nu\circ f,n}(\cdot,\zeta))'(z)} (\nu \circ f)(z) \, dA(z) \\
&= \int_{D_2} \varphi'(w) \, f'(f^{-1}(w)) \, \overline{(M_{D_1,\nu\circ f,n}(\cdot,\zeta))'(f^{-1}(w))} \, \nu(w) \, |(f^{-1})'(w)|^2 \, dA(w) \\
&= \int_{D_2} \varphi'(w) \, \overline{(M_{D_1,\nu\circ f,n}(\cdot,\zeta))'(f^{-1}(w)) \cdot (f^{-1})'(w)} \, \nu(w) \, dA(w) \\
&= \int_{D_2} \varphi'(w) \, \overline{(M_{D_1,\nu\circ f,n}(\cdot,\zeta) \circ f^{-1})'(w)} \, \nu(w) \, dA(w).
\end{aligned}
$$

Since $(M_{D_1,\nu\circ f,n}(\cdot,\zeta) \circ f^{-1}) \in AD^{\nu}(D_2, (f(\zeta))^n)$, it follows by the uniqueness of the reproducing kernel of $AD^{\nu}(D_2, (f(\zeta))^n)$ that $M_{D_2,\nu,n}(w, f(\zeta)) = \overline{(f'(\zeta))}^{-n} M_{D_1,\nu\circ f,n}(f^{-1}(w), \zeta)$ for all $w \in D_2$ and $\zeta \in D_1$. That is,

$$
(\overline{f'(\zeta)})^n \, M_{D_2,\nu,n}(f(z), f(\zeta)) = M_{D_1,\nu\circ f,n}(z,\zeta) \quad \text{for all } z, \zeta \in D_1.
$$

This can be differentiated n times to obtain the other transformation formula. $\qquad \square$

Proof of Theorem 1.12. We will use the scaling principle in the case when D is simply connected. So let us quickly recall the scaling principle:

Let ψ be a C^2 defining function for D near p such that $\frac{\partial \psi}{\partial z}(p) = 1$. Let U be a neighborhood of p in $\mathbb{C}$ where ψ is defined. Choose a sequence p_j in $D \cap U$ converging to the boundary point p. For $j \in \mathbb{Z}^+$, define the affine maps

$$
T_j(z) = \frac{z - p_j}{-\psi(p_j)}, \qquad z \in \mathbb{C}.
$$

Let $D_j := T_j(D)$. Note that $0 \in D_j$ for every j as $T_j(p_j) = 0$. Let $K \subset \mathbb{C}$ be compact set. Since $\psi(p_j) \to 0$ as $j \to \infty$, $\{T_j(U)\}$ is an eventually increasing family of open sets that exhaust $\mathbb{C}$ and hence $K \subset T_j(U)$ for all large j. Taking the Taylor series expansion of ψ near $z = p_j$, the functions

$$
\begin{aligned}
\psi \circ T_j^{-1}(z) &= \psi(p_j + z(-\psi(p_j))) \\
&= \psi(p_j) + 2Re\left(\frac{\partial \psi}{\partial z}(p_j)z\right)(-\psi(p_j)) + \psi(p_j)^2 o(1)
\end{aligned}
$$

are therefore well-defined on K for all large j. Let ψ_j be the defining functions of D_j defined near $T_j(p) \in \partial D_j$ on $T_j(U)$, given by

$$
\begin{aligned}
\psi_j(z) &= \frac{1}{(-\psi(p_j))} \psi \circ T_j^{-1}(z) \\
&= -1 + 2Re\left(\frac{\partial \psi}{\partial z}(p_j)z\right) + (-\psi(p_j))o(1).
\end{aligned}
$$

It is easy to see that ψ_j converges to

$$
\psi_{\infty}(z) = -1 + 2Re\left(\frac{\partial \psi}{\partial z}(p)z\right) = -1 + 2Rez.
$$

uniformly on all the compact subsets of $\mathbb{C}$. Let $\mathcal{H}$ denote the half-space defined by ψ_{∞}, i.e.

$$
\mathcal{H} = \{z : -1 + 2Rez < 0\}.
$$

Since ψ_j converges to ψ_{∞} uniformly on compacts, it can be shown that the closure of the domains $T_j(D \cap U)$ (and therefore $T_j(D) = D_j$) converge to the closure of the half-space $\mathcal{H}$ in

the Hausdorff sense. Therefore, every compact $K \subset \mathcal{H}$ is eventually contained in D_j, and every compact $K \subset \mathbb{C} \setminus \overline{\mathcal{H}}$ is eventually contained in $\mathbb{C} \setminus \overline{D_j}$.

We want to study the behaviour of the sequence $\tilde{K}_{D,n}^{(n-1)}(p_j, p_j)$ as $j \to \infty$. Applying Theorem (1.11) on $T_j : D \longrightarrow D_j$ gives

$$\tilde{K}_{D,n}^{(n-1)}(z, \zeta) = (T_j{}'(z))^n \, \tilde{K}_{D_j,n}^{(n-1)}(T_j(z), T_j(\zeta)) \, (\overline{T_j{}'(\zeta)})^n \quad z, \zeta \in D.$$

Since $T_j' \equiv -1/\psi(p_j)$ and $T_j(p_j) = 0$, we obtain

$$\tilde{K}_{D,n}^{(n-1)}(p_j, p_j) = \frac{1}{(\psi(p_j))^{2n}} \, \tilde{K}_{D_j,n}^{(n-1)}(0, 0), \quad j \in \mathbb{Z}^+. \tag{10}$$

Therefore, it is enough to study the behaviour of $\tilde{K}_{D_j,n}^{(n-1)}(0, 0)$ as $j \to \infty$. This is called the scaling principle where a boundary problem has been converted to an interior problem at the cost of varying the domains (for details, see [4]).

Case 1. Let D be simply connected. Since T_j's are affine maps, the domains D_j's are also simply connected. Let $F_j : D_j \longrightarrow \mathcal{H}$ be the Riemann maps such that $F_j(0) = 0$ and $F_j'(0) > 0$. Applying Theorem (1.11) on the Riemann maps F_j gives

$$\tilde{K}_{D_j,n}^{(n-1)}(z, w) = (F_j'(z))^n \, \tilde{K}_{\mathcal{H},n}^{(n-1)}(F_j(z), F_j(w)) \, (\overline{F_j'(w)})^n, \qquad z, w \in D_j. \tag{11}$$

The problem is therefore reduced to studying the Riemann maps F_j.

Lemma 3.1. *The Riemann maps F_j converge to the identity map $i_{\mathcal{H}}$ locally uniformly on $\mathcal{H}$.*

Proof. Since every compact $K \subset \mathbb{C} \setminus \overline{\mathcal{H}}$ is eventually contained in $\mathbb{C} \setminus \overline{D_j}$, we observe that both $\{F_j\}$ and $\{F_j^{-1}\}$ omit atleast two values in $\mathbb{C}$ and fixes 0. Therefore, $\{F_j\}$ and $\{F_j^{-1}\}$ are normal families of holomorphic functions, i.e. there exists a subsequence $\{j_k\}$ of the sequence of natural numbers such that both $\{F_{j_k}\}$ and $\{F_{j_k}^{-1}\}$ converge locally uniformly on $\mathcal{H}$ to some holomorphic functions $F, G : \mathcal{H} \to \overline{\mathcal{H}}$ respectively. Since $F(0) = 0 = G(0)$, the Open mapping theorem implies that $F, G : \mathcal{H} \longrightarrow \mathcal{H}$.

Now, $F_{j_k} \circ F_{j_k}^{-1} \equiv id_{\mathcal{H}}$ and $F_{j_k}^{-1} \circ F_{j_k} \equiv id_{D_j}$. Since $F_{j_k} \circ F_{j_k}^{-1}$ converges to $F \circ G$ locally uniformly on $\mathcal{H}$, we have $F \circ G \equiv id_{\mathcal{H}}$. Since any compact $K \subset \mathcal{H}$ is eventually contained in D_j, the sequence $F_{j_k}^{-1} \circ F_{j_k}$ converges to $G \circ F$ uniformly on K. So, $(G \circ F)|_K \equiv id_K$ and therefore $G \circ F \equiv id_{\mathcal{H}}$. Thus, F is an automorphism of $\mathcal{H}$ such that $F(0) = 0$ and $F'(0) > 0$.

Let $\varphi : \mathcal{H} \to \mathbb{D}$ be the Riemann map such that $\varphi(0) = 0$ and $\varphi'(0) > 0$. Then, $\phi = \varphi \circ F \circ \varphi^{-1}$ is an automorphism of $\mathbb{D}$ such that $\phi(0) = 0$ and $\phi'(0) > 0$. So, $\phi \equiv id_{\mathbb{D}}$ and therefore $F \equiv id_{\mathcal{H}}$. Hence, the Riemann maps F_j converge to the identity map $id_{\mathcal{H}}$ locally uniformly on $\mathcal{H}$. $\qquad \square$

Applying Theorem 1.11 on a Riemann map $\varphi : \mathcal{H} \to \mathbb{D}$ gives that for $z, w \in \mathcal{H}$,

$$\tilde{K}_{\mathcal{H},n}^{(n-1)}(z, w) = (\varphi'(z))^n \, \tilde{K}_{\mathbb{D},n}^{(n-1)}(\varphi(z), \varphi(w)) \, \overline{(\varphi'(w))^n}.$$

Since $\tilde{K}_{\mathbb{D},n}^{(n-1)}$ is continuous on $\mathbb{D} \times \mathbb{D}$ (see Remark 1.5), the kernel function $\tilde{K}_{\mathcal{H},n}^{(n-1)}$ is continuous on $\mathcal{H} \times \mathcal{H}$. Therefore, we conclude from equation (11) and Lemma 3.1 that

$$\lim_{j \to \infty} \tilde{K}_{D_j,n}^{(n-1)} = \tilde{K}_{\mathcal{H},n}^{(n-1)}$$

locally uniformly on $\mathcal{H} \times \mathcal{H}$. Thus, it follows from equation (10) that

$$\psi(p_j)^{2n} \, \tilde{K}_{D,n}^{(n-1)}(p_j, p_j) \to \tilde{K}_{\mathcal{H},n}^{(n-1)}(0, 0) \quad \text{as } j \to \infty.$$

By a previous calculation, for $z, \zeta \in \mathbb{D}$

$$\tilde{K}_{\mathbb{D},n}(z,\zeta) = \frac{n!}{\pi} \frac{(z-\zeta)^{n-1}}{(1-z\bar{\zeta})^{n+1}(1-|\zeta|^2)^{n-1}}.$$

Upon differentiating $n-1$ times with respect to z, we obtain

$$\begin{aligned}
\tilde{K}_{\mathbb{D},n}^{(n-1)}(z,\zeta) &= \frac{n!}{\pi} \frac{1}{(1-|\zeta|^2)^{n-1}} \sum_{k=0}^{n-1} \binom{n-1}{k} \frac{(n-1)!}{k!} (z-\zeta)^k \frac{(n+k)!}{n!} \frac{\bar{\zeta}^k}{(1-z\bar{\zeta})^{n+k+1}} \\
&= \frac{1}{\pi} \frac{1}{(1-|\zeta|^2)^{n-1}} \sum_{k=0}^{n-1} \binom{n-1}{k}^2 (n+k)!(n-k-1)! \frac{\bar{\zeta}^k(z-\zeta)^k}{(1-z\bar{\zeta})^{n+k+1}}.
\end{aligned}$$

Therefore,

$$\begin{aligned}
\tilde{K}_{\mathbb{D},n}^{(n-1)}(z,z) &= \frac{1}{\pi} \frac{1}{(1-|z|^2)^{n-1}} n!(n-1)! \frac{1}{(1-|z|^2)^{n+1}} \\
&= \frac{n!(n-1)!}{\pi} \frac{1}{(1-|z|^2)^{2n}}.
\end{aligned}$$

Applying Theorem 1.11 on the biholomorphism $f : \mathcal{H} \longrightarrow \mathbb{D}$ defined by

$$f(z) = \frac{2z+1}{-2z+3}$$

gives that

$$\tilde{K}_{\mathcal{H},n}^{(n-1)}(0,0) = |f'(0)|^{2n} \tilde{K}_{\mathbb{D},n}^{(n-1)}(f(0),f(0)) = \left(\frac{8}{9}\right)^{2n} \frac{n!(n-1)!}{\pi} \left(\frac{9}{8}\right)^{2n} = \frac{n!(n-1)!}{\pi}.$$

Hence, we have proved that

$$\tilde{K}_{D,n}^{(n-1)}(z,z) \sim \frac{n!(n-1)!}{\pi} \frac{1}{(\psi(z))^{2n}} \quad \text{as } z \to p. \tag{12}$$

Let $\delta(z) = dist(z, \partial D)$. Since $\frac{\partial \psi}{\partial z}(p) = 1$, $(-\psi(z)) \sim 2\,\delta(z)$ as $z \to p$. Therefore,

$$\tilde{K}_{D,n}^{(n-1)}(z,z) \sim \frac{n!(n-1)!}{\pi\, 2^{2n}} \frac{1}{(\delta(z))^{2n}} \quad \text{as } z \to p. \tag{13}$$

In particular, we have proved that

$$\tilde{K}_{D,n}^{(n-1)}(z,z) \approx \frac{1}{(\delta(z))^{2n}} \quad \text{as } z \to p.$$

This proves the theorem when D is simply connected. $\hfill\square$

Case 2. Let D be a bounded domain in $\mathbb{C}$ and $p \in \partial D$. Assume that ∂D is C^2-smooth near p. Choose a neighborhood U of p such that $D \cap U$ is simply connected.

Since $D \cap U$ is simply connected and $\partial(D \cap U)$ is C^2-smooth near p, we have by Case (1) that

$$\tilde{K}_{D\cap U,n}^{(n-1)}(z,z) \sim \frac{n!(n-1)!}{\pi} \frac{1}{(\psi(z))^{2n}} \approx \frac{1}{(\delta(z))^{2n}} \quad \text{as } z \to p.$$

By Theorem 1.10, we have

$$\lim_{z \to p} \frac{\tilde{K}_{D\cap U,n}^{(n-1)}(z,z)}{\tilde{K}_{D,n}^{(n-1)}(z,z)} = 1.$$

Therefore,

$$\tilde{K}_{D,n}^{(n-1)}(z,z) \sim \frac{n!(n-1)!}{\pi} \frac{1}{(\psi(z))^{2n}} \approx \frac{1}{(\delta(z))^{2n}}$$

as $z \to p$. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\square$

This completes the proof of the theorem. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\square$

Proof of Theorem 1.13. Since ν is continuous at p, for $0 < \epsilon < \nu(p)$, there exits $\delta > 0$ such that whenever $|z - p| < \delta$, we have

$$\nu(p) - \epsilon < \nu(z) < \nu(p) + \epsilon.$$

Set $N_p := B(p, \delta) \cap D$. Now, for all $z \in D$, define $\nu_+(z) = \nu(p) - \epsilon$ and $\nu_-(z) = \nu(p) + \epsilon$. So, we get for all $z \in N_p$,

$$\frac{\tilde{K}^{(n-1)}_{N_p,\nu_-,n}(z,z)}{\tilde{K}^{(n-1)}_{D,\nu,n}(z,z)} \leq \frac{\tilde{K}^{(n-1)}_{N_p,\nu,n}(z,z)}{\tilde{K}^{(n-1)}_{D,\nu,n}(z,z)} \leq \frac{\tilde{K}^{(n-1)}_{N_p,\nu_+,n}(z,z)}{\tilde{K}^{(n-1)}_{D,\nu,n}(z,z)}.$$

Note that by definition, it can seen that $\tilde{K}^{(n-1)}_{N_p,\nu_-,n}(z,z) = \frac{\tilde{K}^{(n-1)}_{N_p,n}(z,z)}{\nu(p)+\epsilon}$ and $\tilde{K}^{(n-1)}_{N_p,\nu_+,n}(z,z) = \frac{\tilde{K}^{(n-1)}_{N_p,n}(z,z)}{\nu(p)-\epsilon}$. Applying the localization for the weighted n-th order reduced Bergman kernel, we get

$$\lim_{z \to p} \frac{\tilde{K}^{(n-1)}_{N_p,\nu,n}(z,z)}{\tilde{K}^{(n-1)}_{D,\nu,n}(z,z)} = 1.$$

Therefore, we obtain from the inequality above,

$$\limsup_{z \to p} \frac{\tilde{K}^{(n-1)}_{N_p,n}(z,z)/(\nu(p)+\epsilon)}{\tilde{K}^{(n-1)}_{D,\nu,n}(z,z)} \leq 1 \text{ and } \liminf_{z \to p} \frac{\tilde{K}^{(n-1)}_{N_p,n}(z,z)/(\nu(p)-\epsilon)}{\tilde{K}^{(n-1)}_{D,\nu,n}(z,z)} \geq 1.$$

So

$$\limsup_{z \to p} \frac{\psi(z)^{2n}\tilde{K}^{(n-1)}_{N_p,n}(z,z)/(\nu(p)+\epsilon)}{\psi(z)^{2n}\tilde{K}^{(n-1)}_{D,\nu,n}(z,z)} \leq 1 \text{ and } \liminf_{z \to p} \frac{\psi(z)^{2n}\tilde{K}^{(n-1)}_{N_p,n}(z,z)/(\nu(p)-\epsilon)}{\psi(z)^{2n}\tilde{K}^{(n-1)}_{D,\nu,n}(z,z)} \geq 1.$$

Now, using the boundary behaviour of the non-weighted n-th order reduced Bergman kernel

$$\tilde{K}^{(n-1)}_{N_p,n}(z,z) \sim \frac{n!(n-1)!}{\pi} \frac{1}{(\psi(z))^{2n}} \quad \text{as } z \to p,$$

we get

$$\liminf_{z \to p} \tilde{K}^{(n-1)}_{D,\nu,n}(z,z) \geq \frac{1}{\nu(p)+\epsilon} \frac{n!(n-1)!}{\pi} \frac{1}{(\psi(z))^{2n}},$$

and

$$\limsup_{z \to p} \tilde{K}^{(n-1)}_{D,\nu,n}(z,z) \leq \frac{1}{\nu(p)-\epsilon} \frac{n!(n-1)!}{\pi} \frac{1}{(\psi(z))^{2n}}.$$

Since ϵ is arbitrarily small, we have

$$\tilde{K}^{(n-1)}_{D,\nu,n}(z,z) \sim \frac{1}{\nu(p)} \frac{n!(n-1)!}{\pi} \frac{1}{(\psi(z))^{2n}} \quad \text{as } z \to p.$$

This proves the required result. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\square$

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