Algebras and Representation Theory
https://doi.org/10.1007/s10468-023-10201-5

On Quasi Steinberg Characters of Complex Reflection Groups

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Received: 3 October 2022 / Accepted: 3 February 2023

Abstract

Let G be a finite group and p be a prime number dividing the order of G. An irreducible character χ of G is called a quasi p-Steinberg character if $\chi(g)$ is nonzero for every p-regular element g in G. In this paper, we classify the quasi p-Steinberg characters of complex reflection groups $G(r, q, n)$ and exceptional complex reflection groups. In particular, we obtain this classification for Weyl groups of type B_n and type D_n.

Keywords Quasi Steinberg characters · Complex reflection groups · Weyl groups · Murnaghan–Nakayama rule

Mathematics Subject Classification (2010) 05E10 · 20F55 · 20C15

1 Introduction

The Steinberg characters and their importance are well known for the finite groups of Lie type; see Steinberg [25, 26], Curtis [6], and Humphreys [10]. Using the intrinsic property of the Steinberg character, Feit [8] defined p-Steinberg character for any finite group G and a prime p dividing the order of G (denoted $|G|$). Recall that an element x of G is called p-regular if order of x is coprime to p. An irreducible character θ of G is called a *p-Steinberg character* of G if $\theta(x) = \pm|C_G(x)|_p$ for every p-regular element x in G. Here

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$C_G(x)$ denotes the centralizer of x in G and $|n|_p$ denotes the p-part of an integer n. Feit [8] conjectured that if a finite simple group G has a p-Steinberg character, then G is isomorphic to a simple group of Lie type in characteristic p. Darafasheh [7] proved this conjecture for alternating and projective special linear groups, and Tiep [28] proved it for the remaining finite simple groups. Consequently, in order to explore the structure of finite groups through their characters, many variants of p-Steinberg characters have been studied recently; see, for example, Pellegrini–Zalesski [20], and Malle–Zalesski [15]. Two authors of this article (DP and PS) introduced the following variants of a p-Steinberg character of a finite group G in [19].

Definition 1.1 Let G be a finite group and p be a prime dividing the order of G. An irreducible character χ of G is called **quasi** p**-Steinberg** if $\chi(g) \neq 0$ for every p-regular element g in G. A quasi p-Steinberg character χ is called **weak** p**-Steinberg** if χ has degree $|G|_p$.

These variants were introduced to answer a question of Dipendra Prasad that asked whether the existence of a weak p-Steinberg character of a finite group G implies that G is a finite group of Lie type. We follow [3] for the definition and other related results regarding the finite groups of Lie type. It is well known that every finite group of Lie type has a p-Steinberg character for a prime p. Therefore, if a group does not have a non-linear quasi p-Steinberg character, then it can not be a finite group of Lie type of characteristic p. This naturally leads to asking for a classification of all quasi p-Steinberg characters of any finite group G.

In [19], a classification of all quasi p-Steinberg characters was obtained for symmetric groups (see Table 2), alternating groups, and their double covers. In this work, we classify the quasi p-Steinberg characters of all finite irreducible complex reflection groups. This includes the infinite family of finite irreducible complex reflection groups, denoted by $G(r, q, n)$, and 34 exceptional groups. We refer [23] for the classification of irreducible complex reflection groups. See Section 2 for the definition and other related results regarding $G(r, q, n)$. In particular, we also classify quasi p-Steinberg characters of Weyl groups of type B_n and type D_n. As mentioned above, the parallel classification for Weyl groups of type A_n was obtained in [19].

We now describe the main results of this paper. Note that every linear character (degree one) of G is a quasi p-Steinberg character of G for $p \mid |G|$. Therefore, we will only focus on non-linear characters of G. In this direction, the following is a general result that is true for all finite groups.

Lemma 1.2 (i) *For $d \in \{2, 3, 4\}$, any irreducible character of a finite group G of degree d is a quasi p-Steinberg character of G for $p \mid d$.*
(ii) *For a finite group G and an automorphism $\mathfrak{a}$ of G, an irreducible character χ of G is quasi p-Steinberg if and only if the irreducible character ${}^{\mathfrak{a}}\chi$, defined by ${}^{\mathfrak{a}}\chi(g) = \chi(\mathfrak{a}(g))$ for $g \in G$, is quasi p-Steinberg.*

See Section 3 for a proof of this result. Lemma 1.2(i) does not hold true for any $d \geq 5$, see Remark 3.1. In view of Lemma 1.2(i), we will focus on characterizing the quasi p-Steinberg characters of $G(r, q, n)$ of degrees greater than or equal to 5.

Let $\mathcal{Y}(r, n)$ denote the *set of all r-partite Young diagrams* with total number of boxes being n. It is well known that the set $\mathcal{Y}(r, n)$ indexes the irreducible characters of $G(r, 1, n)$. For $\boldsymbol{\lambda} \in \mathcal{Y}(r, n)$, we use $\chi^{\boldsymbol{\lambda}}$ to denote the corresponding irreducible character of $G(r, 1, n)$.

Table 1 Quasi p-Steinberg characters of $G(r, 1, n)$

n	$\boldsymbol{\lambda}$	$\boldsymbol{\lambda}_j$	p	$\deg(\chi^{\boldsymbol{\lambda}})$	$(r, p) = 1$
2	$\widehat{\boldsymbol{\lambda}}^{j,k}$	(1)	2	2	weak
3	$\widehat{\boldsymbol{\lambda}}^{j}$	(2, 1)	2	2	weak
3	$\widehat{\boldsymbol{\lambda}}^{j,k}$	(2), (1, 1)	3	3	weak
4	$\widehat{\boldsymbol{\lambda}}^{j}$	(2, 2)	2	2	not weak
4	$\widehat{\boldsymbol{\lambda}}^{j,k}$	(3), (1, 1, 1)	2	4	not weak
4	$\widehat{\boldsymbol{\lambda}}^{j,k}$	(2, 1)	2	8	weak
4	$\widehat{\boldsymbol{\lambda}}^{j}$	(3, 1), (2, 1, 1)	3	3	weak
5	$\widehat{\boldsymbol{\lambda}}^{j}$	(4, 1), (2, 1, 1, 1)	2	4	not weak
5	$\widehat{\boldsymbol{\lambda}}^{j}$	(3, 2), (2, 2, 1)	5	5	weak
6	$\widehat{\boldsymbol{\lambda}}^{j}$	(3, 2, 1)	2	16	weak
6	$\widehat{\boldsymbol{\lambda}}^{j}$	(4, 2), (2, 2, 1, 1)	3	9	weak
8	$\widehat{\boldsymbol{\lambda}}^{j}$	(5, 2, 1), (3, 2, 1, 1, 1)	2	64	not weak

In the following definition, we consider certain special elements of $\mathcal{Y}(r, n)$ that help us to characterize the quasi p-Steinberg characters of $G(r, q, n)$.

Definition 1.3 For $j \in \{0, 1, \cdots r-1\}$, define $\widehat{\boldsymbol{\lambda}}^{j} = (\boldsymbol{\lambda}_0, \boldsymbol{\lambda}_1, \ldots, \boldsymbol{\lambda}_{r-1}) \in \mathcal{Y}(r, n)$ by

$$\boldsymbol{\lambda}_j \vdash n \text{ and } \boldsymbol{\lambda}_l = \emptyset \text{ for } l \neq j,$$

and for $j, k \in \{0, 1, \cdots r-1\}$ such that $j \neq k$, define $\widehat{\boldsymbol{\lambda}}^{j,k} = (\boldsymbol{\lambda}_0, \boldsymbol{\lambda}_1, \ldots, \boldsymbol{\lambda}_{r-1}) \in \mathcal{Y}(r, n)$ by

$$\boldsymbol{\lambda}_j \vdash n-1, \boldsymbol{\lambda}_k = (1) \text{ and } \boldsymbol{\lambda}_l = \emptyset \text{ for } l \notin \{j, k\}.$$

We use this definition to give a complete classification of quasi p-Steinberg characters of $G(r, 1, n)$ in the following theorem.

Theorem 1.4 *For $n \geq 2$, let $\boldsymbol{\lambda} = (\boldsymbol{\lambda}_0, \boldsymbol{\lambda}_1, \ldots, \boldsymbol{\lambda}_{r-1})$ be an r-partite partition of n such that $\boldsymbol{\lambda}_t \notin \{(n), (1^n)\}$ for every $0 \leq t \leq r-1$. All triples $(n, \boldsymbol{\lambda}, p)$ such that $\chi^{\boldsymbol{\lambda}}$ is a quasi p-Steinberg character of $G(r, 1, n)$ are given in* Table 1.

Table 2 Quasi p-Steinberg characters of S_n

n	μ	p
3	(2, 1)	2
4	(2, 2)	2
4	(3, 1), (2, 1, 1)	3
5	(4, 1), (2, 1, 1, 1)	2
5	(3, 2), (2, 2, 1)	5
6	(3, 2, 1)	2
6	(4, 2), (2, 2, 1, 1)	3
8	(5, 2, 1), (3, 2, 1, 1, 1)	2

While describing $\widehat{\boldsymbol{\lambda}}^{j,k}$ in Table 1, we mention only $\boldsymbol{\lambda}_j$ because $\boldsymbol{\lambda}_k = (1)$. We prove the above result (see Section 3) by showing that the problem of classifying quasi p-Steinberg characters of $G(r, 1, n)$ reduces to the $\chi^{\boldsymbol{\lambda}}$, where $\boldsymbol{\lambda}$'s are as described in Definition 1.3 (see Proposition 3.3).

From Theorem 1.4 and by the definition of weak p-Steinberg character, we note that a quasi p-Steinberg character of $G(r, 1, n)$ is a weak p-Steinberg character only if $(r, p) = 1$. For $(r, p) = 1$, it is easy to identify the weak p-Steinberg characters by using the Murnaghan–Nakayama rule (see Theorem 2.7). We indicate the weak p-Steinberg characters of $G(r, 1, n)$, for r such that $(r, p) = 1$, in the last column of Table 1 by writing these as "weak". In particular, we obtain the following result.

Corollary 1.5 (i) *The groups $G(r, 1, n)$ do not have any weak p-Steinberg characters for $p \mid r$.*

(ii) *For $(r, p) = 1$, the weak p-Steinberg characters $\chi^{\boldsymbol{\lambda}}$ of $G(r, 1, n)$ are the ones mentioned in the last column of* Table 1.

Given $\boldsymbol{\lambda} = (\boldsymbol{\lambda}_0, \boldsymbol{\lambda}_1, \ldots, \boldsymbol{\lambda}_{r-1}) \in \mathcal{Y}(r, n)$, we use the notation $(\chi^{\boldsymbol{\lambda}})^*$ to denote an irreducible character of $G(r, q, n)$ which appears in the decomposition of the restriction of the irreducible character $\chi^{\boldsymbol{\lambda}}$ of $G(r, 1, n)$ to $G(r, q, n)$. In our next result, we classify all quasi p-Steinberg characters of $G(r, q, n)$ of degrees greater than or equal to 5.

Theorem 1.6 *Let $n \geq 2$ and $(\chi^{\boldsymbol{\lambda}})^*$ be an irreducible character of $G(r, q, n)$ associated with an r-partite partition $\boldsymbol{\lambda} = (\boldsymbol{\lambda}_0, \boldsymbol{\lambda}_1, \ldots, \boldsymbol{\lambda}_{r-1})$ of n such that* $\deg((\chi^{\boldsymbol{\lambda}})^*) \geq 5$. *Then $(\chi^{\boldsymbol{\lambda}})^*$ is a quasi p-Steinberg characters of $G(r, q, n)$ if and only if $\boldsymbol{\lambda}$ is as given in the* Table 1 *and $\chi^{\boldsymbol{\lambda}}$ is an irreducible character of $G(r, 1, n)$ with* $\deg(\chi^{\boldsymbol{\lambda}}) \geq 5$.

We obtain the above result from Theorem 3.6, which gives the complete classification of quasi p-Steinberg characters of $G(r, q, n)$. We use Clifford theory (see [11, Chapter 6]), Theorems 1.2(i) and 1.4 to prove this result; see Section 3 for the proof. We now list a few corollaries of our main results. The following result extends the parallel known results for $G(1, 1, n)$ (see [19, Corollary 1.7]).

Corollary 1.7 *For $n \geq 9$ and $p \leq n$, every non-linear irreducible character χ of $G(r, q, n)$ has a zero at some p-regular element of $G(r, q, n)$.*

The proof of this result follows directly from Theorem 1.4 for $q = 1$ and from Theorem 3.6 for $q \neq 1$. The next result follows directly from Theorem 1.2(i) and Corollary 1.7.

Corollary 1.8 *For $n \geq 9$, every non-linear irreducible representation of $G(r, q, n)$ has degree greater than or equal to five.*

We also obtain the following result from Theorem 1.4, Table 1, and the Murnaghan–Nakayama rule.

Corollary 1.9 (i) *The groups $G(r, 1, n)$, for $n \geq 9$, do not have a weak p-Steinberg character.*

(ii) *The groups $G(r, 1, 2)$ for odd r have a weak 2-Steinberg character of degree 2, but do not have p-Steinberg characters for every prime p.*

The above corollary shows that the groups $G(r, 1, n)$ for $n \geq 9$ do not have a p-Steinberg character for any prime p. Therefore these are not finite groups of Lie type. It also shows that the groups $G(r, 1, 2)$ for odd r have weak 2-Steinberg characters but are not finite groups of Lie type. Hence, in general, it may be difficult to conclude the structure of a given finite group by the existence of a weak p-Steinberg character.

Now we classify quasi p-Steinberg characters for exceptional finite irreducible complex reflection groups. There are 34 such groups, say $G(n)$, where $n \in \{4, \ldots, 37\}$ is the Shephard–Todd number of the corresponding reflection group. Accessing character tables of $G(n)$ using Sagemath and GAP, we obtained the orders of elements on which a particular irreducible character vanishes. This led us to the following result (see Section 4 for more details).

Theorem 1.10 *For $n \in \{4, \ldots, 37\}$, every quasi p-Steinberg character of $G(n)$ has degree a power of p. Conversely, every irreducible character of $G(n)$ with degree a power of p is quasi p-Steinberg except the following:*

(i) *Any degree 5 irreducible character of $G(32)$, $G(33)$ is not quasi 5-Steinberg.*
(ii) *Any degree 7 irreducible character of $G(36)$ is not quasi 7-Steinberg.*

2 Preliminaries

2.1 Quasi p-Steinberg Characters of Symmetric Groups

It is well known that the set of integer partitions of n indexes conjugacy classes and irreducible characters of S_n. Given a partition $\mu = (\mu_1, \mu_2, \ldots, \mu_{l(\mu)})$ of n (denoted by $\mu \vdash n$), let $\varkappa^\mu$ denote the corresponding irreducible character of S_n. All triples (n, μ, p) such that $\varkappa^\mu$ is a quasi p-Steinberg character of S_n are listed in Table 2. See [19, Theorem 1.3] for the proof.

2.2 Complex Reflection Groups $G(r, q, n)$

Given positive integers r and n, the symmetric group S_n acts on the direct product $\mathbb{Z}_r^n$ of n copies of the additive cyclic group $\mathbb{Z}_r$ by permuting the coordinates. This action gives us the wreath product of $\mathbb{Z}_r$ by S_n, denoted by $G(r, 1, n) := \mathbb{Z}_r^n \rtimes S_n$, i.e.,

$$G(r, 1, n) = \{(z_1, z_2, \ldots, z_n; \sigma) \mid z_i \in \mathbb{Z}_r \text{ for all } 1 \leq i \leq n, \sigma \in S_n\}.$$

For a positive integer q which divides r, we define a subgroup $G(r, q, n)$ of $G(r, 1, n)$ as follows:

$$G(r, q, n) := \{(z_1, z_2, \ldots, z_n; \sigma) \in G(r, 1, n) \mid \sum_{i=1}^{n} z_i \equiv 0 (\bmod q)\}.$$

The group $G(r, q, n)$ is a normal subgroup of $G(r, 1, n)$ of index q. By Shephard–Todd's classification, the family of groups $G(r, q, n)$ for $n > 1$, (except the group $G(2, 2, 2)$), is the only infinite family of finite irreducible imprimitive *complex reflection groups* [23, Section 2]. The group $G(2, 2, 2)$ is imprimitive, but it is not irreducible [5, Theorem (2.4)]. The group $G(r, 1, n)$ is also known as the *generalized symmetric group*.

Some families of groups that are special cases of $G(r, q, n)$ are:

(i) Cyclic group of order r, $\mathbb{Z}/r\mathbb{Z} = G(r, 1, 1)$;

(ii) Dihedral group of order $2r$, $D_{2r} = G(r, r, 2)$;
(iii) symmetric group on n symbols, $S_n = G(1, 1, n)$;
(iv) Weyl group of type B_n (also called hyperoctahedral group) is $G(2, 1, n)$; and
(v) Weyl group of type D_n is $G(2, 2, n)$.

2.3 Conjugacy Classes of $G(r, q, n)$

Let $\pi = (z_1, z_2, \ldots, z_n; \sigma) \in G(r, 1, n)$ be such that the corresponding cycle decomposition of σ be $c = (c_1, c_2, \ldots, c_t)$, where the cycles are written in an arbitrarily fixed order. For $1 \leq i \leq t$, suppose that the cycle c_i of length $l(c_i)$ is written as $(c_{i1}, c_{i2}, \ldots, c_{il(c_i)})$. We define the *color of the cycle* c_i to be the cycle sum $z(c_i) := z_{c_{i1}} + z_{c_{i2}} + \cdots + z_{c_{il(c_i)}} \in \mathbb{Z}_r$. Let $\mathcal{P}$ denote the set of all partitions (by convention, 0 has a unique partition, called empty partition, denoted by $\emptyset$). For $\pi = (z_1, z_2, \ldots, z_n; \sigma) \in G(r, 1, n)$, define a map

$$\tau_\pi : \mathbb{Z}_r \to \mathcal{P}$$

by setting $\tau_\pi(j)$ to be the partition associated to the multiset of lengths of all cycles in σ whose color is j; denote this partition by $\boldsymbol{\lambda}_j$. The map τ_π is called the *type* of π. Thus, the type of π can be written as an r-tuple of partitions, $\boldsymbol{\lambda} = (\boldsymbol{\lambda}_0, \boldsymbol{\lambda}_1, \ldots, \boldsymbol{\lambda}_{r-1})$, such that the total sum of all the parts is n. We call such r-tuple of partitions an *r-partite partition* of size n. Given a partition, we consider the Young diagram associated with it. Throughout this article, we use the notions of partition and the Young diagram interchangeably. Let $\mathcal{Y}(r, n)$ denote the *set of all r-partite Young diagrams* with total number of boxes being n. The following theorem (see [14, p.170] for a proof) states that the conjugacy classes of $G(r, 1, n)$ are indexed by the set $\mathcal{Y}(r, n)$.

Theorem 2.1 *Two elements π_1 and π_2 in $G(r, 1, n)$ are conjugate if and only if the corresponding types are equal, i.e., $\tau_{\pi_1} = \tau_{\pi_2}$.*

Let $z(\pi)$ be the gcd of all nonzero $z(c_i)$ for $1 \leq i \leq t$. Define

$$d(\pi) := \gcd(z(\pi), l(c_1), l(c_2), \ldots, l(c_t), q).$$

The following theorem describes the splitting of conjugacy classes of $G(r, 1, n)$ into conjugacy classes of $G(r, q, n)$; for more details about conjugacy classes of $G(r, q, n)$, see [22, Section 2].

Theorem 2.2 [22, Theorem 3] *If $\pi \in G(r, q, n)$, then the conjugacy class of π in $G(r, 1, n)$ splits into $d(\pi)$ conjugacy classes in $G(r, q, n)$.*

Now, we give a characterization of an element in $G(r, 1, n)$ to be an element of $G(r, q, n)$.

Lemma 2.3 *Given $\pi = (z_1, z_2, \ldots, z_n; \sigma) \in G(r, 1, n)$, let $\boldsymbol{\lambda} = (\boldsymbol{\lambda}_0, \boldsymbol{\lambda}_1, \ldots, \boldsymbol{\lambda}_{r-1})$ be the type of π. Then, π belongs to $G(r, q, n)$ if and only if $\sum_{j=1}^{r-1} jl(\boldsymbol{\lambda}_j) \equiv 0 (mod\ q)$.*

Proof The parts of the partition $\boldsymbol{\lambda}_j$ are the lengths of the cycles in σ whose color is j. For each cycle $(i_1, \ldots, i_k)$ of color j, we have $z_{i_1} + \cdots + z_{i_k} = j$. Thus, j appears $l(\boldsymbol{\lambda}_j)$ times in the sum $z_1 + \cdots + z_n$. Therefore,

$$z_1 + \cdots + z_n = \sum_{j=0}^{r-1} jl(\boldsymbol{\lambda}_j) = \sum_{j=1}^{r-1} jl(\boldsymbol{\lambda}_j).$$

□

2.4 Representation Theory of $G(r, q, n)$

The representation theory of the wreath product of a finite group by symmetric group S_n is a well-studied subject, see [4, 12, 14, 16]. As stated in Section 2.3, the set $\mathcal{Y}(r, n)$ indexes the irreducible representations of $G(r, 1, n)$. The degree of an irreducible representation $V^{\boldsymbol{\lambda}}$ corresponding to $\boldsymbol{\lambda} \in \mathcal{Y}(r, n)$ can be obtained from [16, Theorem 6.7].

Since $G(r, q, n)$ is a normal subgroup of $G(r, 1, n)$, the representation theory of $G(r, q, n)$ can be deduced from the representation theory of $G(r, 1, n)$ using Clifford theory. Let H denote the group of linear characters of $G(r, 1, n)$ which contain $G(r, q, n)$ in their kernel. The group H acts on the set of irreducible representations of $G(r, 1, n)$. Given $\boldsymbol{\lambda} \in \mathcal{Y}(r, n)$, let $[\boldsymbol{\lambda}]$ and $H_{\boldsymbol{\lambda}}$ denote the orbit of $\boldsymbol{\lambda}$ and the stabilizer subgroup of H with respect to $\boldsymbol{\lambda}$, respectively.

Given $m := \frac{r}{q}$, let us define a combinatorial object, an (m, q)-*necklace* [9, p.174], which will be useful in parametrization of irreducible $G(r, q, n)$-modules. Given $\boldsymbol{\lambda} = (\boldsymbol{\lambda}_0, \boldsymbol{\lambda}_1, \ldots, \boldsymbol{\lambda}_{r-1}) \in \mathcal{Y}(r, n)$, consider the q-tuple

$$\tilde{\boldsymbol{\lambda}}_{(i)} := (\boldsymbol{\lambda}_i, \boldsymbol{\lambda}_{m+i}, \boldsymbol{\lambda}_{2m+i}, \ldots, \boldsymbol{\lambda}_{(q-1)m+i}),$$

for each $0 \leq i \leq m-1$. We depict $\tilde{\boldsymbol{\lambda}}_{(i)}$ as a circular q-necklace in (x, y)-plane with the $\boldsymbol{\lambda}_i$ node being placed on the positive y-axis and the j-th node $\boldsymbol{\lambda}_{(j-1)m+i}$, $2 \leq j \leq q$, being placed at a clockwise angle of $2\pi/(j-1)$ with the positive y-axis. An (m, q)-*necklace* of total n boxes obtained from $\boldsymbol{\lambda} \in \mathcal{Y}(r, n)$, denoted by $\tilde{\boldsymbol{\lambda}}$, is an m-tuple of q-necklaces

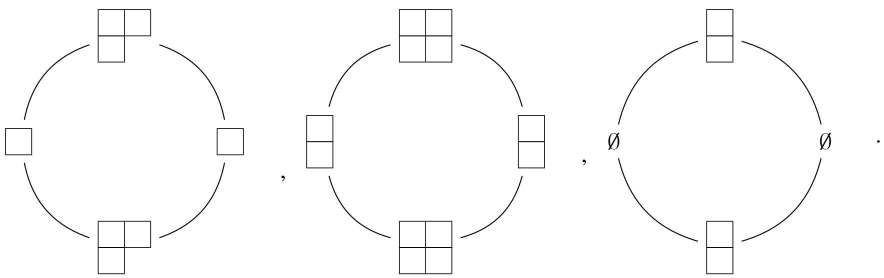
$$\tilde{\boldsymbol{\lambda}} = (\tilde{\boldsymbol{\lambda}}_{(0)}, \tilde{\boldsymbol{\lambda}}_{(1)}, \ldots, \tilde{\boldsymbol{\lambda}}_{(m-1)}).$$

For $1 \leq j \leq q$ and $0 \leq i \leq m-1$, let $\tilde{\boldsymbol{\lambda}}_{(i,j)}$ denote the j-th node in $\tilde{\boldsymbol{\lambda}}_{(i)}$, i.e., $\tilde{\boldsymbol{\lambda}}_{(i,j)} = \lambda_{(j-1)m+i}$. Thus,

$$\sum_{i=0}^{m-1} \sum_{j=1}^{q} \tilde{\boldsymbol{\lambda}}_{(i,j)} = n.$$

Example 2.4 An example of $(3, 4)$-necklace of total 24 boxes obtained from

$$\boldsymbol{\lambda} = ((2, 1), (2, 2), (1, 1), (1), (1, 1), \emptyset, (2, 1), (2, 2), (1, 1), (1), (1, 1), \emptyset) \text{ is :}$$



Two (m, q)-necklaces, $\tilde{\boldsymbol{\lambda}}$ and $\tilde{\boldsymbol{\mu}}$, both with total n boxes, are said to be *equivalent* if for some integer t, $\tilde{\boldsymbol{\lambda}}_{(i,j)} = \tilde{\boldsymbol{\mu}}_{(i,(j+t)(mod\ q))}$ for all $1 \leq j \leq q$ and $0 \leq i \leq m-1$. Let $\mathcal{Y}(m, q, n)$ denote the *set of inequivalent* (m, q)*-necklaces* with total n boxes.

Theorem 2.5 describes the inequivalent irreducible representations of $G(r, q, n)$. For more details and proof of Theorem 2.5, see [2, 17, 18, 27].

Theorem 2.5 *The irreducible $G(r, q, n)$-modules are indexed by the set of ordered pairs $(\tilde{\boldsymbol{\lambda}}, \delta)$, where $\tilde{\boldsymbol{\lambda}} \in \mathcal{Y}(m, q, n)$ and $\delta \in H_{\boldsymbol{\lambda}}$. Given $\boldsymbol{\lambda} \in \mathcal{Y}(r, n)$, the restriction of the corresponding irreducible $G(r, 1, n)$-module $V^{\boldsymbol{\lambda}}$ to $G(r, q, n)$ has multiplicity free decomposition given as:*

$$Res^{G(r,1,n)}_{G(r,q,n)}(V^{\boldsymbol{\lambda}}) = \bigoplus_{\delta \in H_{\boldsymbol{\lambda}}} V^{(\tilde{\boldsymbol{\lambda}},\delta)},$$

where $V^{(\tilde{\boldsymbol{\lambda}},\delta)}$ denotes the irreducible $G(r, q, n)$-module indexed by $(\tilde{\boldsymbol{\lambda}}, \delta)$. Moreover, for $\boldsymbol{\lambda}, \boldsymbol{\mu} \in \mathcal{Y}(r, n)$, we have $Res^{G(r,1,n)}_{G(r,q,n)}(V^{\boldsymbol{\lambda}}) \cong Res^{G(r,1,n)}_{G(r,q,n)}(V^{\boldsymbol{\mu}})$ if and only if $\boldsymbol{\mu} \in [\boldsymbol{\lambda}]$.

Next lemma is an important consequence of Clifford theory and will be helpful in studying the character values of an element in $G(r, 1, n)$ whose conjugacy class does not split in the normal subgroup $G(r, q, n)$.

Lemma 2.6 *Let N be a normal subgroup of a group G. Suppose that ρ and τ are irreducible representations of G and N with characters χ_ρ and χ_τ, respectively, such that $\langle \chi_\rho|_N, \chi_\tau \rangle_N \neq 0$. Assume that x is an element of N such that its conjugacy class in G does not split in N. Then, $\chi_\rho|_N(x)$ is a positive integral multiple of $\chi_\tau(x)$.*

Proof Assume that $\{g_1 = id, g_2, \ldots, g_k\}$ be a set of coset representatives of the inertia group of τ in G. By Clifford theory, $\rho|_N = \oplus_{i=1}^k (\tau^{g_i})^f = (\oplus_{i=1}^k \tau^{g_i})^f$, where τ^{g_i} is a conjugate representation of τ and f is the multiplicity of each conjugate representation. Then the result follows from the following computation:

$$\chi_\rho|_N(x) = f \sum_{i=1}^{k} \chi_\tau^{g_i}(x) = f \sum_{i=1}^{k} \chi_\tau(g_i^{-1} x g_i) = f \sum_{i=1}^{k} \chi_\tau(x) = f k \chi_\tau(x).$$

□

2.5 Murnaghan–Nakayama Rule for $G(r, 1, n)$

The Murnaghan–Nakayama rule [27, Theorem 4.3] is a combinatorial method to compute the irreducible characters of $G(r, 1, n)$. An edge-connected skew shape that does not contain a 2×2 square of boxes is called a *ribbon*. Given a partition μ of n, a *ribbon tableau* of shape μ is obtained by filling the boxes of μ with positive integers such that the entries in each row and column are weakly increasing and each appearing integer forms a distinct ribbon. Let $\boldsymbol{\lambda} = (\boldsymbol{\lambda}_0, \boldsymbol{\lambda}_1, \ldots, \boldsymbol{\lambda}_{r-1})$ be an r-partite partition of n. An r-tuple $T = (T_0, \ldots, T_{r-1})$ is called *r-partite ribbon tableau* of shape $\boldsymbol{\lambda}$ if each T_j is a ribbon tableau of shape $\boldsymbol{\lambda}_j$ and for each positive integer i, the ribbon containing i appears in at most one component of T.

Given an r-partite ribbon tableau $T = (T_0, \dots, T_{r-1})$ we denote its i-th *index*, i-th *length* and i-th *height* by $f_T(i)$, $l_T(i)$ and $ht_T(i)$, respectively, which are defined as follows:

$$
\begin{aligned}
f_T(i) &:= \text{ index of the ribbon containing } i \text{ in } T;\\
l_T(i) &:= \text{ size of the ribbon containing } i \text{ in } T;\\
ht_T(i) &:= \text{ one less than the number of rows in the ribbon containing } i \text{ in } T.
\end{aligned}
$$

Theorem 2.7 [1, Proposition 2.2] *(Murnaghan–Nakayama rule for $G(r, 1, n)$) Suppose that $\chi^{\boldsymbol{\lambda}}$ denotes the irreducible character of $G(r, 1, n)$ corresponding to $\boldsymbol{\lambda} \in \mathcal{Y}(r, n)$. Let $\pi = (z_1, z_2, \dots, z_n; \sigma) \in G(r, 1, n)$ such that the corresponding cycle decomposition of σ in an arbitrarily fixed order be $c = (c_1, c_2, \dots, c_t)$. For $1 \le i \le t$, let $l(c_i)$ and $z(c_i)$ be the length and color of the cycle c_i, respectively. Then,*

$$
\chi^{\boldsymbol{\lambda}}(\pi) = \sum_{T \in RT_c(\boldsymbol{\lambda})} \prod_{i=1}^{t} (-1)^{ht_T(i)} \omega^{f_T(i).z(c_i)}, \tag{2.1}
$$

where $RT_c(\boldsymbol{\lambda})$ is the set of all r-partite ribbon tableaux T of shape $\boldsymbol{\lambda}$ such that $l_T(i) = l(c_i)$ for all $1 \le i \le t$, and $\omega = e^{2\pi\iota/r}$.

Let us illustrate Theorem 2.7 by an example.

Example 2.8 Consider the 3-partite partition $\boldsymbol{\lambda} = ((2, 1), \emptyset, (1, 1, 1)) \in \mathcal{Y}(3, 6)$ and the element π of $G(3, 1, 6)$

$$\pi = (1, 1, 0, 0, 1, 0; (1, 2, 3)(4, 5)).$$

Here $c = (c_1, c_2, c_3)$, where we choose $c_1 = (1, 2, 3)$, $c_2 = (4, 5)$ and $c_3 = (6)$. The set $RT_c(\boldsymbol{\lambda})$ consists of three 3-partite ribbon tableaux $T = (T_0, T_1, T_2)$ as described below

$$
\begin{array}{|c|c|}\hline 1 & 1 \\ \hline 1 & \multicolumn{1}{c}{} \\ \cline{1-1}\end{array},\ \emptyset,\ \begin{array}{|c|}\hline 2\\ \hline 2\\ \hline 3\\ \hline\end{array};\quad
\begin{array}{|c|c|}\hline 2 & 2 \\ \hline 3 & \multicolumn{1}{c}{} \\ \cline{1-1}\end{array},\ \emptyset,\ \begin{array}{|c|}\hline 1\\ \hline 1\\ \hline 1\\ \hline\end{array};\quad
\begin{array}{|c|c|}\hline 2 & 3 \\ \hline 2 & \multicolumn{1}{c}{} \\ \cline{1-1}\end{array},\ \emptyset,\ \begin{array}{|c|}\hline 1\\ \hline 1\\ \hline 1\\ \hline\end{array}.
$$

Consider $\omega = e^{2\pi\iota/3}$. The character value is

$$\chi^{\boldsymbol{\lambda}}(\pi) = (-1)(-1)\omega^{2.1} + (-1)^2\omega^{2.2} + (-1)^2\omega^{2.2}(-1) = \omega^2 + \omega - \omega = \omega^2.$$

Now consider another element of $G(3, 1, 6)$:

$$\pi = (1, 1, 0, 0, 1, 0; (1, 2)(3, 5)(4, 6)).$$

Here $l(c_i) = 2$ for all $i = 1, 2, 3$. Observe that there are no 3-partite ribbon tableaux $T = (T_0, T_1, T_2)$ of shape $\boldsymbol{\lambda}$ such that $l_T(i) = 2$. Hence the character value $\chi^{\boldsymbol{\lambda}}(\pi)$ is zero.

In the following corollary of Theorem 2.7, we relate character values of $G(r, 1, n)$ with those of S_n.

Corollary 2.9 (i) *For an r-partite partition $\boldsymbol{\lambda} = (\boldsymbol{\lambda}_0, \boldsymbol{\lambda}_1, \dots, \boldsymbol{\lambda}_{r-1})$ of n, the character value $\chi^{\boldsymbol{\lambda}}((z_1, z_2, \dots, z_n; (1, 2, \dots, n)))$ is nonzero if and only if $\boldsymbol{\lambda} = \widehat{\boldsymbol{\lambda}}^j$ for some $0 \le j \le r-1$ with $\boldsymbol{\lambda}_j$ being a hook of size n. In such a case, we have*

$$\chi^{\boldsymbol{\lambda}}((z_1, z_2, \dots, z_n; (1, 2, \dots, n))) = \omega^{j(z_1 + \cdots + z_n)} \varkappa^{\boldsymbol{\lambda}_j}((1, 2, \dots, n)).$$

(ii) *For an element σ of S_n and $0 \leq j \leq r-1$, we have*

$$\chi^{\widehat{\boldsymbol{\lambda}}^j}((0,\ldots,0;\sigma)) = \varkappa^{\boldsymbol{\lambda}_j}(\sigma).$$

(iii) *For $\pi = (z_1,\ldots,z_n;\sigma) \in G(r,1,n)$, we have*

$$\chi^{\widehat{\boldsymbol{\lambda}}^0}(\pi) = \varkappa^{\boldsymbol{\lambda}_0}(\sigma).$$

Moreover, for $0 \leq j,k \leq r-1$, $\chi^{\widehat{\boldsymbol{\lambda}}^j}(\pi)$ is nonzero if only if $\chi^{\widehat{\boldsymbol{\lambda}}^k}(\pi)$ is nonzero.

(iv) *The character $\chi^{\widehat{\boldsymbol{\lambda}}^j}$ never vanishes on $(z_1,z_2,\ldots,z_n;(1)) \in G(r,1,n)$.*

Proof (i) Let $\chi^{\boldsymbol{\lambda}}((z_1,z_2,\ldots,z_n;(1,2,\ldots,n)))$ be nonzero. Since $\sigma = (1,2,\ldots,n)$ has a single part of length n, T must have only one nonempty component, say T_j, which must be a single ribbon of size n. That implies $\boldsymbol{\lambda} = \widehat{\boldsymbol{\lambda}}^j$ and $\boldsymbol{\lambda}_j$ is a hook of size n. The converse and the computation of character value are straightforward.

(ii) Here, $\widehat{\boldsymbol{\lambda}}^j$ has only one nonempty component and $z_i = 0$ for $i = 0,1,\ldots,r-1$. We obtain the result by evaluating equation (2.1) and comparing it with Murnaghan–Nakayama rule for S_n [24, Theorem 7.17.3].

(iii) Since $T_j = \emptyset$ for all $j \neq 0$, $f_T(i) = 0$ for all i. So Eq. 2.1 is equivalent to the Murnaghan–Nakayama rule to compute irreducible characters of S_n corresponding to the partition $\boldsymbol{\lambda}_0$. For the next part, a routine calculation shows that

$$\chi^{\widehat{\boldsymbol{\lambda}}^j}(\pi) = \omega^{j-k}\chi^{\widehat{\boldsymbol{\lambda}}^k}(\pi).$$

(iv) When $\sigma = (1)$, the set $RT_c(\boldsymbol{\lambda})$ is nonempty and for each r-partite ribbon tableau $T \in RT_c(\boldsymbol{\lambda})$, we have $ht_T(i) = 0$, $f_T(i) = j$, $z(c_i) = z_i$ for all $i = 1,2,\ldots,n$.

□

Next, we write some observations as a lemma which will be useful later.

Lemma 2.10 (i) *If $\pi = (z_1,z_2,\ldots,z_n;\sigma)$ is a p-regular element in $G(r,1,n)$, then σ is a p-regular element of S_n.*

(ii) *The order of the sum of elements of odd order in an abelian group can never be even.*

Proof (i) This follows from the observation that under the projection map

$$pr : G(r,1,n) \to S_n,$$

the order of $pr(z_1,z_2,\ldots,z_n;\sigma)$ divides the order of $(z_1,z_2,\ldots,z_n;\sigma)$.

(ii) Let $g_1,\ldots,g_s$ be elements of odd orders $l_1,\ldots,l_s$ respectively, in an abelian group. Then the order of $g_1+\cdots+g_s$ divides $l_1\cdots l_s$.

□

3 Proof of Theorems 1.4 and 1.6

We now prove the results stated in Section 1 for the groups $G(r,1,n)$ and $G(r,q,n)$. The groups $G(r,1,1)$ and $G(r,q,1)$ are cyclic, all of whose irreducible characters are linear and, thus, quasi p-Steinberg. Therefore, now onwards we assume that $n \geq 2$ for $G(r,1,n)$ and $G(r,q,n)$.

Proof of Lemma 1.2 (i) For $d \in \{2, 3, 4\}$ with $p \mid d$, consider a degree d irreducible character χ of G and a p-regular element g, say of order t, in G. Then $\chi(g)$ is a sum of t-th roots of unity and there are d-summands in this sum. This sum is always non-zero by [13, Main Theorem, p.2] and therefore our result follows.

(ii) This follows from the fact that $g \in G$ is p-regular if and only if $\mathfrak{a}(g)$ is p-regular. □

Remark 3.1 It is interesting to note that Theorem 1.2(i) is not true in general for any $d \geq 5$. The degree n irreducible character $\varkappa^{(n,1)}$ of S_{n+1} for $n \geq 5$ is not a quasi p-Steinberg character for any prime $p \leq n + 1$ (see Table 2).

The following useful corollary follows directly from Theorem 1.2(ii) and Clifford theory.

Corollary 3.2 *For a normal subgroup N and an irreducible character χ of G, either all the irreducible characters of N appearing in the restriction $\chi|_N$ are quasi p-Steinberg characters or none of these are.*

As an immediate application of Theorem 1.2(i), we list the triples $(n, \boldsymbol{\lambda}_j, p)$ in Table 3 which correspond to some specific quasi p-Steinberg characters $\chi^{\widehat{\boldsymbol{\lambda}}^j}$ of $G(r, 1, n)$.

Proposition 3.3 *Given an r-partite partition $\boldsymbol{\lambda} = (\boldsymbol{\lambda}_0, \boldsymbol{\lambda}_1, \ldots, \boldsymbol{\lambda}_{r-1})$ of n and a prime p, assume that $\chi^{\boldsymbol{\lambda}}$ is a quasi p-Steinberg character of $G(r, 1, n)$. Then, either $\boldsymbol{\lambda} = \widehat{\boldsymbol{\lambda}}^j$ for some $0 \leq j \leq r - 1$ or $\boldsymbol{\lambda} = \widehat{\boldsymbol{\lambda}}^{j,k}$ for some $0 \leq j \neq k \leq r - 1$.*

Proof When $p \nmid n$, the element $\alpha = (0, 0, \ldots, 0; (1, 2, \ldots, n))$ is p-regular in $G(r, 1, n)$. Since $\chi^{\boldsymbol{\lambda}}$ is a quasi p-Steinberg character, we have $\chi^{\boldsymbol{\lambda}}(\pi) \neq 0$. By Corollary 2.9(i), we must have $\boldsymbol{\lambda} = \widehat{\boldsymbol{\lambda}}^j$ for some $0 \leq j \leq r - 1$.

When $p \mid n$, consider the p-regular element $\alpha_1 = (0, 0, \ldots, 0; (1, 2, \ldots, n - 1))$. By Murnaghan–Nakayama rule for $G(r, 1, n)$, the only possible $\boldsymbol{\lambda} = (\boldsymbol{\lambda}_0, \boldsymbol{\lambda}_1, \ldots, \boldsymbol{\lambda}_{r-1})$ such that $\chi^{\boldsymbol{\lambda}}(\pi) \neq 0$ must satisfy $\boldsymbol{\lambda} = \widehat{\boldsymbol{\lambda}}^j$ for some $0 \leq j \leq r - 1$ or $\boldsymbol{\lambda} = \widehat{\boldsymbol{\lambda}}^{j,k}$ for some $0 \leq j \neq k \leq r - 1$. □

Proposition 3.4 *Let μ be a partition of n such that $\mu \notin \{(n), (1^n)\}$. The character $\chi^{\widehat{\boldsymbol{\lambda}}^j}$, where $\boldsymbol{\lambda}_j = \mu$, is a quasi p-Steinberg character of $G(r, 1, n)$ if and only if $\varkappa^{\mu}$ is a quasi p-Steinberg character of S_n.*

Table 3 Some specific quasi p-Steinberg characters of $G(r, 1, n)$

n	$\boldsymbol{\lambda}_j$	$\deg(\chi^{\widehat{\boldsymbol{\lambda}}^j})$	p
3	(2, 1)	2	2
4	(3, 1)	3	3
4	(2, 1, 1)	3	3
4	(2, 2)	2	2
5	(4, 1)	4	2
5	(2, 1, 1, 1)	4	2

Proof Let $\chi^{\widehat{\boldsymbol{\lambda}}^j}$, where $\boldsymbol{\lambda}_j = \mu$, be a quasi p-Steinberg character of $G(r, 1, n)$. Consider a p-regular element σ of S_n. Then, Corollary 2.9(ii) implies that

$$\varkappa^{\mu}(\sigma) = \chi^{\widehat{\boldsymbol{\lambda}}^j}((0, \ldots, 0; \sigma)),$$

which is nonzero because $(0, \ldots, 0; \sigma)$ is a p-regular element of $G(r, 1, n)$.

Conversely, assume that $\varkappa^{\mu}$ is a quasi p-Steinberg character of S_n. Let $\pi = (z_1, \ldots, z_n; \sigma)$ be a p-regular element of $G(r, 1, n)$. Then, σ is a p-regular element of S_n by Lemma 2.10(i), and we have $\varkappa^{\mu}(\sigma) \neq 0$. Now by Corollary 2.9(ii), we have

$$\chi^{\widehat{\boldsymbol{\lambda}}^0}(\pi) = \varkappa^{\mu}(\sigma).$$

and $\chi^{\widehat{\boldsymbol{\lambda}}^j}(\pi)$ is nonzero if and only if $\chi^{\widehat{\boldsymbol{\lambda}}^0}(\pi)$ is nonzero for any j. Therefore, $\chi^{\widehat{\boldsymbol{\lambda}}^j}$ is a quasi p-Steinberg character of $G(r, 1, n)$. □

Proof of Theorem 1.4 Given $\boldsymbol{\lambda} = (\boldsymbol{\lambda}_0, \boldsymbol{\lambda}_1, \ldots, \boldsymbol{\lambda}_{r-1})$, let $\chi^{\boldsymbol{\lambda}}$ be the quasi p-Steinberg character of $G(r, 1, n)$. By Proposition 3.3, we have either $\boldsymbol{\lambda} = \widehat{\boldsymbol{\lambda}}^j$ for some $0 \leq j \leq r-1$ or $\boldsymbol{\lambda} = \widehat{\boldsymbol{\lambda}}^{j,k}$ for some $0 \leq j \neq k \leq r-1$.

Using Proposition 3.4 and Table 2, we get the quasi p-Steinberg characters of $G(r, 1, n)$ corresponding to $\widehat{\boldsymbol{\lambda}}^j$ in Table 1.

Now we shall classify quasi p-Steinberg characters of $G(r, 1, n)$ corresponding to $\widehat{\boldsymbol{\lambda}}^{j,k}$. As we observe in the proof of Proposition 3.3, this kind of character may appear only when $p \mid n$.

When $n \geq 5$, there does not arise a quasi p-Steinberg character of this kind because of the following observations:

(i) Either $\alpha_2 = (0, \ldots, 0; (1, 2, \ldots, n-2)(n-1, n))$ or $\alpha_3 = (0, \ldots, 0; (1, 2, \ldots, n-3)(n-2, n-1, n))$ is a p-regular element of $G(r, 1, n)$;
(ii) $\chi^{\boldsymbol{\lambda}}(\alpha_2) = \chi^{\boldsymbol{\lambda}}(\alpha_3) = 0$.

Now we consider $n < 5$ in which some additional quasi p-Steinberg characters for $G(r, 1, n)$ will arise.

Let us start with $n = 2$, $p = 2$. The irreducible character corresponding to $\boldsymbol{\lambda} = \widehat{\boldsymbol{\lambda}}^{j,k}$, where $\boldsymbol{\lambda}^j = (1)$, is a quasi 2-Steinberg character by Theorem 1.2(i).

Let us consider the case $n = 3$, $p = 3$. The irreducible characters corresponding to $\widehat{\boldsymbol{\lambda}}^{j,k}$, where $\lambda_j \vdash 2$, are of degree three and hence, quasi 3-Steinberg characters of $G(r, 1, 3)$ by Theorem 1.2(i).

Now consider the case $n = 4$, $p = 2$. The irreducible characters corresponding to $\boldsymbol{\lambda} = \widehat{\boldsymbol{\lambda}}^{j,k}$ where $\boldsymbol{\lambda}_j \in \{(3), (1, 1, 1)\}$, are quasi 2-Steinberg characters of $G(r, 1, 4)$ by Theorem 1.2(i) by virtue of being degree four characters.

We now discuss the final case: $n = 4$, $p = 2$ and $\boldsymbol{\lambda} = \widehat{\boldsymbol{\lambda}}^{j,k}$ with $\boldsymbol{\lambda}_j = (2, 1)$. For an r-partite ribbon tableau $T = (T_0, \ldots, T_{r-1})$ of shape $\widehat{\boldsymbol{\lambda}}^{j,k}$, we have $T_l = \emptyset$ for all $l \notin \{j, k\}$. Let $(z_1, \ldots, z_4; \sigma)$ be a p-regular element in $G(r, 1, 4)$, thus, either $\sigma = (1)$ or σ has cycle type $(3, 1)$, and $z_1, \ldots, z_4$ are of odd orders. When $\sigma = (1)$, then there are exactly eight r-partite ribbon tableaux $T = (T_0, \ldots, T_{r-1})$ as we have eight different pairs (T_j, T_k):

$$\begin{array}{|c|c|}\hline 1 & 2 \\ \hline 3 & \multicolumn{1}{c}{} \\ \cline{1-1}\end{array} \quad \begin{array}{|c|}\hline 4 \\ \hline\end{array}\,; \qquad \begin{array}{|c|c|}\hline 1 & 3 \\ \hline 2 & \multicolumn{1}{c}{} \\ \cline{1-1}\end{array} \quad \begin{array}{|c|}\hline 4 \\ \hline\end{array}\,; \qquad \begin{array}{|c|c|}\hline 1 & 2 \\ \hline 4 & \multicolumn{1}{c}{} \\ \cline{1-1}\end{array} \quad \begin{array}{|c|}\hline 3 \\ \hline\end{array}\,; \qquad \begin{array}{|c|c|}\hline 1 & 4 \\ \hline 2 & \multicolumn{1}{c}{} \\ \cline{1-1}\end{array} \quad \begin{array}{|c|}\hline 3 \\ \hline\end{array}\,;$$

$$\begin{array}{|c|c|}\hline 1 & 3 \\ \hline 4 & \multicolumn{1}{c}{} \\ \cline{1-1}\end{array}\ \ \boxed{2}\,; \qquad \begin{array}{|c|c|}\hline 1 & 4 \\ \hline 3 & \multicolumn{1}{c}{} \\ \cline{1-1}\end{array}\ \ \boxed{2}\,; \qquad \begin{array}{|c|c|}\hline 2 & 3 \\ \hline 4 & \multicolumn{1}{c}{} \\ \cline{1-1}\end{array}\ \ \boxed{1}\,; \qquad \begin{array}{|c|c|}\hline 2 & 4 \\ \hline 3 & \multicolumn{1}{c}{} \\ \cline{1-1}\end{array}\ \ \boxed{1}\,.$$

By Murnaghan–Nakayama rule for $G(r, 1, n)$, we have

$$\begin{aligned} &\chi^{\boldsymbol{\lambda}}((z_1, z_2, z_3, z_4, (1))) \\ &= 2(\omega^{jz_1+jz_2+jz_3+kz_4} + \omega^{jz_1+jz_3+jz_4+kz_2} + \omega^{jz_1+jz_2+jz_4+kz_3} + \omega^{kz_1+jz_2+jz_3+jz_4}) \\ &= 2\omega^{kz_1+jz_2+jz_3+jz_4}(\omega^{(j-k)(z_1-z_4)} + \omega^{(j-k)(z_1-z_2)} + \omega^{(j-k)(z_1-z_3)} + 1), \end{aligned}$$

which can be zero only when r is an even integer [13, Main Theorem, p.2]. However, when r is even, the sum above is nonzero by Lemma 2.10(ii) because z_1, z_2, z_3, z_4 have odd orders in $\mathbb{Z}/r\mathbb{Z}$ [21, Table 3.1, p.141].

When σ is of cycle type $(3, 1)$, say $\sigma = (1, 2, 3)$, then there is exactly one r-partite ribbon tableaux T given by

$$T_j = \begin{array}{|c|c|}\hline 1 & 1 \\ \hline 1 & \multicolumn{1}{c}{} \\ \cline{1-1}\end{array} \qquad T_k = \boxed{2} \qquad T_l = \emptyset \text{ for all } l \neq j, k.$$

This implies that $\chi^{\boldsymbol{\lambda}}((z_1, z_2, z_3, z_4, (1, 2, 3))) = -\omega^{jz_1+jz_2+jz_3+kz_4} \neq 0.$ □

We now describe the quasi p-Steinberg characters of $G(r, q, n)$ for $q \neq 1$. Given an r-partite partition $\boldsymbol{\lambda} = (\boldsymbol{\lambda}_0, \boldsymbol{\lambda}_1, \ldots, \boldsymbol{\lambda}_{r-1})$ of n, recall that the notation $(\chi^{\boldsymbol{\lambda}})^*$ denotes an irreducible character of $G(r, q, n)$ which appears in the decomposition of the restriction of the irreducible character $\chi^{\boldsymbol{\lambda}}$ of $G(r, 1, n)$ to $G(r, q, n)$. Note that $(\chi^{\boldsymbol{\lambda}})^*$ may not be unique and is an irreducible character $\chi^{(\tilde{\boldsymbol{\lambda}}, \delta)}$, where $\tilde{\boldsymbol{\lambda}}$ is an (m, q)-necklace obtained from $\boldsymbol{\lambda}$ and $\delta \in H_{\boldsymbol{\lambda}}$.

We need the following notation for Proposition 3.5, which deals with two specific kinds of quasi p-Steinberg characters of $G(r, q, n)$.

- For $n = 3$ and $0 \leq j \neq k \neq l \leq r-1$, define $\boldsymbol{\nu}^{j,k,l} = (\boldsymbol{\nu}_0, \boldsymbol{\nu}_1, \ldots, \boldsymbol{\nu}_{r-1}) \in \mathcal{Y}(r, 3)$ where $\boldsymbol{\nu}_j = \boldsymbol{\nu}_k = \boldsymbol{\nu}_l = (1)$.
- For $n = 4$ and $0 \leq j \neq k \leq r-1$, define $\boldsymbol{\nu}^{j,k} = (\boldsymbol{\nu}_0, \boldsymbol{\nu}_1, \ldots, \boldsymbol{\nu}_{r-1}) \in \mathcal{Y}(r, 4)$ where $\boldsymbol{\nu}_j = \boldsymbol{\nu}_k \vdash 2$.
- For a multiple r of 3, define

$$X_1 = \left\{(j, k, l) \mid 0 \leq j \leq r-1, k = \left(j + \frac{r}{3}\right) (\text{mod } r), l = \left(j + \frac{2r}{3}\right) (\text{mod } r)\right\},$$

and for r even, define

$$X_2 = \left\{(j, k) \mid 0 \leq j \leq r-1, k = \left(j + \frac{r}{2}\right) \pmod r\right\}.$$

Proposition 3.5 (i) *The irreducible character $\chi^{\boldsymbol{\nu}^{j,k,l}}$ of $G(r, 1, 3)$ for $\boldsymbol{\nu}^{j,k,l} \in \mathcal{Y}(r, 3)$ decomposes into three irreducible characters of $G(r, q, 3)$ if and only if r, q are multiples of 3 and $(j, k, l) \in X_1$.*

(ii) *The irreducible character $\chi^{\boldsymbol{\nu}^{j,k}}$ of $G(r, 1, 4)$ for $\boldsymbol{\nu}^{j,k} \in \mathcal{Y}(r, 4)$ decomposes into two irreducible characters of $G(r, q, 4)$ if and only if r, q are even and $(j, k) \in X_2$.*

Proof The proof follows by Theorem 2.5. □

We now prove Theorem 3.6 which characterizes quasi p-Steinberg characters of $G(r, q, n)$. In particular, this gives proof of Theorem 1.6.

Theorem 3.6 *Given an r-partite partition $\boldsymbol{\lambda} = (\boldsymbol{\lambda}_0, \boldsymbol{\lambda}_1, \ldots, \boldsymbol{\lambda}_{r-1})$ of n, a non-linear irreducible character $(\chi^{\boldsymbol{\lambda}})^*$ of $G(r, q, n)$ is a quasi p-Steinberg character if and only if one of the following holds:*

(i) $\boldsymbol{\lambda}$ *corresponds to a quasi p-Steinberg character of $G(r, 1, n)$ which are given in* Table 1. *In this case,*

$$(\chi^{\boldsymbol{\lambda}})^* = Res^{G(r,1,n)}_{G(r,q,n)} \chi^{\boldsymbol{\lambda}}.$$

(ii) *$n = 3$, r, q are multiples of 3 and $\boldsymbol{\lambda} = \boldsymbol{\nu}^{j,k,l}$ for $(j, k, l) \in X_1$ for $p = 2$.*
(iii) *$n = 4$, r, q are even and $\boldsymbol{\lambda} = \boldsymbol{\nu}^{j,k}$ for $(j, k) \in X_2$ for $p = 3$.*

Proof For an r-partite partition $\boldsymbol{\lambda} = (\boldsymbol{\lambda}_0, \boldsymbol{\lambda}_1, \ldots, \boldsymbol{\lambda}_{r-1})$ of n, consider a quasi p-Steinberg character $(\chi^{\boldsymbol{\lambda}})^*$ of $G(r, q, n)$. By Corollary 3.2, either all the irreducible characters $\chi^{(\tilde{\boldsymbol{\lambda}},\delta)}$ appearing in $\text{Res}^{G(r,1,n)}_{G(r,q,n)} \chi^{\boldsymbol{\lambda}}$ are quasi p-Steinberg or none of these are. Therefore, we can arbitrarily choose $(\chi^{\boldsymbol{\lambda}})^*$. We consider the cases $p \nmid n$ and $p \mid n$ separately.

Case 1: $p \nmid n$. Since $G(r, q, n)$ is a normal subgroup of $G(r, 1, n)$, our method is to apply Lemma 2.6 by choosing a p-regular element in $G(r, 1, n)$ whose conjugacy class does not split in $G(r, q, n)$. We consider the two subcases: $p \nmid n - 1$ and $p \mid n - 1$.

Case 1(a): $p \nmid n - 1$. The element $\alpha_1 = (0, \ldots, 0; (1, 2, \ldots, n - 1))$ is a p-regular element of G(r, q, n). Thus $(\chi^{\boldsymbol{\lambda}})^*(\alpha_1) \neq 0$, and by Lemma 2.6 we get that $\chi^{\boldsymbol{\lambda}}(\alpha_1) \neq 0$. By Murnaghan–Nakayama rule, either $\boldsymbol{\lambda} = \widehat{\boldsymbol{\lambda}}^j$ for some $0 \leq j \leq r - 1$ or $\boldsymbol{\lambda} = \widehat{\boldsymbol{\lambda}}^{j,k}$ for some $0 \leq j \neq k \leq r - 1$.

However, we do not get any non-linear quasi p-Steinberg character of $G(r, q, n)$ in this subcase because of the following reason. When $\boldsymbol{\lambda} = \widehat{\boldsymbol{\lambda}}^j$ for some $0 \leq j \leq r - 1$, Theorem 2.5 implies that $(\chi^{\boldsymbol{\lambda}})^* = \text{Res}^{G(r,1,n)}_{G(r,q,n)} \chi^{\boldsymbol{\lambda}}$. When $\boldsymbol{\lambda} = \widehat{\boldsymbol{\lambda}}^{j,k}$ for some $0 \leq j \neq k \leq r - 1$, for $n \geq 3$ we have $(\chi^{\boldsymbol{\lambda}})^* = \text{Res}^{G(r,1,n)}_{G(r,q,n)} \chi^{\boldsymbol{\lambda}}$. From Table 1, we can see that there does not exist a non-linear quasi p-Steinberg character of $G(r, 1, n)$ when $p \nmid n$ and $p \nmid n - 1$.

For $n = 2$ and $\boldsymbol{\lambda} = \widehat{\boldsymbol{\lambda}}^{j,k}$, $\chi^{\boldsymbol{\lambda}}$ is a character of degree two which is not a quasi p-Steinberg character of $G(r, 1, 2)$ for $p \neq 2$. When $\chi^{\boldsymbol{\lambda}}$ decomposes, the two linear characters appearing in $\text{Res}^{G(r,1,2)}_{G(r,q,2)} \chi^{\boldsymbol{\lambda}}$ are quasi p-Steinberg characters for any prime p that divides the order of $G(r, q, 2)$.

Case 1(b): $p \mid n-1$. Then, $p \nmid n-2$ and $\alpha_2 = (0, \ldots, 0; (1, 2, \ldots, n-2))$ is a p-regular element of $G(r, q, n)$. Thus $(\chi^{\boldsymbol{\lambda}})^*(\alpha_2) \neq 0$, and by Lemma 2.6 we have $\chi^{\boldsymbol{\lambda}}(\alpha_2) \neq 0$. Murnaghan–Nakayama rule for $G(r, 1, n)$ implies that $\boldsymbol{\lambda}$ must be of one of the following forms:

(a) $\boldsymbol{\lambda} = \widehat{\boldsymbol{\lambda}}^j$;
(b) $\boldsymbol{\lambda} = \widehat{\boldsymbol{\lambda}}^{j,k}$;
(c) $\boldsymbol{\lambda}_j \vdash n - 2$, $\boldsymbol{\lambda}_k \vdash 2$ for some $k \neq j$, $\boldsymbol{\lambda}_l = \emptyset$ for all $l \notin \{j, k\}$;
(d) $\boldsymbol{\lambda}_j \vdash n - 2$, $\boldsymbol{\lambda}_k = (1)$ for some $k \neq j$, $\boldsymbol{\lambda}_l = (1)$ for some $l \notin \{j, k\}$, and $\boldsymbol{\lambda}_u = \emptyset$ for all $u \notin \{j, k, l\}$;

When $\boldsymbol{\lambda}$ is of one of the forms (a)-(d) and for $n \geq 5$, by Theorem 2.5 we have

$$(\chi^{\boldsymbol{\lambda}})^* = \text{Res}^{G(r,1,n)}_{G(r,q,n)} \chi^{\boldsymbol{\lambda}}.$$

For $\boldsymbol{\lambda}$ of the form (b), (c) or (d), $\chi^{\boldsymbol{\lambda}}$ vanishes on the p-regular element $\alpha = (0, \ldots, 0; (1, 2, \ldots, n))$ which implies that the character $(\chi^{\boldsymbol{\lambda}})^*$ is not a quasi p-Steinberg character of $G(r, q, n)$. Now consider that $\boldsymbol{\lambda}$ is of the form (a), i.e., $\boldsymbol{\lambda} = \widehat{\boldsymbol{\lambda}}^j$. By Corollary 2.9(ii) and by $(\chi^{\widehat{\boldsymbol{\lambda}}^j})^*$ being a quasi p-Steinberg character of $G(r, q, n)$, for a p-regular element $\sigma \in S_n$, we have

$$\varkappa^{\lambda_j}(\sigma) = \chi^{\widehat{\boldsymbol{\lambda}}^j}(0, \ldots, 0; \sigma) = (\chi^{\widehat{\boldsymbol{\lambda}}^j})^*(0, \ldots, 0; \sigma) \neq 0.$$

This implies that $\varkappa^{\lambda_j}$ is a quasi p-Steinberg character of S_n. Thus, $\chi^{\widehat{\boldsymbol{\lambda}}^j}$ is a quasi p-Steinberg character of $G(r, 1, n)$ by Proposition 3.4.

We now consider $n < 5$. Of course, $n \neq 2$ as $p \mid n - 1$. For $n = 3$, $p = 2$, when $\boldsymbol{\lambda}$ is of the form (a), (b) or (c), then it can be seen from Table 1 that $\chi^{\boldsymbol{\lambda}}$ is a quasi 2-Steinberg character of $G(r, 1, 3)$. Thus, $(\chi^{\boldsymbol{\lambda}})^*$ is a quasi 2-Steinberg character of $G(r, q, 3)$ because

$$(\chi^{\boldsymbol{\lambda}})^* = \mathrm{Res}^{G(r,1,3)}_{G(r,q,3)} \chi^{\boldsymbol{\lambda}}.$$

A similar argument holds true when $n = 4$, $p = 3$ and $\boldsymbol{\lambda}$ is of the form (a). From Table 1, we see that $\boldsymbol{\lambda}$ cannot be of the form (b) or (d) for $n = 4$, $p = 3$.

When $\boldsymbol{\lambda}$ is of the form (d) for $n = 3$, $p = 2$, i.e. $\boldsymbol{\lambda} = \boldsymbol{\nu}^{j,k,l}$, then Theorem 1.2(i) and Proposition 3.5 imply that $(\chi^{\boldsymbol{\lambda}})^*$, being irreducible character of degree two, occurring in $\mathrm{Res}^{G(r,1,3)}_{G(r,q,3)} \chi^{\boldsymbol{\lambda}}$ is a quasi 2-Steinberg character if and only if r, q are multiples of 3 and $(j, k, l) \in X_1$. The analogous arguments can be used to deduce that when $\boldsymbol{\lambda}$ is of the form (c) for $n = 4$, $p = 3$, i.e. $\boldsymbol{\lambda} = \boldsymbol{\nu}^{j,k}$, the character $(\chi^{\boldsymbol{\lambda}})^*$ occurring in $\mathrm{Res}^{G(r,1,4)}_{G(r,q,4)} \chi^{\boldsymbol{\lambda}}$ is a quasi 3-Steinberg character if and only if r, q are even and $(j, k) \in X_2$.

Case 2: $p \mid n$. Then, $p \nmid n - 1$ and $\alpha_1 = (0, \ldots, 0; (1, 2, \ldots, n - 1))$ is a p-regular element of $G(r, q, n)$. Thus $(\chi^{\boldsymbol{\lambda}})^*(\alpha_1) \neq 0$, and by Lemma 2.6 we get that $\chi^{\boldsymbol{\lambda}}(\alpha_1) \neq 0$. By Murnaghan–Nakayama rule for $G(r, 1, n)$, either $\boldsymbol{\lambda} = \widehat{\boldsymbol{\lambda}}^j$ for some $0 \leq j \leq r - 1$ or $\boldsymbol{\lambda} = \widehat{\boldsymbol{\lambda}}^{j,k}$ for some $0 \leq j \neq k \leq r - 1$. Using similar arguments as given in Case 1(b), we deduce that if $\boldsymbol{\lambda} = \widehat{\boldsymbol{\lambda}}^j$ for some $0 \leq j \leq r - 1$, then $\chi^{\widehat{\boldsymbol{\lambda}}^j}$ is a quasi p-Steinberg character of $G(r, 1, n)$.

For $\boldsymbol{\lambda} = \widehat{\boldsymbol{\lambda}}^{j,k}$ for some $0 \leq j \neq k \leq r - 1$ and for $n \geq 5$, the character $(\chi^{\boldsymbol{\lambda}})^*$ is not a quasi p-Steinberg character of $G(r, q, n)$ because of the following observations:

(a) $\alpha_2 = (0, \ldots, 0; (1, 2, \ldots, n-2))$ and $\alpha_3 = (0, \ldots, 0; (1, 2, \ldots, n-3))$ are p-regular elements of $G(r, q, n)$ for $p > 3$;
(b) α_2 is 3-regular and α_3 is 2-regular;
(c) $\chi^{\boldsymbol{\lambda}}(\alpha_2) = \chi^{\boldsymbol{\lambda}}(\alpha_3) = 0$;
(d) $(\chi^{\boldsymbol{\lambda}})^* = \mathrm{Res}^{G(r,1,n)}_{G(r,q,n)} \chi^{\boldsymbol{\lambda}}$.

Now, we discuss the case when $\boldsymbol{\lambda} = \widehat{\boldsymbol{\lambda}}^{j,k}$ for some $0 \leq j \neq k \leq r - 1$ and $n < 5$. For $n = 2$, $p = 2$, we have $\lambda_j = (1)$ for some j, $\lambda_k = (1)$ for some $k \neq j$, $\lambda_l = \emptyset$ for all $l \notin \{j, k\}$. Therefore, the character $\chi^{\boldsymbol{\lambda}}$ is a degree two quasi 2-Steinberg character of $G(r, 1, 2)$. The character $(\chi^{\boldsymbol{\lambda}})^*$ is a non-linear quasi 2-Steinberg character of $G(r, q, 2)$ if and only if $(\chi^{\boldsymbol{\lambda}})^* = \mathrm{Res}^{G(r,1,2)}_{G(r,q,2)} \chi^{\boldsymbol{\lambda}}$. The character $\chi^{\boldsymbol{\lambda}}$ decomposes into two linear characters of $G(r, q, 2)$ if and only if both r and q are even, and $k = j + \frac{r}{2} \mod (r)$.

For $n = 3$, $p = 3$ and $n = 4$, $p = 2$, given $\boldsymbol{\lambda} = \widehat{\boldsymbol{\lambda}}^{j,k}$ implies that $(\chi^{\boldsymbol{\lambda}})^* = \mathrm{Res}^{G(r,1,n)}_{G(r,q,n)} \chi^{\boldsymbol{\lambda}}$ (by Theorem 2.5). Also, all the possible $\widehat{\boldsymbol{\lambda}}^{j,k}$ appear in Table 1. Thus, for $n = 3$, $p = 3$ and

Table 4 Quasi p-Steinberg characters of $G(n)$

Group	Order	Degrees of non-linear irreducible characters	p-Steinberg
$G(4)$	24	**2, 3w**	all **3w**
$G(5)$	72	**2, 3**	
$G(6)$	48	**2, 3w**	all **3w**
$G(7)$	144	**2, 3**	
$G(8)$	96	**2, 3w, 4**	two out of four **3w**
$G(9)$	192	**2, 3w, 4**	four out of eight **3w**
$G(10)$	288	**2, 3, 4**	
$G(11)$	576	**2, 3, 4**	
$G(12)$	48	**2, 3w, 4**	all **3w**
$G(13)$	96	**2, 3w, 4**	all **3w**
$G(14)$	144	**2, 3, 4**	
$G(15)$	288	**2, 3, 4**	
$G(16)$	600	**2, 3w, 4, 5**, 6	
$G(17)$	1200	**2, 3w, 4, 5**, 6	
$G(18)$	1800	**2, 3, 4, 5**, 6	
$G(19)$	3600	**2, 3, 4, 5**, 6	
$G(20)$	360	**2, 3, 4, 5w**, 6	one out of three **5w**
$G(21)$	720	**2, 3, 4, 5w**, 6	two out of six **5w**
$G(22)$	240	**2, 3w, 4, 5w**, 6	all **5w**
$G(23)$	120	**3w, 4, 5w**	all **5w**
$G(24)$	336	**3w, 7w, 8**, 6	all **7w**
$G(25)$	648	**2, 3, 8w, 9**, 6	one out of three **8w**
$G(26)$	1296	**2, 3, 8, 9**, 6	
$G(27)$	2160	**3, 5w, 8, 9**, 6, 10, 15	
$G(28)$	1152	**2, 4, 8, 9w, 16**, 6, 12	all **9w**
$G(29)$	7680	**4, 5w, 16**, 6, 10, 15, 20, 24, 30	
$G(30)$	14400	**4, 8, 9w, 16, 25w**, 6, 10, 18, 24, 30, 36, 40, 48	all **25w**
$G(31)$	46080	**4, 5w, 9w, 16, 64**, 6, 10, 15, 20, 24, 30, 36, 40, 45	
$G(32)$	155520	**4, 64, 81**, 5, 6, 10, 15, 20, 24, 30, 36, 40, 45, 60, 80	
$G(33)$	51840	**64, 81w**, 5, 6, 10, 15, 20, 24, 30, 40, 45, 60	
$G(34)$	39191040	**729**, 6, 15, 20, 21, 35, 56, 70, 84, 90, 105, 120, 126, 140, 189, 210, 280, 315, 336, 384, 420, 504, 540, 560, 630, 720, 756, 840, 896, 945, 1260, 1280	
$G(35)$	51840	**64, 81w**, 6, 10, 15, 20, 24, 30, 60, 80, 90	all **81w**
$G(36)$	2903040	**27, 512**, 7, 15, 21, 35, 56, 70, 84, 105, 120 168, 189, 210, 216, 280, 315, 336, 378, 405, 420	
$G(37)$	696729600	**8, 4096**, 28, 35, 50, 56, 70, 84, 112, 160, 168, 175, 210, 300, 350, 400, 420, 448, 525, 560, 567, 700, 840, 972, 1008, 1050, 1134, 1296, 1344, 1400, 1575, 1680, 2016, 2100, 2240, 2268, 2400, 2688, 2800, 2835, 3150, 3200, 3240, 3360, 4200, 4480, 4536, 5600, 5670, 6075, 7168	

$n = 4$, $p = 2$, the character $(\chi^{\widehat{\boldsymbol{\lambda}}^{j,k}})^*$ is a quasi p-Steinberg character of $G(r, q, n)$ if and only if $\chi^{\widehat{\boldsymbol{\lambda}}^{j,k}}$ is a quasi p-Steinberg character of $G(r, 1, n)$. $\square$

4 Exceptional Complex Reflection Groups

In this section, we provide information regarding the quasi p-Steinberg, weak p-Steinberg and p-Steinberg characters of exceptional irreducible complex reflection groups $G(n)$ for $n \in \{4, 5, \ldots, 37\}$. The integer n for the group $G(n)$ denotes its Shephard–Todd number. We have used GAP interface in SAGEMATH online via Cocalc.com to obtain this information and have collected the details in Table 4. The second and third columns of Table 4 denote the order and the degrees of the non-linear irreducible characters of the group, respectively. By accessing character tables of $G(n)$, we observe that for $n \in \{4, \ldots, 37\}$, every quasi p-Steinberg character of $G(n)$ has degree a power of p. However, the converse does not hold in general. The numbers in the bold font (and with suffix **w**) highlight the degrees equal to a prime power p whose corresponding character is a quasi p-Steinberg (and a weak p-Steinberg, respectively) character. We then consider the character values of weak p-Steinberg characters and obtain the information regarding the p-Steinberg characters. The details of this are included in the last column of Table 4.

Acknowledgements The authors are grateful to Dipendra Prasad for his encouragement regarding this project. The authors thank Anupam Singh and Manoj K. Yadav for organizing the Group Theory Sangam seminar series, during which one of the seminars led to this project. The authors would also like to thank the referee for their thorough reading and helpful suggestions, which significantly improved this article. The second named author acknowledges IISc Raman post doctoral fellowship for their support.

Author Contributions All the authors took active part in writing the main manuscript. All authors reviewed the manuscript.

Data Availability All of the material is owned by the authors and/or no permissions are required.

Declarations

Competing interests No, we declare that the authors have no competing interests as defined by Springer, or other interests that might be perceived to influence the results and/or discussion reported in this paper.

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