Obstructions to the existence of solutions of the self-dual Einstein-Maxwell-Higgs equations on a compact surface ☆

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A R T I C L E   I N F O

A B S T R A C T

In this note we present an obstruction to the existence of solutions of the self-dual Einstein-Maxwell-Higgs equations on a compact surface, which depends on the multiplicities of the zeroes of the *Higgs field* and the *vortex number* N. In particular, we exhibit infinitely many examples of Higgs fields for which solutions cannot exist.

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1. Introduction

Let S be a smooth oriented compact surface, and $\{p_1, \cdots, p_d\}$ be a collection a distinct points on S with positive integer multiplicities $n_1, \cdots, n_d \in \mathbb{N}$, respectively. Give a smooth metric g on S and smooth function $u : S \backslash \{p_1, \cdots, p_d\} \to \mathbb{R}$ defined away from the points consider the system of PDE

$$
\begin{aligned}
\Delta_g u &= \frac{1}{\varepsilon^2}(e^u - \tau^2) + 4\pi \sum_{j=1}^{d} n_j \delta_{p_j}, \\
K_g &= -\frac{a}{2\tau^2}\left[\frac{\tau^2}{\varepsilon^2}(e^u - \tau^2) - \Delta_g e^u\right].
\end{aligned}
\tag{1.1}
$$

Here Δ_g is Laplace-Beltrami operator induced by the metric g, K_g is the Gaussian curvature of g, δ_{p_j} is the Dirac distribution on (S, ω) concentrated at p_j, where ω is the area (volume) form of g, and τ, ε, a are positive real constants. With our convention, the Laplacian has negative eigenvalues. As proved by Y. Yang, the system (1.1) is equivalent to the self-dual Einstein-Maxwell-Higgs equations arising in theoretical physics [11,12].

The aim of this note is to present an obstruction to the existence of solutions of (1.1), and hence to the self-dual Einstein-Maxwell-Higgs equations on a compact surface, which depends on the configuration of points p_j and their multiplicities n_j. Observe that, taking $g = e^\eta g_0$ for a smooth function η and fixed background metric g_0 and using the formula for the change in the scalar curvature

$$
2K_g = e^{-\eta}(2K_{g_0} - \Delta_{g_0}\eta),
$$

the previous system is equivalent to (cf. [8, Equation (1.1)])

$$
\begin{aligned}
\Delta_{g_0} u &= \frac{1}{\varepsilon^2} e^\eta (e^u - \tau^2) + 4\pi \sum_{j=1}^{d} n_j \delta_{p_j}, \\
\Delta_{g_0}(\eta + \tfrac{a}{\tau^2} e^u) &= 2K_{g_0} + \frac{a}{\varepsilon^2} e^\eta (e^u - \tau^2).
\end{aligned}
\tag{1.2}
$$

Here δ_{p_j} is the Dirac distribution on (S, ω_0) concentrated at p_j, where ω_0 is the area form of g_0. Define the *vortex number* by

$$
N = \sum_{j=1}^{d} n_j.
$$

Or main result can be stated as follows.

Theorem 1.1. *Suppose that* (1.2) *admits a solution* (u, η) *with* $d \leq 2$ *for some parameter* $\varepsilon > 0$ *and some smooth background metric* g_0 *on* S, *where* η *is a smooth function on* S

and $u : S \backslash \{p_1, \cdots, p_d\} \to \mathbb{R}$ *is a smooth function defined away from the points. Then, there must hold that*

$$d = 2 \quad and \quad n_1 = n_2.$$

The previous theorem was originally proved in previous work by the first four authors of this note, published online in *Mathematische Annalen* in February 2020 (see [3, Theorem 1.3 and Proposition 2.6]) and previously announced in [2]. As a matter of fact, our method of proof here is, as in [3], the evaluation of a *Futaki invariant* for the self-dual Einstein-Maxwell-Higgs equations. This invariant was first constructed in [3] via the general theory for the Kähler-Yang-Mills equations developed in [1], and follows the same principles as the classical Futaki invariant in the Kähler-Einstein theory [5]. The main novelty of the present note is an alternative, independent construction of the Futaki invariant, by direct and explicit calculations. We have deliberatively avoided the use of the general moment map theory in [1,3] as well as any technicalities coming from the geometry of the self-dual Einstein-Maxwell-Higgs equations, trying to make the exposition self-contained and more accessible to the PDE community.

Theorem 1.1 gives infinitely many examples of configurations of points for which there cannot be solutions to the self-dual Einstein-Maxwell-Higgs equations on a compact surface, in apparent contradiction with [8, Theorem 1.2] (see Remark 2.10 and the Erratum [9]). Our proof exploits systematically the complex geometry of the *Higgs field* ϕ in the self-dual Einstein-Maxwell-Higgs equations, which is implicit in the system of equations (2.1) (see Section 2.1).

2. An obstruction to the self-dual Einstein-Maxwell-Higgs equations

2.1. Preliminaries

Let S be a compact connected smooth surface. We fix $\{p_1, \cdots, p_d\}$ a collection of distinct points on S with positive integer multiplicities $n_1, \cdots, n_d \in \mathbb{N}$, respectively. We also fix positive real constants $\tau, \varepsilon, a > 0$. By Gauss-Bonnet Theorem, the existence of solutions of (1.1) implies that

$$\chi(S) = aN > 0.$$

Therefore, S must be diffeomorphic to the two-sphere S^2 and furthermore one has the following 'quantization condition'

$$a = \frac{2}{N}.$$

We fix a background metric g_0 on S^2 with volume V. One of the main methods introduced in [3] for the proof of Theorem 1.1 is the construction of an invariant, called the *Futaki*

invariant, which only depends on the conformal class of g_0, the volume V, and the tuple $(\{p_j\}, \{n_j\}, \tau, a)$, which vanishes identically provided that there exists a solution of (1.2). To recall the definition of the Futaki invariant, in this note we will focus on the special case

$$d \leq 2.$$

Without loss of generality, we assume that p_1 and p_2 are the north and south pole on S^2, respectively, and take g_0 to be the round metric on S^2 with total volume V. Given a positive integer $0 \leq \ell \leq N$, we consider the following special case of equation (1.2)

$$
\begin{aligned}
\Delta_{g_0} u &= \frac{1}{\varepsilon^2} e^\eta (e^u - \tau^2) + 4\pi(N-\ell)\delta_{p_1} + 4\pi\ell\delta_{p_2}, \\
\Delta_{g_0}(\eta + \tfrac{a}{\tau^2} e^u) &= 2K_{g_0} + \frac{a}{\varepsilon^2} e^\eta (e^u - \tau^2).
\end{aligned}
\tag{2.1}
$$

The construction of the Futaki invariant exploits systematically the complex geometry of the *Higgs field* ϕ in the self-dual Einstein-Maxwell-Higgs equations, which is implicit in the system of equations (2.1). In order to use this geometry, in the sequel we identity the two-sphere with the complex Riemann sphere

$$S^2 \cong \mathbb{P}^1.$$

Taking the stereographic projection from the north pole, we have

$$g_0 = \frac{V}{\pi} \frac{dx^2 + dy^2}{(1 + x^2 + y^2)^2}. \tag{2.2}$$

In the complex coordinate $z = x + iy$, the Higgs field of our interest is

$$\phi = z^\ell. \tag{2.3}$$

To define the Futaki invariant, we first write (2.1) in a different form. Define a smooth function Φ by

$$\Phi = \frac{|z|^{2\ell}}{(1 + |z|^2)^N}.$$

By a simple change of holomorphic coordinates $z \to z^{-1}$, it is easy to see that Φ extends to a smooth function on $\mathbb{P}^1$. More invariantly, ϕ defines a global holomorphic section of the line bundle $L = \mathcal{O}_{\mathbb{P}^1}(N)$ over $\mathbb{P}^1$, which vanishes at the points $p_1 \equiv \infty$ and $p_2 \equiv 0$ with multiplicities $N - \ell$ and ℓ, respectively. In the holomorphic coordinate z the Fubini-Study metric on L reads

$$h_{FS}^N = \frac{1}{(1 + |z|^2)^N}$$

and therefore $\Phi = |\phi|^2_{h^N_{FS}}$. For the next formula, notice that with our convention the Laplacian has negative eigenvalues.

Lemma 2.1.

$$\Delta_{g_0} \log \Phi = -\frac{4\pi N}{V} + 4\pi(N - \ell)\delta_{p_1} + 4\pi\ell\delta_{p_2}.$$

Proof. The claim follows now from the Poincaré-Lelong formula

$$i\partial\bar{\partial} \log |\phi|^2_{h^N_{FS}} = -iF_{h^N_{FS}} + 2\pi(N - \ell)\delta_{p_1} + 2\pi\ell\delta_{p_2}$$

$$= -N\frac{idz \wedge d\bar{z}}{(1 + |z|^2)^2} + 2\pi(N - \ell)\delta_{p_1} + 2\pi\ell\delta_{p_2}. \quad \square$$

Using the unknown function u in (2.1), we define a smooth function on the sphere by

$$f = u - \log \Phi.$$

Using the previous Lemma, the system (2.1) now reads

$$\Delta_{g_0} f = \frac{1}{\varepsilon^2} e^\eta (e^f \Phi - \tau^2) + \frac{4\pi N}{V},$$

$$\Delta_{g_0} (\eta + \frac{a}{\tau^2} e^f \Phi) = 2K_{g_0} + \frac{a}{\varepsilon^2} e^\eta (e^f \Phi - \tau^2). \tag{2.4}$$

Consider the radial vector field

$$\boldsymbol{v} = \boldsymbol{v}^{1,0} + \boldsymbol{v}^{0,1} = z\partial_z + \bar{z}\partial_{\bar{z}}.$$

Of course, this is the local expression of a globally defined real holomorphic vector field on $\mathbb{P}^1$ whose flow evolves along the meridians of $S^2 \cong \mathbb{P}^1$, and vanishes at the poles. Let η be a smooth function on $\mathbb{P}^1$. Using that $\boldsymbol{v}$ is holomorphic we have

$$\bar{\partial}(e^\eta \iota_{\boldsymbol{v}^{1,0}} \omega_0) = 0.$$

Since $\mathbb{P}^1$ is simply connected, the Hodge decomposition of the de Rham cohomology ensures the existence of a global $\bar{\partial}$-potential for $e^\eta \iota_{\boldsymbol{v}^{1,0}} \omega_0$.

Definition 2.2. Let η be a smooth real function on $\mathbb{P}^1$. We define φ_η as the unique smooth complex-valued function on $\mathbb{P}^1$ satisfying

$$\bar{\partial}\varphi_\eta = e^\eta \iota_{\boldsymbol{v}^{1,0}} \omega_0, \qquad \int_{\mathbb{P}^1} \varphi_\eta e^\eta \omega_0 = 0.$$

We next construct a complex function ψ_f associated to the other unknown of the equation (2.4). Let ω_{FS} denote the Fubini-Study Kähler form

$$\omega_{FS} := i\bar{\partial}\partial \log h^1_{FS} = \frac{idz \wedge d\bar{z}}{(1+|z|^2)^2}.$$

Given a smooth function f on $\mathbb{P}^1$, similarly as before we have

$$\bar{\partial}(\iota_{\boldsymbol{v}^{1,0}}(N\omega_{FS} + i\bar{\partial}\partial f)) = 0$$

and hence $\iota_{\boldsymbol{v}^{1,0}}(N\omega_{FS} + i\bar{\partial}\partial f)$ admits a smooth global $\bar{\partial}$-potential. The choice of normalization for this potential is a delicate point, which we consider next. Geometrically, this is choosen so that $(\boldsymbol{v}^{1,0}, \psi_f)$ defines an *infinitesimal automorphism* of the Higgs field $\phi = z^\ell$, in the following sense.

Lemma 2.3. *Let f be a smooth real function on $\mathbb{P}^1$. We define ψ_f as the smooth complex-valued function on $\mathbb{P}^1$ given by*

$$\psi_f = \ell - N\frac{|z|^2}{1+|z|^2} + \partial f(\boldsymbol{v}^{1,0}).$$

Then, if we set $h = e^f h^N_{FS}$, the following identities hold

$$-i\bar{\partial}\psi_f = \iota_{\boldsymbol{v}^{1,0}}(i\bar{\partial}\partial \log h),$$
$$\iota_{\boldsymbol{v}^{1,0}}(\partial\phi + (\partial \log h)\phi) = \psi_f\phi. \tag{2.5}$$

Proof. Observe that $\psi_0 = \ell - N\frac{|z|^2}{1+|z|^2}$ defines a smooth global function on $\mathbb{P}^1$, which satisfies

$$\bar{\partial}\psi_0 = -N\frac{zd\bar{z}}{(1+|z|^2)^2} = iN\iota_{\boldsymbol{v}^{1,0}}\omega_{FS}.$$

From this, using that $\boldsymbol{v}$ is holomorphic we obtain

$$\bar{\partial}\psi_f = iN\iota_{\boldsymbol{v}^{1,0}}\omega_{FS} - \iota_{\boldsymbol{v}^{1,0}}\bar{\partial}\partial f.$$

Similarly

$$\iota_{\boldsymbol{v}^{1,0}}(\partial\phi + (\partial \log e^f h^N_{FS})\phi) = \iota_{\boldsymbol{v}^{1,0}}\left(\ell z^{\ell-1}dz - N\frac{\bar{z}dz}{1+|z|^2}\phi + (\partial f)\phi\right) = \psi_f\phi. \quad \square$$

We will also need the following technical formula.

Lemma 2.4. *Set $h = e^f h_{FS}^N$. Then the following identity holds*

$$\bar{\partial}\left(\psi_f(e^f\Phi - \tau^2)\right) = -\iota_{\boldsymbol{v}^{1,0}}\left(\bar{\partial}\partial(e^f\Phi) - \tau^2\bar{\partial}\partial\log h\right).$$

Proof. Applying the structural formulae for ψ_f in Lemma 2.3, we obtain

$$
\begin{aligned}
\bar{\partial}\left(\psi_f(e^f\Phi - \tau^2)\right) &= \bar{\partial}\left(\psi_f\phi h\overline{\phi} - \tau^2\psi_f\right) \\
&= \bar{\partial}\left(\iota_{\boldsymbol{v}^{1,0}}\left(\partial\phi + (\partial\log h)\phi\right)h\overline{\phi}\right) + \tau^2\iota_{\boldsymbol{v}^{1,0}}(\bar{\partial}\partial\log h) \\
&= -\iota_{\boldsymbol{v}^{1,0}}\left(\bar{\partial}\partial(e^f\Phi) - \tau^2\bar{\partial}\partial\log h\right). \quad \square
\end{aligned}
$$

2.2. A Futaki invariant for the self-dual Einstein-Maxwell-Higgs equations

In this section we introduce the definition of the Futaki invariant for the self-dual Einstein-Maxwell-Higgs equations, following [3] (see Remark 2.7). As a warm-up, we recall first the definition of the classical Futaki invariant in our setup. This classical invariant, originally introduced by A. Futaki [5], provides an obstruction to the existence of Kähler-Einstein metrics on a compact complex manifold. Since we are dealing with the Riemann sphere, we will simply prove that the classical Futaki invariant identically vanishes.

Lemma 2.5 *(Classical Futaki invariant). The following expression does not depend on the choice of smooth function η:*

$$\mathcal{F}_0 = -\int_{S^2}\varphi_\eta\left(2K_{g_0} - \Delta_{g_0}\eta\right)\omega_0.$$

Consequently, it vanishes identically.

Proof. We claim that $\mathcal{F}_0 = \mathcal{F}_0(\eta)$ is independent of η, provided that $\int_{S^2} e^\eta\omega_0 = V$. If this is the case, given such $\tilde{\eta}$ and using that g_0 has constant curvature

$$\mathcal{F}_0(\tilde{\eta}) = -2K_{g_0}\int_{S^2}\varphi_0\omega_0 = 0$$

by definition of φ_0. Furthermore,

$$\varphi_{\eta+\log t} = t\varphi_\eta$$

for any η, which implies $\mathcal{F}_0(\tilde{\eta} + \log t) = t\mathcal{F}_0(\tilde{\eta}) = 0$.

To prove our initial claim, we set $g := e^\eta g_0$ and $\varphi_g := \varphi_\eta$, and note that

$$\mathcal{F}_0(g) := \mathcal{F}_0(\eta) = -2\int_{S^2}\varphi_g\left(K_g - \hat{K}\right)\omega = -2\int_{S^2}\varphi_g\left(\rho_g - \hat{K}\omega\right)$$

where $\hat{K} = 4\pi/V$, ω denotes the volume form corresponding to g, and $\rho_g = -i\partial\bar{\partial}\log\det g$ denotes the Ricci form. Using that $\int_{S^2}\omega = V$ we can write

$$\omega = e^\eta\omega_0 = (1 + \Delta_{g_0}u)\omega_0$$

for some smooth function u. Since the space of metrics conformal to g_0 and with fixed volume is path-connected, it suffices to prove that the variation of $\mathcal{F}_0 = \mathcal{F}_0(u)$ with respect to u vanishes identically. Taking a path $\omega_t = \omega + 2i\partial\bar{\partial}u_t$ and denoting $\frac{d}{dt}u_t = \dot{u}$, we calculate

$$\dot{\varphi}_g = 2i\boldsymbol{v}^{1,0}(\dot{u}), \qquad \dot{\rho}_g = i\bar{\partial}\partial\Delta_g\dot{u}.$$

The second identity is straightforward. As for the first identity, from the defining equation (we set $\varphi = \varphi_g$ for simplicity)

$$\varphi_{\bar{\lambda}} = ig_{\mu\bar{\lambda}}v^\mu,$$

taking the derivative with respect to t, we get

$$\dot{\varphi}_{\bar{\lambda}} = 2i\dot{u}_{\mu\bar{\lambda}}v^\mu = 2i(v^\mu\dot{u}_\mu)_{\bar{\lambda}}$$

by the holomorphicity of the vector field $\boldsymbol{v}^{1,0} = v^\mu\partial_\mu$, thus $\dot{\varphi} = 2i\boldsymbol{v}^{1,0}(\dot{u}) + c$ for some constant c. On the other hand, the normalization condition $\int_{S^2}\varphi\omega = 0$ implies

$$0 = \int_{S^2}\dot{\varphi}\omega + \varphi\cdot 2i\partial\bar{\partial}\dot{u} = \int_{S^2}\dot{\varphi}\omega - 2i\partial\dot{u}\wedge\bar{\partial}\varphi$$

$$= \int_{S^2}2iv^\mu\dot{u}_\mu\cdot ig_{z\bar{z}}dz\wedge d\bar{z} + c\omega - 2i\int_{S^2}\dot{u}_z ig_{z\bar{z}}v^z dz\wedge d\bar{z},$$

therefore $c = 0$ and $\dot{\varphi}_g = 2iv^\mu\dot{u}_\mu = 2i\boldsymbol{v}^{1,0}(\dot{u})$.

We will also need the following identity

$$\bar{\partial}\Delta_g\varphi_g = -2\iota_{\boldsymbol{v}^{1,0}}\rho_g.$$

To prove this, we calculate

$$\bar{\partial}\Delta_g\varphi_g = 2\bar{\partial}(g^{\alpha\bar{\beta}}\varphi_{\alpha\bar{\beta}})$$

$$= 2g^{\alpha\bar{\beta}}(\partial_{\bar{\kappa}}\varphi_{\alpha\bar{\beta}} - g^{\alpha\bar{\mu}}g^{\lambda\bar{\beta}}\partial_{\bar{\kappa}}g_{\lambda\bar{\mu}}\varphi_{\alpha\bar{\beta}})d\bar{z}^\kappa$$

$$= 2g^{\alpha\bar{\beta}}\nabla_{\bar{\kappa}}\nabla_{\bar{\beta}}\nabla_\alpha\varphi\cdot d\bar{z}^\kappa$$

$$= 2g^{\alpha\bar{\beta}}\left(-R_{\bar{\kappa}\alpha\bar{\beta}}{}^{\bar{\lambda}}\nabla_{\bar{\lambda}}\varphi + \nabla_\alpha\nabla_{\bar{\kappa}}\nabla_{\bar{\beta}}\varphi\right)d\bar{z}^\kappa$$

$$= -2R_{\mu\bar{\kappa}}g^{\mu\bar{\lambda}}\nabla_{\bar{\lambda}}\varphi d\bar{z}^\kappa,$$

in which we use the relation $\nabla_{\bar{\kappa}}\nabla_{\bar{\beta}}\varphi = \nabla_{\bar{\kappa}}\left(ig_{\alpha\bar{\beta}}v^{\alpha}\right) = ig_{\alpha\bar{\beta}}\nabla_{\bar{\kappa}}v^{\alpha} = 0$ by the holomorphicity of $\boldsymbol{v}^{1,0}$. Finally, using again the formula $\bar{\partial}\varphi = \iota_{\boldsymbol{v}^{1,0}}\omega$, i.e. $\varphi_{\bar{\lambda}} = ig_{\mu\bar{\lambda}}v^{\mu}$, the above equation yields

$$\bar{\partial}\Delta_g\varphi_g = -2iR_{\mu\bar{\kappa}}v^{\mu}d\bar{z}^{\kappa} = -2\iota_{\boldsymbol{v}^{1,0}}\rho_g.$$

Using the various identities above, we conclude with the following calculation

$$
\begin{aligned}
-\frac{d}{dt}\mathcal{F}_0(g_t) &= 2i\int\limits_{S^2}\boldsymbol{v}^{1,0}(\dot{u})\left(\rho_g - \hat{K}\omega\right) + \int\limits_{S^2}\varphi_g\left(-i\partial\bar{\partial}\Delta_g\dot{u} - \hat{K}\cdot2i\partial\bar{\partial}\dot{u}\right) \\
&= 2i\int\limits_{S^2}v^z\dot{u}_z(\rho_g - \hat{K}\omega) + \int\limits_{S^2}\varphi_g\left(-i\partial\bar{\partial}\Delta_g\dot{u} - \hat{K}\cdot2i\partial\bar{\partial}\dot{u}\right) \\
&= 2i\int\limits_{S^2}\dot{u}_zdz \wedge \iota_{\boldsymbol{v}^{1,0}}(\rho_g - \hat{K}\omega) + \int\limits_{S^2}\varphi_g\left(-i\partial\bar{\partial}\Delta_g\dot{u} - \hat{K}\cdot2i\partial\bar{\partial}\dot{u}\right) \\
&= i\int\limits_{S^2}\dot{u}_zdz \wedge (-\bar{\partial}\Delta_g\varphi_g - 2\hat{K}\bar{\partial}\varphi_g) + \int\limits_{S^2}\varphi_g\left(-i\partial\bar{\partial}\Delta_g\dot{u} - \hat{K}\cdot2i\partial\bar{\partial}\dot{u}\right) \\
&= \frac{1}{2}\int\limits_{S^2}\Delta_g\dot{u}\left(\Delta_g\varphi_g + 2\hat{K}\varphi_g\right)\omega - \frac{1}{2}\int\limits_{S^2}\left(\Delta_g\varphi_g\Delta_g\dot{u} + 2\hat{K}\varphi_g\Delta_g\dot{u}\right)\omega \\
&= 0. \quad \square
\end{aligned}
$$

We are ready to recall the definition of the Futaki invariant for the self-dual Einstein-Maxwell-Higgs equations, originally introduced in [3].

Proposition 2.6 *(Einstein-Maxwell-Higgs Futaki invariant). The following expression does not depend on the choice of smooth functions η and f*

$$
\begin{aligned}
\mathcal{F}_{\ell,N,V,\tau} = {}& 2i\frac{a}{\tau^2}\int\limits_{S^2}\psi_f\left(\frac{4\pi N}{V} - \Delta_{g_0}f + e^{\eta}(e^f\Phi - \tau^2)\right)\omega_0 \\
&- \int\limits_{S^2}\varphi_\eta\left(2K_{g_0} - \Delta_{g_0}(\eta + \tfrac{a}{\tau^2}e^f\Phi) - a\left(\frac{4\pi N}{V} - \Delta_{g_0}f\right)\right)\omega_0,
\end{aligned}
$$

provided that $\int_{S^2}e^{\eta}\omega_0 = V$. Consequently, it defines an invariant which vanishes if the equations (2.4) admit a smooth solution (η, f) with $\int_{S^2}e^{\eta}\omega_0 = V$ and $\varepsilon^2 = 1$.

Proof. We denote $\mathcal{F}(\eta, f) = \mathcal{F}_{\ell,N,V,\tau}$, as above. By Lemma 2.5, we have a simplified expression

$$\mathcal{F}(\eta, f) = 2i\frac{a}{\tau^2}\int\limits_{S^2}\psi_f\left(\frac{4\pi N}{V} - \Delta_{g_0}f + e^{\eta}(e^f\Phi - \tau^2)\right)\omega_0$$

$$+ a \int\limits_{S^2} \varphi_\eta \left(\tfrac{1}{\tau^2} \Delta_{g_0}(e^f \Phi) + \frac{4\pi N}{V} - \Delta_{g_0} f \right) \omega_0.$$

Using that $\int_{S^2} e^\eta \omega_0 = V$, we have $e^\eta \omega_0 = (1 + \Delta_{g_0} u)\omega_0$ for some smooth function u. As in the proof of Lemma 2.5, denoting $\omega = (1 + \Delta_{g_0} u)\omega_0$ we can write the previous expression as

$$\mathcal{F}(u, f) = 4i \frac{a}{\tau^2} \int\limits_{S^2} \psi_f \left((N\omega_{FS} + i\bar\partial\partial f) + \frac{1}{2}(e^f \Phi - \tau^2)\omega \right)$$

$$+ a \int\limits_{S^2} \varphi_g \left(\tfrac{1}{\tau^2} \Delta_g(e^f \Phi)\omega + 2(N\omega_{FS} + i\bar\partial\partial f) \right).$$

Taking variations with respect to u and applying Lemma 2.4 we have

$$\delta_u \mathcal{F} = 2i \frac{a}{\tau^2} \int\limits_{S^2} \psi_f \left((e^f \Phi - \tau^2) \cdot 2i\partial\bar\partial \dot{u} \right)$$

$$+ 2i \frac{a}{\tau^2} \int\limits_{S^2} \boldsymbol{v}^{1,0}(\dot{u}) \left(2i\partial\bar\partial(e^f \Phi) + 2\tau^2(N\omega_{FS} + i\bar\partial\partial f) \right)$$

$$= 2i \frac{a}{\tau^2} \int\limits_{S^2} \dot{u} \cdot 2i\partial\bar\partial \left(\psi_f(e^f \Phi - \tau^2) \right)$$

$$+ 2i \frac{a}{\tau^2} \int\limits_{S^2} \partial\dot{u} \wedge \iota_{\boldsymbol{v}^{1,0}} \left(2i\partial\bar\partial(e^f \Phi) + 2\tau^2(N\omega_{FS} + i\bar\partial\partial f) \right)$$

$$= 2i \frac{a}{\tau^2} \int\limits_{S^2} \dot{u} \cdot 2i\partial \left(\bar\partial \left(\psi_f(e^f \Phi - \tau^2) \right) - \iota_{\boldsymbol{v}^{1,0}}(\partial\bar\partial(e^f \Phi) + \tau^2 \bar\partial\partial \log h) \right)$$

$$= 0,$$

where we have used that $i\bar\partial\partial \log h = N\omega_{FS} + i\bar\partial\partial f$. Similarly, taking variations with respect to f, gives

$$\delta_f \mathcal{F} = 4i \frac{a}{\tau^2} \int\limits_{S^2} \partial\dot{f}(\boldsymbol{v}^{1,0}) \left(i\bar\partial\partial \log h + \frac{1}{2}(e^f \Phi - \tau^2)\omega \right) + 4i \frac{a}{\tau^2} \int\limits_{S^2} \psi_f \left(i\bar\partial\partial \dot{f} + \frac{1}{2}\dot{f} e^f \Phi \omega \right)$$

$$+ \frac{a}{\tau^2} \int\limits_{S^2} \varphi_g \cdot 2i\partial\bar\partial \left(\dot{f}(e^f \Phi - \tau^2) \right)$$

$$= -4i \frac{a}{\tau^2} \int\limits_{S^2} \dot{f} \cdot \partial \left(\iota_{\boldsymbol{v}^{1,0}} \left(i\bar\partial\partial \log h + \frac{1}{2}(e^f \Phi - \tau^2)\omega \right) \right)$$

$$+ 4i\frac{a}{\tau^2}\int\limits_{S^2}\left(\dot{f}\cdot i\bar{\partial}\partial\psi_f + \frac{1}{2}\psi_f\dot{f}e^f\Phi\omega\right)$$

$$+ 2i\frac{a}{\tau^2}\int\limits_{S^2}\dot{f}(e^f\Phi - \tau^2)\cdot\partial(\iota_{\boldsymbol{v}^{1,0}}\omega)$$

$$= -4i\frac{a}{\tau^2}\int\limits_{S^2}\dot{f}\cdot\partial\left(\iota_{\boldsymbol{v}^{1,0}}(i\bar{\partial}\partial\log h)\right) + \frac{1}{2}\dot{f}\cdot\partial(e^f\Phi)\wedge\iota_{\boldsymbol{v}^{1,0}}\omega + \frac{1}{2}\dot{f}(e^f\Phi - \tau^2)\partial(\iota_{\boldsymbol{v}^{1,0}}\omega)$$

$$+ 4i\frac{a}{\tau^2}\int\limits_{S^2}\dot{f}\cdot\partial\left(\iota_{\boldsymbol{v}^{1,0}}(i\bar{\partial}\partial\log h)\right) + \frac{1}{2}\dot{f}\cdot\psi_f e^f\Phi\omega + \frac{1}{2}\dot{f}(e^f\Phi - \tau^2)\partial\left(\iota_{\boldsymbol{v}^{1,0}}\omega\right)$$

$$= 2i\frac{a}{\tau^2}\int\limits_{S^2}\dot{f}\left(\psi_f e^f\Phi\omega - \partial(e^f\Phi)\wedge\iota_{\boldsymbol{v}^{1,0}}\omega\right)$$

$$= 2i\frac{a}{\tau^2}\int\limits_{S^2}\dot{f}\left(\psi_f\phi h\overline{\phi} - \iota_{\boldsymbol{v}^{1,0}}\left(\partial\phi + \phi\partial\log h\right)h\overline{\phi}\right)\omega$$

$$= 0,$$

where in the third inequality we used the first item of Equation (2.5) and in the last inequality we used the definition $e^f\Phi = \phi h\overline{\phi}$ and the second item of Equation (2.5). □

Remark 2.7. More precisely, the Futaki invariant for the self-dual Einstein-Maxwell-Higgs equations constructed in [3] is a character of the Lie algebra of infinitesimal automorphisms of the triple $(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(N), \phi)$. The quantity $\mathcal{F}_{\ell,N,V,\tau}$ considered here is the evaluation of this character at the infinitesimal automorphism $(\boldsymbol{v}^{1,0}, \psi_f)$ constructed in Lemma 2.3.

2.3. Proof of Theorem 1.1

We start calculating an explicit formula for the Futaki invariant.

Lemma 2.8. *The Einstein-Maxwell-Higgs Futaki invariant in Proposition 2.6 is*

$$\mathcal{F}_{\ell,N,V,\tau} = ia\left(V - \frac{4\pi N}{\tau^2}\right)(N - 2\ell).$$

Proof. Take $f = \eta = 0$, and $\omega_0 = \frac{V}{2\pi}\omega_{FS}$ in Proposition 2.6, we get that

$$\mathcal{F}_{\ell,N,V,\tau} = 2i\frac{a}{\tau^2}\int\limits_{S^2}\psi_0\left(2N\omega_{FS} + \frac{V}{2\pi}(\Phi - \tau^2)\omega_{FS}\right)$$

$$+ \frac{a}{\tau^2}\int\limits_{S^2}\varphi_0\Delta_{\omega_{FS}}\Phi\omega_{FS}$$

$$= 2i\frac{a}{\tau^2}\int_{S^2}\psi_0\left(2N - \frac{V}{2\pi}\tau^2\right)\omega_{FS} + 2i\frac{a}{\tau^2}\cdot\frac{V}{2\pi}\int_{S^2}\Phi\psi_0\omega_{FS}$$

$$- \frac{4a}{\tau^2}\int_{S^2}\varphi_0\Phi\omega_{FS}$$

$$= 2i\frac{a}{\tau^2}\left(2N - \frac{V}{2\pi}\tau^2\right)\int_{S^2}\psi_0\omega_{FS} + 2i\frac{a}{\tau^2}\frac{V}{2\pi}\int_{S^2}\Phi\left(\psi_0 - \frac{|z|^2-1}{|z|^2+1}\right)\omega_{FS},$$

where $\psi_0 = \ell - N\frac{|z|^2}{1+|z|^2}$, and $\varphi_0 = \frac{V}{2\pi}\cdot\frac{i}{2}\frac{|z|^2-1}{|z|^2+1}$, and we used the fact that $\Delta_{\omega_{FS}}\varphi_0 = -4\varphi_0$. The final formula for the Futaki invariant follows from the calculations

$$\int_{S^2}\psi_0\omega_{FS} = \int_{\mathbb{C}}\left(\ell - N\frac{|z|^2}{|z|^2+1}\right)\frac{idz\wedge d\bar{z}}{(|z|^2+1)^2}$$

$$= 2\pi\int_0^\infty\left((\ell - N)\frac{1}{(1+s)^2} + \frac{N}{(1+s)^3}\right)ds$$

$$= \pi(2\ell - N),$$

and

$$\int_{S^2}\Phi\left(\psi_0 - \frac{|z|^2-1}{|z|^2+1}\right)\omega_{FS} = \int_{\mathbb{C}}\left(\ell - N\frac{|z|^2}{|z|^2+1} + \frac{1-|z|^2}{1+|z|^2}\right)\frac{|z|^{2\ell}}{(1+|z|^2)^{N+2}}idz\wedge d\bar{z}$$

$$= 4\pi\int_0^\infty\left(\ell - N\frac{r^2}{r^2+1} + \frac{1-r^2}{1+r^2}\right)\frac{r^{2\ell+1}}{(1+r^2)^{N+2}}dr$$

$$= 2\pi\int_0^\infty\frac{(\ell+1)s^\ell + (\ell-N-1)s^{\ell+1}}{(1+s)^{N+3}}ds$$

$$= 2\pi\frac{s^{\ell+1}}{(1+s)^{N+2}}\Big|_0^\infty$$

$$= 0. \quad \square$$

We also need the following necessary condition for the existence of solutions to the self-dual Einstein-Maxwell-Higgs equations. This inequality was originally observed by Noguchi, Bradlow, and García-Prada in the context of Abelian vortices on a compact surface [4,6,7,10]. The proof follows directly by integrating the first equation in (2.4) using the volume form $e^\eta\omega_0$.

Lemma 2.9. *Assume that there exists a solution* (f, η) *to* (2.4) *with* $\int_{S^2}e^\eta\omega_0 = V$ *and* $\varepsilon^2 = 1$. *Then, there holds the following inequality*

$$V > \frac{4\pi N}{\tau^2}.$$

We are ready for the proof of our main result.

Proof of Theorem 1.1. Assume $d \leq 2$ and that there exists a solution (u, η) of (1.2) for some smooth background metric g_0' on S^2 and some parameter $\varepsilon > 0$. Notice that in the first equation in (1.2) we use the volume form ω_0' of g_0' to define the distributional equation. That is, for any smooth function ψ on S^2 one has

$$\int_{S^2} \psi \Delta_{g_0'} u \omega_0' = \int_{S^2} \psi \frac{1}{\varepsilon^2} e^\eta (e^u - \tau^2) \omega_0' + 4\pi \sum_{j=1}^d n_j \psi(p_j).$$

Setting $g = \frac{1}{\varepsilon^2} e^\eta g_0'$, and taking now ω the volume form of the metric g, one obtains

$$\int_{S^2} \psi \Delta_g u \omega = \int_{S^2} \psi (e^u - \tau^2) \omega + 4\pi \sum_{j=1}^d n_j \psi(p_j).$$

This means that (u, g) is a solution of

$$\Delta_g u = (e^u - \tau^2) + 4\pi \sum_{j=1}^d n_j \delta_{p_j},$$

$$K_g = -\frac{a}{2\tau^2} \left[\tau^2 (e^u - \tau^2) - \Delta_g e^u \right], \tag{2.6}$$

where δ_{p_j} is the Dirac distribution on (S^2, ω) concentrated at p_j and ω is the volume form of g. Let $V = \int_{S^2} \omega$, and g_0 be the constant scalar curvature metric on S^2 defined by the formula (2.2). Let ω_0 be the area form of the metric g_0. Notice that $\int_{S^2} \omega_0 = V$. Set $f := u - \log \Phi$, where Φ is as in Lemma 2.1. Then, for η' the smooth function on S^2 such that $\omega = e^{\eta'} \omega_0$, the pair (f, η') is a solution of

$$\Delta_{g_0} f = e^{\eta'} (e^f \Phi - \tau^2) + \frac{4\pi N}{V},$$

$$\Delta_{g_0} (\eta' + \frac{a}{\tau^2} e^f \Phi) = 2K_{g_0} + a e^{\eta'} (e^f \Phi - \tau^2),$$

such that $\int_{S^2} e^{\eta'} \omega_0 = V$, i.e. the pair (f, η') is a solution of equations (2.4) with the parameter $\varepsilon^2 = 1$ and $\int_{S^2} e^{\eta'} \omega_0 = V$. Therefore, by Proposition 2.6 and Lemma 2.8 we necessarily have

$$\mathcal{F}_{\ell, N, V, \tau} = ia \left(V - \frac{4\pi N}{\tau^2} \right) (N - 2\ell) = 0.$$

Finally, by Lemma 2.9, we have $N = 2\ell$, and therefore we conclude that

$$d = 2 \qquad \text{and} \qquad n_1 = n_2 = \ell = \frac{N}{2}. \quad \Box$$

Remark 2.10. Our main result in Theorem 1.1 gives infinitely many examples of configurations of points on the sphere for which there cannot be solutions to the self-dual Einstein-Maxwell-Higgs equations, in apparent contradiction with [8, Theorem 1.2] (see the Erratum [9]). For example, one can take $d = 1$ and arbitrary N, or $d = 2$ and $n_1 > n_2$.

Declaration of competing interest

The authors have no competing interests.

Data availability

No data was used for the research described in the article.

References

[1] L. Álvarez-Cónsul, M. Garcia-Fernandez, O. García-Prada, Coupled equations for Kähler metrics and Yang–Mills connections, Geom. Topol. 17 (2013) 2731–2812.
[2] L. Álvarez-Cónsul, M. García-Fernández, O. García-Prada, Gravitating vortices, cosmic strings, and the Kähler–Yang–Mills equations, Commun. Math. Phys. 351 (2017) 361–385.
[3] L. Álvarez-Cónsul, M. García-Fernández, O. García-Prada, V. Pingali, Gravitating vortices and the Einstein-Bogomol'nyi equations, Math. Ann. 379 (2021) 1651–1684.
[4] S. Bradlow, Vortices in holomorphic line bundles over closed Kähler manifolds, Commun. Math. Phys. 135 (1990) 1–17.
[5] A. Futaki, An obstruction to the existence of Einstein Kähler metrics, Invent. Math. 73 (1983) 437–443.
[6] O. García-Prada, Invariant connections and vortices, Commun. Math. Phys. 156 (1993) 527–546.
[7] O. García-Prada, A direct existence proof for the vortex equations over a compact Riemann surface, Bull. Lond. Math. Soc. 26 (1994) 88–96.
[8] Jongmin Han, Youngae Lee, Juhee Sohn, Existence of nontopological solutions of the self-dual Einstein-Maxwell-Higgs equation on compact surfaces, Calc. Var. 60 (2021) 94.
[9] Jongmin Han, Youngae Lee, Juhee Sohn, Erratum to: "Existence of nontopological solutions of the self-dual Einstein-Maxwell-Higgs equations on compact surfaces", 2021.
[10] M. Noguchi, Yang–Mills–Higgs theory on a compact Riemann surface, J. Math. Phys. 28 (1987) 2343–2346.
[11] Y. Yang, An equivalence theorem for string solutions of the Einstein-matter-gauge equations, Lett. Math. Phys. 26 (1992) 79–90.
[12] Y. Yang, Prescribing topological defects for the coupled Einstein and Abelian Higgs equations, Commun. Math. Phys. 170 (1995) 541–582.