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DETERMINING SETS FOR HOLOMORPHIC FUNCTIONS ON THE SYMMETRIZED BIDISK

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ABSTRACT. A subset $\mathcal{D}$ of a domain $\Omega \subset \mathbb{C}^d$ is determining for an analytic function $f : \Omega \to \overline{\mathbb{D}}$ if whenever an analytic function $g : \Omega \to \overline{\mathbb{D}}$ coincides with f on $\mathcal{D}$, equals to f on whole Ω. This note finds several sufficient conditions for a subset of the symmetrized bidisk to be determining. For any $N \geq 1$, a set consisting of $N^2 - N + 1$ many points is constructed which is determining for any rational inner function with a degree constraint. We also investigate when the intersection of the symmetrized bidisk intersected with some special algebraic varieties can be determining for rational inner functions.

1. Introduction

1.1. **Motivation.** For a domain Ω in $\mathbb{C}^d$ $(d \geq 1)$, let $\mathbb{S}(\Omega)$ denote the set of analytic functions $f : \Omega \to \overline{\mathbb{D}}$, where $\mathbb{D}$ denotes the open unit disk in $\mathbb{C}$. Given a function $f \in \mathbb{S}(\Omega)$, this paper revolves around the question when a given subset $\mathcal{D}$ of Ω has the property that whenever $g \in \mathbb{S}(\Omega)$ coincides with f on $\mathcal{D}$, equals to f on whole Ω. When a subset has this property we call it a *determining set* for (f, Ω), or just f when the domain is clear from the context. For example, $\{0, 1/2\}$ is a determining set for the identity map (by the Schwarz Lemma); any open subset of Ω is determining for any analytic function on Ω (by the Identity Theorem). See Rudin [32, Chapter 5] for some interesting results related to a similar concept for $\Omega = \mathbb{D}^d$.

The motivation behind the study of determining sets comes from the Pick interpolation problem. It corresponds to the case when $\mathcal{D}$ is a finite set. Given a finite subset $\mathcal{D} = \{\lambda_1, \lambda_2, \ldots, \lambda_N\}$ of Ω and points $w_1, w_2, \ldots, w_N$ in the open unit disk $\mathbb{D}$, the Pick interpolation problem asks if there is an analytic function $f : \Omega \to \mathbb{D}$ such that $f(\lambda_j) = w_j$ for $j = 1, 2, \ldots, N$. Therefore in this case, $\mathcal{D}$ being a determining set for (f, Ω) means that the (solvable) Pick problem $\lambda_j \mapsto f(\lambda_j)$ has a unique solution. In view of Pick's pioneering work [31], it is therefore clear that when $\Omega = \mathbb{D}$, then $\mathcal{D}$ is determining for f if and only if the Pick matrix

$$\left[\frac{1-f(\lambda_i)\overline{f(\lambda_j)}}{1-\lambda_i\overline{\lambda_j}}\right]_{i,j=1}^N$$

has rank less than N, which is further equivalent to the existence of a Blaschke function of degree less than N solving the data. The classical Pick interpolation problem has seen a wide range of generalizations. To mention a few, a necessary and sufficient condition for the solvability of a given Pick data is known when Ω is the polydisk $\mathbb{D}^d$ [2], the Euclidian ball $\mathbb{B}_d$ [26], the symmetrized bidisk [10, 14], an affine variety [22] and in more general setting of test functions [19, 20]. However, unlike the classical

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case, it is rather obscure in higher dimension when it comes to understanding when a given solvable Pick problem has a unique solution, and usually one has to settle with either necessary or sufficient conditions; see for example [4, 33, 34, 35].

1.2. **The main results.** The purpose of this article is to explore this direction where the domain under consideration is the *symmetrized bidisk*

$$\mathbb{G} := \{(z_1 + z_2, z_1 z_2) : (z_1, z_2) \in \mathbb{D}^2\}. \tag{1.1}$$

Following the work [7] of Agler and Young, this domain has remained a field of extensive research in operator theory and complex geometry constituting examples and counter-examples to celebrated problems in these areas such as the rational dilation problem [8, 12] and the Lempert Theorem [18]. In quest of understanding the determining sets we shall actually consider the following more general situation.

Definition 1.1. Let $\Omega \subset \mathbb{C}^d$ be a domain, $E \subset \Omega$ and $f \in \mathbb{S}(\Omega)$. We say that a subset $\mathcal{D}$ of E is *determining* for (f, E) if for every $g \in \mathbb{S}(\Omega)$, $g = f$ on $\mathcal{D}$ implies $g = f$ on E. If $\mathcal{D}$ is determining for (f, E) for all $f \in \mathbb{S}(\Omega)$, then we say that $\mathcal{D}$ is determining for E. Moreover, when E is the largest set in Ω such that $\mathcal{D}$ is determining for (f, E), we say that E is the *uniqueness set* for $(f, \mathcal{D})$, i.e., in this case

$$E = \bigcap \{Z(g - f) : g \in \mathbb{S}(\Omega) \text{ and } g = f \text{ on } \mathcal{D}\}.$$

Here, for a function f, we use the standard notation $Z(f)$ for the zero set of f.

Note that if E is the uniqueness set for $(f, \mathcal{D})$, then for every $z \in \Omega \setminus E$, there exists a function $g \in \mathbb{S}(\Omega)$ such that $g = f$ on $\mathcal{D}$ but $f(z) \neq g(z)$. Remarkably, when $\mathcal{D}$ is a finite subset of $\mathbb{G}$, then for any function $f \in \mathbb{S}(\mathbb{G})$, the uniqueness set for $(f, \mathcal{D})$ is an affine variety (see [16], [3]). This is owing to the fact that every solvable Pick data in $\mathbb{G}$ always has a rational inner solution (see [16], [5]). Also note that if f and g agree on $\mathcal{D}$, then $\mathcal{D}$ is determining for (f, E) if and only if $\mathcal{D}$ is determining for (g, E) also. In view of these facts, we shall mostly be concerned with the case when the function f in Definition 1.1 is rational and inner. Here, a function f in $\mathbb{S}(\mathbb{G})$ is called *inner*, if $\lim_{r \to 1-} |f(r\zeta_1 + r\zeta_2, r^2\zeta_1\zeta_2)| = 1$ for almost all ζ_1, ζ_2 in $\mathbb{T}$.

Note that $\mathbb{G}$ is the image of $\mathbb{D}^2$ under the (proper) holomorphic map $\pi : (z_1, z_2) \mapsto (z_1 + z_2, z_1 z_2)$. The topological boundary of $\mathbb{G}$ is $\partial\mathbb{G} := \pi(\overline{\mathbb{D}} \times \mathbb{T}) \cup \pi(\mathbb{T} \times \mathbb{D})$ and the distinguished boundary of $\mathbb{G}$ is $b\mathbb{G} := \pi(\mathbb{T} \times \mathbb{T})$ (see [9]). Here the *distinguished boundary* of a bounded domain $\Omega \subset \mathbb{C}^d$ is the Šilov boundary with respect to the algebra of complex-valued functions continuous on $\overline{\Omega}$ and holomorphic in Ω. A special type of algebraic varieties has been prevalent in the study of uniqueness of the solutions of a Pick interpolation problem (see [3, 16, 24, 25, 26, 27]). We define it below. Throughout the paper, the notation ξ stands for a polynomial in two variables.

Definition 1.2. An algebraic variety $Z(\xi)$ in $\mathbb{C}^2$ is said to be *distinguished* with respect to a bounded domain Ω, if

$$Z(\xi) \cap \Omega \neq \emptyset \quad \text{and} \quad Z(\xi) \cap \partial\Omega = Z(\xi) \cap b\Omega.$$

An example of a distinguished variety with respect to $\mathbb{G}$ is $\{(2z, z^2) : z \in \mathbb{C}\}$. We refer the readers to the papers [3, 11, 16, 17, 29] for results concerning these varieties and their connection to interpolation problems.

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We now state the main results of this paper in the order they are proved.

(1) In §2.1 we reformulate the notion of determining set in the more general setting of reproducing kernel Hilbert spaces and find a sufficient condition for a finite subset of a general domain to be determining. This is Theorem 2.1. We also show by an example that the sufficient condition need not be necessary, in general.
(2) Starting with a natural number N, §2.2 constructs a finite subset of $\mathbb{G}$ consisting exactly of $N^2 - N + 1$ many points which is determining for any rational inner function with a natural degree constraint on it. This is Theorem 2.5. Proposition 2.4 is an intermediate step of the construction and is interesting on its own right.
(3) Given a distinguished variety $\mathcal{W} = Z(\xi)$, we investigate in §2.3 when the intersection $\mathcal{W} \cap \mathbb{G}$ can be the uniqueness set for $(f, \mathcal{D})$, where f is a rational inner function and $\mathcal{D}$ a finite subset of $\mathbb{G}$ – see Theorem 2.10. The preparatory results Propositions 2.7 and 2.8 are interesting in their own rights. Propositions 2.7 states that if f is a rational inner function with some regularity assumption, then there is a natural number N depending on f large enough so that *any* subset of $\mathcal{W} \cap \mathbb{G}$ consisting of N points is determining for $(f, \mathcal{W} \cap \mathbb{G})$. This section then goes on to find (in Theorem 2.12) a sufficient condition for $\mathcal{W} \cap \mathbb{G}$ to be determining for a rational inner function f with a regularity assumption on it. The condition is just that the inequality

$$2\,\mathrm{Re}\langle f, \xi h\rangle_{H^2} < \|\xi h\|_2^2$$

holds, whenever h is a non-zero analytic function on $\mathbb{G}$ and ξh is bounded on $\mathbb{G}$. Here the inner product is the Hardy space inner product, briefly discussed in §2.3.
(4) §3 proves a bounded extension theorem for distinguished varieties with no singularities on $b\mathbb{G}$. More precisely, given a distinguished variety $\mathcal{W}$, we show that corresponding to every two-variable polynomial f, there is a rational function F on $\mathbb{G}$ such that $F|_{\mathcal{W}\cap\mathbb{G}} = f$ and that $\sup_{\mathbb{G}} |F(s,p)| \le \alpha \sup_{\mathcal{W}\cap\mathbb{G}} |f|$, for some constant α depending only on the distinguished variety $\mathcal{W}$.

2. Determining and the uniqueness sets

2.1. **A result for a general domain.** We begin by proving a sufficient condition for a finite subset of a general domain to be determining. The concept of determining set can be formulated in a general setup of reproducing kernel Hilbert spaces. Here a *kernel* on a domain Ω in $\mathbb{C}^d$ $(d \ge 1)$ is a function $k : \Omega \times \Omega \to \mathbb{C}$ such that for every choice of points $\lambda_1, \lambda_2, \dots, \lambda_N$ in Ω, the $N \times N$ matrix $[k(\lambda_i, \lambda_j)]$ is positive definite. Given a kernel k, there is a unique Hilbert space $H(k)$ associated to it, called the reproducing kernel Hilbert space; we refer the uninitiated reader to the book [30]. For the purpose of this paper, all that is needed to know is that elements of the form $\{\sum_{j=1}^n c_j k(\cdot, \lambda_j) : c_j \in \mathbb{C} \text{ and } \lambda_j \in \Omega\}$ constitute a dense set of $H(k)$. A kernel k is said to be a holomorphic kernel, if it is holomorphic in the first and conjugate holomorphic in the second variable. Note that when k is holomorphic, then so are the elements of $H(k)$. Let us denote by Mult $H(k)$ the algebra of all bounded holomorphic functions

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φ on Ω such that $\varphi \cdot f \in H(k)$ whenever $f \in H(k)$. Such a holomorphic function is generally referred to as a *multiplier* for $H(k)$. Let $\mathrm{Mult}_1 H(k)$ denote the set of all multipliers φ such that the operator norm of $M_\varphi : f \mapsto \varphi \cdot f$ for all f in $H(k)$ is no greater than one. A subset $\mathcal{D} \subset \Omega$ is said to be *determining* for a function φ in $\mathrm{Mult}_1 H(k)$ if whenever $\psi \in \mathrm{Mult}_1 H(k)$ such that $\varphi = \psi$ on $\mathcal{D}$, then $\varphi = \psi$ on Ω.

Theorem 2.1. *Let k be a holomorphic kernel on a domain Ω in $\mathbb{C}^d$, $\varphi \in \mathrm{Mult}_1 H(k)$ and $\mathcal{D} = \{\lambda_1, \lambda_2, \ldots, \lambda_N\} \subset \Omega$. If the matrix*

$$[(1 - \varphi(\lambda_i)\overline{\varphi(\lambda_j)})k(\lambda_i, \lambda_j)]_{i,j=1}^N \tag{2.1}$$

is singular, then $\mathcal{D}$ is determining for φ.

Proof. Since the matrix (2.1) is singular, there is a non-zero vector in its kernel; let us denote it by γ. Let λ_{N+1} be any point in $\Omega \setminus \mathcal{D}$, and $\psi \in \mathrm{Mult}_1 H(k)$ be any function such that $\varphi = \psi$ on $\mathcal{D}$. Since $\psi \in \mathrm{Mult}_1 H(k)$, the operator $M_\psi : f \mapsto \psi \cdot f$ is a contractive operator on $H(k)$ and therefore for every $z \in \mathbb{C}$,

$$\langle [(1 - \psi(\lambda_i)\overline{\psi(\lambda_j)})k(\lambda_i, \lambda_j)]_{i,j=1}^{N+1} \begin{bmatrix} \gamma \\ z \end{bmatrix}, \begin{bmatrix} \gamma \\ z \end{bmatrix} \rangle \geq 0.$$

Since $\gamma \in \mathrm{Ker}[(1 - \varphi(\lambda_i)\overline{\varphi(\lambda_j)})k(\lambda_i, \lambda_j)]$ and $\varphi = \psi$ on $\mathcal{D}$, the above inequality collapses to

$$2\,\mathrm{Re}[\overline{z} \sum_{j=1}^N (1 - \overline{\psi(\lambda_j)}\psi(\lambda_{N+1}))\gamma_j k(\lambda_{N+1}, \lambda_j)] + |z|^2(1 - |\psi(\lambda_{N+1})|^2)||k_{\lambda_{N+1}}||^2 \geq 0.$$

Since the above inequality is true for all $z \in \mathbb{C}$, we have

$$\sum_{j=1}^N (1 - \overline{\psi(\lambda_j)}\psi(\lambda_{N+1}))\gamma_j k_{N+1,j} = 0,$$

which, after a rearrangement of terms, gives

$$\psi(\lambda_{N+1}) \left(\sum_{j=1}^N \overline{\psi(\lambda_j)}\gamma_j k(\lambda_{N+1}, \lambda_j) \right) = \sum_{j=1}^N \gamma_j k(\lambda_{N+1}, \lambda_j). \tag{2.2}$$

Define for z in Ω,

$$L(z) = \sum_{j=1}^N \gamma_j k_{\lambda_j}(z) = \sum_{j=1}^N \gamma_j k(z, \lambda_j).$$

By definition, it is clear that $L \in H(k)$. Consider the open set $\mathcal{O} = \Omega \setminus Z(L)$. Note that if $\lambda_{N+1} \in \mathcal{O}$, then the right hand side of (2.2) does not vanish, and therefore $\psi(\lambda_{N+1})$ is uniquely determined.

Now suppose $\phi = \psi$ on $\mathcal{O}$. By the assumption that $\mathcal{O}$ is a set of uniqueness for $\mathrm{Mult}_1(H(k))$, it follows that $\phi = \psi$. □

The converse of the above result is not true as the simple example below demonstrates.

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Example 2.2. Let k be the Bergman kernel on $\Omega = \mathbb{D}$, i.e., $k(z,w) = (1 - z\overline{w})^{-2}$. Then it is well-known that $\operatorname{Mult}_1 H(k) = \mathbb{S}(\mathbb{D})$, see for example [6, section 2.3]. By the Schwarz Lemma, $\mathcal{D} = \{0, 1/2\}$ is determining for the identity function. However, the matrix $\left[\begin{smallmatrix} 1 & 1 \\ 1 & 4/3 \end{smallmatrix}\right]$ is non-singular.

The rest of the paper specializes to the symmetrized bidisk.

2.2. **Finite sets as a determining set.** Given a natural number N, this subsection constructs a finite subset $\mathcal{D}$ of $\mathbb{G}$ consisting exactly of $N^2 - N + 1$ many points, which is determining for any rational inner function on $\mathbb{G}$ with a degree constraint on it. This is inspired by the work of Scheinker [33] which extends the following classical result for the unit disk to the polydisks.

Lemma 2.3 (Pick [31])**.** *Let $\mathcal{D} = \{\lambda_1, \lambda_2, \dots, \lambda_N\} \subset \mathbb{D}$ and f be a rational inner function on $\mathbb{D}$ with degree strictly less than N. Then if $g \in \mathbb{S}(\mathbb{D})$ is such that $f = g$ on $\mathcal{D}$, then $f = g$ on $\mathbb{D}$.*

For $\epsilon > 0$ and $z \in \mathbb{C}$, let $D(z;\epsilon) := \{w \in \mathbb{D} : |z - w| < \epsilon\}$. For $\zeta \in \mathbb{T}$ and $a \in \mathbb{D}$, let $m_{\zeta,a}$ be the Möbius map

$$m_{\zeta,a}(z) = \zeta \frac{z - a}{1 - \overline{a}z}.$$

We shall have use of two notions of degree for a polynomial in two variables. The one used in this subsection is the following. For a polynomial $\xi(z,w) = \sum_{i,j} a_{i,j} z^i w^j$, we define $\deg \xi := \max(i + j)$ such that $a_{i,j} \neq 0$. The degree of a rational function in its reduced fractional representation is defined to be the degree of the numerator polynomial. The following is an intermediate step to proving Theorem 2.5.

Proposition 2.4. *Let N be a positive integer and for each $j = 1, 2, \dots, N$, β_j be distinct points in $\mathbb{T}$ and D_j be the analytic disks $D_j = \{(z + \beta_j z, \beta_j z^2) : z \in \mathbb{D}\}$. Then*

(a) *there exist $\beta \in \mathbb{T}$ and $\epsilon > 0$ such that for every fixed $\zeta \in D(\beta;\epsilon) \cap \mathbb{T}$ and $a \in D(0;\epsilon)$, the analytic disk*

$$\mathcal{D}_{\zeta,a} = \{(z + m_{\zeta,a}(z), z m_{\zeta,a}(z)) : z \in \mathbb{D}\}$$

intersects each of the analytic disks D_j at a non-zero point;

(b) *for each $\zeta \in \mathbb{T}$ and $\epsilon > 0$, the set*

$$\mathcal{D}_\zeta = \{(z + m_{\zeta,a}(z), z m_{\zeta,a}(z)) : z \in \mathbb{D} \text{ and } a \in D(0;\epsilon)\}$$

is a determining set for any function in $\mathbb{S}(\mathbb{G})$; and

(c) *the set*

$$E = \{(z + \beta_j z, \beta_j z^2) : z \in \mathbb{D} \text{ and } j = 1, 2, \dots, N\} = \cup_{j=1}^N D_j$$

is a determining set for any rational inner function of degree less than N.

Proof. For part (a), note that given a $\zeta \in \mathbb{T}$ and $a \in \mathbb{D}$, the analytic disk $\mathcal{D}_{\zeta,a}$ intersects each D_j at a non-zero point if and only if there exist $0 \neq z \in \mathbb{D}$ such that for each j, $\beta_j z = m_{\zeta,a}(z)$, which is equivalent to having $\overline{a}\beta_j z^2 + (\beta_j - \zeta)z - a\zeta = 0$. Therefore ζ must belong to $\mathbb{T} \setminus \{\beta_j : j = 1, 2, \dots, N\}$. Now fix one such ζ and j. Let $\lambda_1(a), \lambda_2(a)$ be the roots of the polynomial above. Then clearly $\lambda_1(0) = 0 = \lambda_2(0)$. Therefore by continuity of the roots, there exists $\epsilon > 0$ such that whenever $a \in D(0;\epsilon)$, $\lambda_1(a)$ and

$\lambda_2(a)$ belong to $\mathbb{D}$. This ϵ will of course depend on j but since there are only finitely many j, we can find an $\epsilon > 0$ so that (a) holds.

For part (b) we have to show that if $f : \mathbb{G} \to \overline{\mathbb{D}}$ is any analytic function such that $f|_{\mathcal{D}_\zeta} = 0$, then $f = 0$ on $\mathbb{G}$. Fix $z \in \mathbb{D}$ and consider $f_z : \mathbb{D} \to \overline{\mathbb{D}}$ defined by $f_z : w \mapsto f(z+w, zw)$. Since f vanishes on $\mathcal{D}_\zeta$, f_z vanishes on $\{m_{\zeta,a}(z) : a \in D(0;\epsilon)\}$ which shows that $f_z = 0$ on $\mathbb{D}$. Since $z \in \mathbb{D}$ is arbitrary, $f = 0$ on $\mathbb{G}$.

For (c), let f be a rational inner function of degree less than N and $g \in \mathbb{S}(\mathbb{G})$ be such that $g = f$ on each D_j. For each ζ and a as in part (a), $\mathcal{D}_{\zeta,a}$ intersects each D_j at say $(s_j, p_j) = (\lambda_j + m_{\zeta,a}(\lambda_j), \lambda_j m_{\zeta,a}(\lambda_j))$. Restrict f and g to $\mathcal{D}_{\zeta,a}$ to get $f_{\zeta,a}(z) = f(z + m_{\zeta,a}(z), zm_{\zeta,a}(z))$ and $g_{\zeta,a}(z) = g(z + m_{\zeta,a}(z), zm_{\zeta,a}(z))$. Then clearly $f_{\zeta,a}$ is a rational inner function on $\mathbb{D}$ of degree less than N and $g_{\zeta,a} \in \mathbb{S}(\mathbb{D})$. Then for each $j = 1, 2, \ldots, N$, $g_{\zeta,a}(\lambda_j) = f_{\zeta,a}(\lambda_j)$. Therefore by Lemma (2.3), we have $g_{\zeta,a} = f_{\zeta,a}$ on $\mathbb{D}$ for each ζ and a as in part (a). Hence $g = f$ on $\mathcal{D}$, which by part (b) gives $g = f$ on $\mathbb{G}$. This completes the proof. □

Theorem 2.5. *For any $N \geq 1$, there exists a set D consisting of $(N^2 - N + 1)$ points in $\mathbb{G}$ such that $\mathcal{D}$ is a determining set for any rational inner function of degree less than N.*

Proof. For $N = 1$, it is trivial because then a rational inner function of degree less than 1 is identically constant. So suppose $N > 1$. Let $\lambda_1 := 0, \lambda_2, ..., \lambda_N$ be distinct points in $\mathbb{D}$, $\beta_1, ..., \beta_N$ be distinct points in $\mathbb{T}$ and $D_1, ..., D_N$ be the analytic disks as in Proposition 2.4. Consider the set

$$\mathcal{D} = \{(\lambda_j + \beta_k\lambda_j, \beta_k\lambda_j^2) : k, j = 1, 2, ..., N\}.$$

Since β_j and λ_j are distinct, $\mathcal{D}$ consists of precisely $N^2 - N + 1$ many points. Let f be a rational inner function on $\mathbb{G}$ and $g \in \mathbb{S}(\mathbb{G})$ be such that g agrees with f on $\mathcal{D}$. As before restrict f and g to each D_k to obtain rational inner functions $f_k(z) = f(z + \beta_k z, z^2\beta_k)$ and $g_k(z) = g(z + \beta_k z, z^2\beta_k)$ on the unit disk $\mathbb{D}$. We then have $f_k(\lambda_j) = g_k(\lambda_j)$ for each $j = 1, 2, \ldots, N$. Thus by Lemma 2.3, $f_k(z) = g_k(z)$ on $\mathbb{D}$ for each $k = 1, 2, \ldots, N$, which is same as saying that $f = g$ on $\cup_{k=1}^N D_k$. Consequently, by part (c) of Proposition 2.4, $f = g$ on $\mathbb{G}$. □

2.3. **Distinguished varieties as a determining and the uniqueness set.** A rational function $f = g/h$ with relatively prime polynomials g and h, is called *regular* if $h \neq 0$ on $\overline{\mathbb{G}}$. For example, note that while the rational function $(3p - s)/(3 - s)$ is regular, $(2p - s)/(2 - s)$ is not.

We first recall the known results that will be used later. Let $\mathcal{W} = Z(\xi)$ be a distinguished variety with respect to $\mathbb{G}$. Then it follows easily that $\mathcal{V} = Z(\xi \circ \pi)$ defines a distinguished variety with respect to $\mathbb{D}^2$. Lemma 1.2 of [3] produces a regular Borel measure ν on $\partial\mathcal{V} := \mathcal{V} \cap \mathbb{T}^2$ such that ν gives rise to a Hardy-type Hilbert function space on $\mathcal{V} \cap \mathbb{D}^2$, denoted by $H^2(\nu)$, i.e., $H^2(\nu)$ is the closure in $L^2(\nu)$ of polynomials such that evaluation at every point in $\mathcal{V} \cap \mathbb{D}^2$ is a bounded linear functional on $H^2(\nu)$. It was then shown in [29, Lemma 3.2] that the push-forward measure $\mu(E) = \nu(\pi^{-1}(E))$ for every Borel subset E of $\partial\mathcal{W} := \mathcal{W} \cap b\Gamma$ has all the properties that ν has. Furthermore, the spaces $H^2(\mu)$ and $H^2(\nu)$ are unitary equivalent via the isomorphism given by

$$U : H^2(\mu) \to H^2(\nu) \quad \text{by} \quad U : f \mapsto f \circ \pi. \tag{2.3}$$

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Note that if k^μ and k^ν are the Szegö-type reproducing kernels for $H^2(\mu)$ and $H^2(\nu)$, respectively, then for every $(z,w) \in \mathcal{V} \cap \mathbb{D}^2$ and $f \in H^2(\mu)$

$$\langle U^* k^\nu_{(z,w)}, f\rangle_{H^2(\mu)} = \langle k^\nu_{(z,w)}, Uf\rangle_{H^2(\nu)} = f \circ \pi(z,w) = \langle k^\mu_{\pi(z,w)}, f\rangle_{H^2(\mu)}.$$

We observe the following.

Lemma 2.6. *Let $\mathcal{W}$ be a distinguished variety with respect to $\mathbb{G}$ and μ be the regular Borel measure on $\partial\mathcal{W}$ as in the preceding discussion. Then for every regular rational inner function f on $\mathbb{G}$, the multiplication operator M_f on $H^2(\mu)$ has a finite dimensional kernel.*

Proof. We note that for every $(z,w) \in \mathcal{V} \cap \mathbb{D}^2$,

$$U^* M^*_{f\circ\pi} k^\nu_{(z,w)} = \overline{f \circ \pi(z,w)} U^* k^\nu_{(z,w)} = \overline{f \circ \pi(z,w)} k^\mu_{\pi(z,w)} = M^*_f k^\mu_{\pi(z,w)} = M^*_f U^* k^\nu_{(z,w)}.$$

Thus M_f on $H^2(\mu)$ and $M_{f\circ\pi}$ on $H^2(\nu)$ are unitarily equivalent via the unitary U as in 2.3. Now the lemma follows from [34, Theorem 3.6], which states that $\operatorname{Ker} M_{f\circ\pi}$ is finite dimensional. □

Proposition 2.7. *Let $\mathcal{W} = Z(\xi)$ be a distinguished variety with respect to $\mathbb{G}$ and f be a regular rational inner function on $\mathbb{G}$. If $\dim \operatorname{Ker} M^*_f < N$, then any N distinct points in $\mathcal{W} \cap \mathbb{G}$ is a determining set for $(f, \mathcal{W} \cap \mathbb{G})$.*

Proof. Let $\{w_1, w_2, \ldots, w_N\}$ be distinct points in $\mathcal{W} \cap \mathbb{G}$ and $g \in \mathbb{S}(\mathbb{G})$ be such that $g(w_j) = f(w_j)$ for each $j = 1, 2, \ldots, N$. Let $\mathcal{V} = Z(\xi \circ \pi)$ and $\{v_1, v_2, \ldots, v_N\}$ be in $\mathcal{V} \cap \mathbb{D}^2$ such that $\pi(v_j) = w_j$ for all $j = 1, 2, ..., N$. Thus $g \circ \pi(v_j) = f \circ \pi(v_j)$ for each $j = 1, 2, \ldots, N$. Theorem 1.7 of [34] yields $g \circ \pi = f \circ \pi$ on $\mathcal{V} \cap \mathbb{D}^2$ which is same as $g = f$ on $\mathcal{W} \cap \mathbb{G}$. This completes the proof. □

The 2-degree of a two-variable polynomial $\xi \in \mathbb{C}[z,w]$ is defined as $(d_1, d_2) =:$ 2-deg ξ, where d_1 and d_2 are the largest power of z and w, respectively in the expansion of $\xi(z,w)$. The reflection of a two-variable polynomial $\xi \in \mathbb{C}[z,w]$ is defined as

$$\widetilde{\xi}(z,w) = z^{d_1} w^{d_2} \overline{\xi\left(\frac{1}{\bar{z}}, \frac{1}{\bar{w}}\right)}.$$

For a rational function $f(z,w) = \xi(z,w)/\eta(z,w)$ with ξ and η having no common factor, the 2-degree of f is defined to be the 2-degree of the numerator. For two pairs of non-negative integers (p,q) and (m,n), we write $(p,q) \leq (m,n)$ to indicate that $p \leq m$ and $q \leq n$.

Proposition 2.8. *Let $\mathcal{W} = Z(\xi)$ be an irreducible distinguished variety and f be a regular rational inner function on $\mathbb{G}$ of the form*

$$f \circ \pi(z,w) = (zw)^m \frac{\widetilde{\eta \circ \pi}(z,w)}{\eta \circ \pi(z,w)}. \tag{2.4}$$

If 2-deg $\xi \circ \pi \leq$ 2-deg $f \circ \pi$, then for each $(s,p) \in \mathbb{G} \setminus (\mathbb{G} \cap \mathcal{W})$ there exists a regular rational inner function g on $\mathbb{G}$ such that g coincides with f on $\mathcal{W} \cap \mathbb{G}$ but $g(s,p) \neq f(s,p)$.

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Proof. Let 2-deg $\eta\circ\pi = (l,l)$ and 2-deg $\xi\circ\pi = (n,n)$. The hypothesis then is that $m+l-n$ is non-negative. For $\epsilon > 0$, define a symmetric function g_ϵ on $\mathbb{D}^2$ as

$$g_\epsilon(z,w) = \frac{(zw)^m \widetilde{\eta\circ\pi}(z,w) + \epsilon\widetilde{\xi\circ\pi}(z,w)}{\eta\circ\pi(z,w) + \epsilon(zw)^{m+l-n}\xi\circ\pi(z,w)}. \tag{2.5}$$

Simple computation shows that the reflection of the denominator of g_ϵ is equal to the numerator of g_ϵ, which implies that each each g_ϵ is a rational inner function on $\mathbb{D}^2$ provided that the denominator does not vanish on $\mathbb{D}^2$. Since $\eta\circ\pi$ does not vanish on $\overline{\mathbb{D}}^2$, we can always find a sufficiently small ϵ so that the denominator of each g_ϵ does not vanish in $\overline{\mathbb{D}}^2$, thus making g_ϵ regular.

By Proposition 4.3 of [23], $\xi\circ\pi = c\widetilde{\xi\circ\pi}$ for some $c\in\mathbb{T}$. This ensures that each g_ϵ coincides with f on $\mathcal{W}\cap\mathbb{G}$. Now let $(z_0,w_0)\in\mathbb{D}^2$ be such that $\pi(z_0,w_0)\in\mathbb{G}\setminus\mathcal{W}$. Then $g_\epsilon(z_0,w_0) = f\circ\pi(z_0,w_0)$ if and only if

$$\frac{(z_0w_0)^m\widetilde{\eta\circ\pi}(z_0,w_0) + \epsilon\bar{c}\xi\circ\pi(z_0,w_0)}{\eta\circ\pi(z_0,w_0) + \epsilon(z_0w_0)^{m+l-n}\xi\circ\pi(z_0,w_0)} = (z_0w_0)^m\frac{\widetilde{\eta\circ\pi}(z_0,w_0)}{\eta\circ\pi(z_0,w_0)},$$

which, after cross-multiplication and using the fact that $\xi\circ\pi(z_0,w_0)\neq 0$, leads to

$$\bar{c}\eta\circ\pi(z_0,w_0) = (z_0w_0)^{2m+l-n}\widetilde{\eta\circ\pi}(z_0,w_0). \tag{2.6}$$

Since $\eta\circ\pi$ does not vanish on $\overline{\mathbb{D}}^2$, we have $z_0w_0\neq 0$. Therefore the above equation holds if and only if

$$f\circ\pi(z_0,w_0) = (z_0w_0)^m\frac{\widetilde{\eta\circ\pi}(z_0,w_0)}{\eta\circ\pi(z_0,w_0)} = \frac{\bar{c}}{(z_0w_0)^{m+l-n}}. \tag{2.7}$$

If $m+l-n=0$, then f is a constant function. The hypothesis on the 2-degrees of ξ and f then implies that ξ must be constant. This is not possible because ξ defines a distinguished variety. Therefore $m+l-n\geq 1$, in which case, equation (2.7) implies that $|f\circ\pi(z_0,w_0)|>1$. This again is a contradiction because f is a rational inner function and so by the Maximum Modulus Principle, $|f\circ\pi(z)|\leq 1$ for every $(z,w)\in\mathbb{D}^2$. Consequently, $g_\epsilon(s,p)\neq f(s,p)$ for every $(s,p)\in\mathbb{G}\setminus(\mathcal{W}\cap\mathbb{G})$. □

Remark 2.9. In a forthcoming paper [15] it is shown that any rational inner function on $\mathbb{G}$ is of the form (2.4) possibly multiplied by a unimodular constant.

Theorem 2.10. *Let $\mathcal{W}=Z(\xi)$ be an irreducible distinguished variety with respect to $\mathbb{G}$, f be a regular rational inner function on $\mathbb{G}$ of the form* (2.4) *such that* 2-deg $\xi\circ\pi\leq$ 2-deg $f\circ\pi$, *and $\mathcal{D}$ be any subset of $\mathcal{W}\cap\mathbb{G}$ consisting of at least $1+\dim\operatorname{Ker}M_f^*$ many points. Then $\mathcal{W}\cap\mathbb{G}$ is the uniqueness set for $(f,\mathcal{D})$.*

Proof. Consider the multiplication operator M_f on $H^2(\mu)$, where $H^2(\mu)$ is the Hilbert space corresponding to $\mathcal{W}$ as mentioned in Lemma 2.6. By this lemma, $\dim\operatorname{Ker}(M_f^*)$ is finite. So let N be such that $\dim\operatorname{Ker}(M_f^*)<N$ and $\mathcal{D}=\{\lambda_1,\lambda_2,...,\lambda_N\}\subset\mathcal{W}\cap\mathbb{G}$. By Proposition 2.7, $\mathcal{D}$ is determining for $(f,\mathcal{W}\cap\mathbb{G})$. We use Proposition 2.8 to show that $\mathcal{W}\cap\mathbb{G}$ is the uniqueness set. Toward that end, pick $(s,p)\in\mathbb{G}\setminus\mathcal{W}\cap\mathbb{G}$. Proposition 2.8 guarantees the existence of a (regular) rational inner function g that coincides with f on $\mathcal{W}\cap\mathbb{G}$ but $g(s,p)\neq f(s,p)$. This proves that $\mathcal{W}\cap\mathbb{G}$ is the uniqueness set for the interpolation problem. This completes the proof of the theorem. □

Remark 2.11. An *extremal* interpolation problem in $\mathbb{G}$ is a solvable problem with no solution of supremum norm less than 1. Let $\mathcal{D} = \{\lambda_1, \lambda_2, \ldots, \lambda_N\}$ be a subset of $\mathbb{G}$ and f be a rational inner function on $\mathbb{G}$ such that the N-point Pick problem $\lambda_j \mapsto f(\lambda_j)$ is extremal and that none of the $(N-1)$-point subproblems is extremel. Then it is shown in [16] that the uniqueness set for $(f, \mathcal{D})$ contains a distinguished variety. Theorem 2.10 can be seen as a converse to this result. Indeed, Theorem 2.10 starts with a distinguished variety $\mathcal{W} = Z(\xi)$ and produces a regular rational inner function f and a finite set $\mathcal{D}$ depending on $\mathcal{W}$ such that $\mathcal{W} \cap \mathbb{G}$ is the uniqueness set for $(f, \mathcal{D})$. In addition, we note that the problem $\lambda_j \mapsto f(\lambda_j)$ is an extremal problem. This is because if g is any solution of the problem, then by Proposition 2.7 $g = f$ on $\mathcal{W} \cap \mathbb{G}$. Thus

$$\|g\|_{\infty,\mathbb{G}} \geq \|g\|_{\infty,\mathcal{W}\cap\mathbb{G}} = \|f\|_{\infty,\mathcal{W}\cap\mathbb{G}} = 1.$$

The last equality follows because f is a regular rational inner function.

There is a sufficient condition for a distinguished variety to be determining. In the theorem below and in its proof, the inner product $\langle , \rangle_{H^2}$ for analytic functions $f, g : \mathbb{G} \to \mathbb{C}$ is defined to be

$$\langle f, g\rangle_{H^2} = \sup_{0<r<1} \int_{\mathbb{T}\times\mathbb{T}} f \circ \pi(r\zeta_1, r\zeta_2)\overline{g \circ \pi(r\zeta_1, r\zeta_2)}|J(r\zeta_1, r\zeta_2)|^2 dm(\zeta_1, \zeta_2), \tag{2.8}$$

where m is the standard normalized Lebesgue measure on $\mathbb{T} \times \mathbb{T}$, and $J(z, w) = z - w$ is the Jacobian of the map $\pi : (z, w) \mapsto (z + w, zw)$. See the papers [13, 14, 28] for some motivation for and operator theory on the spaces of analytic functions for which $\|f\|_2 := \sqrt{\langle f, f\rangle_{H^2}} < \infty$. Note here that if f is an inner function on $\mathbb{G}$, then $\|f\|_2 = 1$.

Theorem 2.12. *Let $\mathcal{W} = Z(\xi)$ be a distinguished variety such that $\xi = \xi_1.\xi_2 \ldots \xi_l$ where ξ_i are irrudicible polynomials with ξ_i and ξ_j are co-prime for each $i \neq j$ and f be a regular rational inner function on $\mathbb{G}$. If for each analytic function $h(\not\equiv 0)$ on $\mathbb{G}$,*

$$2\,\mathrm{Re}\langle f, \xi h\rangle_{H^2} < \|\xi h\|_2^2$$

holds, whenever ξh is bounded on $\mathbb{G}$, then $\mathcal{W} \cap \mathbb{G}$ is a determining set for f.

Proof. We shall use contrapositive argument. So suppose that there exists $g \in \mathbb{S}(\mathbb{G})$ such that g coincides with f on $\mathcal{W} \cap \mathbb{G}$ but $g \neq f$. Choose an integer N so that $\dim \mathrm{Ker}\, M_f^* < N$ and pick N distinct points $\lambda_1, \ldots., \lambda_N \in \mathcal{W}$. Consider the N-point (solvable) Nevanlinna-Pick problem $\lambda_j \mapsto f(\lambda_j)$. By Proposition 2.7 all the solutions to this problem agree on $\mathcal{W} \cap \mathbb{G}$. Since $g \neq f$, there exists a $\lambda_{N+1} \in \mathbb{G} \setminus \mathcal{W}$ such that $g(\lambda_{N+1}) \neq f(\lambda_{N+1})$. Now consider the $(N + 1)$-point Nevanlinna-Pick problem $\lambda_j \mapsto g(\lambda_j)$ on $\mathbb{G}$. By [16, Theorem 5.3], every solvable Nevanlinna-Pick problem in $\mathbb{G}$ has a rational inner solution. Let ψ be a rational inner solution to the $(N + 1)$-point problem $\lambda_j \mapsto g(\lambda_j)$. Since ψ, in particular, solves the problem $\lambda_j \mapsto f(\lambda_j)$ for each $j = 1, 2, \ldots, N$, $\psi = f$ on $\mathcal{W} \cap \mathbb{G}$. But since $\psi(\lambda_{N+1}) = g(\lambda_{N+1}) \neq f(\lambda_{N+1})$, ψ is distinct from f. Since $\psi = f$ on $\mathcal{W} \cap \mathbb{G}$, by the Study Lemma there exists a rational function h such that $f - \psi = \xi h$, see [21, chapter 1]. Since ψ is inner,

$$1 = \|\psi\|_2^2 = \|f - \xi h\|_2^2 = \|f\|_2^2 - 2\,\mathrm{Re}\langle f, \xi h\rangle_{H^2} + \|\xi h\|_2^2.$$

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Since f is an inner function, $\|f\|_2 = 1$, and therefore the above computation leads to $2\operatorname{Re}\langle f, \xi h\rangle = \|\xi h\|_2^2$. This contradicts the hypothesis because $\xi h = f - \psi$ is bounded. Consequently, g must coincide with f on $\mathbb{G}$. □

One can easily find examples of distinguished varieties and regular rational inner functions such that the stringent hypothesis of the above result is satisfied.

Example 2.13. Let $f \circ \pi(z, w) = (zw)^d$ and $\mathcal{W} = Z(\xi)$ be such that

$$\xi \circ \pi(z, w) = (z^m - w^n)(z^n - w^m),$$

where m, n are mutually prime integers bigger than d. Then it follows that $\mathcal{W}$ is a distinguished variety with respect to $\mathbb{G}$ because $Z(z^m - w^n)$ is a distinguished variety with respect to $\mathbb{D}^2$. For concrete example, one can take $d = 1$ and $(m, n) = (2, 3)$ – the corresponding distinguished variety then is the Neil parabole. Note that the inner product $\langle , \rangle$ as defined in (2.8) can be expressed in terms of the inner product on the Hardy space of the bidisk $H^2(\mathbb{D}^2)$ as

$$\langle f, \xi h\rangle_{H^2(\mathbb{G})} = \frac{1}{\|J\|^2}\langle J(f \circ \pi), J\big((\xi \circ \pi)(h \circ \pi)\big)\rangle_{H^2(\mathbb{D}^2)}. \tag{2.9}$$

Let $h : \mathbb{G} \to \mathbb{C}$ be an analytic function such that $\|\xi h\|_2 < \infty$. Since $\{z^i w^j : i, j \geq 0\}$ forms an orthonormal basis for $H^2(\mathbb{D}^2)$, it is easy to read off from (2.9) that $\langle f, \xi h\rangle = 0$. Therefore, by Theorem 2.12, $\mathcal{W} \cap \mathbb{G}$ is a determining set for f as chosen above.

3. A bounded extension theorem

We end with a bounded extension theorem for distinguished varieties with no singularities on the distinguished boundary of Γ. Here singularity of an algebraic variety $Z(\xi)$ at a point means that both the partial derivatives of ξ vanish at that point. Note that the substance of the following theorem is not that there is a rational extension of every polynomial, it is that the supremum of the rational extension over $\mathbb{G}$ does not exceed the supremum of the polynomial over the variety intersected with $\mathbb{G}$ multiplied by a constant that only depends on the variety. See the papers [1, 23, 36] for similar results in other contexts.

Theorem 3.1. *Let $\mathcal{W}$ be a distinguished variety with respect to $\mathbb{G}$ such that it has no singularities on $b\Gamma$. Then for every polynomial $f \in \mathbb{C}[s, p]$, there exists a rational extension F of f such that*

$$|F(s, p)| \leq \alpha \sup_{\mathcal{W}\cap\mathbb{G}} |f|$$

for all $(s, p) \in \mathbb{G}$, where α is a constant depends only on $\mathcal{W}$.

Proof. Let $\mathcal{V}$ be a distinguished variety with respect to $\mathbb{D}^2$ such that $\mathcal{W} = \pi(\mathcal{V})$. Since $\mathcal{W}$ has no singularities on $b\Gamma$, it follows that $\mathcal{V}$ has no singularities on $\mathbb{T}^2$. Invoke Theorem 2.20 of [23] to obtain a rational extension G of the polynomial $f \circ \pi \in \mathbb{C}[z, w]$ such that

$$|G(z, w)| \leq \alpha \sup_{\mathcal{V}\cap\mathbb{D}^2} |f \circ \pi| \tag{3.1}$$

for all $(z,w) \in \mathbb{D}^2$, where α is a constant depends only on $\mathcal{V}$. Now, define a rational function H on $\mathbb{D}^2$ as follows

$$H(z,w) = \frac{G(z,w) + G(w,z)}{2}. \tag{3.2}$$

Clearly, H is also a rational extension of $f \circ \pi$ with

$$|H(z,w)| \leq \alpha \sup_{\mathcal{V}\cap\mathbb{D}^2} |f \circ \pi| \quad \text{for all } (z,w) \in \mathbb{D}^2.$$

Note that H is a symmetric rational function on $\mathbb{D}^2$. So, there is a rational function F on $\mathbb{G}$ such that

$$H(z,w) = (F \circ \pi)(z,w) = F(z+w, zw) \quad \text{for all } (z,w) \in \mathbb{D}^2.$$

Now we will show that this F will do our job. It is easy to see that F is a rational extension of f. Let $(s,p) \in \mathbb{G}$. Then there exists a point $(z,w) \in \mathbb{D}^2$ such that $(s,p) = (z+w, zw)$. Now,

$$|F(s,p)| = |(F \circ \pi)(z,w)| = |H(z,w)| \leq \alpha \sup_{\mathcal{V}\cap\mathbb{D}^2} |f \circ \pi| = \alpha \sup_{\mathcal{W}\cap\mathbb{G}} |f|.$$

This complete the proof. □

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