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The lexicographic method for the threshold cover problem ☆

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ABSTRACT

Threshold graphs are a class of graphs that have many equivalent definitions and have applications in integer programming and set packing problems. A graph is said to have a threshold cover of size k if its edges can be covered using k threshold graphs. Chvátal and Hammer, in 1977, defined the *threshold dimension* $\mathrm{th}(G)$ of a graph G to be the least integer k such that G has a threshold cover of size k and observed that $\mathrm{th}(G) \geq \chi(G^*)$, where G^* is a suitably constructed auxiliary graph. Raschle and Simon (1995) [9] proved that $\mathrm{th}(G) = \chi(G^*)$ whenever G^* is bipartite. We show how the lexicographic method of Hell and Huang can be used to obtain a completely new and, we believe, simpler proof for this result. For the case when G is a split graph, our method yields a proof that is much shorter than the ones known in the literature. Our methods give rise to a simple and straightforward algorithm to generate a 2-threshold cover of an input graph, if one exists.

1. Introduction

We consider only simple, undirected and finite graphs. We denote an edge between two vertices u and v of a graph by the two-element set $\{u, v\}$, which is usually abbreviated to just uv. Two edges ab, cd in a graph G are said to form an *alternating 4-cycle* if $ad, bc \in E(\overline{G})$. A graph G that does not contain any pair of edges that form an alternating 4-cycle is called a *threshold graph*; or equivalently, G is $(2K_2, P_4, C_4)$-free [1]. A graph $G = (V, E)$ is said to be *covered* by the graphs $H_1, H_2, \ldots, H_k$ if $E(G) = E(H_1) \cup E(H_2) \cup \cdots \cup E(H_k)$.

Definition 1 *(Threshold cover and threshold dimension).* A graph G is said to have a *threshold cover* of size k if it can be covered by k threshold graphs. The *threshold dimension* of a graph G, denoted as $\mathrm{th}(G)$, is defined to be the smallest integer k such that G has a threshold cover of size k.

Mahadev and Peled [8] give a comprehensive survey of threshold graphs and their applications.

Chvátal and Hammer [1] showed that the fact that a graph G has $\mathrm{th}(G) \leq k$ is equivalent to the following: there exist k linear inequalities on $|V(G)|$ variables such that the characteristic vector of a set $S \subseteq V(G)$ satisfies all the inequalities if

☆ A preliminary version of this paper, claiming the same result as proved in this work, appeared in the proceedings of the conference CALDAM 2020. But the proof in that version contains a serious error, and the algorithm mentioned in that paper may fail to produce a 2-threshold cover if the input graph contains a paraglider as an induced subgraph.

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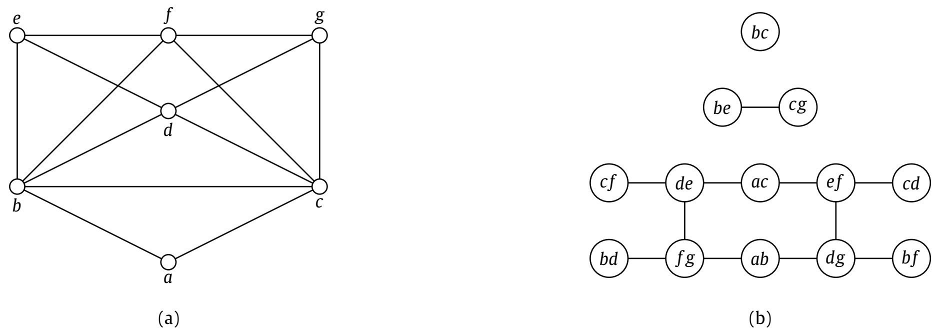

Fig. 1. (a) A graph G, and (b) the auxiliary graph G^* of G.

and only if S is an independent set of G (see [9] for details). They further defined the auxiliary graph G^* corresponding to a graph G as follows.

Definition 2 *(Auxiliary graph).* Given a graph G, the graph G^* has vertex set $V(G^*) = E(G)$ and edge set $E(G^*) = \{\{ab, cd\}: ab, cd \in E(G)$ such that ab, cd form an alternating 4-cycle in $G\}$.

A graph G and the auxiliary graph G^* corresponding to it is shown in Fig. 1. Chvátal and Hammer observed that since for any subgraph H of G that is a threshold graph, $E(H)$ is an independent set in G^*, the following lower bound on $\mathrm{th}(G)$ holds.

Lemma 1 *(Chvátal-Hammer).* $\mathrm{th}(G) \geq \chi(G^*)$.

This gave rise to the question of whether there is any graph G such that $\mathrm{th}(G) > \chi(G^*)$. Cozzens and Leibowitz [4] showed the existence of such graphs. In particular, they showed that for every $k \geq 4$, there exists a graph G such that $\chi(G^*) = k$ but $\mathrm{th}(G) > k$. While the question of whether such graphs exist for $k = 3$ does not seem to have received much attention and still remains open (to the best of our knowledge), the question for the case $k = 2$ was studied quite intensively (see [7]). Ibaraki and Peled [6] showed, by means of some very involved proofs, that if G is a split graph or if G^* contains at most two non-trivial components, then $\chi(G^*) = 2$ if and only if $\mathrm{th}(G) = 2$. They further conjectured that for any graph G, $\chi(G^*) = 2 \Leftrightarrow \mathrm{th}(G) = 2$. If the conjecture held, it would show immediately that graphs having a threshold cover of size 2 can be recognized in polynomial time, since the auxiliary graph G^* can be constructed and its bipartiteness checked in polynomial time. In contrast, Yannakakis [12] showed that it is NP-complete to recognize graphs having a threshold cover of size 3. Cozzens and Halsey [3] studied some properties of graphs having a threshold cover of size 2 and showed that it can be decided in polynomial time whether the complement of a bipartite graph has a threshold cover of size 2. Finally, more than a decade after the question was first posed, Raschle and Simon [9] proved the conjecture of Ibaraki and Peled by extending the methods in [6].

Theorem 1 *(Raschle-Simon). For any graph G, $\chi(G^*) = 2$ if and only if* $\mathrm{th}(G) = 2$.

This proof of Raschle and Simon is very technical and involves the use of a number of complicated reductions and previously known results. In this paper, we provide a new, and we believe, simpler, proof for Theorem 1.

We construct an algorithm that generates a 2-threshold cover of an input graph G if G^* is bipartite. In Sections 3 and 4, we describe the algorithm and its proof of correctness for the case when the input graph belongs to the class of "paraglider-free" graphs, which is a superclass of the class of chordal graphs. Since all split graphs are also chordal graphs, we have a proof of Theorem 1 for the case of split graphs in Section 4 itself. We believe that this proof for split graphs is much simpler and shorter than the proof for split graphs given by Ibaraki and Peled [6] (or the proof of Raschle and Simon [9] for general graphs that builds upon the work of Ibaraki and Peled). In Section 5, we show how our algorithm can be modified to work for general graphs. Note that for the case of general graphs, even though the algorithm remains simple, the proof of its correctness becomes more involved.

Outline of the algorithm. Let G be any graph such that G^* is bipartite. We would like to construct a threshold cover of size 2 for G. A natural way to approach the problem is to compute a 2-coloring of G^*, which corresponds to a partition of the edge set of G into two sets, say E_1 and E_2, and try to show that $G_1 = (V(G), E_1)$ and $G_2 = (V(G), E_2)$ are threshold graphs (and so $\{G_1, G_2\}$ is a 2-threshold cover of G). But this approach does not work since if we take an arbitrary 2-coloring of G^*, the graphs G_1 and G_2 need not necessarily be threshold graphs (this can be easily seen in the case when G is a complete graph, as then G^* contains only isolated vertices). Instead, our algorithm generates a special kind of 2-coloring of

G^*, which is then used to construct a 2-threshold cover of G. Let X denote the set of isolated vertices in G^*. Our algorithm computes a set $Y \subseteq X$ and a 2-coloring of $G^* - Y$ (we call this a "partial 2-coloring" of G^*) which partitions $E(G) \setminus Y$ into two sets E_1 and E_2 such that the graphs $G_1 = (V(G), E_1 \cup Y)$ and $G_2 = (V(G), E_2 \cup Y)$ are both threshold graphs, thereby yielding a 2-threshold cover of G. Note that as G^* need not be connected, even if $X = \emptyset$, there can be an exponential number of 2-colorings of G^* and as noted above, not every 2-coloring gives rise to a 2-threshold cover of G. The algorithm runs in time $O(|V(G^*)| + |E(G^*)|) = O(|E(G)|^2)$. In the case of split graphs, and more generally paraglider-free graphs, our algorithm does not need to process the isolated vertices in G^* at all; instead it just takes $Y = X$. In other words, it computes a 2-coloring of $G^* - X$ which partitions $E(G) \setminus X$ into two sets E_1 and E_2 such that the graphs $G_1 = (V(G), E_1 \cup X)$ and $G_2 = (V(G), E_2 \cup X)$ are both threshold graphs.

The Chain Subgraph Cover Problem. A bipartite graph $G = (A, B, E)$ is called a *chain graph* if it does not contain a pair of edges whose endpoints induce a $2K_2$ in G. Let $\hat{G}$ be the split graph obtained from G by adding edges between every pair of vertices in A (or B). It can be seen that G is a chain graph if and only if $\hat{G}$ is a threshold graph. A collection of chain graphs $\{H_1, H_2, \dots, H_k\}$ is said to be a *k-chain subgraph cover* of a bipartite graph G if it is covered by $H_1, H_2, \dots, H_k$. Yannakakis [12] credits Martin Golumbic for observing that a bipartite graph G has a k-chain subgraph cover if and only if $\hat{G}$ has a k-threshold cover. The problem of deciding whether a bipartite graph G can be covered by k chain graphs, i.e. whether G has a k-chain subgraph cover, is known as the *k-chain subgraph cover (k-CSC)* problem. Yannakakis [12] showed that 3-CSC is NP-complete, which implies that the problem of deciding whether $\mathrm{th}(G) \le 3$ for an input graph G is also NP-complete. He also pointed out that using Golumbic's observation and the results of Ibaraki and Peled [6], the 2-CSC problem can be solved in polynomial time, as it can be reduced to the problem of determining whether a split graph can be covered by two threshold graphs. Thus our algorithm for split graphs can also be used to compute a 2-chain subgraph cover, if one exists, for an input bipartite graph G in time $O(|E(G)|^2)$ (note that even though $|E(\hat{G})| > |E(G)|$, the vertices in $\hat{G}^*$ corresponding to the edges in $E(\hat{G}) \setminus E(G)$ are all isolated vertices and hence do not need to be put in G^* since our algorithm for split graphs does not take them into account anyway). Note that Ma and Spinrad [7] propose a more involved $O(|V(G)|^2)$ algorithm for the problem. However, our algorithm for split graphs, and hence the algorithm for computing a 2-chain subgraph cover that it yields, is considerably simpler to implement than the algorithms of [6,7,9,11].

Lex-BFS orderings. A *Lex-BFS* ordering of a graph is an ordering of the vertices of the graph having the property that it is possible for a *Lexicographic Breadth First Search (Lex-BFS)* algorithm to visit the vertices of the graph in that order. A Lex-BFS ordering is also a BFS ordering—i.e., a breadth-first search algorithm can also visit the vertices in that order—but it has some additional properties. Lex-BFS can be implemented to run in time linear in the size of the input graph and was introduced by Rose, Tarjan and Lueker [10] to construct a linear-time algorithm for recognizing chordal graphs. Later, Lex-BFS based algorithms were discovered for the recognition of many different graph classes (see [2] for a survey).

The Lexicographic Method. We use a technique called the *lexicographic method* introduced by Hell and Huang [5], who demonstrated how this method can lead to shorter proofs and simpler recognition algorithms for certain problems that involve constructing a specific 2-coloring of an auxiliary bipartite graph that captures certain relationships among the edges of the graph. The method involves fixing an ordering $<$ of the vertices of the graph, and then processing the edges in the "lexicographic order" implied by the ordering $<$. We adapt this technique to construct a partial 2-coloring of G^* that can be used to generate a 2-threshold cover of G. Hell and Huang [5] start with an arbitrary ordering of the vertices of the graph in their recognition algorithms for comparability graphs and proper circular-arc graphs, but for the case of proper interval graphs, they start with a "perfect elimination ordering" of the given graph, which should necessarily exist when the graph is chordal (note that proper interval graphs form a subclass of chordal graphs). From the work of Rose, Tarjan and Lueker [10], it is known that for chordal graphs, the perfect elimination orderings are exactly the reversals of *Lex-BFS orderings*. Thus the recognition algorithm for proper interval graphs based on the lexicographic method that is given in [5] starts with the reversal of a Lex-BFS ordering of the input graph. As we shall see, our recognition algorithm for graphs having a 2-threshold cover starts with a Lex-BFS order of the input graph. When it is known that the input graph is a "paraglider-free graph" (defined in Section 4), we can even start with an arbitrary ordering of the vertices of the input graph.

2. Preliminaries

Let $G = (V, E)$ be any graph. Recall that edges $ab, cd \in E(G)$ form an alternating 4-cycle if $bc, da \in E(\overline{G})$. In this case, we also say that a, b, c, d, a is an alternating 4-cycle in G (alternating 4-cycles are called AC_4s in [9]). The edges ab and cd are said to be the *opposite edges* of the alternating 4-cycle a, b, c, d, a. Thus for a graph G, the auxiliary graph G^* is the graph with $V(G^*) = E(G)$ and $E(G^*) = \{\{ab, cd\} \colon ab, cd \in E(G)$ are the opposite edges of an alternating 4-cycle in $G\}$. Note that it follows from the definition of an alternating 4-cycle that if a, b, c, d, a is an alternating 4-cycle, then the vertices a, b, c, d are pairwise distinct. We shall refer to the vertex of G^* corresponding to an edge $ab \in E(G)$ alternatively as $\{a, b\}$ or ab, depending upon the context.

Our goal is to provide a new proof for Theorem 1.

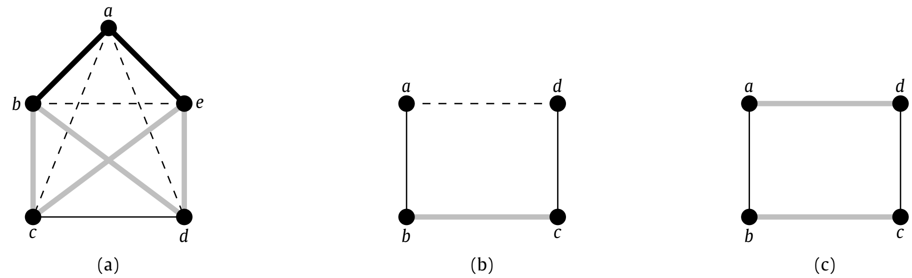

Fig. 2. (a) An A_1-pentagon, (b) an A_1-switching path, and (c) an A_1-switching cycle. In each of the figures: a dashed line between two vertices indicates that they are non-adjacent, a thin line represents an edge that may be in either A_1 or A_0, a thick line indicates an edge in A_1, and a gray line represents an edge in A_2.

It is easy to see that $\chi(G^*) = 1$ if and only if $\text{th}(G) = 1$. Therefore, by Lemma 1, it is enough to prove that if G^* is bipartite, then G can be covered by two threshold graphs. In order to prove this, we find a specific 2-coloring of the non-trivial components of G^* (components of size at least 2) using the lexicographic method of Hell and Huang [5].

We say that (A_0, A_1, A_2) is a *valid 3-partition* of $E(G)$ if $\{A_0, A_1, A_2\}$ is a partition of $E(G)$ with the property that in any alternating 4-cycle in G, one of the opposite edges belongs to A_1 and the other to A_2. In other words, for any edge $\{ab, cd\} \in E(G^*)$, one of ab, cd is in A_1 and the other in A_2. Note that this means while some edges in A_1 and A_2 may have the property that they do not form an alternating 4-cycle with any other edge, every edge in A_0 definitely has this property.

Given a valid 3-partition (A_0, A_1, A_2) of $E(G)$ and $A \in \{A_1, A_2\}$, we say that a, b, c, d is an *alternating A-path* if $a \neq d$, $ab, cd \in A \cup A_0$, and $bc \in E(\overline{G})$. Further, we say that a, b, c, d, e, f, a is an *alternating A-circuit* if $a \neq d$, $ab, cd, ef \in A \cup A_0$, and $bc, de, fa \in E(\overline{G})$.

Observation 1. *Let (A_0, A_1, A_2) be a valid 3-partition of $E(G)$ and let $\{A, \overline{A}\} = \{A_1, A_2\}$.*

(a) If a, b, c, d is an alternating A-path, then $ad \in E(G)$.
(b) If a, b, c, d, e, f, a is an alternating A-circuit, then $ef \in A$ and $ad \in \overline{A}$.

Proof. To prove (a), it just needs to be observed that if $ad \in E(\overline{G})$, then a, b, c, d, a would be an alternating 4-cycle in G whose opposite edges both belong to $A \cup A_0$, which contradicts the fact that (A_0, A_1, A_2) is a valid 3-partition of $E(G)$. To prove (b), suppose that a, b, c, d, e, f, a is an alternating A-circuit. Since a, b, c, d is an alternating A-path, we have by (a) that $ad \in E(G)$. Then since a, d, e, f, a is an alternating 4-cycle in G and $ef \in A \cup A_0$, it follows that $ef \in A$ and $ad \in \overline{A}$. □

We shall use the above observation throughout this paper without referring to it explicitly.

Let (A_0, A_1, A_2) be a valid 3-partition of $E(G)$ and let $\{A, \overline{A}\} = \{A_1, A_2\}$. We say that (a, b, c, d, e) is an *A-pentagon in G with respect to* (A_0, A_1, A_2) if $a, b, c, d, e \in V(G)$, $ac, ad, be \in E(\overline{G})$, $ab, ae \in A$, $bc, bd, ec, ed \in \overline{A}$ and $cd \in A \cup A_0$. We abbreviate this to just "A-pentagon" when the graph G and the 3-partition (A_0, A_1, A_2) of G are clear from the context. We say that an A-pentagon (a, b, c, d, e) is a *strict A-pentagon* if $cd \in A$. We say that (a, b, c, d, e) is a *pentagon* (resp. *strict pentagon*) if it is an A-pentagon (resp. strict A-pentagon) for some $A \in \{A_1, A_2\}$. (Pentagons are similar to the "AP_5-s" in [9].)

We say that (a, b, c, d) is an *A-switching path in G with respect to* (A_0, A_1, A_2) if $a, b, c, d \in V(G)$, $ad \in E(\overline{G})$, $ab, cd \in A \cup A_0$, and $bc \in \overline{A}$. When the graph G and the 3-partition (A_0, A_1, A_2) of G are clear from the context, we abbreviate this to just "A-switching path". We say that (a, b, c, d) is a *strict A-switching path* if it is an A-switching path and in addition, $ab, cd \in A$. We say that (a, b, c, d) is a *switching path* (resp. *strict switching path*) if it is an A-switching path (resp. strict A-switching path) for some $A \in \{A_1, A_2\}$. We say that (a, b, c, d) is an *A-switching cycle* in G with respect to (A_0, A_1, A_2) if $ab, cd \in A \cup A_0$ and $bc, ad \in \overline{A}$. As before, we say that (a, b, c, d) is a *switching cycle* in G with respect to (A_0, A_1, A_2) if there exists $A \in \{A_1, A_2\}$ such that (a, b, c, d) is an A-switching cycle.

Note that from the definitions of pentagons, switching paths and switching cycles, it follows that if (a, b, c, d, e) is a pentagon, then the vertices a, b, c, d, e are pairwise distinct, and if (a, b, c, d) is a switching path or a switching cycle, then the vertices a, b, c, d are pairwise distinct. Fig. 2 illustrates an A_1-pentagon, an A_1-switching path, and an A_1-switching cycle.

Lemma 2. *Let (A_0, A_1, A_2) be a valid 3-partition of $E(G)$. If there are no switching paths and no switching cycles in G with respect to (A_0, A_1, A_2) then* $\text{th}(G) = 2$.

Proof. Consider the graphs H_1, H_2, having $V(H_1) = V(H_2) = V(G)$, $E(H_1) = A_1 \cup A_0$ and $E(H_2) = A_2 \cup A_0$. We claim that H_1 and H_2 are both threshold graphs. Suppose for the sake of contradiction that H_i is not a threshold graph for some

$i \in \{1, 2\}$. Then there exist edges $ab, cd \in E(H_i)$ such that $bc, ad \in E(\overline{H_i})$. If $bc, ad \in E(\overline{G})$, then a, b, c, d, a is an alternating 4-cycle in G whose opposite edges both belong to $A_i \cup A_0$, which contradicts the fact that (A_0, A_1, A_2) is a valid 3-partition. So we can assume by symmetry that $bc \in E(G)$. Since $bc \in E(\overline{H_i})$, $bc \notin A_i \cup A_0$, which implies that $bc \in A_{3-i}$. Now if $ad \in E(\overline{G})$, then (a, b, c, d) is an A_i-switching path in G with respect to (A_0, A_1, A_2), which is a contradiction. On the other hand, if $ad \in E(G)$, then $ad \in A_{3-i}$ (since $ad \in E(\overline{H_i})$), which implies that (a, b, c, d) is an A_i-switching cycle in G with respect to (A_0, A_1, A_2), which is again a contradiction. Thus we can conclude that both H_1 and H_2 are threshold graphs. □

Lemma 3. *Let (A_0, A_1, A_2) be a valid 3-partition of $E(G)$. Let $\{A, \overline{A}\} = \{A_1, A_2\}$. Let (x, y, z, w) be an A-switching path in G and let $y'z' \in E(G)$ be such that $yz', zy' \in E(\overline{G})$. Then,*

(*a*) *if $x = y'$, then $(x = y', y, z, w, z')$ is an A-pentagon and*
(*b*) *if $w = z'$, then $(w = z', z, y, x, y')$ is an A-pentagon.*

Proof. Since $yz \in \overline{A}$ and $\{yz, y'z'\} \in E(G^*)$ we have that $y'z' \in A$. Suppose that $x = y'$. Then $y, (x = y'), z, w, (x = y'), z', y$ is an alternating A-circuit (note that $y \neq w$ as $x \in N(y) \setminus N(w)$), implying that $yw \in \overline{A}$. This further implies that $z' \neq w$. Then we also have alternating A-circuits z', y', z, w, x, y, z' and $z', (y' = x), w, z, (y' = x), y, z'$, implying that $xy \in A$ and $z'w, z'z \in \overline{A}$. Consequently, $(x = y', y, z, w, z')$ is an A-pentagon. Since (w, z, y, x) is also an A-switching path, we can similarly conclude that if $w = z'$, then $(w = z', z, y, x, y')$ is an A-pentagon. □

We then have the following corollary.

Corollary 1. *Let (A_0, A_1, A_2) be a valid 3-partition of $E(G)$. Suppose that there are no pentagons (resp. strict pentagons) in G with respect to (A_0, A_1, A_2). Let (x, y, z, w) be a switching path (resp. strict switching path) with respect to (A_0, A_1, A_2). Let $y'z' \in E(G)$ be such that $yz', zy' \in E(\overline{G})$. Then, $y' \neq x$ and $z' \neq w$.*

Proof. Let $\{A, \overline{A}\} = \{A_1, A_2\}$. Suppose that there are no pentagons (resp. strict pentagons) in G with respect to (A_0, A_1, A_2). Let (x, y, z, w) be an A-switching path (resp. strict A-switching path, and therefore $xy, zw \in A$). By Lemma 3, we know that if $y' = x$ then $(x = y', y, z, w, z')$ is an A-pentagon (resp. a strict A-pentagon, as $zw \in A$), and if $z' = w$ then $(w = z', z, y, x, y')$ is an A-pentagon (resp. a strict A-pentagon, as $xy \in A$). Since there are no pentagons (resp. strict pentagons), we can conclude that $y' \neq x$ and $z' \neq w$. □

Let $<$ be an ordering of the vertices of G. Given two k-element subsets $S = \{s_1, s_2, \ldots, s_k\}$ and $T = \{t_1, t_2, \ldots, t_k\}$ of $V(G)$, where $s_1 < s_2 < \cdots < s_k$ and $t_1 < t_2 < \cdots < t_k$, S is said to be *lexicographically smaller* than T, denoted by $S < T$, if $s_j < t_j$ for some $j \in \{1, 2, \ldots, k\}$, and $s_i = t_i$ for all $1 \leq i < j$. In the usual way, we let $S \leq T$ denote the fact that either $S < T$ or $S = T$. For a set $S \subseteq V(G)$, we abbreviate $\min_< S$ to just $\min S$. Note that the relation $<$ ("is lexicographically smaller than") that we have defined on k-element subsets of $V(G)$ is a total order. Therefore, given a collection of k-element subsets of $V(G)$, the lexicographically smallest one among them is well-defined.

The following observation states a well-known property of Lex-BFS orderings [2].

Observation 2. *Let $<$ denote a Lex-BFS ordering of a graph G. For $a, b, c \in V(G)$, if $a < b < c$, $ab \notin E(G)$ and $ac \in E(G)$, then there exists $x \in V(G)$ such that $x < a < b < c$, $xb \in E(G)$ and $xc \notin E(G)$.*

3. The algorithm

Let G be a graph such that G^* is bipartite. For two vertices $ab, a'b' \in V(G^*)$ (i.e. $ab, a'b' \in E(G)$), we say that ab is lexicographically smaller than $a'b'$ with respect to an ordering $<$ of $V(G)$, if $\{a, b\} < \{a', b'\}$.

We shall now construct a partial 2-coloring of the vertices of G^* using the colors $\{1, 2\}$ by means of an algorithm, and then construct a valid 3-partition of $E(G)$ using this partial 2-coloring.

Phase I. Construct a Lex-BFS ordering $<$ of G.

Recall that every vertex of G^* is a two-element subset of $V(G)$.

Phase II. For every non-trivial component C of G^*, perform the following operation:

> Choose the lexicographically smallest vertex in C (with respect to the ordering $<$) and assign the color 1 to it. Extend this to a proper coloring of C using the colors $\{1, 2\}$.

Note that after Phase II, every vertex of G^* that is in a non-trivial component has been colored either 1 or 2. For $i \in \{1, 2\}$, let $F_i = \{e \in V(G^*) \colon e \text{ is colored } i\}$. Further, let F_0 denote the set of all isolated vertices (trivial components) in

G^*. Clearly, F_0 is exactly the set of uncolored vertices of G^* and we have $V(G^*) = F_0 \cup F_1 \cup F_2$. Note that since the opposite edges of any alternating 4-cycle in G correspond to adjacent vertices in G^*, one of them receives color 1 and the other color 2 in the partial 2-coloring of G^* constructed in Phase II. It follows that (F_0, F_1, F_2) is a valid 3-partition of $E(G)$.

First we note the following lemma.

Lemma 4. *If there are no strict pentagons in G with respect to (F_0, F_1, F_2), then there are no strict switching paths in G with respect to (F_0, F_1, F_2).*

Proof. Suppose that G contains a strict switching path. We say that a strict switching path (a, b, c, d) is lexicographically smaller than a strict switching path (a', b', c', d') if $\{a, b, c, d\} < \{a', b', c', d'\}$. Let (a, b, c, d) be the lexicographically smallest strict switching path in G.

Claim. *(a, b, c, d) is not a strict F_1-switching path.*

Suppose for the sake of contradiction that (a, b, c, d) is a strict F_1-switching path. Let C be the component of G^* containing bc. Let $b_0c_0, b_1c_1, \ldots, b_kc_k$, where $b_0 = b$ and $c_0 = c$, be a path in C between bc and the lexicographically smallest vertex b_kc_k in C. We assume that for each $i \in \{0, 1, \ldots, k-1\}$, $b_ic_{i+1}, c_ib_{i+1} \in E(\overline{G})$. As $b_0c_0 \in F_2$, it follows that $b_ic_i \in F_2$ for each even i and $b_ic_i \in F_1$ for each odd i. Since b_kc_k is the lexicographically smallest vertex in its component in G^*, we know that $b_kc_k \in F_1$, which implies that k is odd.

We claim that $b_ia, c_id \in F_1$ for each even i and $b_ia, c_id \in F_2$ for each odd i, where $0 \le i \le k$. We prove this by induction on i. The case where $i = 0$ is trivial as $b_0 = b$ and $c_0 = c$. So let us assume that $i > 0$. Consider the case where i is odd. As $i-1$ is even, by the induction hypothesis we have $b_{i-1}a, c_{i-1}d \in F_1$. Since $b_{i-1}c_{i-1} \in F_2$, we can observe that, (a, b_{i-1}, c_{i-1}, d) is a strict F_1-switching path. Then by Corollary 1, we have that $a \ne b_i$ and $d \ne c_i$. Now the alternating F_1-circuits $b_i, c_i, b_{i-1}, a, d, c_{i-1}, b_i$ and $c_i, b_i, c_{i-1}, d, a, b_{i-1}, c_i$ imply that $b_ia, c_id \in F_2$. The case where i is even is symmetric and hence the claim.

By the above claim, $b_ka, c_kd \in F_2$. Since $b_kc_k \in F_1$, we now have that (a, b_k, c_k, d) is a strict F_2-switching path. Since $b_kc_k < bc$, we have that $\{a, b_k, c_k, d\} < \{a, b, c, d\}$, which is a contradiction to our assumption that (a, b, c, d) is the lexicographically smallest strict switching path in G. This proves the claim.

By the above claim, we have that (a, b, c, d) is a strict F_2-switching path. By the symmetry between a and d, we can assume without loss of generality that $a < d$.

As $bc \in F_1$, the vertex bc belongs to a non-trivial component of G^*. Then there exists a neighbor uv of bc in G^* such that $bv, uc \in E(\overline{G})$. As $bc \in F_1$, we have $uv \in F_2$. By Corollary 1, we have that $u \ne a$. Then a, b, v, u, c, d, a is an alternating F_2-circuit, implying that $au \in F_1$. As $ab \in F_2$, we know that ab is not the lexicographically smallest vertex in its component. Let $a_0b_0, a_1b_1, \ldots, a_kb_k$ be a path in G^* between ab and the lexicographically smallest vertex a_kb_k in its component, where $a_0 = a$, $b_0 = b$, and for $0 \le i < k$, $a_ib_{i+1}, a_{i+1}b_i \in E(\overline{G})$. Note that for $0 \le i \le k$, $a_ib_i \in F_2$ if i is even and $a_ib_i \in F_1$ if i is odd. Since $a_kb_k \in F_1$ (as it is the lexicographically smallest vertex in its component in G^*), this implies that k is odd.

We claim that for $0 \le i \le k$, $a_iu, b_ic \in F_1$ if i is even and $a_iu, b_ic \in F_2$ if i is odd. We prove this by induction on i. The base case when $i = 0$ is trivial, since $au, bc \in F_1$. Let $i > 0$ be odd. By the induction hypothesis we have that $a_{i-1}u, b_{i-1}c \in F_1$. Since $a_{i-1}b_{i-1} \in F_2$ we can observe that (u, a_{i-1}, b_{i-1}, c) is a strict F_1-switching path. Therefore by Corollary 1, we have that $a_i \ne u$ and $b_i \ne c$. Then we have alternating F_1-circuits $a_i, b_i, a_{i-1}, u, c, b_{i-1}, a_i$ and $b_i, a_i, b_{i-1}, c, u, a_{i-1}, b_i$, implying that $a_iu, b_ic \in F_2$. The case when i is even is symmetric. This proves our claim. Since k is odd, we now have that $a_ku, b_kc \in F_2$. Note that now (c, b_k, a_k, u) is a strict F_2-switching path.

Suppose that $d < b$. Then we have that $a < d < b$, where $ad \in E(\overline{G})$ and $ab \in E(G)$. Therefore by Observation 2, there exists $x < a$ such that $xd \in E(G)$ and $xb \in E(\overline{G})$. Then x, d, a, b, x is an alternating 4-cycle in which $ab \in F_2$, implying that $xd \in F_1$. Then we have a strict F_1-switching path (x, d, c, b) such that $\{x, d, c, b\} < \{a, b, c, d\}$, which is a contradiction to the choice of (a, b, c, d). Therefore we can assume that $b < d$. Since $a_kb_k < ab$ and $a, b < d$, we have that $\{c, b_k, a_k, u\} < \{a, b, c, d\}$. As (c, b_k, a_k, u) is a strict switching path, this contradicts the choice of (a, b, c, d). □

4. Proof of Theorem 1 for split graphs

Let G be any graph such that G^* is bipartite. Let (F_0, F_1, F_2) be a valid 3-partition of $E(G)$ obtained by running the algorithm of Section 3 on the graph G.

Lemma 5. *If there are no pentagons in G with respect to (F_0, F_1, F_2), then there are no switching paths or switching cycles in G with respect to (F_0, F_1, F_2).*

Proof. Suppose not. Let (a, b, c, d) be a switching path in G with respect to (F_0, F_1, F_2). Let $i \in \{1, 2\}$ such that (a, b, c, d) is an F_i-switching path. Then we have $ad \in E(\overline{G})$, $ab, cd \in F_i \cup F_0$, and $bc \in F_{3-i}$. Since $bc \in F_{3-i}$, there exists $uv \in E(G)$

such that $bv, cu \in E(\overline{G})$. Since there are no pentagons in G, by Corollary 1 we have that $a \neq u$ and $d \neq v$. Notice that as $bc \in F_{3-i}$ and b, c, u, v, b is an alternating 4-cycle, we have $uv \in F_i$. Then d, c, u, v, b, a, d and a, b, v, u, c, d, a are alternating F_i-circuits, implying that $dv, au \in F_{3-i}$ and $ab, cd \in F_i$. This further implies that (a, b, c, d) is a strict F_i-switching path which is a contradiction to Lemma 4. This proves that there are no switching paths in G with respect to (F_0, F_1, F_2).

Suppose that (a, b, c, d) is a switching cycle in G with respect to (F_0, F_1, F_2). Let $i \in \{1, 2\}$ such that (a, b, c, d) is an F_i-switching cycle. Then we have $ab, cd \in F_i \cup F_0$ and $ad, bc \in F_{3-i}$. As $bc \in F_{3-i}$, there exists $uv \in E(G)$ such that $bv, cu \in E(\overline{G})$. Since b, c, u, v, b is an alternating 4-cycle and $bc \in F_{3-i}$, we have that $uv \in F_i$. If $u = a$ and $v = d$, then $b, (a = u), c, (d = v), b$ is an alternating 4-cycle in which both the opposite edges belong to $F_i \cup F_0$, which is a contradiction. Therefore, either $u \neq a$ or $v \neq d$. Because of symmetry, we can assume without loss of generality that $u \neq a$ (by renaming (a, b, c, d) as (d, c, b, a) and interchanging the labels of u and v if necessary). Then a, b, v, u is an alternating F_i-path, implying that $au \in E(G)$. If $au \in F_i \cup F_0$ then (c, d, a, u) is an F_i-switching path, and if not, then $au \in F_{3-i}$, in which case (b, a, u, v) is an F_i-switching path. In both cases, we have a contradiction to our observation above that there are no switching paths in G with respect to (F_0, F_1, F_2). □

Corollary 2. *If there are no pentagons in G with respect to (F_0, F_1, F_2), then* $\mathrm{th}(G) \leq 2$.

Proof. The proof follows from Lemma 2 and Lemma 5. □

A *paraglider* is the graph $\overline{P_3 \cup K_2}$. Note that the subgraph induced by the vertices of a pentagon in a graph is a paraglider. A graph is said to be *paraglider-free* if it contains no induced subgraph isomorphic to a paraglider. Thus, paraglider-free graphs cannot contain any pentagons with respect to any valid 3-partition its edge set. We then have the following theorem from Lemma 1 and Corollary 2.

Theorem 2. *If G is a paraglider-free graph, then $\chi(G^*) \leq 2$ if and only if* $\mathrm{th}(G) \leq 2$.

A graph $G = (X, Y, E)$ is said to be a *split graph* if X is a clique in G, Y is an independent set in G and $V(G) = X \cup Y$. It is also known that split graphs are precisely $(2K_2, C_4, C_5)$-free graphs. As the paraglider contains an induced C_4, split graphs are paraglider-free.

Corollary 3. *If G is a split graph, then $\chi(G^*) \leq 2$ if and only if* $\mathrm{th}(G) \leq 2$.

Ibaraki and Peled [6] were the first to show that if G is a split graph, then G has a 2-threshold cover if and only if G^* is bipartite. We believe that our proof of Theorem 1 for the case of split graphs is much simpler than the proofs in [6] or [9].

Note. Suppose that G is a split graph such that G^* is bipartite. Then clearly by the proof of Theorem 2, the algorithm from Section 3 can be used to obtain two threshold graphs that cover G. In fact, as we show below, for the case of split graphs we can additionally also skip Phase I of our algorithm.

Let $G = (X, Y, E)$ be a split graph such that G^* is bipartite. We start with an arbitrary ordering $<$ of the vertices of G, and once we get the valid 3-partition (F_0, F_1, F_2) after running Phase II of the algorithm, we can output $H_1 = (V(G), F_1 \cup F_0)$ and $H_2 = (V(G), F_2 \cup F_0)$ as the two threshold graphs that form a 2-threshold cover of G. We follow the same proof as the one for paraglider-free graphs, with the only change being made to the last paragraph of the proof of Lemma 4, where Observation 2 is used (note that Observation 2 no longer holds as $<$ is not necessarily a Lex-BFS ordering of G). We replace this paragraph with the following:

> Recall that $a_0b_0, a_1b_1, \ldots, a_kb_k$ is a path in G^*, such that for any $i \in \{0, 1, \ldots, k-1\}$, $a_ib_{i+1} \in E(\overline{G})$ and $b_ia_{i+1} \in E(\overline{G})$. Let $i \in \{0, 1, \ldots, k-1\}$. If a_i and b_{i+1} both belong to one of X or Y, then it should be the case that $a_i, b_{i+1} \in Y$ (recall that X is a clique in G). Since $a_ib_i, a_{i+1}b_{i+1} \in E(G)$ and Y is an independent set in G, we then have $b_i, a_{i+1} \in X$. Since X is a clique, this contradicts the fact that $b_ia_{i+1} \in E(\overline{G})$. Therefore we can conclude that for each $i \in \{0, 1, \ldots, k-1\}$, one of a_i, b_{i+1} belongs to X and the other to Y. By the same argument, we can also show that for each $i \in \{0, 1, \ldots, k-1\}$, one of b_i, a_{i+1} belongs to X and the other to Y. Since k is odd, it now follows that one of $(a = a_0), b_k$ belongs to X and the other to Y, and similarly, one of $(b = b_0), a_k$ belongs to X and the other to Y. We can therefore conclude that $a \neq b_k$ and $b \neq a_k$. Recall that $a_kb_k < ab$, $a < d$, (c, b_k, a_k, u) is a strict F_2-switching path, and $a_iu, b_ic \in F_1$ (resp. $a_iu, b_ic \in F_2$) for each even i (resp. odd i). Then we have $a_ku, b_kc \in F_2$ and $au, bc \in F_1$, which implies that $a_k \neq a$ and $b_k \neq b$. We now have $\{a, b\} \cap \{a_k, b_k\} = \emptyset$, and therefore $\min\{a_k, b_k\} < \min\{a, b\}$. But then as $a < d$, we have $\{c, b_k, a_k, u\} < \{a, b, c, d\}$, which is a contradiction to the choice of (a, b, c, d).

Thus, the algorithm to construct a 2-threshold cover of a split graph (whose auxiliary graph is bipartite) does not even require a Lex-BFS to be run on the input graph. We believe that this algorithm for generating a 2-threshold cover of a split graph, if one exists, is much simpler and more direct than algorithms for this task that are known in the literature (see [6], [9]). Also, as noted before, this algorithm can be easily adapted to make a simple and straightforward algorithm that constructs a 2-chain subgraph cover for an input bipartite graph, if one exists.

5. Proof of Theorem 1 for general graphs

Let G be any graph such that G^* is bipartite. Let (F_0, F_1, F_2) be a valid 3-partition of $E(G)$ that we obtained at the end of Phase I and Phase II as defined before. In order to extend our approach to general graphs, first we shall prove the following lemma.

Lemma 6. *There are no strict pentagons in G (with respect to (F_0, F_1, F_2)).*

5.1. Proof of Lemma 6

Let $\{F, \overline{F}\} = \{F_1, F_2\}$. Let (a, b, c, d, e) be a strict F-pentagon and $c_0d_0, c_1d_1, \ldots, c_kd_k$ be a path in G^*, where $c_0 = c$, $d_0 = d$, $k \geq 0$, and for each $i \in \{0, 1, \ldots, k-1\}$, $c_id_{i+1}, d_ic_{i+1} \in E(\overline{G})$. Since $cd = c_0d_0 \in F$, it follows that $c_id_i \in F$ for all even i and $c_id_i \in \overline{F}$ for all odd i.

Observation 3. *For each $i \in \{0, 1, \ldots, k\}$, the edges c_ib, c_ie, d_ib, d_ie exist and they belong to F when i is odd and to $\overline{F}$ when i is even.*

Proof. We prove this by induction on i. This is easily seen to be true when $i = 0$. Suppose that $i > 0$. We shall assume without loss of generality that i is odd as the other case is symmetric. Then by the induction hypothesis, $c_{i-1}b, d_{i-1}b, c_{i-1}e, d_{i-1}e \in \overline{F}$. Then $c_i, d_i, c_{i-1}, b, e, d_{i-1}, c_i$ is an alternating $\overline{F}$-circuit (note that $c_i \neq b$ as $d_{i-1} \in N(b) \setminus N(c_i)$), implying that $c_ib \in F$. By symmetric arguments, we get $c_ie, d_ib, d_ie \in F$. □

Remark 1. By the above observation, we have that:

(a) for each $i \in \{0, 1, \ldots, k\}$, $c_i, d_i \notin \{b, e\}$,
(b) if $\{c_i, d_i\} \cap \{c_j, d_j\} \neq \emptyset$ for some $0 \leq i, j \leq k$, then $i \equiv j \mod 2$, and
(c) for each even $i \in \{0, 1, \ldots, k\}$, we have $a \notin \{c_i, d_i\}$.

Observation 4. *If $c_1 \neq a$, then (d, b, c_1, a, e) is a strict $\overline{F}$-pentagon. Similarly, if $d_1 \neq a$, then (c, b, d_1, a, e) is a strict $\overline{F}$-pentagon.*

Proof. By Observation 3, we have $c_1b, c_1e, d_1b, d_1e \in F$. Suppose that $c_1 \neq a$. Then c_1, b, e, a, c, d, c_1 is an alternating F-circuit, and therefore we have that $ac_1 \in \overline{F}$. It now follows that (d, b, c_1, a, e) is a strict $\overline{F}$-pentagon. By similar arguments, it can be seen that if $d_1 \neq a$, then $ad_1 \in \overline{F}$ and therefore (c, b, d_1, a, e) is a strict $\overline{F}$-pentagon. □

Observation 5. *Let $S_0 = \{a, c_0, d_0\}$ and for $1 \leq i \leq k$, let $S_i = S_{i-1} \cup \{c_i, d_i\}$. Let $i \in \{0, 1, \ldots, k\}$. For each $z \in \{c_i, d_i\}$, there exist $x_z, y_z \in S_i$ such that (x_z, b, y_z, z, e) is a strict F-pentagon when i is even and a strict $\overline{F}$-pentagon when i is odd.*

Proof. We are given an $i \in \{0, 1, \ldots, k\}$ and a vertex z that is either c_i or d_i. First let us consider the case when $z = a$. Since $z \in \{c_i, d_i\}$, we have by Remark 1(c), that i is odd, which implies that $i \geq 1$. Note that we have either $c_1 \neq a$ or $d_1 \neq a$. If $c_1 \neq a$, we define $x_z = d$, $y_z = c_1$ and if $d_1 \neq a$, we define $x_z = c$, $y_z = d_1$. Clearly, $x_z, y_z \in S_1 \subseteq S_i$, since $i \geq 1$. By Observation 4, we get that (x_z, b, y_z, z, e) is a strict $\overline{F}$-pentagon, and so we are done. Therefore, we shall now assume that $z \neq a$.

We shall now prove the statement of the observation by induction on i. Clearly, when $i = 0$, $z \in \{c_0, d_0\}$, so we can choose $x_z = a$, $y_z \in \{c, d\} \setminus \{z\}$ such that (x_z, b, y_z, z, e) is a strict F-pentagon (note that $x_z, y_z \in S_0$ as required). So let us assume that $i \geq 1$. If $z \in \{c_j, d_j\}$ for some $j < i$, then by Remark 1(b) we have that $j \equiv i \mod 2$ and by the induction hypothesis applied to j and z, there exist $x_z, y_z \in S_j \subseteq S_i$ (as $j < i$) such that (x_z, b, y_z, z, e) is a strict F-pentagon if i is even and a strict $\overline{F}$-pentagon if i is odd, completing the proof. Therefore, we assume that there is no $j < i$ such that $z \in \{c_j, d_j\}$. Since we have already assumed that $z \neq a$, we now have $z \notin S_{i-1}$.

Observe that there exists $z' \in \{c_{i-1}, d_{i-1}\}$ such that $z'z \in E(\overline{G})$. Then by the induction hypothesis, there exist $x_{z'}, y_{z'} \in S_{i-1}$ such that $(x_{z'}, b, y_{z'}, z', e)$ is a strict F-pentagon if $i-1$ is even and a strict $\overline{F}$-pentagon if $i-1$ is odd. Define $x_z = z'$ and $y_z = x_{z'}$. Then we have $x_z, y_z \in S_{i-1} \subseteq S_i$. Since $y_z \in S_{i-1}$ and $z \notin S_{i-1}$, we also have that $y_z \neq z$. Using Observation 3 and the fact that $(x_{z'}, b, y_{z'}, z', e)$ is a strict F-pentagon (resp. $\overline{F}$-pentagon) if i is odd (resp. even), we now have that $(y_z = x_{z'}), b, e, z, z', y_{z'}, (x_{z'} = y_z)$ is an alternating F-circuit (resp. $\overline{F}$-circuit). Therefore, $y_zz \in \overline{F}$ if i is odd and $y_zz \in F$ if i is even. Consequently we get that (x_z, b, y_z, z, e) is a strict F-pentagon when i is even and a strict $\overline{F}$-pentagon when i is odd. □

It is easy to see that Observation 5 implies the following.

Remark 2. Let $\{F, \overline{F}\} = \{F_1, F_2\}$ and let (a, b, c, d, e) be any strict F-pentagon in G with respect to (F_0, F_1, F_2). Let $c'd'$ be a vertex in the same component as cd in G^*. Then for each $z \in \{c', d'\}$, there exist $x_z, y_z \in V(G)$ such that (x_z, b, y_z, z, e) is a strict F-pentagon if $c'd' \in F$ and a strict $\overline{F}$-pentagon if $c'd' \in \overline{F}$.

Suppose that there is at least one strict pentagon in G with respect to (F_0, F_1, F_2). We say that a pentagon (a, b, c, d, e) is lexicographically smaller than a pentagon (a', b', c', d', e') if $\{a, b, c, d, e\} < \{a', b', c', d', e'\}$. Consider the lexicographically smallest strict pentagon (a, b, c, d, e) in G. Let $\{F, \overline{F}\} = \{F_1, F_2\}$ such that (a, b, c, d, e) is a strict F-pentagon. Since $cd \in F$, it belongs to a non-trivial component C of G^*. Therefore, there exists $uv \in E(G)$ such that $cv, du \in E(\overline{G})$ (so that $\{cd, uv\} \in E(G^*)$). Clearly, at least one of u, v is distinct from a. We assume without loss of generality that $u \neq a$ (by interchanging the labels of c and d if necessary). By applying Observation 4 to the path $(c_0d_0 = cd), (c_1d_1 = uv)$ in G^*, we get that (d, b, u, a, e) is a strict $\overline{F}$-pentagon, which implies that $au \in \overline{F}$. By Observation 3 applied to the same path, we get that $ub, ue \in F$.

Observation 6. $a > \min\{c, d\}$.

Proof. Suppose for the sake of contradiction that $a < \min\{c, d\}$. If $u < c$, then (d, b, u, a, e) is a strict $\overline{F}$-pentagon that is lexicographically smaller than (a, b, c, d, e), which is a contradiction. So we can assume that $c < u$, which gives us $a < c < u$. As $ac \in E(\overline{G})$ and $au \in E(G)$, by Observation 2, there exists a vertex x such that $x < a < c < u$, $xc \in E(G)$ and $xu \in E(\overline{G})$. Since a, u, x, c, a is an alternating 4-cycle in which $au \in \overline{F}$, we have that $xc \in F$. Then b, a, c, x, u, e, b is an alternating F-circuit (note that $b \neq x$ as $u \in N(b) \setminus N(x)$), and therefore $xb \in \overline{F}$. Symmetrically, we also get that $xe \in \overline{F}$. Then d, b, e, x, u, a, d is an alternating $\overline{F}$-circuit (note that $x \neq d$ as $x < a < \min\{c, d\}$), and therefore we have $xd \in F$. Now (u, b, x, d, e) is a strict F-pentagon that is lexicographically smaller than (a, b, c, d, e), which is a contradiction. □

Let $c'd'$ be the lexicographically smallest vertex in C.

Observation 7. $\min\{c', d'\} = \min\{c, d\}$.

Proof. We know that $c'd' \leq cd$, and therefore $\min\{c', d'\} \leq \min\{c, d\}$. Suppose that $z = \min\{c', d'\} < \min\{c, d\}$. From Remark 2, we have that for each $z \in \{c', d'\}$, there exist vertices $x_z, y_z \in V(G)$ such that (x_z, b, y_z, z, e) is a strict pentagon. Since $a > \min\{c, d\}$ by Observation 6, we have $a > z$. Then (x_z, b, y_z, z, e) is a lexicographically smaller strict pentagon than (a, b, c, d, e) which is a contradiction. □

Observation 8. $a > \max\{c, d\}$.

Proof. Let $\{y, \overline{y}\} = \{c, d\}$ such that $y < \overline{y}$. By Observation 6 it is now enough to show that $y < a < \overline{y}$ is not possible. Since $ya \in E(\overline{G})$ and $y\overline{y} \in E(G)$, $y < a < \overline{y}$ implies by Observation 2 that there exists $x < y$ such that $xa \in E(G)$ but $x\overline{y} \in E(\overline{G})$. Then $x, a, y, \overline{y}, x$ is an alternating 4-cycle, and therefore xa and $y\overline{y} = cd$ belong to the same component C of G^*. Thus $c'd' \leq xa$, which implies that $\min\{c', d'\} \leq \min\{x, a\}$. Since $\min\{x, a\} = x < y = \min\{c, d\}$, we now have $\min\{c', d'\} \leq \min\{x, a\} < \min\{c, d\}$. This contradicts Observation 7. □

Since $c'd'$ is the lexicographically smallest vertex in C, our algorithm would have colored it with the color 1. Therefore, we have $c'd' \in F_1$. Consider a path $c_0d_0, c_1d_1, \ldots, c_kd_k$ in G^*, where $c_0 = c$, $d_0 = d$, $c_k = c'$ and $d_k = d'$, in which for each $i \in \{0, 1, \ldots, k-1\}$, $c_id_{i+1}, d_ic_{i+1} \in E(\overline{G})$. Suppose that $cd \in F_2$. Then since $c_kd_k = c'd' \in F_1$, we have that k is odd. Now by Remark 1(b), we have that $\{c_0, d_0\} \cap \{c_k, d_k\} = \emptyset$. But this contradicts Observation 7. Thus we have that $cd \in F_1$. Therefore, (a, b, c, d, e) is a strict F_1-pentagon, or in other words, $F = F_1$. Then, our earlier observations imply that $ub, ue \in F_1$ and $au \in F_2$.

Since ec, ab, ed and bc, ae, bd are paths in G^*, it follows that ec, ed lie in one component of G^* and bc, bd also lie in one component of G^*. Let D be the component containing bc, bd and D' the component containing ec, ed in G^*. Consider the lexicographically smallest vertex in $D \cup D'$. Let us assume without loss of generality that this vertex is in D (we can interchange the labels of b and e if required). Define $p_0 = b$, $q_0 = c$. Then in G^*, there exists a path $p_0q_0, p_1q_1, \ldots, p_tq_t$ between bc and the lexicographically smallest vertex p_tq_t in D. As before, for $0 \leq i \leq t-1$, we have $p_iq_{i+1}, q_ip_{i+1} \in E(\overline{G})$ and for $0 \leq i \leq t$, we have $p_iq_i \in F_1$ when i is odd and $p_iq_i \in F_2$ when i is even. Also, since p_tq_t is the lexicographically smallest vertex in its component in G^*, we know that $p_tq_t \in F_1$, which implies that t is odd.

Observation 9. *Let $i \in \{0, 1, \ldots, t\}$. Then if i is odd, we have*

(*a*) $p_i \notin \{b, e\}$,
(*b*) $q_i \notin \{a, c, d\}$,
(*c*) $p_ib, p_ie \in F_1$,
(*d*) *Either* $p_i = a$ *or* $p_ia \in F_2$, *and*
(*e*) *Either* $q_ic \in F_2$ *or* $q_id \in F_2$,

and if i is even, we have

(*a*) $q_i \notin \{b, e\}$,
(*b*) $p_i \notin \{a, c, d\}$,
(*c*) $q_i b, q_i e \in F_2$,
(*d*) *Either* $q_i = d$ *or* $q_i d \in F_1$, *and*
(*e*) *Either* $p_i u \in F_1$ *or* $p_i a \in F_1$.

Proof. We shall prove this by induction on i. If $i = 0$, then the statement of the lemma can be easily seen to be true. Suppose that $i > 0$. We give a proof for the case when i is odd (the case when i is even is symmetric and can be proved using similar arguments). By the induction hypothesis, $q_{i-1}b, q_{i-1}e \in F_2$, and therefore since $p_i q_{i-1} \in E(\overline{G})$, we have $p_i \notin \{b, e\}$. We now prove the following claim.

Claim 1. *For* $x \in \{a, u\}$, *if* $p_i = x$ *or* $p_i x \in F_2$, *then* $p_i b$, $p_i e \in F_1$.

If $p_i = x$ then there is nothing to prove as we already know that $ab, ae, ub, ue \in F_1$. So assume that $p_i x \in F_2$. Let $\{z, \bar{z}\} = \{b, e\}$. Then $p_i, x, d, z, \bar{z}, q_{i-1}, p_i$ is an alternating F_2-circuit (recall that $p_i \notin \{b, e\}$), which implies that $p_i z \in F_1$. We thus get that $p_i b$, $p_i e \in F_1$. This proves the claim.

By the induction hypothesis we know that either $p_{i-1}a \in F_1$ or $p_{i-1}u \in F_1$, and also that $p_{i-1} \notin \{c, d\}$. First suppose that $p_{i-1}a \in F_1$. This implies that $q_i \neq a$. Let $\{y, \bar{y}\} = \{c, d\}$. Then we have that $p_{i-1}, a, \bar{y}, y$ is an alternating F_1-path implying that $p_{i-1}y \in E(G)$. Thus, $p_{i-1}c$, $p_{i-1}d \in E(G)$. This implies that $q_i \notin \{c, d\}$. By the induction hypothesis we also have that $q_{i-1}y \in F_1$ for some $y \in \{c, d\}$. Then $q_i, p_i, q_{i-1}, y, a, p_{i-1}, q_i$ is an alternating F_1-circuit, which implies that $q_i y \in F_2$. If $p_i \neq a$, then $p_i, q_i, p_{i-1}, a, y, q_{i-1}, p_i$ is an alternating F_1-circuit, implying that $p_i a \in F_2$. Since we have either $p_i = a$ or $p_i a \in F_2$ we are done by Claim 1.

Therefore we can assume that $p_{i-1}a \notin F_1$. If $i = 1$, then we know that $p_{i-1}a = ba \in F_1$, so we can assume that $i \geq 2$. By the induction hypothesis, we have that for some $y \in \{c, d\}$, $q_{i-2}y \in F_2$. Therefore if $p_{i-1}a \in E(G)$, then we have that $p_{i-1}, a, y, q_{i-2}, p_{i-1}$ is an alternating 4-cycle in which $q_{i-2}y \in F_2$, implying that $p_{i-1}a \in F_1$ which is a contradiction. Since we know that $p_{i-1} \neq a$ by the induction hypothesis, we can assume that $p_{i-1}a \in E(\overline{G})$. Note that since $p_{i-1}a \notin F_1$, we have by the induction hypothesis that $p_{i-1}u \in F_1$. If $q_{i-1} = d$, then $p_{i-1}, (q_{i-1} = d), u, a, p_{i-1}$ is an alternating 4-cycle whose opposite edges both belong to F_2, which is a contradiction. Therefore by the induction hypothesis we have $q_{i-1}d \in F_1$. If $q_i = a$ (resp. $q_i = c$) then $p_i, (q_i = a), d, q_{i-1}, p_i$ (resp. $p_{i-1}, u, d, (c = q_i), p_{i-1}$) is an alternating 4-cycle whose opposite edges are both in F_1, which is a contradiction. Therefore, $q_i \notin \{a, c\}$. If $p_i a \in F_2$ then we have that $a, p_i, q_{i-1}, p_{i-1}, a$ is an alternating 4-cycle whose opposite edges are both in F_2, which is a contradiction. This implies that $p_i a \notin F_2$ and therefore $p_i \neq u$. Then $p_i, q_i, p_{i-1}, u, d, q_{i-1}, p_i$ is an alternating F_1-circuit, implying that $p_i u \in F_2$. Therefore by Claim 1, we have that $p_i b$, $p_i e \in F_1$. Now if $a \neq p_i$, then $p_i, b, e, a, d, q_{i-1}, p_i$ is an alternating F_1-circuit, which implies that $p_i a \in F_2$ which is a contradiction. This implies that $a = p_i$, which further implies that $q_i \neq d$. Then $q_i, p_i, q_{i-1}, d, u, p_{i-1}, q_i$ is an alternating F_1-circuit, which implies that $q_i d \in F_2$ and we are done. □

Observation 10. *For each even* $i \in \{0, 1, 2, \dots, t\}$, *either* $ap_i \in E(G)$ *or both* $dq_{i-1}, dq_{i+1} \in E(G)$.

Proof. Suppose that there exists an even $i \in \{0, 1, 2, \dots, t\}$ and $j \in \{i-1, i+1\}$ such that $ap_i, dq_j \notin E(G)$. By Observation 9, we know that $p_i \neq a$ and $q_j \neq d$. So we have $ap_i, dq_j \in E(\overline{G})$. Now if $d \neq q_i$, then we have by Observation 9 that $q_i d \in F_1$. Then p_j, q_j, d, q_i, p_j is an alternating 4-cycle whose both opposite edges belong to F_1, which is a contradiction. Therefore we can assume that $d = q_i$. Then $(d = q_i), p_i, a, u, (d = q_i)$ is an alternating 4-cycle whose opposite edges both belong to F_2, which is again a contradiction. □

Recall that D' is the component containing ec in G^*.

Observation 11. *For any odd* $i \in \{0, 1, \dots, t\}$, *if* $ap_{i-1} \in E(G)$, *then for each* $y \in \{c, d\}$ *for which* $yq_i \in E(G)$, *we have* $yq_i \in D'$. *On the other hand, if* $ap_{i-1} \notin E(G)$, *then* $dq_i \in D'$.

Proof. We prove this by induction on i. When $i = 1$, we have $ap_0 = ab \in E(G)$ and for each $y \in \{c, d\}$ such that $yq_1 \in E(G)$, we have that $ec, (ab = ap_0), yq_1$ is a path in G^*. We thus have the base case. We shall now prove the claim for $i \geq 3$ assuming that the claim is true for $i - 2$. Suppose that $ap_{i-1} \in E(G)$. By Observation 9, there exists $y'' \in \{c, d\}$ such that $y''q_{i-2} \in E(G)$. By the induction hypothesis, either $y''q_{i-2} \in D'$ or $dq_{i-2} \in D'$ (depending upon whether ap_{i-3} is an edge or not). Thus in any case, we have that there exists $y' \in \{c, d\}$ such that $y'q_{i-2} \in D'$. Now for each $y \in \{c, d\}$ such that $yq_i \in E(G)$, since $y'q_{i-2}, ap_{i-1}, yq_i$ is a path in G^*, we get that $yq_i \in D'$, so we are done. Next, suppose that $ap_{i-1} \notin E(G)$. Then by Observation 9, we have $up_{i-1} \in E(G)$ and by Observation 10, we have $dq_{i-2}, dq_i \in E(G)$. We then have by the induction hypothesis that $dq_{i-2} \in D'$. Since dq_{i-2}, up_{i-1}, dq_i is a path in G^*, we have $dq_i \in D'$. □

Recall that C is the component of G^* containing the vertex cd.

Observation 12. *For each odd $i \in \{0, 1, \ldots, t\}$, if $a \neq p_i$ then $ap_i \in C$.*

Proof. We prove this by induction on i. The base case when $i = 1$ is true since if $a \neq p_1$ then by Observation 9, $ap_1 \in E(G)$, and since $\{ap_1, (q_0 = c)d\} \in E(G^*)$, we have $ap_1 \in C$. Assume that $i \geq 3$ and the claim is true for $i - 2$. Suppose that $a \neq p_i$. Then we have $ap_i \in E(G)$ by Observation 9. If $d = q_{i-1}$ then we have $\{ap_i, c(q_{i-1} = d)\} \in E(G^*)$, so we have $ap_i \in C$. So we assume that $d \neq q_{i-1}$. Then by Observation 9, we have that $dq_{i-1} \in E(G)$. By the induction hypothesis, we have that either $ap_{i-2} \in C$ or $a = p_{i-2}$. If $ap_{i-2} \in C$, then since ap_i, dq_{i-1}, ap_{i-2} is a path in G^*, we have $ap_i \in C$. On the other hand, if $a = p_{i-2}$ then we again have $ap_i \in C$ as $ap_i, dq_{i-1}, u(p_{i-2} = a), cd$ is a path in G^*. $\square$

Recall that t is odd, $p_tq_t \in D$, and p_tq_t is the lexicographically smallest vertex in $D \cup D'$.

Observation 13. $p_t < \min\{c, d\}$

Proof. Let $\{y, \bar{y}\} = \{c, d\}$, where $y < \bar{y}$. Note that $p_t \notin \{c, d\}$, since by Observation 9, $p_tb \in F_1$, but we know that $cb, db \in F_2$. By the same lemma, we also have that $q_t \notin \{c, d\}$. Therefore as $\min\{p_t, q_t\} \leq \min\{c, d\}$ (since $p_tq_t < bc, bd$), we have that $\min\{p_t, q_t\} < \min\{c, d\} = y$. Now if $p_t = \min\{p_t, q_t\}$ then we are done. Therefore let us assume that $q_t = \min\{p_t, q_t\}$, and so $q_t < y$.

Suppose that $yq_t \in E(G)$. If $yq_t \notin D'$, then by Observation 11, we have that $ap_{t-1} \notin E(G)$ and $\overline{y}q_t \in D'$. By Observation 9, we know that $p_{t-1} \neq a$, which implies that $ap_{t-1} \in E(\overline{G})$. By our choice of p_tq_t, we now have that $p_tq_t < \overline{y}q_t$, which implies that $p_t < \overline{y}$. Now by Observation 8, $p_t \neq a$, which implies by Observation 9 that $p_ta \in F_2$. Then $a, p_t, q_{t-1}, p_{t-1}, a$ is an alternating 4-cycle in which both opposite edges belong to F_2, which is a contradiction. We can thus conclude that $yq_t \in D'$. Then by our choice of p_tq_t, we have that $p_t < y$, and we are done. So we assume that $yq_t \notin E(G)$.

Recall that $q_t < y$ (and therefore $yq_t \in E(\overline{G})$). Now if $y < p_t$ then we have $q_t < y < p_t$ where $q_ty \notin E(G)$ and $q_tp_t \in E(G)$. By Observation 2, this implies that there exists $x < q_t$ such that $xy \in E(G)$ and $xp_t \notin E(G)$ (which means that $xp_t \in E(\overline{G})$ since $x < p_t$). Then $\{xy, p_tq_t\} \in E(G^*)$, which implies that $xy \in D$. But $xy < p_tq_t$, which contradicts our choice of p_tq_t. We can thus conclude that $p_t < y$ (recall that $p_t \neq y$ as $p_t \notin \{c, d\}$) and we are done. $\square$

Note that by Observation 13 and Observation 8 we have that $a \neq p_t$. Then by Observation 12, we have $ap_t \in C$. By Observation 13 and Observation 7, $p_t < \min\{c', d'\}$, which implies that $ap_t < c'd'$. This is a contradiction to our choice of $c'd'$. This completes the proof of Lemma 6.

By Lemma 6 and Lemma 4, we have the following corollary.

Corollary 4. *There are no strict switching paths in G (with respect to (F_0, F_1, F_2)).*

Note. Given a 2-coloring of G^* in which the color classes are denoted by E_1 and E_2, Raschle and Simon [9] define an "AP_6" in G to be a sequence $v_0, v_1, \ldots, v_5, v_0$ of distinct vertices of G such that $v_0v_1, v_2v_3, v_4v_5 \in E_i$ for some $i \in \{1, 2\}$ and $v_1v_2, v_3v_4, v_5v_0 \in E(\overline{G})$. A 2-coloring of G^* is said to be "AP_6-free" if there is no AP_6 in G with respect to that coloring. Raschle and Simon observe that if G^* has an AP_6-free 2-coloring, then G has a 2-threshold cover and it can be computed in time $O(|E(G)|^2)$ (using Theorem 3.1, Theorem 2.5, Fact 2 and Fact 1 in [9]). The major part of the work of Raschle and Simon is to show that an AP_6-free 2-coloring of G^* always exists if G^* is bipartite and that it can be computed in time $O(|E(G)|^2)$ (Sections 3.2 and 3.3 of [9]). It can be seen that any 2-coloring of G^* obtained by extending the partial 2-coloring of G^* computed after Phases I and II of our algorithm is in fact an AP_6-free 2-coloring of G^* as follows. Let E_1 and E_2 be the color classes of such a 2-coloring of G^*. We can assume without loss of generality that $F_1 \subseteq E_1$ and $F_2 \subseteq E_2$. Note that $F_0 \subseteq E_1 \cup E_2$. Suppose that there is an AP_6 $v_0, v_1, \ldots, v_5, v_0$ in G with respect to this coloring where the edges $v_0v_1, v_2v_3, v_4v_5 \in E_i$, where $i \in \{1, 2\}$. Note that $(\emptyset, E_1, E_2)$ is a valid 3-partition of $E(G)$. For each even $j \in \{0, 1, \ldots, 5\}$, since $v_j, v_{j+1}, v_{j+2}, v_{j+3}$ (subscripts modulo 6) is an alternating E_i-path, we have that $v_jv_{j+3} \in E(G)$. This implies that for each even $j \in \{0, 1, \ldots, 5\}$, $v_j, v_{j+1}, v_{j+2}, (v_{j+5} = v_{j-1}), v_j$ is an alternating 4-cycle in G (note that from the previous observation, we have $v_{j+2}v_{j+5} \in E(G)$), from which it follows that v_jv_{j+1} is in a non-trivial component of G^*. Therefore, $v_0v_1, v_2v_3, v_4v_5 \notin F_0$. Since these edges belong to E_i, it follows that $v_0v_1, v_2v_3, v_4v_5 \in F_i$. Then $v_0, v_1, \ldots, v_5, v_0$ is an alternating F_i-circuit, and therefore $v_0v_3 \in F_{3-i}$. This implies that (v_2, v_3, v_0, v_1) is a strict F_i-switching path in G, which contradicts Corollary 4. Thus the proof of Theorem 1 can already be completed using the observations in [9]. In the next section, we nevertheless give a self-contained proof that shows that G has a 2-threshold cover without using the "threshold completion" method used in [6,9]. Also note that since it is clear that Phases I and II of the algorithm, and also the initial construction of G^*, can be done in time $O(|E(G)|^2)$, we have an algorithm with the same time complexity that computes the 2-threshold cover of a graph G whose auxiliary graph G^* is bipartite (note however that there is a faster algorithm for computing a 2-threshold cover due to Sterbini and Raschle [11]).

5.2. Extending the algorithm

Note that there may be pentagons in G with respect to the valid 3-partition (F_0, F_1, F_2) that is generated after the first two phases of the algorithm given in Section 3 (even though there are no strict pentagons). We now introduce a third phase for the algorithm so as to eliminate all these pentagons to obtain a new valid 3-partition of $E(G)$ that does not contain any pentagons. We then show that there are no switching paths or switching cycles with respect to this valid 3-partition, which completes the proof.

Observation 14. *There does not exist $a_1, a_2, b_1, b_2, e_1, e_2, c, d \in V(G)$ such that (a_1, b_1, c, d, e_1) is an F_1-pentagon and (a_2, b_2, c, d, e_2) is an F_2-pentagon.*

Proof. Suppose not. Then as $b_1c, e_1c \in F_2$ and $b_2c, e_2c \in F_1$, we have $\{b_1, e_1\} \cap \{b_2, e_2\} = \emptyset$. Then b_1, a_1, c, b_2 and e_1, a_1, c, e_2 are alternating F_1-paths, implying that $b_1b_2, e_1e_2 \in E(G)$. As b_1, b_2, e_2, e_1, b_1 is an alternating 4-cycle, we have $\{b_1b_2, e_1e_2\} \in E(G^*)$. Thus, $b_1b_2 \notin F_0$, or in other words, $b_1b_2 \in F_1 \cup F_2$. If $b_1b_2 \in F_1$, then (c, b_1, b_2, a_2) is a strict F_2-switching path, which contradicts Corollary 4. On the other hand, if $b_1b_2 \in F_2$, then (c, b_2, b_1, a_1) is a strict F_1-switching path, which again gives a contradiction to Corollary 4. □

We shall now describe a Phase III that can be added to the algorithm of Section 3 to construct a partial 2-coloring of G^* that can be used to construct a valid 3-partition of $E(G)$ that contains no pentagons.

Phase III. For each $i \in \{1, 2\}$, let

$S_i = \{cd \in F_0 : \exists a, b, e \in V(G)$ such that (a, b, c, d, e) is an F_i-pentagon in G with respect to $(F_0, F_1, F_2)\}$.

Color every vertex in S_1 with 2 and every vertex in S_2 with 1.

Let F'_0 be the set of vertices of G^* that are uncolored after Phase III, and for $i \in \{1, 2\}$, let F'_i be the set of vertices of G^* that are colored i. Clearly, $F'_0 = F_0 \setminus (S_1 \cup S_2)$, $F'_1 = F_1 \cup S_2$ and $F'_2 = F_2 \cup S_1$. Note that $S_1, S_2 \subseteq F_0$ and that $S_1 \cap S_2 = \emptyset$ by Observation 14. It is easy to see that $\{F'_0, F'_1, F'_2\}$ is a partition of $E(G)$. Further, since $F_1 \subseteq F'_1$, $F_2 \subseteq F'_2$ and (F_0, F_1, F_2) is a valid 3-partition of $E(G)$, it follows that (F'_0, F'_1, F'_2) is also a valid 3-partition of $E(G)$. From here onward, we use the terms "pentagons" and "switching paths" with respect to (F'_0, F'_1, F'_2) unless otherwise mentioned.

Lemma 7. *There are no pentagons in G with respect to (F'_0, F'_1, F'_2).*

Proof. Suppose for the sake of contradiction that (a, b, c, d, e) is a pentagon in G with respect to (F'_0, F'_1, F'_2). Let $i \in \{1, 2\}$ such that (a, b, c, d, e) is an F'_i-pentagon. Recall that ec, ab, ed and bc, ae, bd are paths in G^* and hence each of ab, ae, bc, bd, ec, ed is in a non-trivial component of G^*. Thus none of them is in F_0. Since $ab, ae \in F'_i$ and $bc, bd, ec, ed \in F'_{3-i}$, this implies that $ab, ae \in F_i$ and $bc, bd, ec, ed \in F_{3-i}$. Since (a, b, c, d, e) is an F'_i-pentagon, we have $cd \in F'_0 \cup F'_i$. This implies that $cd \notin F'_{3-i}$ and that $cd \in F_0 \cup F_i$. If $cd \in F_0$, then (a, b, c, d, e) is an F_i-pentagon in G with respect to (F_0, F_1, F_2), which implies that $cd \in S_i$, and therefore $cd \in F'_{3-i}$. Since this is a contradiction, we can assume that $cd \in F_i$. Then (a, b, c, d, e) is a strict F_i-pentagon in G with respect to (F_0, F_1, F_2), contradicting Lemma 6. □

Lemma 8. *There are no switching paths in G with respect to (F'_0, F'_1, F'_2).*

Proof. Suppose not. Let (a, b, c, d) be a switching path in G with respect to (F'_0, F'_1, F'_2). Let $i \in \{1, 2\}$ such that (a, b, c, d) is an F'_i-switching path. Then we have $ad \in E(\overline{G})$, $ab, cd \in F'_i \cup F'_0$, and $bc \in F'_{3-i}$. Suppose that bc belongs to a non-trivial component of G^*. Then there exists $uv \in E(G)$ such that $bv, cu \in E(\overline{G})$. By Lemma 3 and Lemma 7, we have that $a \neq u$ and $d \neq v$. Notice that since $bc \in F'_{3-i}$ and b, c, u, v, b is an alternating 4-cycle, we have $uv \in F'_i$. Then d, c, u, v, b, a, d and a, b, v, u, c, d, a are alternating F'_i-circuits, implying that $dv, au \in F'_{3-i}$ and $ab, cd \in F'_i$. This further implies that (a, b, c, d) is a strict F'_i-switching path with respect to (F'_0, F'_1, F'_2). Since b, a, d, v, b and c, d, a, u, c and b, c, u, v, b are alternating 4-cycles, we also have that $ab, cd, bc \notin F_0$, which further implies that $ab, cd \in F_i$ and $bc \in F_{3-i}$. Then (a, b, c, d) is also a strict F_i-switching path with respect to (F_0, F_1, F_2), which is a contradiction to Corollary 4.

Therefore we can assume that bc belongs to a trivial component in G^*, i.e. $bc \in F_0$. Since $bc \in F'_{3-i}$, it should be the case that $bc \in S_i$, which implies that there exists an F_i-pentagon (x, y, b, c, z) in G with respect to (F_0, F_1, F_2). Since $ab, cd \in F'_i \cup F'_0 \subseteq F_i \cup F_0$, we know that $a, d \notin \{x, y, z\}$. Since a, b, x, y and d, c, x, z are alternating F_i-paths, we have that $ay, dz \in E(G)$. Since a, y, z, d, a is an alternating 4-cycle, we know that one of ay, dz is in F_i and the other in F_{3-i}. Because of symmetry, we can assume without loss of generality that $ay \in F_i$ and $dz \in F_{3-i}$ (by renaming (a, b, c, d) as (d, c, b, a) and interchanging the labels of y and z if necessary). Then a, y, z, x, c, d, a is an alternating F_i-circuit, implying that $ax \in F_{3-i}$. Then (a, x, z, d) is a strict F_{3-i}-switching path in G with respect to (F_0, F_1, F_2), which again contradicts Corollary 4. □

Lemma 9. *There are no switching cycles in G with respect to (F'_0, F'_1, F'_2).*

Proof. Suppose not. Let (a, b, c, d) be a switching cycle in G with respect to (F'_0, F'_1, F'_2). Let $i \in \{1, 2\}$ such that (a, b, c, d) is an F'_i-switching cycle. Then we have $ab, cd \in F'_i \cup F'_0$ and $ad, bc \in F'_{3-i}$. Suppose that bc belongs to a non-trivial component of G^*. Then there exists $uv \in E(G)$ such that $bv, cu \in E(\overline{G})$. Since b, c, u, v, b is an alternating 4-cycle and $bc \in F'_{3-i}$, we have that $uv \in F'_i$. If $u = a$ and $v = d$, then $b, (a = u), c, (d = v), b$ is an alternating 4-cycle in which both the opposite edges belong to $F'_i \cup F'_0$, which is a contradiction. Therefore, either $u \neq a$ or $v \neq d$. Because of symmetry, we can assume without loss of generality that $u \neq a$ (by renaming (a, b, c, d) as (d, c, b, a) and interchanging the labels of u and v if necessary). Then a, b, v, u is an alternating F'_i-path, implying that $au \in E(G)$. If $au \in F'_i \cup F'_0$ then (c, d, a, u) is an F'_i-switching path, and if not, then $au \in F'_{3-i}$, in which case (b, a, u, v) is an F'_i-switching path. In both cases, we have a contradiction to Lemma 8.

Therefore we can assume that bc belongs to a trivial component of G^*, i.e. $bc \in F_0$. Since $bc \in F'_{3-i}$, it should be the case that $bc \in S_i$, which implies that there exists an F_i-pentagon (x, y, b, c, z) in G with respect to (F_0, F_1, F_2). Since $ab, cd \in F'_i \cup F'_0 \subseteq F_i \cup F_0$, $a, d \notin \{x, y, z\}$. As y, x, b, a and z, x, c, d are alternating F_i-paths, we have that $ya, zd \in E(G)$. Now if both $ya, zd \in F'_i \cup F'_0$ we have that (y, a, d, z) is an F'_i-switching path, which is a contradiction to Lemma 8. On the other hand, if $ya \in F'_{3-i}$ or $zd \in F'_{3-i}$, then since $xy, xz \in F_i \subseteq F'_i$, we have that either (x, y, a, b) or (x, z, d, c) is an F'_i-switching path, which again contradicts Lemma 8. $\square$

Now from Lemma 1, Lemma 2, Lemma 8, and Lemma 9, we have Theorem 1.

6. Time complexity of the algorithm

We now show that our overall algorithm consisting of Phases I, II, and III can be implemented to run in time $O(|E(G)|^2)$. Since Lex-BFS takes only linear time, Phase I of our algorithm takes only time $O(|V(G)| + |E(G)|) = O(|E(G)|)$ (since we can assume that the graph G does not contain any isolated vertices). We assume that in addition to the adjacency list representation of G, we also have an adjacency matrix representation of G using which we can check in $O(1)$ time whether any two given vertices are adjacent or not (since our algorithm needs to run only in time $O(|E(G)|^2)$, we can just construct this adjacency matrix representation as a preprocessing step). The graph G^* can be constructed in time $O(|V(G^*)| + |E(G^*)|) = O(|E(G)|^2)$. It is not hard to see that given an ordering of the vertices generated by Phase I, the coloring procedure of Phase II can be implemented as a BFS or DFS through every non-trivial component C of G^* to find the lexicographically smallest vertex in C, and hence can be done in time $O(|V(G^*)| + |E(G^*)|) = O(|E(G)|^2)$. Thus, the time complexity of our algorithm for paraglider-free graphs (and therefore also split graphs) is $O(|E(G)|^2)$. For the case of general graphs, we have to implement Phase III as well. We now show that this step can also be implemented to run in time $O(|E(G)|^2)$. Assume that we have constructed the sets F_0, F_1 and F_2 after Phases I and II have completed, and this information is stored in a form such that not only can the sets be enumerated in time proportional to their sizes, given any edge, it can be determined in $O(1)$ time which of the sets F_0, F_1 or F_2 it belongs to. We shall be done if we show how for each edge cd in F_0, we can check in $O(|E(G)|)$ time whether there exists a F_i-pentagon (a, b, c, d, e) in G with respect to (F_0, F_1, F_2), for each $i \in \{1, 2\}$. We shall describe the algorithm only for the case $i = 1$ as the other case is similar; i.e. we describe how, given an edge $cd \in F_0$, one can check for the presence of an F_1-pentagon (a, b, c, d, e) in G in $O(|E(G)|)$ time.

We first construct the set $Q = \{b \in V(G) : bc, bd \in F_2\}$. Since this can be done by just inspecting the edges incident on c and the edges incident on d, it can be done in $O(|E(G)|)$ time. We now construct the set $P = \{a \in V(G) : ac, ad \notin E(G)$ and $\exists b \in Q$ such that $ba \in F_1\}$. Clearly, this can be done by inspecting the edges incident on the vertices in Q, and hence also takes just $O(|E(G)|)$ time. While doing this, for each vertex $b \in Q$ for which we find that the set $\{a \in P : ab \in F_1\}$ is not empty, we store as $f(b)$ an arbitrary vertex in the set. It now only needs to be checked whether there exist $b, e \in Q$ and $a \in P$ such that $be \notin E(G)$ and $ab, ae \in F_1$. For each vertex $a \in P$, we construct the set $N'(a) = \{b \in Q : ab \in F_1\}$, which also takes time $O(|E(G)|)$ as this can be done by inspecting the edges incident on a. We claim that the following procedure now checks if there exists an F_1-pentagon (a, b, c, d, e).

1. $S \leftarrow \emptyset$
2. For each $a \in P$,
3.   For each $b \in N'(a)$,
4.   For each $e \in S$,
5.   If $be \notin E(G)$
6.   If $ae \in F_1$
7.   Report that (a, b, c, d, e) is an F_1-pentagon and stop.
8.   Else
9.   Report that $(f(e), b, c, d, e)$ is an F_1-pentagon and stop.
10. $S \leftarrow S \cup \{b\}$

We shall first analyze the running time of the above procedure. Note that at every point of time during the execution of the procedure, the set S is a clique in G. Line 2 gets executed at most $O(|P|) = O(|E(G)|)$ times. Lines 3 and 10 get

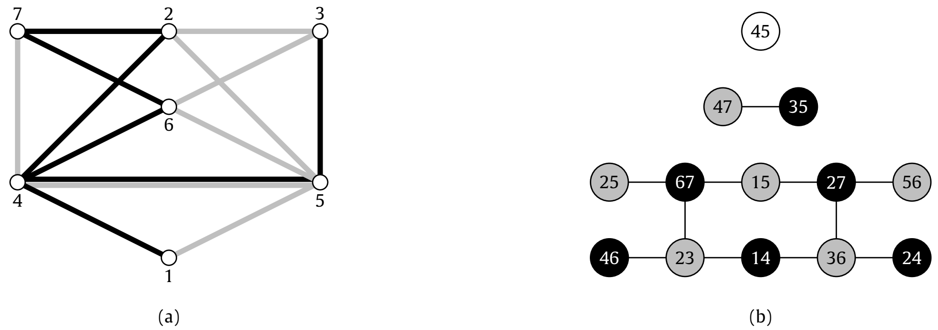

Fig. 3. (a) The graph G from Fig. 1, with its vertices numbered according to a non-Lex-BFS ordering, and (b) the graph G^* and its partial 2-coloring after Phases II and III.

executed at most $O(|Q|) = O(|E(G)|)$ times. Let t denote the cardinality of the set S after the completion of the procedure. Lines 4 and 5 get executed at most $O(\binom{t}{2})$ times. Since the set S constructed by the procedure is a clique in G, we have that $O(\binom{t}{2}) = O(|E(G[S])|) = O(|E(G)|)$. Clearly, lines 6 to 9 get executed at most once. Thus the total running time of the above procedure is $O(|E(G)|)$, as required.

We shall now prove that the procedure is correct, for which the following observation will be useful.

Observation 15. *Let $a, a' \in P$ and $b, b' \in Q$ such that $ab, a'b' \in F_1$ and $ab', a'b \notin F_1$. Then $bb' \in E(G)$.*

Proof. Suppose that $bb' \notin E(G)$. Recall that (F_0, F_1, F_2) is a valid 3-partition of $E(G)$. Suppose that $a'b, ab' \notin E(G)$. As (a, b, a', b') is an alternating F_1-path, we have $ab' \in E(G)$, which is a contradiction. So at least one of $a'b, ab'$ is in $E(G)$. We assume that $a'b \in E(G)$, as the other case is symmetric. Since $a'b \notin F_1$, we have $a'b \in F_2 \cup F_0$. Now (a', b, b', d) is an alternating F_2-path (recall that as $b' \in Q$, we have $b'd \in F_2$), which implies that $a'd \in E(G)$, a contradiction to the fact that $a' \in P$. $\square$

Suppose that the procedure reports that (a, b, c, d, e) is an F_1-pentagon in line 7. Then it is clear that $ab, ae \in F_1$, $bc, bd, ec, ed \in F_2$ (since $b \in N'(a) \subseteq Q$ and $e \in S \subseteq Q$), $cd \in F_0$, and $ac, ad, be \notin E(G)$, which means that the procedure's output is correct. Suppose instead that the procedure reports that $(f(e), b, c, d, e)$ is an F_1-pentagon in line 9. Then as before, it is clear that $cd \in F_0$, $ab \in F_1$, $bc, bd, ec, ed \in F_2$, and $be \notin E(G)$. It follows from the definition of $f(e)$ that $f(e)e \in F_1$ and that $f(e) \in P$, which further implies that $f(e)c, f(e)d \notin E(G)$. Since the procedure has reached line 9, we know that $ae \notin F_1$. Now from Observation 15 applied to $a, f(e), b, e$, we can conclude that $f(e)b \in F_1$. Thus $(f(e), b, c, d, e)$ is indeed an F_1-pentagon.

Next, suppose that there is an F_1-pentagon (a', b', c, d, e') in G, but our procedure fails to detect any pentagon. Clearly, we have $a' \in P$ and $b', e' \in N'(a') \subseteq Q$. As the procedure never detects any pentagon, every vertex in $N'(a')$ gets added to S at some point during the execution of the procedure. We shall assume without loss of generality that e' gets added to S before b'. Then it is clear that line 5 eventually gets executed with $b = b'$ and $e = e'$, and since $b'e' \notin E(G)$, the procedure will report a pentagon, which contradicts our assumption that it did not find any pentagon.

7. Conclusion

Would running just Phases II and III of our algorithm always produce a valid 2-threshold cover of G for any graph G? That is, could we have started with an arbitrary ordering of the vertices of G instead of a Lex-BFS ordering? We show that the algorithm may fail to produce a 2-threshold cover of the graph G shown in Fig. 1 if the algorithm starts by taking an arbitrary ordering of vertices in Phase I. Suppose that the vertices of the graph are ordered according to their labels as shown in Fig. 3(a). Clearly, it is not a Lex-BFS ordering, as since the vertex in the second position is not a neighbor of the vertex in the first position, it is not even a BFS ordering. The sets F'_0, F'_1, F'_2 computed by our algorithm after Phases II and III will be as shown in Fig. 3(b)—the vertices of G^* in the set F'_1 are shown as black, the ones in F'_2 as gray and the ones in F'_0 as white. In Fig. 3(a), the black edges form the graph H_1 and the gray edges form the graph H_2. Clearly, neither is a threshold graph (for example, both contain a C_4). On the other hand, Fig. 4 shows the 2-threshold cover of G computed by our algorithm if it starts with the Lex-BFS ordering of the vertices of G as indicated by the labels of the vertices in Fig. 4(a). Note that starting with a BFS ordering instead of a Lex-BFS ordering will also not work, since we can always add a universal vertex to the graph G shown in Fig. 3(a) and number it 0, so that the vertex ordering is now a BFS ordering. It is not difficult to see that the graphs H_1 and H_2 computed in this case also fail to be threshold graphs (in fact, the edges incident on the vertex labelled 0 are all isolated vertices in the auxiliary graph, and none of them belong to any pentagons; hence they all belong to F'_0, and the sets F'_1 and F'_2 will be exactly the same as before).

Thus the graph G shown in Fig. 1 demonstrates that even though Phase I is optional for split graphs, for general graphs, our algorithm may not produce a 2-threshold cover of the input graph if Phase I is skipped. Note that the graph G is not a

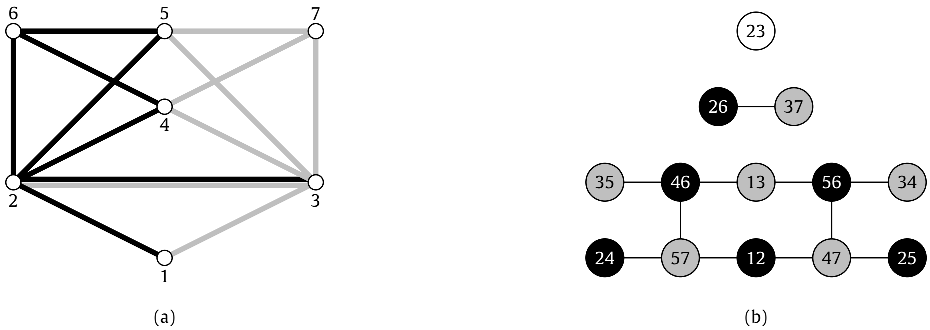

Fig. 4. (a) The graph G from Fig. 1, with its vertices numbered according to a Lex-BFS ordering, and (b) the graph G^* and its partial 2-coloring after Phases II and III.

paraglider-free graph. We have not found an example of a paraglider-free graph for which our algorithm will fail if Phase I is skipped.

Declaration of competing interest

The authors have no conflict of interest to declare with respect to this manuscript.

Data availability

No data was used for the research described in the article.

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