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Triangle-free projective-planar graphs with diameter two: Domination and characterization[☆]

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ABSTRACT

In 1975, Plesník characterized all triangle-free planar graphs as having a diameter 2. We characterize all triangle-free projective-planar graphs having a diameter 2 and discuss some applications. In particular, the main result is applied to calculate the analogue of clique numbers for graphs, namely, colored mixed graphs, having different types of arcs and edges.

1. Introduction and main results

In 1975, Plesník [19] characterized all triangle-free planar graphs[1] having diameter 2 by proving the following result.

Theorem 1 (*Plesník 1975 [19]*)**.** *A triangle-free planar graph G has a diameter 2 if and only if it is isomorphic to one of the following graphs:*

(i) $K_{1,n}$ *for* $n \geq 2$,
(ii) $K_{2,n}$ *for* $n \geq 2$,
(iii) *the graph* $C_5(m, n)$ *obtained by adding* $(m + n)$ *degree-2 vertices to the 5-cycle* $C_5 = v_1v_2v_3v_4v_5v_1$, *for* $m, n \geq 0$, *in such a way that* m *of the vertices are adjacent to* v_1, v_3 *and* n *of the vertices are adjacent to* v_1, v_4.

We prove the analogue of Plesník's result for *projective-planar graphs*, that is, graphs that can be embedded on the non-orientable surface of Euler genus one (also known as the real projective plane) without their edges crossing each other except, maybe, on the vertices. For convenience, let us refer to the graphs listed in Theorem 1 as *Plesník graphs*.

Theorem 2. *A triangle-free projective-planar graph G has diameter 2 if and only if it is isomorphic to one of the following:*

(i) *a Plesník graph,*
(ii) $K_{3,3}$ *or* $K_{3,4}$,

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1 In this article, we use the notation and terminology of "Introduction to Graph Theory" by D. B. West [22].

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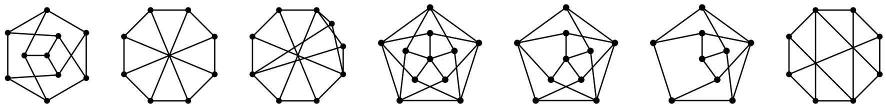

Fig. 1. (i) The Petersen graph P_{10}, (ii) The Wagner graph W_8, (iii) The graph W_8^+, (iv) The Grötzsch graph M_{11}, (v) The graph M_{11}^-, (vi) The graph $M_{11}^=$, (vii) The graph $K_{3,4}^*$.

(iii) *the graph $K_{3,3}(n)$ obtained by adding $(n-1)$ parallel edges $e_2, e_3, \dots, e_n$ to one of the edges e_1 of $K_{3,3}$ and subdividing each e_i exactly once for $n \geq 1$,*
(iv) *the graph $K_{3,4}(n)$ obtained by adding $(n-1)$ parallel edges $e_2, e_3, \dots, e_n$ to one of the edges e_1 of $K_{3,4}$ and subdividing each e_i exactly once for $n \geq 1$,*
(v) *one of the seven graphs depicted in Fig. 1.*

Let us now discuss a few more results regarding properties of graphs having small diameters that can be embedded on a given surface $\mathbb{S}$ to place our work into context. Since $\mathbb{S}$ has an Euler characteristic, it follows from the Euler's formula [16] that any graph embedded in $\mathbb{S}$ has a bounded minimum degree. This implies that if we consider such a graph with diameter 2, then its domination number is at most its minimum degree.

In 1996, MacGillivray and Seyffarth [14] proved that planar graphs with diameter 2 have domination numbers at most 3. In 2002, Goddard and Henning [9] showed that there is exactly one planar graph having diameter 2 that has a domination number equal to 3. They also proved that for each surface (orientable or non-orientable) $\mathbb{S}$, there are finitely many graphs having diameter 2 and domination number at least 3 that can be embedded in $\mathbb{S}$. A natural question to ask in this context is the following.

Question 1. *Given a surface $\mathbb{S}$, can you find the list of all graphs having diameter 2 and domination number at least 3 that can be embedded on $\mathbb{S}$?*

As we just mentioned, Goddard and Henning [9] answered Question 1 when $\mathbb{S}$ is the sphere (or equivalently, the Euclidean plane). However, it seems that the question can be very difficult to answer in general as the tight upper bounds on the domination number for a family of graphs that can be embedded on a surface, other than the sphere, is yet to be found. Therefore, it makes sense to ask the following natural restriction instead.

Question 2. *Given a surface $\mathbb{S}$, can you find the list of all triangle-free graphs having a diameter 2 and domination number at least 3 that can be embedded on $\mathbb{S}$?*

Notice that, Plesník's characterization implies that the answer for Question 2 is the empty list when $\mathbb{S}$ is the sphere. On the other hand, the following immediate corollary of Theorem 2 answers the question when $\mathbb{S}$ is the projective plane, along with implying that the domination number of triangle-free projective-planar graphs having diameter 2 is at most three (following our earlier discussions on Euler's characteristic). Note that the domination number of graphs shown in Fig. 1 is three.

Theorem 3. *Let G be a triangle-free projective-planar graph having a diameter 2. Then*

(a) *The domination number $\gamma(G)$ of G is at most 3.*
(b) *If $\gamma(G) = 3$, then G is isomorphic to one of the seven graphs depicted in Fig. 1.*

As Theorem 3 follows directly from Theorem 2, we will focus on proving Theorem 2. This is done in Section 2. In Section 3, we provide some direct implications of our results in determining the absolute clique number of the families of triangle-free projective-planar graphs, which is an important parameter in the theory of homomorphisms of colored mixed graphs.[2]

2. Proof of Theorem 2

It is known, due to Euler's formula [16] for projective-planar graphs, that any triangle-free projective-planar graph G has minimum degree $\delta(G) \leq 3$. Therefore, any triangle-free projective-planar graph having a diameter 2 has a domination number at most 3.

Notice that as the family of projective-planar graphs is minor-closed, due to The Graph Minor Theorem [20], there exists a finite set $\mathcal{S}$ of graphs such that a graph G is projective-planar if and only if G does not contain a minor from $\mathcal{S}$. Actually, an explicit description of the set $\mathcal{S}$ is provided in [2] (see [16]) and it contains 35 graphs. However, we will not need the full list for our proof — to be precise, we will use only three graphs from that list: (1) $K_{3,5}$, (2) $K_{4,4}^-$, i.e., the graph obtained from $K_{4,4}$ by deleting exactly one edge, and (3) the graph F_0 depicted in Fig. 2.

[2] The related definitions are deferred to Section 3.

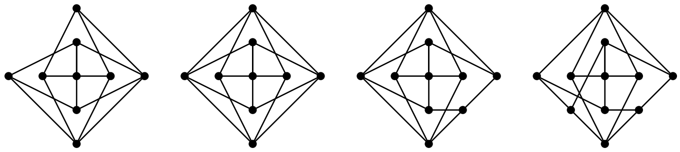

Fig. 2. The graphs (a) F_0, (b) F_1, (c) F_2, and (d) F_3 form the graph family $\mathcal{D}'$.

Observation 1 ([2]; see [16]). *The graphs $K_{3,5}$, $K_{4,4}^-$, and F_0 are not projective-planar.*

Thus any graph containing $K_{3,5}$, $K_{4,4}^-$, or F_0 as a minor is not projective-planar as well. Even though the previous statement is obvious, we will present it as another observation as it will be frequently used in our proofs.

Observation 2 ([2]; see [16]). *If G contains $K_{3,5}$, $K_{4,4}^-$, or F_0 as a minor, then G is not a projective-planar graph.*

Now we get into the more technical part of our proof. First of all, for convenience, let us denote the family of all triangle-free projective-planar graphs having diameter 2 by $\mathcal{PP}_2$. Therefore, what we are trying to do here is to provide a list of all graphs in $\mathcal{PP}_2$. We already know that if $G \in \mathcal{PP}_2$, then its minimum degree $\delta(G)$ is at most 3. We will use this as the basis of our case analysis. Observe that any $G \in \mathcal{PP}_2$ is connected. Therefore, the logical first step is to handle the graphs having a degree one vertex.

2.1. Characterizing graphs in $\mathcal{PP}_2$ having minimum degree at most 2

Lemma 1. *If $\delta(G) = 1$ for a graph $G \in \mathcal{PP}_2$, then G is isomorphic to $K_{1,n}$ for some $n \geq 2$.*

Proof. Let v be a degree-1 vertex in G having v_1 as its only neighbor. As G has diameter 2, v_1 must be adjacent to all the vertices in $V(G) \setminus \{v, v_1\}$. Moreover, as G is triangle-free, the set of all neighbors $N(v_1)$ of v_1 is an independent set. □

The next natural step is to consider the graphs having minimum degree equal to 2.

Lemma 2. *If $\delta(G) = 2$ for a graph $G \in \mathcal{PP}_2$, then G is isomorphic to $K_{2,n+2}$, $C_5(m, n)$, $K_{3,4}(n)$, or $K_{3,3}(n)$ for some $m, n \geq 0$.*

Proof. Let v be a degree-2 vertex having $N(v) = \{v_1, v_2\}$. Let

$$C = (N(v_1) \cap N(v_2)) \setminus \{v\} \text{ and } S_i = N(v_i) \setminus (C \cup \{v\})$$

for $i \in \{1, 2\}$.

$S_1 \cup S_2$ induces a complete bipartite graph (else the end vertices of any non-edge would be at a distance greater than 2, a contradiction) and as $\delta(G) \geq 2$, both sets are nonempty.

If $|S_1| \geq |S_2| \geq 3$, then we find a $K_{4,4}^-$ by taking the graph induced by $S_1 \cup S_2 \cup \{v_1, v_2\}$. This is a contradiction due to Observation 2. Thus we must have $|S_2| \leq 2$.

If $|S_2| = 2$, then $|S_1| \leq 3$ as otherwise we can contract the edge vv_1 to find a $K_{3,5}$ induced by $S_1 \cup S_2 \cup \{v_1, v_2\}$. This is a contradiction due to Observation 2.

Now observe that $|S_1| = 3$ and $|S_2| = 2$ implies G is isomorphic to $K_{3,4}(n)$, where $n = |C| + 1$. Similarly, $|S_1| = 2$ and $|S_2| = 2$ implies G is isomorphic to $K_{3,3}(n)$, where $n = |C| + 1$.

If $|S_2| = 1$, $|S_1|$ can have any value greater than or equal to 1. In this case, G is isomorphic to $C_5(m, n)$, where $m = |C|$ and $n = |S_1| - 1$. □

2.2. Characterizing graphs in $\mathcal{PP}_2$ that are 3-regular

This leaves us with the final case: considering the graphs having minimum degree equal to 3. We break this case into two parts, namely, when G is 3-regular and when G is not 3-regular, and tackle them separately. Also we will use some new terminologies.

A vertex u *reaches* a vertex v if they are adjacent or they have a common neighbor. In particular, if w is a common neighbor of u and v, then we use the term u reaches v *via* w.

Lemma 3. *If a 3-regular graph $G \in \mathcal{PP}_2$, then G is isomorphic to either $K_{3,3}$ or W_8 or P_{10}.*

Proof. Let $v \in V(G)$ be any vertex having neighbors $\{v_1, v_2, v_3\}$. Moreover, let S_i denote the set of vertices in $G\setminus\{v\}$ which are adjacent to exactly i vertices among $\{v_1, v_2, v_3\}$. Note that, as G has diameter 2, every vertex in $G\setminus\{v, v_1, v_2, v_3\}$ belongs to exactly one of S_1, S_2, and S_3.

Observe that as G is 3-regular, we must have $|S_3| \leq 2$. Moreover, if $|S_3| = 2$, then G is isomorphic to $K_{3,3}$.

If $|S_3| = 1$, then note that we must have $|S_2| \leq 1$ as otherwise, it will force one of the vertices among $\{v_1, v_2, v_3\}$ to have at least two neighbors in S_2, and hence have at least four neighbors in G, contradicting the 3-regularity of G.

Furthermore, if $|S_3| = |S_2| = 1$, then without loss of generality we may assume that $S_2 = \{u\} \subseteq N(v_1) \cap N(v_2)$. Thus u must reach v_3 via some vertex $u' \in S_1 \cap N(v_3)$. Now each vertex among $\{v_1, v_2, v_3\}$ already has three neighbors and thus there cannot be any other vertex in G. Also all vertices except u' have degree 3 at present. Thus the 3-regularity of G forces us to include a new vertex adjacent to u' in the graph, a contradiction. Therefore, G cannot have $|S_3| = |S_2| = 1$.

Thus if $|S_3| = 1$, then we must have $S_2 = \emptyset$. However, due to the 3-regularity of G, each vertex among $\{v_1, v_2, v_3\}$ has exactly one neighbor in S_1. Let us assume that the neighbors of v_1, v_2, and v_3 in S_1 are w_1, w_2, and w_3, respectively. Now, as each vertex among $\{v_1, v_2, v_3\}$ already has three neighbors, there cannot be any other vertex in G. In fact, each vertex of G other than w_1, w_2, and w_3 has degree 3 already. Thus w_1, w_2, and w_3 must reach all of v_1, v_2, and v_3 either directly or via themselves. That forces w_1, w_2, and w_3 to create a triangle, contradicting the triangle-free property of G. Therefore, G cannot have $|S_3| = 1$.

Now let us consider the situation where $|S_3| = 0$. First observe that $|S_2| \leq 3$ in this case, as otherwise one vertex among $\{v_1, v_2, v_3\}$ has degree at least 4.

If $|S_2| = 3$, then without loss of generality, we may assume that $S_2 = \{u_1, u_2, u_3\}$ where $u_i \in S_2\setminus N(v_i)$ for all $i \in \{1, 2, 3\}$. Now, as every vertex among $\{v_1, v_2, v_3\}$ already has three neighbors each, there cannot be any other vertex in G. Hence $S_1 = \emptyset$. Thus, in particular, the vertex u_1 must reach v_1 via some vertex of S_2. That will create a triangle, a contradiction. So $|S_2| \leq 2$.

If $|S_2| = 2$ with $S_2 = \{u_1, u_2\}$, then without loss of generality, assume that v_3 is a common neighbor of u_1 and u_2. Moreover, the sum of the degrees of v_1, v_2, and v_3 at the moment is 7 and therefore we must have exactly two more vertices in G. Thus, $|S_1| = 2$ and we may assume that $S_1 = \{w_1, w_2\}$. Observe that u_1 and u_2 must have exactly one neighbor each in S_1 as they cannot be adjacent to each other in order to avoid creating a triangle. This implies that there are exactly two edges between the sets S_1 and S_2. Thus at least one vertex of S_1 does not have degree 3 unless w_1 and w_2 are adjacents. Hence w_1 and w_2 must be adjacent. This implies that w_1 and w_2 do not have a common neighbor. Therefore, without loss of generality, we may assume that w_i and u_i are adjacent to v_i for all $i \in \{1, 2\}$. Hence the edges u_1w_2 and u_2w_1 are in G. Observe that G is isomorphic to W_8 in this case.

We have a total of nine vertices, not possible for a 3-regular graph (as nine is an odd number).

If $|S_2| = 0$, then each v_i has exactly two neighbors w_i and w'_i in S_1 for all $i \in \{1, 2, 3\}$. Without loss of generality, w_1 reaches v_2 and v_3 via w_2 and w_3, respectively. As w_1 already has three neighbors, w'_2 and w'_3 must reach v_1 via w'_1. Note that, w_2 cannot reach v_3 via w_3 in order to avoid a triangle. Therefore, w_2 reaches v_3 via w'_3. Similarly, w_3 reaches v_2 via w'_2. This implies that G is isomorphic to P_{10}. □

2.3. Characterizing not regular graphs in $\mathcal{PP}_2$ having minimum degree 3

Finally, the case where $\delta(G) = 3$ and $\Delta(G) \geq 4$ is handled. The proof of Lemma 4 is lengthy; in order to make the proof easier to follow, we have divided it into several claims and lemmas and presented it in a separate subsection.

Lemma 4. *If $\delta(G) = 3$ and $\Delta(G) \geq 4$ for a graph $G \in \mathcal{PP}_2$, then G is isomorphic to $K_{3,4}$, $K^+_{3,4}$, W^+_8, M_{11}, M^-_{11}, or $M^=_{11}$.*

We will begin by presenting some basic conventions to be used throughout this section.

2.3.1. Conventions used in the proof of Lemma 4

Let $v \in V(G)$ be a vertex with maximum degree and let $N(v) = \{v_1, v_2, v_3, v_4\} \cup X$ where X may or may not be $\emptyset$. Moreover, let S_i denote the set of vertices in $G \setminus \{v\}$ which are adjacent to exactly i vertices among $\{v_1, v_2, v_3, v_4\}$ (see Fig. 3). Furthermore, let m_2 be the cardinality of a maximum matching in S_2.

2.3.2. Basic structural properties

The proof of Lemma 4 runs via a series of claims and lemmas. In the case of the claims, we always assume that $\delta(G) = 3$ and $d(v) = \Delta(G) \geq 4$ for a graph $G \in \mathcal{PP}_2$.

This brings us to our first two observations.

Claim 1. *A vertex in $S_3 \cup S_4$ is not adjacent to a vertex in $S_2 \cup S_3 \cup S_4$.*

Proof. Any vertex in $S_4 \cup S_3$ has at least one neighbor in common with any vertex in $S_2 \cup S_3 \cup S_4$. Hence, any edge between $S_4 \cup S_3$ and $S_2 \cup S_3 \cup S_4$ will create a triangle. □

Claim 2. *The value of $|S_4| + |S_3| + m_2$ is at most 2.*

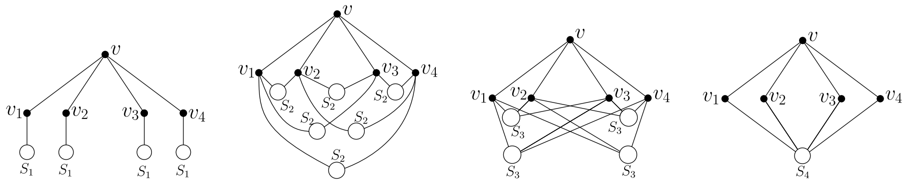

Fig. 3. Illustrations of our convention.

Proof. Observe that each vertex of S_3 reaches three of the four vertices among $\{v_1, v_2, v_3, v_4\}$ directly and one of them via a vertex of S_1 (by Claim 1). Therefore, if we contract all the edges between $\{v_1, v_2, v_3, v_4\}$ and S_1, then each vertex of S_3 becomes adjacent to every vertex of $\{v_1, v_2, v_3, v_4\}$. Moreover, suppose there is an edge having both its end vertices in S_2, then these end vertices cannot have a common neighbor, else a triangle is induced. Contracting this edge, the new resulting vertex will correspond to a vertex adjacent to each of $\{v_1, v_2, v_3, v_4\}$. Therefore, if $|S_4| + |S_3| + m_2 \geq 3$, then G will contain a $K_{4,4}$-minor. □

Therefore, in particular, the above claim implies that $|S_4| + |S_3| \leq 2$. This bound will be the basis of our case study.

2.3.3. Case: $|S_4| + |S_3| = 2$

Claim 3. *If $|S_4| + |S_3| = 2$, then $S_2 = \emptyset$.*

Proof. Recall that each vertex of S_3 reaches three of the four vertices among $\{v_1, v_2, v_3, v_4\}$ directly and one of them via a vertex of S_1 (by Claim 1). If $|S_4| + |S_3| = 2$, then $m_2 = 0$ (by Claim 2). Now if $S_2 \neq \emptyset$, then each vertex of S_2 reaches two of the four vertices among $\{v_1, v_2, v_3, v_4\}$ directly and two of them via vertices of S_1. This is implied by Claim 1 and the following: any vertex in S_2 cannot reach the two non-adjacent vertices in $\{v_1, v_2, v_3, v_4\}$ via a vertex in S_2 (as $m_2 = 0$). Hence, contract all edges between S_1 and $\{v_1, v_2, v_3, v_4\}$ to obtain a $K_{4,4}$, a contradiction. □

Claim 4. *If $S_4 \cup S_3 = \{u_1, u_2\}$ and v_i is a common neighbor of u_1 and u_2 for some $i \in \{1, 2, 3, 4\}$, then no vertex $w \in S_1$ is adjacent to v_i.*

Proof. By Claim 3, we know that $S_2 = \emptyset$. Therefore, each vertex in $S_4 \cup S_3$ would reach non-adjacent vertices in $\{v_1, v_2, v_3, v_4\}$ via some vertices in S_1. If there exists a vertex $w \in S_1 \cap N(v_i)$, where v_i is as defined in the lemma statement, then w cannot be adjacent to any of $\{u_1, u_2\}$ (else a triangle is induced). Since $S_2 = \emptyset$ (by Claim 3), the vertex w reaches vertices in $\{v_1, v_2, v_3, v_4\} \setminus \{v_i\}$ via vertices in S_1. Contract all the edges between S_1 and $\{v_1, v_2, v_3, v_4\}$ except wv_i to obtain a $K_{4,4}$, a contradiction. □

Claim 5. *If $|S_4| + |S_3| = 2$, then $X = \emptyset$.*

Proof. Assume that $S_4 \cup S_3 = \{u_1, u_2\}$ and there exists an $x \in X$.

If $|S_4| = 2$, then x reaches u_1 and u_2 directly or via some other vertices not adjacent to $\{v_1, v_2, v_3, v_4\}$ (else a triangle is induced). Contract the edges between x and those vertices (if they exist) via which x reaches u_1, u_2 to obtain a $K_{3,5}$, a contradiction.

If $|S_4| = 1$ and without loss of generality $u_1 \in S_4$ and $u_2 \in S_3$, then x reaches u_1 directly or via some other vertex w_1 not adjacent to $\{v_1, v_2, v_3, v_4\}$ (else a triangle is induced). Contract the edge xw_1 if it exists. Observe that u_2 is adjacent to some vertex $w_2 \in S_1$ in order to reach all of $\{v_1, v_2, v_3, v_4\}$. Contract u_2w_2. Note that x reaches u_2 directly, or via w_2 or via some other vertex w_3 which is not adjacent to $\{v_1, v_2, v_3, v_4\}$. Contract xw_3, if it exists, to obtain a $K_{3,5}$, a contradiction.

If $|S_4| = 0$ and thus $u_1, u_2 \in S_3$, then u_1 and u_2 may be non-adjacent to the same or different vertices in $\{v_1, v_2, v_3, v_4\}$.

Case 1. If they are non-adjacent to different vertices of $\{v_1, v_2, v_3, v_4\}$, then without loss of generality assume that u_i is not adjacent to v_i for $i \in \{1, 2\}$. In this case, u_i must reach v_i via some $w_i \in S_1$ (by Claim 1) and x must reach u_i directly or via w_i or via some $w'_i \in S_1$ with $w'_i \notin \{w_1, w_2\}$. Contract the edges u_iw_i and xw'_i (if they exist) for all $i \in \{1, 2\}$ to obtain a $K_{3,5}$, a contradiction.

Case 2. If they are non-adjacent to the same vertex, without loss of generality say, v_4 of $\{v_1, v_2, v_3, v_4\}$, then v_4 must have at least two neighbors w_1, w_2 as the degree of v_4 is at least three. Also $w_1, w_2 \in S_1$ (by Claim 1). Due to Claims 3 and 4, we know that the only way for w_i to reach v_1 is via u_1 or u_2. Moreover, for each $i \in \{1, 2\}$, u_i must reach v_4 via some vertex of S_1 (by Claim 1). Therefore, we must have a perfect matching between $\{u_1, u_2\}$ and $\{w_1, w_2\}$. Without loss of generality, assume that perfect matching be $\{u_1w_1, u_2w_2\}$. Furthermore, observe that if x is adjacent to any one of u_1 or u_2, we can rename the vertices of $N(v)$ to reduce this to a case where $|S_4| \geq 1$, which we have already handled earlier in the proof of this lemma. Therefore, x must reach u_i via w_i or via some vertex $w'_i \notin \{u_1, u_2, w_1, w_2\}$ for all $i \in \{1, 2\}$. Contract the edges $u_i w_i$ and xw'_i (if they exist) for all $i \in \{1, 2\}$ to obtain a $K_{3,5}$, a contradiction.

This concludes the proof. □

Now that we have shown the set $X = \emptyset$ when $|S_4| + |S_3| = 2$, we can try to characterize the graphs for this case.

Lemma 5. *If $\delta(G) = 3$ and $d(v) = \Delta(G) \geq 4$ for a graph $G \in \mathcal{PP}_2$, then the following holds: if $|S_4| = 2$, then G is isomorphic to $K_{3,4}$.*

Proof. If $|S_4| = 2$, then $X = S_3 = S_2 = S_1 = \emptyset$ due to Claims 2–5. The only vertices in the graphs other than $\{v, v_1, v_2, v_3, v_4\}$ are the two vertices in S_4. Thus G is isomorphic to $K_{3,4}$. □

Claim 6. *It is not possible to have $|S_4| = |S_3| = 1$.*

Proof. If $|S_4| = |S_3| = 1$, then without loss of generality assume that $u_1 \in S_4$, $u_2 \in S_3 \setminus N(v_4)$. Observe that $X = S_3 = S_2 = \emptyset$ due to Claims 2, 3 and 5. Moreover, all the vertices in S_1 are adjacent to v_4 due to Claim 4. As u_2 must reach v_4 via some vertex of S_1, the set S_1 is not empty. However, any vertex of S_1 has exactly two neighbors, namely v_4 and u_2, contradicting $\delta(G) = 3$. □

Lemma 6. *If $\delta(G) = 3$ and $d(v) = \Delta(G) \geq 4$ for a graph $G \in \mathcal{PP}_2$, then the following holds: if $|S_3| = 2$, then G is isomorphic to W_8^+.*

Proof. Assume that $S_3 = \{u_1, u_2\}$. Thus $X = S_4 = S_2 = \emptyset$ due to Claims 2, 3 and 5. Observe that it is enough to consider the following two cases: (i) u_1 and u_2 are both non-adjacent to v_4, and (ii) u_i is non-adjacent to v_i for $i \in \{1, 2\}$.

Case (i). If u_1 and u_2 are both non-adjacent to v_4, then all the vertices in S_1 are adjacent to v_4 due to Claim 4. The vertices $v, v_1, v_2, v_3, v_4, u_1$, and u_2 and the vertices belonging to S_1 are all of the vertices of G. As $\delta(G) = 3$, each vertex of S_1 must be adjacent to both u_1 and u_2. Furthermore, S_1 must have at least two vertices, say w_1 and w_2, as $\delta(G) = 3$. Now contract vv_4 to obtain a $K_{3,5}$ induced by $\{u_1, u_2, (vv_4)\} \sqcup \{v_1, v_2, v_3, w_1, w_2\}$ where (vv_4) denotes the new vertex obtained by contracting the edge vv_4. Thus this case is not possible since $K_{3,5}$ is not projective planar.

Case (ii). If u_i is non-adjacent to v_i for $i \in \{1, 2\}$, then all the vertices in S_1 are adjacent to either v_1 or v_2 due to Claim 4. Therefore, the vertices $v, v_1, v_2, v_3, v_4, u_1$, and u_2 and the vertices belonging to S_1 are all of the vertices of G. Suppose that $S_1 = \{w_1, w_2, \dots, w_k, w'_1, w'_2, \dots, w'_r\}$ where w_is are adjacent to v_1 and w'_js are adjacent to v_2 where $(i, j) \in \{1, 2, \dots, k\} \times \{1, 2, \dots, r\}$. Each w_i reaches v_3 via u_1 and each w'_j is reaches v_3 via u_2 for all $(i, j) \in \{1, 2, \dots, k\} \times \{1, 2, \dots, r\}$. Furthermore as $\delta(G) = 3$, each w_i must be adjacent to some w'_j and each w'_p must be adjacent to some w_q. Without loss of generality and due to symmetry, we may assume that $1 \leq r \leq k$. If $k \geq 2$, then contract the edges $u_2 w'_j$ for all $j \in \{1, 2, \dots, r\}$ to obtain the new vertex $(u_2 w'_j)$, and contract the edge vv_1 to obtain the new vertex (vv_1). Observe that, in this contracted graph, the vertices $\{u_1, (vv_1), (u_2 w'_j)\} \sqcup \{w_1, w_2, v_2, v_3, v_4\}$ induce a $K_{3,5}$ subgraph, a contradiction. Therefore, $k = r = 1$. Thus G is isomorphic to W_8^+.

This ends the proof of the lemma. □

This concludes the case when we have $|S_4| + |S_3| = 2$. We will present the summary of it in the following lemma.

Lemma 7. *If $\delta(G) = 3$ and $d(v) = \Delta(G) \geq 4$ for a graph $G \in \mathcal{PP}_2$, then the following holds: if $|S_4| + |S_3| = 2$, then G is isomorphic to $K_{3,4}$ or W_8^+.*

Proof. Follows directly from Lemma 5, Claim 6, and Lemma 6. □

It remains to analyze the situations when $|S_4| + |S_3| \leq 1$. The first case is when $|S_4| + |S_3| = 1$.

2.3.4. Case: $|S_4| + |S_3| = 1$

Claim 7. *If $|S_4| + |S_3| = 1$, then $S_2 \neq \emptyset$.*

Proof. Let us assume the contrary and suppose that $S_2 = \emptyset$. Furthermore, assume that $S_4 \cup S_3 = \{u_1\}$ and that u_1 is adjacent to $\{v_1, v_2, v_3\}$. First we will show that it is not possible for any v_i, for $i \in \{1, 2, 3\}$, to have two neighbors in S_1.

Hence without loss of generality assume that v_1 is adjacent to $w_1, w_2 \in S_1$. Observe that both w_1 and w_2 must reach v_2, v_3, and v_4 via some vertices from $S_1 \setminus \{w_1, w_2\}$ and u_1 must reach v_4 via some vertex from $S_1 \setminus \{w_1, w_2\}$ (all due to

Claim 1). Now contract all the edges between $\{v_2, v_3, v_4\}$ and $S_1 \setminus \{w_1, w_2\}$ to obtain a $K_{4,4}$-minor, a contradiction. Thus each v_is, for $i \in \{1, 2, 3\}$, can have at most one neighbor in S_1.

However, $\delta(G) = 3$ implies that each v_i must be adjacent to exactly one vertex (say) w_i from S_1 for $i \in \{1, 2, 3\}$. Now note that w_1 must reach v_2 and v_3 via w_2 and w_3, respectively and w_2 must reach v_3 via w_3. This creates a triangle induced by $\{w_1, w_2, w_3\}$ in G, a contradiction. □

Claim 8. *If $|S_4| = 1$ and $|S_3| = 0$, then $X = \emptyset$.*

Proof. Let $S_4 = \{u_1\}$ and let $x \in X$. Claim 7 implies the existence of a vertex $w_1 \in S_2$. Without loss of generality assume that w_1 is adjacent to v_1 and v_2.

Note that x reaches u_1 directly or via some vertex a (say) not adjacent to any of $\{v_1, v_2, v_3, v_4\}$ in order to avoid creating a triangle in G. Moreover, w_1 reaches x, v_3, and v_4 via some vertices not adjacent to any of $\{v_1, v_2\}$. Let A denote the set of vertices via which w_1 reaches x, v_3, and v_4. Contract the edges between $A \setminus \{a\}$ and w_1 and the edge xa. The vertices $\{v, w_1, u_1\} \sqcup \{x, v_1, v_2, v_3, v_4\}$ form the partition of a $K_{3,5}$-minor, a contradiction. □

Claim 9. *If $|S_4| = 1$ and $|S_3| = 0$, then $m_2 = 1$.*

Proof. By Claim 2, $m_2 \le 1$. Next, suppose that $S_4 = \{u_1\}$ and $m_2 = 0$. That means every vertex of S_2 (which is non-empty by Claim 7) must reach its non-adjacent v_is via vertices of S_1 (by Claim 1 and $m_2 = 0$). Therefore, if $|S_2| \ge 2$, then contracting the edges between S_1 and $\{v_1, v_2, v_3, v_4\}$ will create a $K_{4,4}$-minor.

Thus, we have $|S_2| = 1$. Without loss of generality assume that $S_2 = \{w\}$ and that w is adjacent to v_1 and v_2.

Notice that w must reach v_3 and v_4 via $w_3, w_4 \in S_1$, respectively (by Claim 1 and $m_2 = 0$). Thus to avoid creating a triangle, w_3 must reach v_4 via $w_4' \in S_1$. Moreover, to avoid creating a triangle, w_4' must reach v_1 via some $w_1 \in S_1$.

Notice that, w_1 reaches v_2, v_3, and v_4 via some elements of S_1. Thus, if we contract all the edges between S_1 and $\{v_2, v_3, v_4\}$, we will create a $K_{4,4}$-minor.

Therefore, $m_2 \ge 1$. Since $m_2 \le 1$ by Claim 2, we have $m_2 = 1$. □

Lemma 8. *If $\delta(G) = 3$ and $\Delta(G) \ge 4$ for a graph $G \in \mathcal{PP}_2$, then the following holds: if $|S_4| = 1$ and $|S_3| = 0$, then G is isomorphic to $K^*_{3,4}$.*

Proof. Let $S_4 = \{u_1\}$ and, thus, by Claim 9 we know that $m_2 = 1$. Then without loss of generality we may assume the existence of an edge w_1w_2 such that $w_1, w_2 \in S_2$, w_1 is adjacent to v_1 and v_2, and w_2 is adjacent to v_3 and v_4. If there are no other vertex or edge in G, then G is isomorphic to $K^*_{3,4}$.

However, if there is another vertex $w_3 \in S_2$ and if w_3 reaches $\{v_1, v_2, v_3, v_4\}$ directly or via some vertices except w_1 and w_2, then contract the edge w_1w_2. Also contract the edges between w_3 and the vertices via which w_3 reaches v_is, for $i \in \{1, 2, 3, 4\}$. This will result in a $K_{4,4}$-minor.

On the other hand, if there is a $w_3 \in S_2$ and if w_3 reaches $\{v_1, v_2, v_3, v_4\}$ directly or via w_i for some $i \in \{1, 2\}$, then G is the graph F_1 (depicted in Fig. 2(b)) that contains the graph F_0 (depicted in Fig. 2(a)) as a subgraph, and thus as a minor. □

So far we were dealing with the case when G is a graph with $|S_4| = 1$ and $|S_3| = 0$. Now we turn our attention towards the case when G is a graph with $|S_4| = 0$ and $|S_3| = 1$. Initially, we will observe some properties that this condition implies. However, finally, the satisfaction of those properties will turn out to be impossible, thereby proving that there are no required graphs with $|S_4| = 0$ and $|S_3| = 1$.

Claim 10. *If $|S_4| = 0$ and $|S_3| = 1$, then $X = \emptyset$.*

Proof. Suppose that $X \neq \emptyset$ and $x \in X$. Let $S_3 = \{u_1\}$ and without loss of generality let u_1 be adjacent to v_1, v_2, v_3. Therefore, u_1 must reach v_4 via $w_4 \in S_1$ (by Claim 1). We know that $S_2 \neq \emptyset$ due to Claim 7. Also let $u_2 \in S_2$.

If u_2 reaches v_4 directly or via any vertex other than w_4, then contract the edge u_1w_4, all the edges connecting u_2 to its neighbors via which u_2 reaches v_1, v_2, v_3 or v_4, and all the edges connecting x to its neighbors via which x reaches u_1 and u_2, in order to obtain a $K_{3,5}$-minor (see the vertices in the partition $\{v, u_1, u_2\} \sqcup \{x, v_1, v_2, v_3, v_4\}$).

Thus u_2 reaches v_4 via w_4 and nothing else. Hence without loss of generality, we may assume that u_2 is adjacent to v_1 and v_2 and that u_2 reaches v_3 via $w_3 \in S_1$ (if $w_3 \in S_2$, then w_3 must be adjacent to v_3, v_4 and by similar arguments as the previous case existence $K_{3,5}$ minor can be shown). Moreover, as $\delta(G) = 3$, there must be a $w_4' \in S_1$ adjacent to v_4. On the other hand, v_1 cannot have a neighbor in S_1, as otherwise we may contract all the edges between S_1 and $\{v_2, v_3, v_4\}$ to obtain a $K_{4,4}$-minor. Therefore, w_4' must reach v_1 via u_1 (if w_4' reaches v_1 via u_2, then a $K_{3,5}$-minor is formed: see the vertices in the partition $\{v, u_1, u_2\} \sqcup \{x, v_1, v_2, v_3, v_4\}$). Now contract the edges u_1w_4', u_2w_3, u_2w_4, and all the edges connecting x to its neighbors via which x reaches u_1, u_2. If x is adjacent to w_4, then also contract the edge connecting x to its neighbor via which x reaches w_4'. This creates a $K_{3,5}$-minor (see the vertices in the partition $\{v, u_1, u_2\} \sqcup \{x, v_1, v_2, v_3, v_4\}$). □

Claim 11. *If $|S_4| = 0$ and $|S_3| = 1$, then $m_2 = 1$.*

Proof. We already know that $m_2 = 0$ or 1 due to Claim 2.

If $m_2 = 0$, then $|S_2| \leq 1$ as otherwise we can contract all the edges between S_1 and $\{v_1, v_2, v_3, v_4\}$ to obtain a $K_{4,4}$-minor. Thus $|S_2| = 1$ due to Claim 7. Without loss of generality assume that $S_3 = \{u_1\}$, $S_2 = \{u_2\}$, u_1 is adjacent to v_1, v_2, v_3 and u_2 is adjacent to v_1. If v_1 is adjacent to some vertex of S_1, then we can contract all the edges between S_1 and $\{v_2, v_3, v_4\}$ and obtain a $K_{4,4}$-minor. Thus v_1 cannot have a neighbor in S_1. Hence every vertex of S_1 must reach v_1 via u_1 or u_2.

If u_2 is adjacent to v_4 as well, then u_2 must reach v_2, v_3 via some $w_2, w_3 \in S_1$, respectively. Now w_2 must reach v_3 via some vertex $w'_3 \in S_1$. Observe that it is not possible to have $w_3 = w'_3$ as G is triangle-free. Now w'_3 must reach v_1 via u_1 or u_2. In any case, this will create a triangle. Therefore, u_2 is not adjacent to v_4. Thus we may assume without loss of generality that u_2 is adjacent to v_2.

If u_2 is adjacent to v_2, then u_2 must reach v_3 via some $w_3 \in S_1$ and w_3 must reach v_4 via some $w_4 \in S_1$. Observe that u_2 cannot be adjacent to w_4 in order to avoid creating a triangle. Therefore, u_2 must reach v_4 via some $w'_4 \in S_1$. Note that w'_4 must reach v_3 via some distinct $w'_3 \in S_1$ in order to avoid creating triangle.

By what we have already noted above in this proof, we know that the only way for w'_3 to reach v_1 is via u_1 or u_2. In each case a triangle will be created. □

Claim 12. *If $|S_4| = 0$, then it is not possible to have $|S_3| = 1$.*

Proof. Suppose the contrary. We already know that $m_2 = 1$ due to Claim 11. Thus note that without loss of generality we may assume that $S_3 = \{u_1\}$, $u_2, u_3 \in S_2$, $u_2u_3 \in E(G)$, u_1 is adjacent to $\{v_1, v_2, v_3\}$, u_2 is adjacent to $\{v_1, v_2\}$, u_3 is adjacent to $\{v_3, v_4\}$ and u_1 reaches v_4 via $w_1 \in S_1$.

Observe that any vertex in $S_2 \setminus \{u_2, u_3\}$ will force a $K_{4,4}$-minor or a F_2 (depicted in Fig. 2(c) which contains F_0, depicted in Fig. 2(a), as a minor) as a subgraph of G (this is similar to the second half of the proof of Lemma 8). Thus we may infer that $S_2 = \{u_2, u_3\}$. Note that, according to the partial description of G till now w_1 has two neighbors. Due to the minimum degree requirement, it must have another neighbor. If u_2 is a neighbor of w_1, then note that v_1, v_2, v_3, and w_1 are neighbors of u_1 such that v is adjacent to three of them and u_2 is adjacent to three of them. This reduces the case to where $|S_3| = 2$, which is already taken care of.

Therefore, w_1 is adjacent to another vertex $w_2 \in S_1$. Note that, if w_2 reaches v_1, v_2, v_3, and v_4 directly or via vertices from S_1, then contracting the edge u_2u_3 and all the edges between S_1 and $\{v_1, v_2, v_3, v_4\}$ except for the edge having w_2 as an endpoint creates a $K_{4,4}$-minor. If w_2 is adjacent to either of v_1 or v_2 (without loss of generality assume it is adjacent to v_1), then it reaches v_2 and v_3 via vertices of $A \subseteq S_1$. Now contracting edges w_1v_4, u_2u_3 and edges between w_2 and A, we get a $K_{4,4}$-minor. Thus, the following situation is forced: w_2 is adjacent to v_3 and u_2. This creates the subgraph F_2 (depicted in Fig. 2(c)) in G which contains F_0 (depicted in Fig. 2(a)) as a minor, a contradiction. □

This concludes the case when we have $|S_4| + |S_3| = 1$. We will present the summary of it in the following lemma.

Lemma 9. *If $\delta(G) = 3$ and $d(v) = \Delta(G) \geq 4$ for a graph $G \in \mathcal{PP}_2$, then the following holds: if $|S_4| + |S_3| = 1$, then G is isomorphic to $K^*_{3,4}$ or W^+_8.*

Proof. Follows directly from Lemmas 8 and 12. □

This brings us to the case where $|S_4| + |S_3| = 0$.

2.3.5. Case: $|S_4| + |S_3| = 0$

Claim 13. *It is not possible to have $|S_4| = |S_3| = |S_2| = 0$.*

Proof. As $\delta(G) \geq 3$, each v_i must have at least two neighbors in S_1. Thus without loss of generality assume that v_i is adjacent to $w_i, w'_i \in S_1$ for all $i \in \{1, 2, 3, 4\}$. Moreover, without loss of generality, we may suppose that w_1 reaches v_i via w_i for all $i \in \{2, 3, 4\}$. Note that as G is triangle-free, $\{w_2, w_3, w_4\}$ is an independent set. Therefore, contracting the edges between the vertices of $\{v_1, v_2, v_3, v_4\}$ and the vertices of $(S_1 \setminus \{w_2, w_3, w_4\})$ creates a $K_{4,4}$-minor. □

Now we will consider the case when $|S_2| \geq 1$.

Claim 14. *If $|S_4| = |S_3| = 0$, then it is not possible for v_i, for all $i \in \{1, 2, 3, 4\}$, to have three or more neighbors in S_2.*

Proof. Let us assume the contrary. Without loss of generality suppose that v_1 is adjacent to $u_1, u_2, u_3 \in S_2$. Furthermore suppose that u_1 is adjacent to v_2 as well. Observe that $\{u_1, u_2, u_3\}$ is an independent set as G is triangle-free.

If u_2 or u_3 is also adjacent to v_2, then by renaming v_1 as v, the case reduces to $|S_3| + |S_4| \geq 1$ which has been handled before.

Thus without loss of generality, we may assume that u_2 is adjacent to v_3 and u_3 is adjacent to v_4. Then u_1 must reach v_3 and v_4; u_2 must reach v_2 and v_4; and u_3 must reach v_2 and v_3, via some vertices of $S_1 \cup S_2$. If they use vertices from S_2, then those vertices must be distinct. Let A be the vertices via which u_1, u_2, and u_3 reach v_2, v_3, and v_4. Contract the edges between $A \cap S_2$ and $\{u_1, u_2, u_3\}$. Also, contract the edges between $A \cap S_1$ and $\{v_2, v_3, v_4\}$. We will obtain a $K_{4,4}$-minor, a contradiction.

Thus we have considered all the cases up to symmetry and have proved the claim. □

Claim 15. *If $|S_4| = |S_3| = 0$, then it is not possible to have $u_1, u_2 \in S_2$ having $N(u_1) \cap \{v_1, v_2, v_3, v_4\} = N(u_2) \cap \{v_1, v_2, v_3, v_4\}$.*

Proof. Let us assume the contrary. Without loss of generality suppose that u_1 and u_2 are adjacent to both v_1 and v_2.

Note that it is not possible to have any vertex other than v, u_1, and u_2 adjacent to v_1 (or v_2) as otherwise our case will get reduced to the case where $|S_3| + |S_4| \geq 1$ by renaming v_1 as v which we have already taken care of.

Therefore, every vertex from $V(G) \setminus (N[v] \cup \{u_1, u_2\}) = A$ (say) is adjacent to either u_1 or u_2 in order to reach v_1 and v_2. Since $\delta(G) \geq 3$, v_3 has two more neighbors. Both these neighbors are adjacent to u_1 or u_2. If any vertex from A is adjacent to both u_1 and u_2, then one of u_1 or u_2, without loss of generality assumes u_1, has degree 4. Then our case will get reduced to the case where $|S_3| + |S_4| \geq 1$ by renaming u_1 as v which we have already taken care of.

As diameter of G is 2, u_1, u_2 must reach $N(v) \setminus \{v_1, v_2\}$ via some vertices from A. Let A_i be the set of vertices from A via which u_i reaches the vertices of $N(v) \setminus \{v_1, v_2\}$ where $i \in \{1, 2\}$. Due to the observation made in the previous paragraph, we know that the sets A_1 and A_2 are disjoint.

If $X \neq \emptyset$, then contract the edges between A_i and $\{u_i\}$ for each $i \in \{1, 2\}$ to obtain a $K_{3,5}$-minor, a contradiction (see the vertices in the partition $\{v, u_1, u_2\} \sqcup \{x, v_1, v_2, v_3, v_4\}$, where $x \in X$). Thus we may assume that $X = \emptyset$.

If there exists $u_3 \in S_2$, it must be adjacent to both u_1 and u_2 in order to reach them. This follows from Claim 14. But we have already shown that this is not possible. Thus there are no vertices in S_2 other than u_1 and u_2.

However as $\delta(G) \geq 3$, there are at least two neighbors $w_{i1}, w_{i2} \in S_1$ of v_i for $i \in \{3, 4\}$. Without loss of generality suppose that w_{31} reaches v_1 and v_2 via u_1. Therefore, w_{31} have to reach v_4 via a vertex of $S_1 \cap N(v_4)$, say w_{41}. Now as G is triangle free, w_{41} reaches v_1 and v_2 via u_2. If w_{32} is adjacent to w_{41}, then a triangle is induced as w_{32} has to be adjacent to u_1 or u_2 in order to reach v_1 and v_2. Thus w_{32} is not adjacent to w_{41}. Next, observe that w_{32} must reach w_{41} via u_2. Finally, w_{32} must reach v_4 via some vertex in $S_1 \cap N(v_4)$, say w_{42} and, then w_{42} must reach v_1 and v_2 via u_1.

This so-obtained graph is isomorphic to the graph F_3 depicted in Fig. 1. □

Claim 16. *If $|S_4| = |S_3| = 0$, then it is not possible to have three vertices of S_2 non-adjacent to v_i, for all $i \in \{1, 2, 3, 4\}$.*

Proof. Assume the contrary and let $u_1, u_2, u_3 \in S_2$ be non-adjacent to v_4. Contract all edges between $S_2 \setminus \{u_1, u_2, u_3\}$ and $\{u_1, u_2, u_3\}$. Also, contract the edges between S_1 and $\{v_1, v_2, v_3, v_4\}$. This will create a $K_{4,4}$-minor, a contradiction. □

Claim 17. *If $|S_4| = |S_3| = 0$, then $|S_2| \leq 4$.*

Proof. Follows directly from Claim 14, 15 and 16. □

Lemma 10. *If $\delta(G) = 3$ and $\Delta(G) \geq 4$ for a graph $G \in \mathcal{PP}_2$, then the following holds: if $|S_4| = |S_3| = 0$ and $|S_2| = 4$, then G is isomorphic to $M_{11}^{=}$, M_{11}^{-}, or M_{11}.*

Proof. Assume that $S_2 = \{u_1, u_2, u_3, u_4\}$. Thus due to Claims 14–16 without loss of we may suppose that u_i is adjacent to v_i and v_{i+1}, for all $i \in \{1, 2, 3, 4\}$ and the + operation on the indices is taken modulo 4. If S_2 does not have a perfect matching, then it will force a $K_{4,4}$-minor. Thus we must have the edges u_1u_3 and u_2u_4. Also $X = \emptyset$, as otherwise there will be a $K_{3,5}$-minor in G (see the partition $\{v, (u_1u_3), (u_2u_4)\} \sqcup \{x, v_1, v_2, v_3, v_4\}$, where $x \in X$).

Next, we claim that for all $i \in \{1, 2, 3, 4\}$, $|N(v_i) \cap S_1| \leq 1$. Suppose $|N(v_1) \cap S_1| \geq 2$. Let $w_1, w_2 \in N(v_1) \cap S_1$. Then w_1 reaches v_2 either via u_2 or some vertex in $N(v_2) \cap S_1$, and w_1 reaches v_4 either via u_3 or some vertex in $N(v_4) \cap S_1$. Similarly w_2 reaches v_2 and v_4. Contract the edge v_2u_2 and v_4u_3 and the edges between S_1 and $\{v_2, v_4\}$ to obtain a $K_{3,5}$-minor, a contradiction (see the partition $\{v_1, (v_2u_2), (v_4u_3)\} \sqcup \{w_1, w_2, u_1, u_4, v\}$). Thus $|N(v_1) \cap S_1| \leq 1$. A similar analysis holds for v_2, v_3, and v_4. Hence, for all $i \in \{1, 2, 3, 4\}$, $|N(v_i) \cap S_1| \leq 1$.

Next, we claim that $|S_1| \leq 2$. If $|S_1| \geq 3$, then without loss of generality assume that $w_1 \in N(v_1) \cap S_1$. If w_1 does not reach $\{v_2, v_3, v_4\}$ via u_2, u_3, then it uses vertices from S_1, forcing a $K_{4,4}$-minor in G. Thus w_1 has to use at least one of u_2, u_3 to reach $\{v_2, v_3, v_4\}$. Suppose w_1 is not adjacent to u_2, then it is adjacent to u_3. Then w_1 reaches v_2 via $w_2 \in S_1$. Now w_2 cannot reach v_3 via u_2 or u_3, else a triangle is induced in G. Thus w_2 reaches v_3 via $w_3 \in S_1$. This forms a $K_{4,4}^{-}$-minor in G (see the partition $\{v, (u_1u_3), (u_2u_4), (w_1w_2w_3)\} \sqcup \{v_1, v_2, v_3, v_4\}$). Next, suppose w_1 is not adjacent to u_3, then it is adjacent to u_2. Then, similarly, a $K_{4,4}^{-}$-minor is obtained in G. Thus $|S_1| \leq 2$.

Observe that if $|S_1| = 0, 1$, or 2, then G is isomorphic to $M_{11}^{=}$, M_{11}^{-}, or M_{11}, respectively. It is easy to observe the cases when $|S_1| = 0, 1$. For $|S_1| = 2$, without loss of generality we will have two cases: when the two vertices of S_1 are adjacent to v_1 and v_2; and when the two vertices of S_1 are adjacent to v_1 and v_3. In the first case, we get a graph isomorphic to M_{11}. In the second case, we get a $K_{4,4}$-minor in G. We briefly describe the second case. Let $N(v_1) \cap S_1 = \{w_1\}$ and

$N(v_1) \cap S_3 = \{w_2\}$. Then w_1 reaches v_2 via u_2, and v_4 via u_3. And w_2 reaches v_4 via u_4, and v_2 via u_1. All these edges are forced, or else a triangle is induced in G. The only possible way that w_1 reaches w_2, without inducing a triangle in G, is directly by an edge. This forms a $K_{4,4}$-minor in G (see the partition $\{w_1, (vv_3), u_1, u_4)\} \sqcup \{w_2, v_1, (u_3v_4), (u_2v_2)\}$). □

Lemma 11. *If $\delta(G) = 3$ and $d(v) = \Delta(G) \geq 4$ for a graph $G \in \mathcal{PP}_2$, then the following holds: if $|S_4| = |S_3| = 0$ and $|S_2| = 3$, then G is isomorphic to $M_{11}^{=}$, M_{11}^{-}, or M_{11}.*

Proof. Assume that $S_2 = \{u_1, u_2, u_3\}$. Thus due to Claim 14, 15 and 16, without loss of generality, we may suppose that u_i is adjacent to v_i and v_{i+1}, for all $i \in \{1, 2, 3\}$. If u_1 is not adjacent to u_3, then it will force a $K_{4,4}$-minor. Thus we must have the edge u_1u_3.

If $N(v_2) \cap S_1 \neq \emptyset$, then let $w_1 \in N(v_2) \cap S_1$. Now w_1 reaches v_3 via $w_2 \in S_1$. This forces a $K_{4,4}$-minor (see the partition $\{v, (u_1u_3), u_2, (w_1w_2)\} \sqcup \{v_1, v_2, v_3, v_4\}$). Thus $N(v_2) \cap S_1 = \emptyset$. Similarly, by symmetry, $N(v_3) \cap S_1 = \emptyset$.

Therefore, every vertex in S_1 is adjacent to u_2 to reach v_2 and v_3. Also, every vertex in $N(v_1) \cap S_1$ is adjacent to u_3 to reach v_3, and every vertex in $N(v_4) \cap S_1$ is adjacent to u_1 to reach v_2.

Next to satisfy $\delta(G) \geq 3$, $N(v_1) \cap S_1$ and $N(v_4) \cap S_1$ have at least one vertex. If $|N(v_1) \cap S_1| \geq 2$, then let $w_1, w_1' \in N(v_1) \cap S_1$ and $w_2 \in N(v_4) \cap S_1$. This forces a $K_{3,5}$-minor (see the vertices in the partition $\{(u_1u_3), u_2, (v_1vv_4)\} \sqcup \{w_1, w_1', v_2, v_3, w_2\}$). Thus $N(v_1) \cap S_1 = \{w_1\}$ and $N(v_4) \cap S_1 = \{w_2\}$.

Now let us consider X. Suppose $X \neq \emptyset$: let $x \in X$. Both u_1 and u_3 are not adjacent to x, else $|S_3| \geq 1$ which we have dealt earlier. If u_1 and u_3 do not use w_2 and w_1, respectively, to reach x, then a $K_{3,5}$-minor is forced (see the vertices in the partition $\{(u_1u_3), (w_1u_2w_2), v\} \sqcup \{x, v_1, v_2, v_3, v_4\}$). Hence x is adjacent to w_1, w_2.

If $|X| \geq 2$, then replacing v by w_1 we have $|S_3| \geq 1$ which we have dealt earlier. Hence $|X| \leq 1$.

Observe that if $|X| = 0$ or 1, then G is M_{11}^{-} or M_{11}, respectively. □

Lemma 12. *If $\delta(G) = 3$ and $d(v) = \Delta(G) \geq 4$ for a graph $G \in \mathcal{PP}_2$, then the following holds: if $|S_4| = |S_3| = 0$ and $|S_2| = 2$, then G is isomorphic to $K_{3,4}$, $K_{3,4}^*$, or W_8^+.*

Proof. Assume that $S_2 = \{u_1, u_2\}$. Thus due to Claims 14–16 without loss of generality we may suppose one of the two scenarios: (i) u_1 is adjacent to v_1, v_2, and u_2 is adjacent to v_2 and v_3, (ii) u_1 is adjacent to v_1, v_2, and u_2 is adjacent to v_3 and v_4.

Case (i): First suppose u_1 is adjacent to v_1, v_2, and u_2 is adjacent to v_2 and v_3. Observe that v_2 cannot be adjacent to any vertex of S_1, as otherwise a $K_{4,4}$-minor will be created. Also $|N(v_4) \cap S_1| \geq 2$ as $\delta(G) \geq 3$. Every vertex in $N(v_1) \cap S_1$ reaches v_2 via u_2, and every vertex in $N(v_3) \cap S_1$ reaches v_2 via u_1.

Now u_1 reaches v_4 via $w_4 \in N(v_4) \cap S_1$. w_4 cannot reach v_3 via any vertex in $N(v_3) \cap S_1$, as a triangle is induced. Thus w_4 reaches v_3 via u_2.

Now $w_4' \in (N(v_4) \cap S_1) \setminus \{w_4\}$ reaches v_2 via u_1 or u_2. Next, w_4' cannot use $N(v_3) \cap S_1$ or $N(v_1) \cap S_1$ to reach the yet unreached vertex in $\{v_1, v_3\}$, else a triangle is induced. Thus w_4' is adjacent to both u_1 and u_2. Similarly, every vertex in $N(v_4) \cap S_1$ is adjacent to u_1 and u_2.

Thus any vertex in $N(v_1) \cap S_1$ cannot reach any vertex in $N(v_3) \cap S_1$ via any vertex of $N(v_4) \cap S_1$. Hence every vertex in $N(v_1) \cap S_1$ is adjacent to every vertex in $N(v_3) \cap S_1$. This will reduce the case to the case of $|S_3| + |S_4| \geq 1$ where u_1 plays the role of v (note that u_1 and u_2 share at least three common neighbors).

Case (ii): Next assume that u_1 is adjacent to v_1, v_2, and u_2 is adjacent to v_3 and v_4. Here u_1 is adjacent to u_2, as otherwise a $K_{4,4}$-minor will be created.

Note that $|N(v_i) \cap S_1| \geq 1$ as $\delta(G) \geq 3$. All vertices of $N(v_1) \cap S_1$ and $N(v_2) \cap S_1$ cannot be adjacent to u_2 as a vertex in $N(v_1) \cap S_1$ reaches v_2 via a vertex in $N(v_2) \cap S_1$. Without loss of generality assume that $w_1 \in N(v_1) \cap S_1$ is not adjacent to u_2. If $x \in X$, then a $K_{3,5}$-minor will be created (see the partition $\{v, w_1, (u_1u_2)\} \sqcup \{x, v_1, v_2, v_3, v_4\}$). Hence $X = \emptyset$.

Next, we claim that $|N(v_i) \cap S_1| \leq 1$. Suppose $w_1, w_1' \in N(v_1) \cap S_1$. Both of them cannot be adjacent to u_2, else this case reduces to $|S_3| + |S_4| \geq 1$ where v_1 plays the role of v (note that v_1 and u_2 have at least three neighbors in common). Without loss of generality assume that $w_1 \in N(v_1) \cap S_1$ is not adjacent to u_2. Thus it reaches v_2, v_3, v_4 via vertices in S_1. If w_1' also reaches v_2, v_3, v_4 via vertices in S_1, then a $K_{4,4}$-minor will be created (see the partition $\{v, w_1, w_1', (u_1u_2)\} \sqcup \{v_1, v_2, v_3, v_4\}$). If w_1' reaches v_3, v_4 via u_2, then let w_1' reach v_2 via w_2. Now w_2 cannot be adjacent to u_2, else a triangle is induced. Thus w_2 reaches v_3, v_4 via vertices in S_1. This forces a $K_{4,4}$-minor (see the partition $\{v, w_1, (w_1'w_2), (u_1u_2)\} \sqcup \{v_1, v_2, v_3, v_4\}$). Hence $|N(v_i) \cap S_1| \leq 1$; and since $|N(v_i) \cap S_1| \geq 1$ as $\delta(G) \geq 3$, we have $|N(v_i) \cap S_1| = 1$.

Suppose $N(v_i) \cap S_1 = \{w_i\}$, for all $i \in \{1, 2, 3, 4\}$. Then w_1 reaches v_2 via w_2, and w_3 reaches v_4 via w_4. Thus w_1w_2 and w_3w_4 are edges in G. Moreover, due to symmetry, without loss of generality, we may assume the edges w_2u_2, w_3u_1 as forced as well.

This reduces the case to Case (i) of this proof where u_1 plays the role of v. □

Claim 18. *If $|S_4| = |S_3| = 0$, then it is not possible to have $|S_2| = 1$.*

Proof. Assume that $S_2 = \{u_1\}$ and u_1 is adjacent to v_1, v_2. Note that as $|S_2| = 1$, u_1 must reach v_3, v_4 via (say) $w_{31}, w_{41} \in S_1$, respectively.

Notice that u_1 and v_i cannot have two common neighbors from S_1, as otherwise the case will be reduced to $|S_2| \geq 2$ where u_1 plays the role of v.

Furthermore as $\delta(G) \geq 3$, v_3 must have another neighbor $w_{32} \in S_1$. As w_{32} cannot be adjacent to u_1, it must reach v_1, v_2 via $w_{11}, w_{21} \in S_1$, respectively. Observe that w_{11} is not adjacent to w_{21} in order to avoid creating a triangle. Thus w_{21} must reach v_1 via some $w_{12} \in S_1$.

Now contract all the edges between S_1 and $\{v_2, v_3, v_4\}$. This will create a $K_{4,4,}$-minor, a contradiction (see the partition $\{v, u_1, w_{11}, w_{12}\} \sqcup \{v_1, v_2, v_3, v_4\}$). □

This concludes the case when we have $|S_4| + |S_3| = 0$. We will present the summary of it in the following lemma.

Lemma 13. *If $\delta(G) = 3$ and $d(v) = \Delta(G) \geq 4$ for a graph $G \in \mathcal{PP}_2$, then the following holds: if $|S_4| + |S_3| = 0$, then G is isomorphic to $K_{3,4}$ $K^*_{3,4}$, W_8^+, $M_{11}^=$, M_{11}^-, or M_{11}.*

Proof. Follows directly from Lemmas 10, 11, and 12, and Claims 13 and 18. □

Finally, we are ready to prove Lemma 4.

Proof of Lemma 4. The result readily follows from Lemmas 7, 9, and 13. □

2.4. Concluding the proof of Theorem 1

At last, we can conclude the proof of Theorem 2.

Proof of Theorem 2. The result readily follows from Lemmas 1, 2, 3, and 4. □

3. Direct implications

In Theorem 2, we have characterized all triangle-free projective-planar graphs having diameter 2. This has an immediate theoretical implication in the theory of graph homomorphisms of colored mixed graphs, signed graphs, and oriented graphs. We are going to discuss them here.

First let us start with colored mixed graphs which were introduced by Nešetřil and Raspaud [15]. An (m, n)*-colored mixed graph* G is a graph having m different types of arcs and n different types of edges. Moreover, *colored homomorphism* from an (m, n)-colored mixed graph G to another (m, n)-colored mixed graph H is a vertex mapping $f : V(G) \to V(H)$ such that for any arc (resp., edge) uv of G, the induced image $f(u)f(v)$ is also an arc (resp., edge) of the same type in H. Observe that for $(m, n) = (0, 1), (1, 0), (0, 2)$, and $(0, k)$ the study of colored homomorphism of (m, n)-colored mixed graphs is the same as studying homomorphisms of undirected graphs [11], oriented graphs [21], 2-edge-colored graphs [18], and k-edge-colored graphs [1], respectively. Each of these is a well-studied topic.

Generalizing the notion of oriented absolute cliques and oriented absolute clique number[3] [12], Bensmail, Duffy, and Sen [5] introduced the notion of (m, n)-clique and (m, n)-absolute clique number. An (m, n)*-clique* C is an (m, n)-colored mixed graph that does not admit a colored homomorphism to any other (m, n)-colored mixed graph having strictly fewer vertices. Given a family $\mathcal{F}$ of (m, n)-colored mixed graphs,

$$\omega_{a(m,n)}(\mathcal{F}) = \max\{|V(G)| : G \in \mathcal{F} \text{ is an } (m, n)\text{-clique}\}.$$

A handy characterization of an (m, n)-clique is proved by Bensmail, Duffy, and Sen [5].

Proposition 1 ([5])**.** *An (m, n)-colored mixed graph C is an (m, n)-clique if and only if every pair of non-adjacent vertices u, w of C are connected by a 2-path uvw of one of the following types:*

(i) uv and vw are edges of different colors,
(ii) uv and vw are arcs (possibly of the same color),
(iii) vu and wv are arcs (possibly of the same color),
(iv) uv and wv are arcs of different colors,
(v) vu and vw are arcs of different colors,
(vi) exactly one of uv and vw is an edge.

A 2-path in an (m, n)-graph is a *special* 2*-path* if it is one among the six types of path listed in Proposition 1. If a 2-path uvw is a special 2-path, then we say that u *sees* w *via* v and that u and w *disagrees* on v. If uvw is not a special 2-path, then we say that u and w *agrees* on v. Due to the above proposition, we know that any underlying graph of an (m, n)-clique must have a diameter of at most 2. Moreover, the underlying graph of an (m, n)-clique is called an *underlying* (m, n)*-clique*.

3 The same is also known as *oriented cliques* or *ocliques* and *oriented clique number* or *oclique number*.

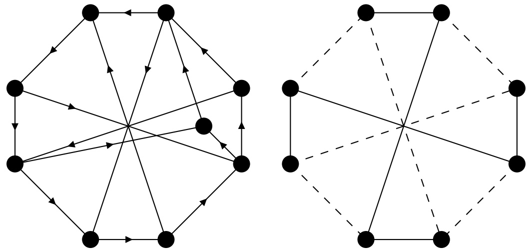

Fig. 4. (a) A $(1, 0)$-clique on 9 vertices. (b) A $(0, 2)$-clique on 8 vertices.

Observation 3. *An underlying* (m, n)*-clique has a diameter at most* 2.

Theorem 2 and Proposition 1 directly imply the following.

Theorem 4. *For the family* $\mathcal{PP}_2$ *of* (m, n)*-colored mixed triangle-free projective-planar graphs*

(i) $\omega_{a(1,0)}(\mathcal{PP}_2) = 9$
(ii) $\omega_{a(0,2)}(\mathcal{PP}_2) = 8$
(iii) $\omega_{a(m,n)}(\mathcal{PP}_2) = (2m + n)^2 + 2,$ *for all* $2m + n \geq 3$.

Proof. (i) Observe that the graph W_8^+ is an underlying $(1, 0)$-clique on 9 vertices (see Fig. 4(a) for the relevant instance). This implies $\omega_{a(1,0)}(\mathcal{PP}_2) \geq 9$.

Let G^* be a triangle-free projective planar $(1, 0)$-clique having at least 9 vertices. Thus, by Observation 3 its underlying graph, say G, must belong to $\mathcal{PP}_2$. We are going to show that such a G^* does not exist. Due to Theorem 2, it is enough to restrict ourselves to checking whether any graph listed in the theorem can be G or not.

As all Plesník graphs are triangle-free planar graphs, and as it is known [7] that the largest triangle-free planar underlying $(1, 0)$-clique has six vertices. Moreover, $K_{3,3}, K_{3,4}, W_8, W_8^+, M_{11}^=$, and $K_{3,4}^*$ has less or equal to nine vertices. Furthermore, using Proposition 1 it is possible to verify that P_{10}, M_{11}, and M_{11}^- are not underlying $(1, 0)$-cliques. Hence, we are only left with verifying whether it is possible to have $K_{3,4}(t)$ or $K_{3,3}(t)$ as G or not, where $t \geq 3$.

Let us suppose that G is either $K_{3,4}(t)$ or $K_{3,3}(t)$ for some $t \geq 2$, and $a_1, a_2, \dots, a_t$ are its vertices of degree two. Notice that, all the a_is are adjacent to exactly two vertices, say b_1, b_2. Note that, there are vertices b'_1 and b'_2 such that b'_j is non-adjacent to a_i and the only 2-path connecting b'_j to a_i is b_j, for all $j \in \{1, 2\}$. Thus in G^*, all a_is must see b'_j via b_j, which implies that all a_is must agree with each other on both b_1 and b_2. Thus, a_1 is neither adjacent to a_2 nor there is a special 2 path among them. Hence G cannot be an underlying $(1, 0)$-clique if it is either $K_{3,4}(t)$ or $K_{3,3}(t)$ for some $t \geq 2$.

(ii) Observe that the graph W_8 is an underlying $(0, 2)$-clique on 8 vertices (see Fig. 4(b) for the relevant instance). This implies $\omega_{a(0,2)}(\mathcal{PP}_2) \geq 8$. The proof of the upper bound can be done similarly to the proof of (i).

(iii) It is known [7] that $\omega_{a(m,n)}(\mathcal{P}') = (2m + n)^2 + 2$, for all $2m + n \geq 3$ for the family of triangle-free planar graphs. This implies $\omega_{a(m,n)}(\mathcal{PP}_2) = (2m + n)^2 + 2$, for all $2m + n \geq 3$.

For the upper bound, as $(2m + n)^2 + 2 \geq 11$ for all $2m + n \geq 3$, and as every non-planar graphs except $K_{3,4}(t)$ and $K_{3,3}(t)$ for $t \geq (2m + n)^2 - 4$ from the graphs listed in Theorem 2 has less than or equal to 11 vertices, it is enough to show that $K_{3,4}(t)$ and $K_{3,3}(t)$ are not underlying (m, n)-cliques for $t \geq (2m + n)^2 - 4$.

Let G^* be an (m, n)-clique having at least $(2m+n)^2+3$ vertices such that its underlying graph G is $K_{3,4}(t)$ and $K_{3,3}(t)$ for some $t \geq (2m+n)^2 - 4$. Let $a_1, a_2, \dots, a_t$ be the vertices of degree two of G. Notice that, all the a_is are adjacent to exactly two vertices, say b_1, b_2. Note that, there are vertices b'_1 and b'_2 such that b'_j is non-adjacent to a_i and the only 2-path connecting b'_j to a_i is b_j, for all $j \in \{1, 2\}$. Thus in G^*, all a_is must see b'_j via b_j. This implies that the adjacency between a_i and b_j is different from the adjacency between b'_j and b_j, for each i and j. This implies that the adjacency between a_i and b_j can be one of the $(2m + n) - 1$ types (each type of arc gives two adjacency options due to directions, and each type of edge gives one adjacency option). As a_i also must see each other via b_1 or b_2, the number of a_is is bounded by

$$t \leq (2m + n - 1)^2 = (2m + n)^2 - 2(2m + n) + 1 \leq (2m + n)^2 - 5$$

for all $(2m + n) \geq 3$. However, this is a contradiction as we assumed $t \geq (2m + n)^2 - 4$. □

Now we turn our focus towards variants of colored homomorphism of $(0, 2)$ and $(1, 0)$-colored mixed graphs (that is, 2-edge-colored graphs and oriented graphs). The variants are known as homomorphisms of signed graphs [17] and pushable homomorphisms of oriented graphs [13], respectively.

Homomorphisms of signed graphs were introduced by Naserasr, Rollová, and Sopena [17] who also defined and characterized signed absolute clique and signed absolute clique numbers. Naserasr, Rollová, and Sopena [17] also showed

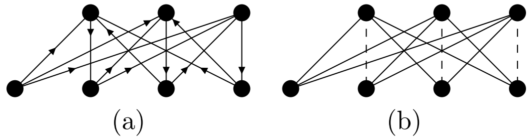

Fig. 5. (a) A pushable absolute clique on 7 vertices. (b) A signed absolute clique on 7 vertices.

how using the notion of homomorphism of signed graph one can capture, as well as extend, many of the classical graph theory results and conjectures including the Four-Color Theorem and Hadwiger's Conjecture [10]. It motivated a number of research works and generated a lot of interest within a short span of time [3,4,8,17,18]. In order to avoid a long series of definitions, we would like to define the notion using its equivalent characterization.

A *signed graph* (G, Σ) is a graph with either positive or negative sign assigned to its edges. A *signed absolute clique* (C, Λ) is a signed graph whose any two non-adjacent vertices are part of a 4 cycle having an odd number of negative edges. Given a family $\mathcal{F}$ of signed graphs,

$$\omega_{as}(\mathcal{F}) = \max\{|V(G)| : G \in \mathcal{F} \text{ and } (G, \Sigma) \text{ is a signed absolute clique}\}.$$

On the other hand, pushable homomorphism of oriented graphs were introduced by Klostermeyer and MacGillivray [12] which motivated some further research works on that topic. These include one work due to Bensmail, Nandi, and Sen [6] that introduced and characterized the notion of pushable absolute clique of oriented graphs. In order to avoid a long series of definitions, we would like define the notion using its equivalent characterization.

An *oriented graph* $\overrightarrow{G}$ is a directed graph without any directed cycle of length 1 or 2. A *pushable absolute clique* $\overrightarrow{C}$ is an oriented graph whose two non-adjacent vertices are part of a 4 cycle having an odd number of arcs in a clockwise direction. Given a family $\mathcal{F}$ of oriented graphs,

$$\omega_{ap}(\mathcal{F}) = \max\{|V(\overrightarrow{G})| : \overrightarrow{G} \in \mathcal{F} \text{ is a pushable absolute clique}\}.$$

Thus from Theorem 2 and the two characterizations (definitions), we have the following theorem.

Theorem 5. *For the families $\mathcal{PP}_2$ (resp. $\mathcal{PP}_2$) of oriented (resp. signed) triangle-free projective-planar graphs*

$$\omega_{ap}(\mathcal{PP}_2) = \omega_{as}(\mathcal{PP}_2) = 7.$$

Proof. Observe that there exist a signed absolute clique and a pushable absolute clique having $K_{3,4}$ as their underlying graphs (see Fig. 5 for the relevant instances). This implies $\omega_{as}(\mathcal{PP}_2) \geq 7$ and $\omega_{ap}(\mathcal{PP}_2) \geq 7$.

For the upper bound, note that if G is the underlying graph of a pushable absolute clique or a signed absolute clique, then it must have the following property: any two non-adjacent vertices of G must be connected by two internally disjoint 2-paths. Observe that, among the graphs listed in Theorem 2, the only graphs that have this property are $K_{3,3}$, $K_{3,4}$, and $K_{2,t}$ for $t \geq 2$. However, notice that a pushable absolute clique (resp., signed absolute clique) is, in particular, a $(1, 0)$-clique (resp., $(0, 2)$-clique). Moreover, the $(1, 0)$-absolute clique number (resp., $(0, 2)$-absolute clique number) for the family of triangle-free planar graphs is at most 6 [7]. As $K_{2,t}$ is a triangle-free planar graph, for all $t \geq 2$, we are done. □

4. Conclusions

In this paper, we gave a characterization of triangle-free projective planar graphs of diameter 2 and proved that the domination number of this class of graphs is at most 3. Moreover, there are only seven triangle-free projective planar graphs with a diameter 2 for which the equality holds. This raises a natural question.

Question 3. *Given a surface $\mathbb{S}$, can you find a tight upper bound on the domination number of triangle-free graphs with diameter 2 that can be embedded on $\mathbb{S}$?*

Goddard and Henning [9] proved that for any integer $g \geq 0$ the number of graphs with orientable genus g, diameter 2, and domination number greater than 2 is finite. For the $g = 0$ case, that is for planar graphs, they reported [9] that there exists only one planar graph of diameter two and domination number greater than two. However, the analogous problem for higher genus surfaces is still unsolved. This motivates the following question.

Question 4. *Given a positive integer g, how many triangle-free graphs of genus g and diameter 2 are there with a domination number greater than or equal to 3?*

We observed that the maximum order of a triangle-free projective planar graph with diameter 2 and domination number 3 is 11. This motivates the following.

Question 5. *Given a positive integer g, what is the highest order of a triangle-free graph of genus g and diameter 2 with a domination number greater than 3?*

Data availability

No data was used for the research described in the article.

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