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# Combinatorial proofs of multivariate Cayley–Hamilton theorems

Arvind Ayyer [a,*], Naren Sundaravaradan [b]

[a] *Department of Mathematics, Indian Institute of Science, Bangalore 560012, India*
[b] *HFN Inc., Bengaluru, 560076, India*

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We give combinatorial proofs of two multivariate Cayley–Hamilton type theorems. The first one is due to Phillips (1919) [10] involving $2k$ matrices, of which $k$ commute pairwise. The second one uses the mixed discriminant, a matrix function which has generated a lot of interest in recent times. Recently, the Cayley–Hamilton theorem for mixed discriminants was proved by Bapat and Roy (2017) [3]. We prove a Phillips-type generalization of the Bapat–Roy theorem, which involves $2nk$ matrices, where $n$ is the size of the matrices, among which $nk$ commute pairwise. Our proofs generalize the univariate proof of Straubing (1983) [11] for the original Cayley–Hamilton theorem in a nontrivial way, and involve decorated permutations and decorated paths.



## 1. Introduction

Suppose $A$ is an $n \times n$ matrix with entries in a commutative ring. Then the Cayley–Hamilton theorem says that $p(A) = 0$, where $p(x) = \det(xI_n - A)$ is the characteristic polynomial of $A$ and $I_n$ is the $n \times n$ identity matrix. The Cayley–Hamilton theorem is

* Corresponding author.
*E-mail addresses:* arvind@iisc.ac.in (A. Ayyer), nano.naren@gmx.com (N. Sundaravaradan).

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probably the first deep theorem one sees in linear algebra. It was first proved for linear functions of quaternions (corresponding to real $4 \times 4$ or complex $2 \times 2$ matrices) by Hamilton [8]. Cayley [5] stated it for sizes 2 and 3, but gave a demonstration only in the former case. Sylvester immediately realised its importance and popularized it, calling it the *no-little-marvellous Hamilton–Cayley theorem* [12].

The first proof was given by Buchheim [4] assuming invertibility of the matrix, but the first general proof was given by Frobenius [7]. For more on the history of this remarkable theorem, see [6]. Several proofs are now known at various levels of abstraction.[1] Relevant to this work is an elegant combinatorial proof due to Straubing [11,13].

H. B. Phillips [10] proved the following generalization of the Cayley–Hamilton theorem. Suppose $A_1, \dots, A_k$ and $B_1, \dots, B_k$ are two families of $n \times n$ matrices such that $B_i B_j = B_j B_i$ for all $1 \le i < j \le k$ and

$$A_1 B_1 + \cdots + A_k B_k = 0. \tag{1}$$

**Theorem 1** *([10, Theorem I]). Define the polynomial* $p(x_1, \dots, x_k) = \det(A_1 x_1 + \cdots + A_k x_k)$*. Then* $p(B_1, \dots, B_k) = 0$*.*

We will give a combinatorial proof of Theorem 1 in Section 2. For our proof, we will think of the entries in these matrices as formal commuting indeterminates. An instructive special case about a pair of commuting matrices $A, B$ arises by setting $k = 2, A_1 = A, B_1 = B, A_2 = -B, B_2 = A$ as follows.

**Corollary 2.** *Let* $A, B$ *be commuting matrices. Define the bivariate polynomial* $q(x, y) = \det(xA - yB)$*. Then* $q(B, A) = 0$*.*

Setting $A$ equal to the identity matrix in Corollary 2 reduces to the Cayley–Hamilton theorem.

We now move on to an important generalization of the determinant. For an integer $n$, $[n] = \{1, \dots, n\}$ and $S_n$ stands for the set of permutations $[n]$.

**Definition 3.** The *mixed discriminant* of an $n$-tuple $(A_1, \dots, A_n)$ of $n \times n$ matrices is defined as

$$\mathfrak{D}(A_1, \dots, A_n) = \frac{1}{n!} \sum_{\alpha \in S_n} \det\left(A_{\alpha_1}^{(1)} \mid \cdots \mid A_{\alpha_n}^{(n)}\right),$$

where $A^{(i)}$ denotes the $i$'th column of the matrix $A$.

The basic properties of the mixed discriminant are given in [1]. From the combinatorial point of view, it has been used to enumerate coloured spanning forests [2]. It

[1] The Wikipedia article on this topic itself gives four distinct proofs.

simultaneously generalizes both the determinant and the permanent. For a fixed matrix $B$, $\mathfrak{D}(B, \dots, B) = \det(B)$, and if we set $B_i$ to be the diagonal matrix with entries $B_{i,1}, \dots, B_{i,n}$, then $\mathfrak{D}(B_1, \dots, B_n) = \sum_{\sigma \in S_n} B_{1,\sigma_1} \cdots B_{n,\sigma_n}$, which is the permanent of $B$.

We will use $I$ for the identity matrix whenever the size is clear from the context. Bapat and Roy [3] generalized the Cayley–Hamilton theorem for mixed discriminants by adapting Straubing's proof [11].

**Theorem 4** *([3, Theorem 1.1]). For an $n$-tuple of $n \times n$ matrices $(A_1, \dots, A_n)$, define the polynomial*

$$f(x_1, \dots, x_n) = \mathfrak{D}(x_1 I - A_1, \dots, x_n I - A_n).$$

*Then $f(A_1, \dots, A_n) = 0$.*

We note in passing that $f(x, \dots, x)$ is also known as the *mixed characteristic polynomial* and was an important ingredient in the recent proof of the Kadison-Singer theorem [9].

For some positive integers $n, k$, let $(A_{i,j})_{i \in [n], j \in [k]}$ and $(B_{i,j})_{i \in [n], j \in [k]}$ be two families of $n \times n$ matrices, where $B_{i,j} B_{i',j'} = B_{i',j'} B_{i,j}$ for all $1 \le i < i' \le n, 1 \le j, j' \le k$. In addition, suppose

$$A_{i,1} B_{i,1} + \cdots + A_{i,k} B_{i,k} = 0, \quad 1 \le i \le n. \tag{2}$$

**Theorem 5.** *For indeterminates $(x_{i,j})_{i \in [n], j \in [k]}$, define the polynomial*

$$\hat{p}\left((x_{i,j})_{i \in [n], j \in [k]}\right) = \mathfrak{D}(A_{1,1} x_{1,1} + \cdots + A_{1,k} x_{1,k}, \dots, A_{n,1} x_{n,1} + \cdots + A_{n,k} x_{n,k}). \tag{3}$$

*Then*

$$\hat{p}\left((B_{i,j})_{i \in [n], j \in [k]}\right) = 0.$$

We will give a combinatorial proof of Theorem 5 in Section 3. Even for this proof, we will think of the entries in these matrices as formal commuting indeterminates. We now discuss a special case of Theorem 5 for $k = 2$. Let $M_1, \dots, M_n$ be a family of matrices. We then set $A_{i,1} = -B_{i,2} = I$ and $A_{i,2} = B_{i,1} = M_i$ for $i \in [n]$. Then, this family of matrices automatically satisfies (2). For convenience, we will set $x_{i,1} = x_i$ and $x_{i,2} = y_i$. Then the polynomial in (3) becomes

$$\hat{p}_2\left(x_1, \dots, x_n; y_1, \dots, y_n\right) = \mathfrak{D}(x_1 I - M_1 y_1, \dots, x_n I - M_n y_1).$$

**Corollary 6.** *Suppose $M_1, \dots, M_n$ are a pairwise commuting family of matrices. Then*

$$\hat{p}_2\left(M_1, \dots, M_n; -I_1, \dots, -I_n\right) = 0.$$

Corollary 6 bears the same relation to Theorem 5 as Corollary 2 does to Theorem 1. If we compare this result with Theorem 4, we see that the extra set of variables $y_i$ forces $M_i$'s to be pairwise commuting in order for the Cayley–Hamilton theorem to apply.

**Remark 7.** Suppose we choose matrices such that $A_{i,j} = A_j$ and $B_{i,j} = B_j$ as well as set variables $x_{i,j} = x_j$ for all $i$. Then Theorem 5 reduces to Theorem 1.

The plan of the rest of the paper is as follows. We first give a combinatorial proof for Theorem 1 in Section 2. We will illustrate the key ideas of the proof using $2 \times 2$ matrices in Section 2.1. We show how the proof relates to Straubing's proof of the Cayley–Hamilton theorem in Section 2.2. We also compare our proof to Phillips' original proof in Section 2.3. We then give a proof of Theorem 5 in Section 3 using a naturally generalization of our proof strategy for Theorem 1. We illustrate the proof ideas again for $2 \times 2$ matrices in Section 3.1.

## 2. Proof of Phillips' theorem

Throughout this section, we will fix $k$ and $n \times n$ matrices $A_1, \dots, A_k$ and $B_1, \dots, B_k$ where $B_i$'s commute pairwise and the matrices satisfy

$$A_1 B_1 + \cdots + A_k B_k = 0. \tag{4}$$

We will first define the key combinatorial objects involved in the proof.

**Definition 8.** Let $\pi = (\pi_1, \dots, \pi_n) \in S_n$. A *decorated permutation* $\bar{\pi}$ of $\pi$ is an $n$-tuple of triples $\bar{\pi}_i = (i, \pi_i, \ell_i)$ for $i \in [n]$, where each $\ell_i \in [k]$ is called a *label*. We will denote $\bar{\pi}_i$ as $\quad i \xrightarrow{\ell_i} \pi_i \quad$ which has *weight* $(A_{\ell_i})_{i,\pi_i}$. The *signed weight* of the decorated permutation is given by

$$\operatorname{swgt}(\bar{\pi}) = \operatorname{sgn}(\pi) \prod_{i=1}^{n} (A_{\ell_i})_{i,\pi_i}.$$

The set of all decorated permutations is denoted $\bar{S}_{n,k}$.

Since there are $n!$ permutations and all labels are independently chosen, the cardinality of $\bar{S}_{n,k}$ is $n! k^n$. Let $n = 3$, $k = 2$, and $\pi = (3, 1, 2)$. Then an example of a decorated permutation is

$$\bar{\pi}: \quad 1 \xrightarrow{1} 3 \quad 2 \xrightarrow{2} 1 \quad 3 \xrightarrow{2} 2 \tag{5}$$

with $\operatorname{swgt}(\bar{\pi}) = +(A_1)_{1,3}(A_2)_{2,1}(A_2)_{3,2}$.

$$
\begin{array}{llllll}
\bar{\pi}: & 1 \xrightarrow[\mathrm{A}]{\ell_1} \pi_1 & \cdots & s \xrightarrow[\mathrm{A}]{\ell_s} \pi_s & \cdots & n \xrightarrow[\mathrm{A}]{\ell_n} \pi_n \\
\bar{q}: & q_1 \xrightarrow[\mathrm{B}]{\ell_1} q_2 & \cdots & q_s \xrightarrow[\mathrm{B}]{\ell_s} q_{s+1} & \cdots & q_n \xrightarrow[\mathrm{B}]{\ell_n} q_{n+1}
\end{array}
$$

**Fig. 1.** An illustration of a generic pathmutation $(\bar{\pi}, \bar{q})$.

**Definition 9.** A *decorated path* of length $n$ is a tuple $\bar{q} = (q_1, \dots, q_{n+1})$, where each $q_i \in [n]$. For $i \in [n]$, the $i$'th labeled edge is denoted $\bar{q}_i = \quad q_i \xrightarrow[\mathrm{B}]{\ell_i} q_{i+1}$ and has weight $(B_{\ell_i})_{q_i,q_{i+1}}$, where the *label* $\ell_i \in [k]$. The *weight* of the decorated path is

$$\mathrm{wgt}(\bar{q}) = \prod_{i=1}^{n} (B_{\ell_i})_{q_i,q_{i+1}}.$$

The set of all decorated paths is denoted $\overline{Q}_{n,k}$.

For instance, with $n = 3$ and $k = 2$,

$$\bar{q}: \quad 3 \xrightarrow[\mathrm{B}]{1} 1 \quad 1 \xrightarrow[\mathrm{B}]{2} 2 \quad 2 \xrightarrow[\mathrm{B}]{2} 1 \tag{6}$$

is a decorated path with $\mathrm{wgt}(\bar{q}) = (B_1)_{3,1}(B_2)_{1,2}(B_2)_{2,1}$.

**Definition 10.** A *pathmutation* is a pair $(\bar{\pi}, \bar{q})$ where $\bar{\pi} \in \bar{S}_{n,k}, \bar{q} \in \overline{Q}_{n,k}$ such that the labels of the $i$'th element of the permutation and the $i$'th edge of the path are the same for all $i \in [n]$. The *signed weight* of a pathmutation is

$$\mathrm{wgt}(\bar{\pi}, \bar{q}) = \mathrm{swgt}(\bar{\pi})\,\mathrm{wgt}(\bar{q}).$$

The set of pathmutations beginning with $q_1 = b$ and ending with $q_{n+1} = e$ is denoted $\mathcal{A}(b, e)$.

The cardinality of $\mathcal{A}(b, e)$ is $n!k^n n^{n-1}$ for every $b, e \in [n]$ because we can choose $q_2, \dots, q_{n-1}$ arbitrarily.

See Fig. 1 for a generic pathmutation. We then set

$$\mathrm{swgt}(\mathcal{A}(b, e)) = \sum_{(\bar{\pi},\bar{q}) \in \mathcal{A}(b,e)} \mathrm{swgt}(\bar{\pi})\,\mathrm{wgt}(\bar{q}).$$

We will need more general objects than decorated permutations in our proofs, which we now define.

**Definition 11.** A *decorated map* $\overline{m}$ is an $n$-tuple of triples $\overline{m}_i = (\sigma_i, \tau_i, \ell_i)$, where $\sigma = (\sigma_1, \dots, \sigma_n) \in S_n$ is either the identity or a single transposition, $\tau_i \in [n]$ and $\ell_i \in [k]$ for all $i$ such that

- $\#\{\tau_1, \dots, \tau_n\} \geq n-1$,
- if $\#\{\tau_1, \dots, \tau_n\} = n$, then $\sigma$ is the identity permutation,
- if $\tau_i = \tau_j$ for some $(i, j)$, then either $\sigma_i = i, \sigma_j = j$ or $\sigma_i = j, \sigma_j = i$.

The *weight* of $\overline{m}_i$ is $(A_{\ell_i})_{\sigma_i,\tau_i}$ and is denoted $\sigma_i \xrightarrow{\ell_i} \tau_i$. The weight of $\overline{m}$ is then

$$\mathrm{wgt}(\overline{m}) = \prod_{i=1}^{n} (A_{\ell_i})_{\sigma_i,\tau_i}.$$

The set of all decorated maps is denoted $\overline{M}_{n,k}$.

When $\#\{\tau_1, \dots, \tau_n\} = n$, we get exactly decorated permutations. When $\#\{\tau_1, \dots, \tau_n\} = n-1$, there are $n(n-1) \times n!/2$ possibilities for $\tau$ and 2 possibilities for $\sigma$ in each case so that the cardinality of $\overline{M}_{n,k}$ is

$$k^n(n! + n(n-1)n!) = n!k^n(n^2 - n + 1).$$

For example, we can view (5) as the decorated map

$$\overline{m} : ((1, 3, 1), (2, 1, 2), (3, 2, 2)),$$

where the first component $\sigma$ is the identity permutation and the second component $\tau$ is the permutation $\pi = (3, 1, 2)$. Now suppose we fix $\tau_2 = 1$, $\tau_1 = \tau_3$, and the same labels as above. Then $\sigma$ is forced to be either $(1, 2, 3)$ (i.e. the identity) or $(3, 2, 1)$, and the four possible decorated maps are

$$1 \xrightarrow{1} 3 \quad 2 \xrightarrow{2} 1 \quad 3 \xrightarrow{2} 3 \ , \tag{7}$$

$$3 \xrightarrow{1} 3 \quad 2 \xrightarrow{2} 1 \quad 1 \xrightarrow{2} 3 \ , \tag{8}$$

$$1 \xrightarrow{1} 2 \quad 2 \xrightarrow{2} 1 \quad 3 \xrightarrow{2} 2 \ , \tag{9}$$

$$3 \xrightarrow{1} 2 \quad 2 \xrightarrow{2} 1 \quad 1 \xrightarrow{2} 2 \ . \tag{10}$$

**Definition 12.** A *pathmap* is a pair $(\overline{m}, \bar{q})$ where $\overline{m} \in \overline{M}_{n,k}, \bar{q} \in \overline{Q}_{n,k}$ such that

- If $\{\tau_1, \dots, \tau_n\} = [n]$, then $(\overline{m}, \bar{q})$ is a pathmutation.
- If $\#\{\tau_1, \dots, \tau_n\} = n-1$ and $\tau_s = \tau_t$ for some $s < t$, then $q_1 = \tau_s = \tau_t$. In this case, the labels of $\overline{m}_k$ and $\bar{q}_k$ must match for all $k \neq s, t$. In addition, if $\sigma_s = s, \sigma_t = t$

(resp. $\sigma_s = t, \sigma_t = s$), then the labels of $\overline{m}_s$ and $\overline{m}_t$ are equal to those of $\bar{q}_s, \bar{q}_t$ (resp. $\bar{q}_t, \bar{q}_s$) respectively.

The weight of the pathmap $(\overline{m}, \bar{q})$ is

$$\mathrm{wgt}(\overline{m}, \bar{q}) = \mathrm{wgt}(\overline{m})\,\mathrm{wgt}(\bar{q}).$$

The set of pathmaps with $\#\{\tau_1, \dots, \tau_n\} = n-1$ such that $\{\tau_1, \dots, \tau_n\} = [n] \setminus \{b\}$ and ending with $q_{n+1} = e$ is denoted $\mathcal{H}(b,e)$. In addition, let $\mathcal{G}(b,e) = \mathcal{H}(b,e) \cup \mathcal{A}(b,e)$.

For instance, we may combine the decorated permutation (5) and the decorated path (6) to get a pathmutation in $\mathcal{A}(3,1)$:

$$\begin{array}{llll}
\bar{\pi}: & 1 \xrightarrow[\text{A}]{1} 3 & 2 \xrightarrow[\text{A}]{2} 1 & 3 \xrightarrow[\text{A}]{2} 2 \\
\bar{q}: & 3 \xrightarrow[\text{B}]{1} 1 & 1 \xrightarrow[\text{B}]{2} 2 & 2 \xrightarrow[\text{B}]{2} 1
\end{array}$$

where the labels match. We can also combine the decorated maps in (7) and the same decorated path $\bar{q}$ to get the pathmap

$$\begin{array}{llll}
\bar{\pi}: & 1 \xrightarrow[\text{A}]{1} 3 & 2 \xrightarrow[\text{A}]{2} 1 & 3 \xrightarrow[\text{A}]{2} 3 \\
\bar{q}: & 3 \xrightarrow[\text{B}]{1} 1 & 1 \xrightarrow[\text{B}]{2} 2 & 2 \xrightarrow[\text{B}]{2} 1
\end{array}$$

However, the combination of the decorated map (8) with $\bar{q}$ is not a pathmap because the condition on the labels is not satisfied. Further, the decorated map (9) with $\bar{q}$ does not form a pathmap because $3 = q_1 \neq \tau_1 = \tau_3 = 2$. Lastly, (10) with $\bar{q}$ fails both conditions.

In other words $\mathcal{G}(b,e)$ consists of two kinds of elements $(\overline{m}, \bar{q})$. Those with $q_1 = b$ are pathmutations and the remaining are elements of $\mathcal{H}(b,e)$, which we count now. For every fixed $b$ and $e$, there are $n-1$ possibilities for $q_1$, $n$ possibilities each for $q_1, \dots, q_n$, $k$ possibilities each for $\ell_1, \dots, \ell_n$, $n!/2$ arrangements of $\tau$ and 2 arrangements for $\sigma$. Therefore, $\#\mathcal{H}(b,e) = (n-1)n!k^n n^{n-1}$ and the cardinality of $\mathcal{H}(b,e)$ is $n-1$ times that of $\mathcal{A}(b,e)$.

Fig. 2 illustrates the two kinds of elements in $\mathcal{H}(b,e)$ in the second condition in Definition 12.

To assign a sign to the elements of $\mathcal{G}(b,e)$, we define a map $\phi : \mathcal{A}(b,e) \times [n] \to \mathcal{G}(b,e)$ defined by $\phi((\bar{\pi}, \bar{q}), j) = (\overline{m}', \bar{q}')$ as follows. First, define $\bar{q}'$ by

$$\bar{q}'_r = \begin{cases} j \xrightarrow[\text{B}]{\ell_1} q_2 & r = 1, \\ \bar{q}_r & \text{otherwise.} \end{cases}$$

$$\begin{array}{llllll}
\overline{m}: & \cdots & s \xrightarrow[\text{A}]{\ell_s} q_1 & \cdots & t \xrightarrow[\text{A}]{\ell_t} q_1 & \cdots \\
\bar{q}: & \cdots & q_s \xrightarrow[\text{B}]{\ell_s} q_{s+1} & \cdots & q_t \xrightarrow[\text{B}]{\ell_t} q_{t+1} & \cdots
\end{array}$$

(a) $(\overline{m}, \bar{q})$

$$\begin{array}{llllll}
\overline{m}: & \cdots & t \xrightarrow[\text{A}]{\ell_t} q_1 & \cdots & s \xrightarrow[\text{A}]{\ell_s} q_1 & \cdots \\
\bar{q}: & \cdots & q_s \xrightarrow[\text{B}]{\ell_s} q_{s+1} & \cdots & q_t \xrightarrow[\text{B}]{\ell_t} q_{t+1} & \cdots
\end{array}$$

(b) $(\overline{m}', \bar{q})$

**Fig. 2.** Two elements $(\overline{m}, \bar{q})$ and $(\overline{m}', \bar{q})$ of $\mathcal{H}(b, e)$ such that $\overline{m}_k = \overline{m}'_k$ for $k \neq s, t$. Note that $\overline{m}'_s = \overline{m}_t$ and $\overline{m}'_t = \overline{m}_s$. If we write $\overline{m}_i = (\sigma_i, \tau_i, \ell_i)$ and $\overline{m}'_i = (\sigma'_i, \tau'_i, \ell_i)$, then $\sigma$ is the identity permutation, $\sigma'$ is the transposition $(s, t)$, and $\tau_s = \tau_t = \tau'_s = \tau'_t = q_1$.

Next, set $s = \pi_b^{-1}$ and $t = \pi_j^{-1}$. Then let

$$\overline{m}'_r = \begin{cases} \min(s,t) \xrightarrow[\text{A}]{\ell_{\min(s,t)}} j & \text{if } r = s, \\ \max(s,t) \xrightarrow[\text{A}]{\ell_{\max(s,t)}} j & \text{if } r = t, \\ \bar{\pi}_r & \text{otherwise.} \end{cases} \tag{11}$$

**Proposition 13.** *$\phi$ is a bijection.*

**Proof.** We prove this by constructing the inverse map. Let $(\overline{m}', \bar{q}') \in \mathcal{G}(b, e)$ and $\overline{m}'_i = (\sigma'_i, \tau'_i, \ell'_i)$, $i \in [n]$.

$$\bar{q}_r = \begin{cases} b \xrightarrow[\text{B}]{\ell_1} q'_2 & \text{if } r = 1 \\ \bar{q}'_r & \text{otherwise.} \end{cases}$$

If $q'_1 = b$, then set $\bar{\pi}_r = \overline{m}'_r$; otherwise, there exists $1 \leq s < s' \leq n$ such that $\tau'_s = \tau'_{s'} = q'_1$. In this case, set

$$\bar{\pi}_r = \begin{cases} s \xrightarrow[\text{A}]{\ell_s} b & \text{if } r = s, \sigma'_r = s, \\ s \xrightarrow[\text{A}]{\ell_s} q'_1 & \text{if } r = s, \sigma'_r = s', \\ s' \xrightarrow[\text{A}]{\ell_{s'}} b & \text{if } r = s', \sigma'_r = s, \\ s' \xrightarrow[\text{A}]{\ell_{s'}} q'_1 & \text{if } r = s', \sigma'_r = s', \\ \overline{m}'_r & \text{otherwise.} \end{cases}$$

Clearly, $(\bar{\pi}, \bar{q}) \in \mathcal{A}(b, e)$. It is routine to check that $\phi((\bar{\pi}, \bar{q}), q_1') = (\overline{m}', \bar{q}')$. $\square$

Note also that $\phi((\bar{\pi}, \bar{q}), b) = (\bar{\pi}, \bar{q})$ for $(\bar{\pi}, \bar{q}) \in \mathcal{A}(b, e)$. We now use Proposition 13 to give a signed weight to a pathmap $(\overline{m}', \bar{q}')$. Suppose $\phi^{-1}(\overline{m}', \bar{q}') = ((\bar{\pi}, \bar{q}), k)$. Then set

$$\operatorname{swgt}(\overline{m}', \bar{q}') = \operatorname{sgn}(\pi) \operatorname{wgt}(\overline{m}', \bar{q}'). \tag{12}$$

**Lemma 14.** *Let $A_1, \dots, A_k$, $B_1, \dots, B_k$ be $n \times n$ matrices satisfying (4) and where the $B_i$'s commute pairwise, and let $b, e \in [n]$. Then*

$$\sum_{(\overline{m}, \bar{q}) \in \mathcal{G}(b,e)} \operatorname{swgt}(\overline{m}, \bar{q}) = 0.$$

**Proof.** By definition of (12),

$$\operatorname{swgt}(\mathcal{G}(b, e)) = \sum_{(\bar{\pi}, \bar{q}) \in \mathcal{A}(b,e)} \sum_{a=1}^{n} \operatorname{swgt}\left(\phi((\bar{\pi}, \bar{q}), a)\right).$$

We will refine the sum according to the underlying permutation $\pi$ and all the labels except $\ell_s$, where $\pi_s = b$. Thus,

$$\operatorname{swgt}(\mathcal{G}(b, e)) = \sum_{\pi \in S_n} \sum_{\substack{(\bar{\pi}, \bar{q}) \in \mathcal{A}(b,e) \\ 1 \le \ell_1, \dots, \ell_{s-1}, \ell_{s+1}, \dots, \ell_n \le k}} \sum_{1 \le q_2, \dots, q_n \le n} \sum_{\ell_s=1}^{k} \sum_{a=1}^{n} \operatorname{swgt}\left(\phi((\bar{\pi}, \bar{q}), a)\right).$$

We will now perform the three inner sums. The common factor for these sums is

$$\operatorname{sgn}(\pi) \prod_{\substack{i=1 \\ i \neq s}}^{n} \operatorname{wgt}\left( i \xrightarrow[\text{A}]{\ell_i} \pi_i \right) = \operatorname{sgn}(\pi) \prod_{\substack{i=1 \\ i \neq s}}^{n} (A_{\ell_i})_{i, \pi_i}.$$

Since all three are independent, we can perform them in any order. We first perform

$$\begin{aligned} &\sum_{\ell_s=1}^{k} \sum_{a=1}^{n} \operatorname{wgt}\left( s \xrightarrow[\text{A}]{\ell_s} a \right) \\ &\qquad \times \sum_{1 \le q_2, \dots, q_n \le n} \operatorname{wgt}\left( a \xrightarrow[\text{B}]{\ell_1} q_2 \right) \cdots \operatorname{wgt}\left( q_n \xrightarrow[\text{B}]{\ell_n} e \right). \end{aligned} \tag{13}$$

Using the pairwise commutativity of $B_1, \dots, B_k$, cycle the labels $\ell_1, \dots, \ell_s$ in the path to bring $\ell_s$ to the front so that we have

$$\sum_{\ell_s=1}^{k}\sum_{a=1}^{n}\mathrm{wgt}\left( s \xrightarrow[\text{A}]{\ell_s} a \right) \sum_{1\le q_2,\dots,q_n\le n} \mathrm{wgt}\left( a \xrightarrow[\text{B}]{\ell_s} q_2 \right)$$
$$\times\, \mathrm{wgt}\left( q_2 \xrightarrow[\text{B}]{\ell_1} q_3 \right) \cdots \mathrm{wgt}\left( q_s \xrightarrow[\text{B}]{\ell_{s-1}} q_{s+1} \right)$$
$$\times\, \mathrm{wgt}\left( q_{s+1} \xrightarrow[\text{B}]{\ell_{s+1}} q_{s+2} \right) \cdots \mathrm{wgt}\left( q_n \xrightarrow[\text{B}]{\ell_n} e \right).$$

We now perform the sum over $a$ and $\ell_s$ first. This amounts to

$$\sum_{\ell_s=1}^{k}\sum_{a=1}^{n}\mathrm{wgt}\left( s \xrightarrow[\text{A}]{\ell_s} a \right)\mathrm{wgt}\left( a \xrightarrow[\text{B}]{\ell_s} q_2 \right) = \sum_{\ell_s=1}^{k}\sum_{a=1}^{n}(A_{\ell_s})_{s,a}(B_{\ell_s})_{a,q_2}, \tag{14}$$

which, by matrix multiplication is the $(s, q_2)$'th entry of $A_1B_1 + \cdots + A_kB_k$, which is zero by (4). This completes the proof. □

**Lemma 15.** *Let $A_1, \dots, A_k$, $B_1, \dots, B_k$ be $n \times n$ matrices satisfying* (4) *and where the $B_i$'s commute pairwise, and let $b, e \in [n]$. Then*

$$\sum_{(\overline{m},\bar{q})\in\mathcal{H}(b,e)} \mathrm{swgt}(\overline{m}, \bar{q}) = 0.$$

**Proof.** By Proposition 13, every pair in $\mathcal{H}(b, e)$ is equal to $\phi((\bar{\pi}, \bar{q}), j)$ for some $(\bar{\pi}, \bar{q}) \in \mathcal{A}(b, e)$ and $1 \le j \le n$, $j \ne b$. Define a map $f : \mathcal{H}(b, e) \to \mathcal{H}(b, e)$ such that if $f(\phi((\bar{\pi}, \bar{q}), j)) = \phi((\bar{\pi}', \bar{q}), j)$, then

$$\bar{\pi}'_r = \begin{cases} r \xrightarrow[\text{A}]{\ell_r} j & \text{if } \pi_r = b, \\ r \xrightarrow[\text{A}]{\ell_r} b & \text{if } \pi_r = j, \\ \bar{\pi}_r & \text{otherwise.} \end{cases}$$

Clearly, $f$ is an involution. We claim that it is sign-reversing and weight-preserving. Let $(\overline{m}, \bar{q}) = \phi((\bar{\pi}, \bar{q}), j)$ and suppose that $s = \pi_b^{-1} < t = \pi_j^{-1}$. Then, by (11), we have

$$\bar{\pi}_s = s \xrightarrow[\text{A}]{\ell_s} b \,, \quad \bar{\pi}_t = t \xrightarrow[\text{A}]{\ell_t} j \,,$$
$$\overline{m}_s = s \xrightarrow[\text{A}]{\ell_s} j \,, \quad \overline{m}_t = t \xrightarrow[\text{A}]{\ell_t} j \,.$$

By the definition of $f$, $\bar{\pi}'$ will have

$$\bar{\pi}'_s = \quad s \xrightarrow[\mathrm{A}]{\ell_s} j \quad , \quad \bar{\pi}'_t = \quad t \xrightarrow[\mathrm{A}]{\ell_t} b \quad .$$

Let $(\overline{m}', \bar{q}) = \phi((\bar{\pi}', \bar{q}), j)$, then

$$\overline{m}'_s : \quad t \xrightarrow[\mathrm{A}]{\ell_t} j \quad , \quad \overline{m}'_t : \quad s \xrightarrow[\mathrm{A}]{\ell_s} j \quad .$$

Thus, the weights of $\overline{m}$ and $\overline{m}'$ are the same, and $\pi$ and $\pi'$ differ by a single transposition. Hence, $\operatorname{swgt}(\overline{m}', \bar{q}) = -\operatorname{swgt}(\overline{m}, \bar{q})$ by (12). The case of $s > t$ proceeds in a very similar manner. $\square$

**Proof of Theorem 1.** We first claim that

$$\operatorname{swgt}(\mathcal{A}(b,e)) = p(B_1, \ldots, B_k)_{b,e}.$$

To see this, begin by expanding the polynomial $p$ as

$$p(x_1, \ldots, x_k) = \sum_{\sigma \in S_n} \operatorname{sgn}(\sigma) \prod_{r=1}^{n} \left((A_1)_{r,\sigma_r} x_1 + \cdots + (A_k)_{r,\sigma_r} x_k\right).$$

Now, substitute $x_i$ by $B_i$ and use the fact that $B_i$'s commute pairwise to obtain

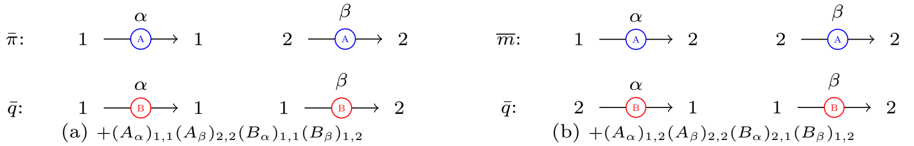
$$\begin{aligned} p(B_1, \ldots, B_k) &= \sum_{\sigma \in S_n} \operatorname{sgn}(\sigma) \prod_{r=1}^{n} \left((A_1)_{r,\sigma_r} B_1 + \cdots + (A_k)_{r,\sigma_r} B_k\right) \\ &= \sum_{\sigma \in S_n} \operatorname{sgn}(\sigma) \prod_{(z_1,\ldots,z_n) \in [k]^n} (A_{z_1})_{1,\sigma_1} \cdots (A_{z_n})_{n,\sigma_n} B_{z_1} \cdots B_{z_n}. \end{aligned}$$

Now consider the $(b,e)$'th entry of this sum. For each permutation $\sigma$ and each element $z = (z_1, \ldots, z_n) \in [k]^n$, we obtain a decorated permutation $\bar{\sigma}$, the label of whose $i$'th element is $z_i$ as seen above. Now expand the product of $B_{z_i}$'s on the right hand side. The $(b,e)$'th entry is a sum of terms, each of which corresponds exactly to a decorated path with initial vertex $b$ and final vertex $e$. This proves the claim above.

Now, we have by construction, $\mathcal{G}(b,e) = \mathcal{A}(b,e) \cup \mathcal{H}(b,e)$. We have proved that $\operatorname{swgt}(\mathcal{G}(b,e)) = 0$ in Lemma 14 and that $\operatorname{swgt}(\mathcal{H}(b,e)) = 0$ in Lemma 15. Therefore, we have shown $\operatorname{swgt}(\mathcal{A}(b,e)) = 0$, completing the proof. $\square$

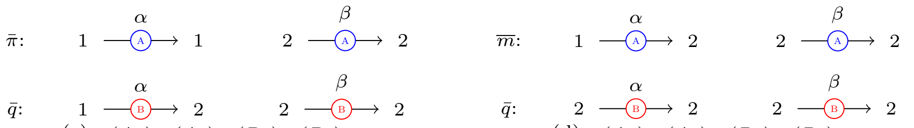
### *2.1. Illustration for $n = 2$*

The essence of the proof of Theorem 1 is contained in Lemmas 14 and 15. We illustrate the ideas behind the proofs of these lemmas by looking at the case of $n = k = 2$ in detail



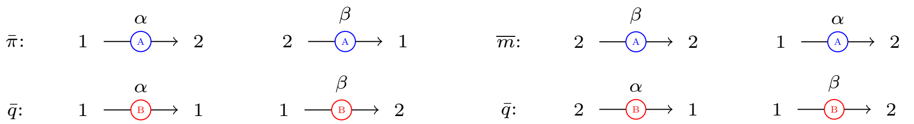
(a) $+(A_\alpha)_{1,1}(A_\beta)_{2,2}(B_\alpha)_{1,1}(B_\beta)_{1,2}$ (b) $+(A_\alpha)_{1,2}(A_\beta)_{2,2}(B_\alpha)_{2,1}(B_\beta)_{1,2}$

**Fig. 3.** The terms proportional to $(A_\beta)_{2,2}(B_\beta)_{1,2}$ along with their signed weights.



(c) $+(A_\alpha)_{1,1}(A_\beta)_{2,2}(B_\alpha)_{1,2}(B_\beta)_{2,2}$ (d) $+(A_\alpha)_{1,2}(A_\beta)_{2,2}(B_\alpha)_{2,2}(B_\beta)_{2,2}$

**Fig. 4.** The terms proportional to $(A_\beta)_{2,2}(B_\beta)_{2,2}$ along with their signed weights.



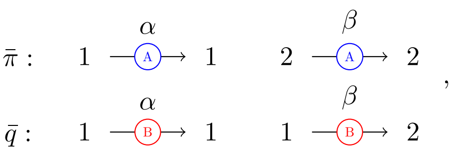
(e) $-(A_\alpha)_{1,2}(A_\beta)_{2,1}(B_\alpha)_{1,1}(B_\beta)_{1,2}$ (f) $-(A_\alpha)_{1,2}(A_\beta)_{2,2}(B_\alpha)_{2,1}(B_\beta)_{1,2}$

(g) $-(A_\alpha)_{1,2}(A_\beta)_{2,1}(B_\alpha)_{1,2}(B_\beta)_{2,2}$ (h) $-(A_\alpha)_{1,2}(A_\beta)_{2,2}(B_\alpha)_{2,2}(B_\beta)_{2,2}$

**Fig. 5.** The terms proportional to $(A_\alpha)_{1,2}$ along with their signed weights.

for $b = 1$ and $e = 2$. We will keep the labels $\ell_1 = \alpha$ and $\ell_2 = \beta$ arbitrary, so that we have 4 pathmutations, which are shown in the left columns of Figs. 3, 4 and 5. Similarly, there are $(2 - 1)2!2^1 = 4$ such pathmaps in $\mathcal{H}(1, 2)$, which are shown in the right columns of Figs. 3, 4 and 5.

We now illustrate Lemma 14 for $s = 1$. This will amount to summing over all configurations in Figs. 3 and 4. First compare the pathmutation $(\bar{\pi}, \bar{q})$ in Fig. 3(a) and the pathmap $(\overline{m}', \bar{q}')$ in Fig. 3(b). To explain the sign of the pathmap, note that $\phi^{-1}(\overline{m}', \bar{q}')$ is given by



using Proposition 13. Thus the corresponding permutation according to (12) is $(1, 2)$. Now, the sum of weights of these are

$$\sum_{r=1}^{2}\sum_{s=1}^{2}(A_\beta)_{2,2}(B_\beta)_{1,2}\Big((A_\alpha)_{1,1}(B_\alpha)_{1,1}+(A_\alpha)_{1,2}(B_\alpha)_{2,1}\Big)$$
$$=\sum_{s=1}^{2}(A_\beta)_{2,2}(B_\beta)_{1,2}\sum_{r=1}^{2}(A_\alpha B_\alpha)_{1,1},$$

which is zero by (4). A very similar computation goes through for the terms in Fig. 4(c) and (d).

We now illustrate Lemma 14 for $s=2$. This will amount to summing over all possible configurations in Fig. 5. Complications arise in the remaining terms shown in Fig. 5(e), (f), (g) and (h). The sign for the terms in (f) and (h) are computed as described above. In this case, combining terms (e) and (g), we get

$$-\sum_{r=1}^{2}\sum_{s=1}^{2}(A_\alpha)_{1,2}(A_\beta)_{2,1}\Big((B_\alpha)_{1,1}(B_\beta)_{1,2}+(B_\alpha)_{1,2}(B_\beta)_{2,2}\Big)$$
$$=-\sum_{r=1}^{2}\sum_{s=1}^{2}(A_\alpha)_{1,2}(A_\beta)_{2,1}\sum_{r=1}^{2}(B_\alpha B_\beta)_{1,2}$$
$$=-\sum_{r=1}^{2}\sum_{s=1}^{2}(A_\alpha)_{1,2}(A_\beta)_{2,1}\sum_{r=1}^{2}(B_\beta B_\alpha)_{1,2}$$
$$=-\sum_{r=1}^{2}\sum_{s=1}^{2}(A_\alpha)_{1,2}(A_\beta)_{2,1}\Big((B_\beta)_{1,1}(B_\alpha)_{1,2}+(B_\beta)_{1,2}(B_\alpha)_{2,2}\Big),$$

where we have used the commutativity of $B_\alpha$ and $B_\beta$ in the third line. Similarly, combining terms (f) and (h), we get

$$-\sum_{r=1}^{2}\sum_{s=1}^{2}(A_\alpha)_{1,2}(A_\beta)_{2,2}\Big((B_\beta)_{2,1}(B_\alpha)_{1,2}+(B_\beta)_{2,2}(B_\alpha)_{2,2}\Big).$$

Now, add the first summands in both the above equations to obtain

$$-\sum_{r=1}^{2}\sum_{s=1}^{2}(A_\alpha)_{1,2}(B_\alpha)_{1,2}\Big((A_\beta)_{2,1}(B_\beta)_{1,1}+(A_\beta)_{2,2}(B_\beta)_{2,1}\Big)$$
$$=\sum_{r=1}^{2}(A_\alpha)_{1,2}(B_\alpha)_{1,2}\sum_{s=1}^{2}(A_\beta B_\beta)_{2,1},$$

which is now 0 by (4). A similar computation goes through for the sums involving the second and fourth summands. This computation is what is essentially carried out in Lemma 14.

Now focus on the pathmap terms, namely (b), (d), (f) and (h). The (b) and (f) terms have the same weights but opposite signs. Ditto for (d) and (h) terms. This is an illustration of the sign-reversing involution in the proof of Lemma 15.

### 2.2. Reduction to the Cayley–Hamilton theorem

The Cayley–Hamilton theorem is a specialization of Theorem 1 when $k = 2$ and $A_1 = -I, A_2 = M, B_1 = M, B_2 = I$. Straubing's proof of the Cayley–Hamilton theorem [11] gives a weight-preserving and sign-reversing involution on $\mathcal{A}(b, e)$. Our proof when specialized to the Cayley–Hamilton theorem presents a weight-preserving and sign-reversing involution directly on $\mathcal{G}(b, e)$.

The constraint $A_1B_1 + A_2B_2 = 0$, in this case, is $(-I)M + M(I) = 0$ which means

$$
\begin{aligned}
\operatorname{swgt}\left( x \xrightarrow[\text{A}]{1} y \right) &= -\operatorname{swgt}\left( x \xrightarrow[\text{B}]{2} y \right) = -\delta_{x,y},\\
\operatorname{swgt}\left( x \xrightarrow[\text{A}]{2} y \right) &= \operatorname{swgt}\left( x \xrightarrow[\text{B}]{1} y \right) = M_{x,y}.
\end{aligned}
\tag{15}
$$

Therefore, we also have

$$
\begin{aligned}
&\operatorname{swgt}\left( y \xrightarrow[\text{B}]{1} z \right) \operatorname{swgt}\left( z \xrightarrow[\text{A}]{1} z \right)\\
&= \operatorname{swgt}\left( y \xrightarrow[\text{A}]{1} y \right) \operatorname{swgt}\left( y \xrightarrow[\text{B}]{1} z \right).
\end{aligned}
\tag{16}
$$

Now consider the sum over $a$ in the left hand side of (14). For example

$$
\operatorname{wgt}\left( s \xrightarrow[\text{A}]{1} a \right) \operatorname{wgt}\left( a \xrightarrow[\text{B}]{1} q_2 \right) = \delta_{s,a} M_{a,q_2},
$$

and therefore $a = s$. In that case

$$
\begin{aligned}
&\operatorname{swgt}\left( s \xrightarrow[\text{A}]{1} s \right) \operatorname{swgt}\left( s \xrightarrow[\text{B}]{1} q_2 \right)\\
&= \operatorname{swgt}\left( s \xrightarrow[\text{B}]{1} q_2 \right) \operatorname{swgt}\left( q_2 \xrightarrow[\text{A}]{1} q_2 \right)
\end{aligned}
$$

$$
\begin{aligned}
&= -\operatorname{swgt}\left( s \overset{2}{-\!\textcircled{\scriptsize B}\!\!\rightarrow} s \right) \operatorname{swgt}\left( s \overset{2}{-\!\textcircled{\scriptsize A}\!\!\rightarrow} q_2 \right) \\
&= -\operatorname{swgt}\left( s \overset{2}{-\!\textcircled{\scriptsize A}\!\!\rightarrow} q_2 \right) \operatorname{swgt}\left( q_2 \overset{2}{-\!\textcircled{\scriptsize B}\!\!\rightarrow} q_2 \right),
\end{aligned}
$$

where the first equality follows by (16), and the second and third by (15). This shows that the two terms in (14) cancel pairwise for $\ell_s = 1, 2$, and demonstrates the involution on $\mathcal{G}(b, e)$.

Notice that our proof strategy does not reduce to an involution on $\mathcal{A}(b, e)$. Therefore, we have a different combinatorial proof of the Cayley–Hamilton theorem as compared to the one by Straubing [11].

### *2.3. Relation to the proof by Phillips*

We show now that our combinatorial proof is a reinterpretation of the algebraic proof of Theorem 1 by Phillips [10]. Recall that we have matrices $A_1, \dots, A_k$, $B_1, \dots, B_k$ satisfying (4), where the $B_i$'s commute pairwise. Let $M(x_1, \dots, x_k) = (A_1 x_1 + \cdots + A_k x_k)_{1 \le i,j \le n}$ be an $n \times n$ matrix and $M_{i,j}(x_1, \dots, x_k)$ be its $(i, j)$'th entry. Then, let

$$
M^B_{i,j} = M_{i,j}(B_1, \dots, B_k) = (A_1)_{i,j} B_1 + \cdots + (A_k)_{i,j} B_k \tag{17}
$$

be the $n \times n$ matrix obtained by setting $B_i$ in place of $x_i$ for $i \in [k]$. For a matrix $A$, let $A[i|j]$ be the matrix $A$ with row $i$ and column $j$ removed, and denote $\det_B M[i|j]$ to be the matrix obtained by substituting $B_i$ in place of $x_i$ for $i \in [k]$ in $\det\big(M(x_1, \dots, x_k)[i|j]\big)$ so that

$$
\det\nolimits_B M[i|j] = (-1)^{i+j} \sum_{\substack{\sigma \in S_n \\ \sigma_i = j}} \operatorname{sgn}(\sigma) \prod_{\substack{r=1 \\ r \ne i}}^{n} M^B_{r,\sigma_r}, \tag{18}
$$

using (17).

Let us compute the signed weight of $\mathcal{G}(b, e)$, which we know by Lemma 14 to be 0.

$$
\begin{aligned}
\sum_{(\overline{m}, \bar{q}) \in \mathcal{G}(b,e)} \operatorname{swgt}(\overline{m}, \bar{q}) &= \sum_{j=1}^{n} \sum_{(\bar{\pi}, \bar{q}) \in \mathcal{A}(b,e)} \operatorname{swgt}(\phi((\bar{\pi}, \bar{q}), j)) \\
&= \sum_{j=1}^{n} \sum_{\sigma \in S_n} \operatorname{sgn}(\sigma) \sum_{(\bar{\sigma}, \bar{q}) \in \mathcal{A}(b,e)} \operatorname{wgt}(\phi((\bar{\sigma}, \bar{q}), j)) \\
&= \sum_{j=1}^{n} \sum_{s=1}^{n} \sum_{\substack{\sigma \in S_n \\ \sigma_s = b}} \operatorname{sgn}(\sigma) \sum_{\ell_1, \dots, \ell_n = 1}^{k} (A_{\ell_s})_{s,j}
\end{aligned}
$$

$$\times \left( \prod_{\substack{r=1 \\ r\neq s}}^{n} (A_{\ell_r})_{r,\sigma_r} \right) \left( \prod_{r=1}^{n} B_{\ell_r} \right)_{j,e}.$$

Next we rely on the commutativity of the $B_i$'s to write this as

$$\sum_{j=1}^{n} \sum_{s=1}^{n} \sum_{\substack{\sigma\in S_n \\ \sigma_s=b}} \operatorname{sgn}(\sigma) \sum_{\ell_1,\dots,\ell_n=1}^{k} \Big( (A_{\ell_s})_{s,j} B_{\ell_s} \prod_{\substack{r=1 \\ r\neq s}}^{n} (A_{\ell_r})_{r,\sigma_r} B_{\ell_r} \Big)_{j,e}$$

$$= \sum_{j=1}^{n} \sum_{s=1}^{n} \sum_{\substack{\sigma\in S_n \\ \sigma_s=b}} \operatorname{sgn}(\sigma) \left( M^B_{s,j} \prod_{\substack{r=1 \\ r\neq s}}^{n} M^B_{r,\sigma_r} \right)_{j,e},$$

where we have first performed the $\ell_1, \dots, \ell_n$ sums before taking the $(j, e)$'th entry and used (17) in the last step. Now let us perform the inner sum. Since the product over $r \neq s$ is not dependent on $j$, we can use (18) to arrive at

$$\sum_{j=1}^{n} \sum_{s=1}^{n} (-1)^{s+j} \left( M^B_{s,j} \det_B M[s|b] \right)_{j,e}.$$

By the standard Laplace expansion, the only contribution to the $j$ sum comes from $j = b$, giving

$$\sum_{s=1}^{n} (-1)^{s+b} \left( M^B_{s,b} \det_B M[s|b] \right)_{b,e}.$$

This is precisely what Phillips shows to be 0 in [10, Theorem I].

## 3. Application to mixed discriminants

In this section, we will prove Theorem 5 using the same strategy as for the proof of Theorem 1 in Section 2. We recall the setup. We have $2nk$ matrices, which we call $(A_{i,j})_{i\in[n],j\in[k]}$ and $(B_{i,j})_{i\in[n],j\in[k]}$ which satisfy the conditions:

- $B_{i,j} B_{i',j'} = B_{i',j'} B_{i,j}$ for all $1 \le i < i' \le n, 1 \le j, j' \le k$, and
- $$A_{i,1} B_{i,1} + \cdots + A_{i,k} B_{i,k} = 0, \quad 1 \le i \le n. \tag{19}$$

Recall the polynomial $\hat{p}\left((x_{i,j})_{i\in[n],j\in[k]}\right)$ from (3),

$$\hat{p}\left((x_{i,j})_{i\in[n],j\in[k]}\right) = \mathfrak{D}(A_{1,1}x_{1,1} + \cdots + A_{1,k}x_{1,k}, \dots, A_{n,1}x_{n,1} + \cdots + A_{n,k}x_{n,k}),$$

in $nk$ variables $(x_{i,j})_{i\in[n],j\in[k]}$, where $\mathfrak{D}$ is the mixed discriminant given in Definition 3.

**Definition 16.** A *decorated 2-permutation* $\hat{\alpha}_\pi$ is an $n$-tuple of quadruples $(\hat{\alpha}_\pi)_i = (i, \alpha_i, \pi_i, \ell_i)$ for $i \in [n], \ell_i \in [k]$, where $\pi, \alpha \in S_n$ and the pairs $\alpha_{\pi_i}, \ell_i$ are called *labels*. We will denote $(\hat{\alpha}_\pi)_i$ as $i \xrightarrow[\text{A}]{\alpha_{\pi_i},\, \ell_i} \pi_i$ , which has *weight* $(A_{\alpha_{\pi_i},\ell_i})_{i,\pi_i}$. The *signed weight* of the decorated 2-permutation is given by

$$\operatorname{swgt}(\hat{\alpha}_\pi) = \operatorname{sgn}(\pi) \prod_{i=1}^{n} (A_{\alpha_{\pi_i},\ell_i})_{i,\pi_i}.$$

The set of all decorated 2-permutations is denoted $\hat{S}^2_{n,k}$.

Since there are $n!$ permutations and all labels are independently chosen, the cardinality of $\hat{S}^2_{n,k}$ is $n!^2 k^n$.

**Definition 17.** A *decorated 2-path* of length $n$ is a tuple $\hat{q} = (q_1, \dots, q_{n+1})$, where each $q_i \in [n]$. For $i \in [n]$, the $i$'th labeled edge is denoted $\hat{q}_i = q_i \xrightarrow[\text{B}]{\alpha_i,\, \ell_i} q_{i+1}$ and has weight $(B_{\alpha_i,\ell_i})_{q_i,q_{i+1}}$, where $\alpha \in S_n$ and the *label* $\ell_i \in [k]$. The *weight* of the decorated 2-path is

$$\operatorname{wgt}(\hat{q}) = \prod_{i=1}^{n} (B_{\alpha_i,\ell_i})_{q_i,q_{i+1}}.$$

The set of all decorated 2-paths is denoted $\widehat{Q}^2_{n,k}$.

**Definition 18.** A *2-pathmutation* is a pair $(\hat{\alpha}_\pi, \hat{q})$ where $\hat{\alpha}_\pi \in \hat{S}^2_{n,k}, \hat{q} \in \widehat{Q}^2_{n,k}$ such that the labels of the $i$'th element of the permutation and the $i$'th edge of the path are the same for all $i \in [n]$. The *signed weight* of a 2-pathmutation is

$$\operatorname{wgt}(\hat{\alpha}_\pi, \hat{q}) = \operatorname{swgt}(\hat{\alpha}_\pi) \operatorname{wgt}(\hat{q}).$$

The set of 2-pathmutations beginning with $q_1 = b$ and ending with $q_{n+1} = e$ is denoted $\mathcal{A}^2(b, e)$.

The cardinality of $\mathcal{A}^2(b, e)$ is $n!^2 k^n n^{n-1}$ for every $b, e \in [n]$ because we can choose $q_2, \dots, q_{n-1}$ arbitrarily.

Fig. 6 shows a 2-pathmutation $(\hat{\alpha}_\pi, \hat{q})$.

**Definition 19.** A *decorated 2-map* $\widehat{m}$ is an $n$-tuple of quadruples $\widehat{m}_i = (\sigma_i, \tau_i, \alpha_i, \ell_i)$, where $\sigma \in S_n$ is either the identity or a single transposition, $\alpha \in S_n$, $\tau_i \in [n]$ and $\ell_i \in [k]$ for all $i$ such that

$$\begin{array}{llllll}
\hat{\alpha}_\pi: & 1 \xrightarrow[\text{A}]{\alpha_{\pi_1}, \ell_1} \pi_1 & \cdots & s \xrightarrow[\text{A}]{\alpha_b, \ell_s} b & \cdots & n \xrightarrow[\text{A}]{\alpha_{\pi_n}, \ell_n} \pi_n \\
\hat{q}: & b \xrightarrow[\text{B}]{\alpha_{\pi_1}, \ell_1} q_2 & \cdots & q_s \xrightarrow[\text{B}]{\alpha_b, \ell_s} q_{s+1} & \cdots & q_n \xrightarrow[\text{B}]{\alpha_{\pi_n}, \ell_n} e
\end{array}$$

**Fig. 6.** A 2-pathmutation $(\hat{\alpha}_\pi, \hat{q}) \in \mathcal{A}^2(b, e)$ where $\pi_s = b$.

- $\#\{\tau_1, \dots, \tau_n\} \geq n - 1$,
- if $\#\{\tau_1, \dots, \tau_n\} = n$, then $\sigma$ is the identity permutation,
- if $\tau_i = \tau_j$ for some $(i, j)$, then either $\sigma_i = i, \sigma_j = j$ or $\sigma_i = j, \sigma_j = i$.

The *weight* of $\widehat{m}_i$ is $(A_{\alpha_i,\ell_i})_{\sigma_i,\tau_i}$ and is denoted $\sigma_i \xrightarrow[\text{A}]{\alpha_i, \ell_i} \tau_i$. The weight of $\widehat{m}$ is then

$$\mathrm{wgt}(\widehat{m}) = \prod_{i=1}^{n} (A_{\alpha_i,\ell_i})_{\sigma_i,\tau_i}.$$

The set of all decorated 2-maps is denoted $\widehat{M}^2_{n,k}$.

When $\#\{\tau_1, \dots, \tau_n\} = n$, we get exactly decorated 2-permutations. When $\#\{\tau_1, \dots, \tau_n\} = n - 1$, there are $n(n-1) \times n!/2$ possibilities for $\tau$ and 2 possibilities for $\sigma$ so that the cardinality of $\widehat{M}^2_{n,k}$ is

$$k^n(n! + n(n-1)n!) = n!^2 k^n (n^2 - n + 1).$$

**Definition 20.** A 2-*pathmap* is a pair $(\widehat{m}, \hat{q})$ where $\widehat{m} \in \widehat{M}^2_{n,k}, \hat{q} \in \widehat{Q}^2_{n,k}$ such that

- If $\{\tau_1, \dots, \tau_n\} = [n]$, then $(\widehat{m}, \hat{q})$ is a 2-pathmutation.
- If $\#\{\tau_1, \dots, \tau_n\} = n - 1$ and $\tau_i = \tau_j$ for some $i \neq j$, then $q_1 = \tau_i$. In this case, the labels of $\widehat{m}_k$ and $\hat{q}_k$ must match for all $k \neq i, j$. In addition, if $\sigma_i = i, \sigma_j = j$ (resp. $\sigma_i = j, \sigma_j = i$), then the labels of $\widehat{m}_i$ and $\widehat{m}_j$ are equal to those of $\hat{q}_i, \hat{q}_j$ (resp. $\hat{q}_j, \hat{q}_i$) respectively.

The weight of the 2-pathmap $(\widehat{m}, \hat{q})$ is

$$\mathrm{wgt}(\widehat{m}, \hat{q}) = \mathrm{wgt}(\widehat{m})\, \mathrm{wgt}(\hat{q}).$$

The set of 2-pathmaps with $\#\{\tau_1, \dots, \tau_n\} = n-1$ such that $\{\tau_1, \dots, \tau_n\} = [n] \setminus \{b\}$ and ending with $q_{n+1} = e$ is denoted $\mathcal{H}^2(b, e)$. In addition, let $\mathcal{G}^2(b, e) = \mathcal{H}^2(b, e) \cup \mathcal{A}^2(b, e)$.

Analogous to the enumeration of pathmaps, the cardinality of $\mathcal{H}^2(b, e)$ is again $n - 1$ times that of $\mathcal{A}^2(b, e)$. As in the proof of Phillips' theorem, we will need to attach a sign to a 2-pathmap in $\mathcal{G}^2(b,e)$. As before, define a map $\hat{\phi} : \mathcal{A}^2(b,e) \times [n] \to \mathcal{G}^2(b,e)$. Set $\hat{\phi}((\hat{\alpha}_\pi, \hat{q}), j) = (\widehat{m}', \hat{q}')$ as follows. First, set

$$\hat{q}'_r = \begin{cases} i' \xrightarrow[\text{B}]{\alpha_{\pi_1}, \ell_1} q_2 & r = 1, \\ \hat{q}_r & \text{otherwise.} \end{cases}$$

Next, set $s = \pi_b^{-1}$ and $t = \pi_j^{-1}$. Then let

$$\widehat{m}'_r = \begin{cases} \min(s,t) \xrightarrow[\text{A}]{\alpha_{\pi_{\min(s,t)}}, \ell_{\min(s,t)}} j & \text{if } r = s, \\ \max(s,t) \xrightarrow[\text{A}]{\alpha_{\pi_{\max(s,t)}}, \ell_{\max(s,t)}} j & \text{if } r = t, \\ (\hat{\alpha}_\pi)_r & \text{otherwise.} \end{cases} \tag{20}$$

The sign of an element $(\widehat{m}, \hat{q}) \in \mathcal{G}^2(b,e)$ can be defined in the same way as before. If $\hat{\phi}^{-1}(\widehat{m}, \hat{q}) = ((\hat{\alpha}_\pi, \hat{q}), j)$, then the signed weight of $(\widehat{m}, \hat{q})$ is given by

$$\operatorname{swgt}(\widehat{m}, \hat{q}) = \operatorname{sgn}(\pi) \operatorname{wgt}(\widehat{m}, \hat{q}). \tag{21}$$

**Proof of Theorem 5.** We first claim that

$$\sum_{(\hat{\alpha}_\pi, \hat{q}) \in \mathcal{A}^2(b,e)} \operatorname{swgt}(\hat{\alpha}_\pi, \hat{q}) = \hat{p}\left((B_{i,j})_{i\in[n], j\in[k]}\right)_{b,e}. \tag{22}$$

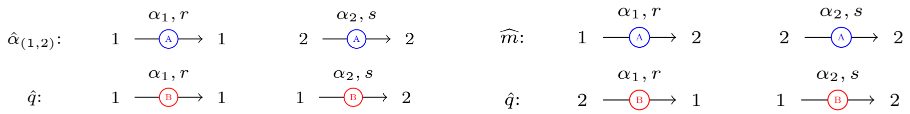
To see this, begin by expanding the polynomial $q$ in (3) as

$$\hat{p}\left((x_{i,j})_{i\in[n],j\in[k]}\right) = \frac{1}{n!} \sum_{\alpha \in S_n} \sum_{\pi \in S_n} \operatorname{sgn}(\pi) \times \prod_{i=1}^{n} \Big((A_{\alpha_{\pi_i},1})_{i,\pi_i} x_{\alpha_{\pi_i},1} + \cdots + (A_{\alpha_{\pi_i},k})_{i,\pi_i} x_{\alpha_{\pi_i},k}\Big).$$

Now, substitute $x_{i,j}$ by $B_{i,j}$ and use the fact that $B_{i,j}$'s commute pairwise to obtain

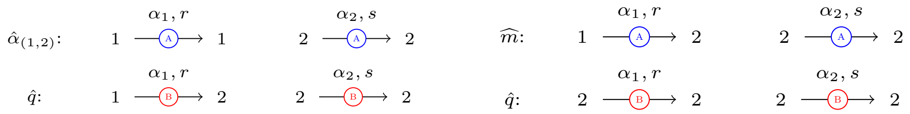
$$\hat{p}\left((B_{i,j})_{i\in[n],j\in[k]}\right) = \frac{1}{n!} \sum_{\alpha \in S_n} \sum_{\pi \in S_n} \operatorname{sgn}(\pi) \times \prod_{i=1}^{n} \Big((A_{\alpha_{\pi_i},1})_{i,\pi_i} B_{\alpha_{\pi_i},1} + \cdots + (A_{\alpha_{\pi_i},k})_{i,\pi_i} B_{\alpha_{\pi_i},k}\Big),$$

which now simplifies to



(a) $+(A_{\alpha_1,r})_{1,1}(A_{\alpha_2,s})_{2,2}(B_{\alpha_1,r})_{1,1}(B_{\alpha_2,s})_{1,2}$ (b) $+(A_{\alpha_1,r})_{1,2}(A_{\alpha_2,s})_{2,2}(B_{\alpha_1,r})_{2,1}(B_{\alpha_2,s})_{1,2}$

**Fig. 7.** The 2-pathmap terms proportional to $(A_{\alpha_2,s})_{2,2}(B_{\alpha_2,s})_{1,2}$ along with their signed weights.



(c) $+(A_{\alpha_1,r})_{1,1}(A_{\alpha_2,s})_{2,2}(B_{\alpha_1,r})_{1,2}(B_{\alpha_2,s})_{2,2}$ (d) $+(A_{\alpha_1,r})_{1,2}(A_{\alpha_2,s})_{2,2}(B_{\alpha_1,r})_{2,2}(B_{\alpha_2,s})_{2,2}$

**Fig. 8.** The 2-pathmap terms proportional to $(A_{\alpha_2,s})_{2,2}(B_{\alpha_2,s})_{2,2}$ along with their signed weights.

$$\hat{p}\left((B_{i,j})_{i\in[n],j\in[k]}\right) = \frac{1}{n!}\sum_{\alpha\in S_n}\sum_{\pi\in S_n}\operatorname{sgn}(\pi)\sum_{(\ell_1,\dots,\ell_n)\in[k]^n}\prod_{i=1}^{n}(A_{\alpha_{\pi_i},\ell_i})_{i,\pi_i}B_{\alpha_{\pi_i},\ell_i}.$$

Now consider the $(b,e)$'th entry of this sum. For each pair of permutations $\alpha,\pi$ and each element $\ell=(\ell_1,\dots,\ell_n)\in[k]^n$, we can represent the product of $A_{\alpha_{\pi_i},\ell_i}$ over $i$ as the weight of the decorated 2-permutation $\hat{\alpha}_\pi$, the label of whose $i$'th element is $(\alpha_{\pi_i},\ell_i)$ as seen above. Now expand the product of $B_{\alpha_{\pi_i},\ell_i}$'s on the right hand side. The $(b,e)$'th entry is a sum of terms, each of which corresponds exactly to a decorated 2-path with initial vertex $b$ and final vertex $e$. The $i$'th edge in the decorated 2-path has the same label, $(\alpha_{\pi_i},\ell_i)$. Therefore, each term corresponds to a 2-pathmutation, whose weight is equal to the term. This proves the claim above.

We now have the analogues of Lemma 14 and Lemma 15.

$$\sum_{(\widehat{m},\hat{q})\in\mathcal{G}^2(b,e)}\operatorname{swgt}(\widehat{m},\hat{q}) = 0, \tag{23}$$

$$\sum_{(\widehat{m},\hat{q})\in\mathcal{H}^2(b,e)}\operatorname{swgt}(\widehat{m},\hat{q}) = 0. \tag{24}$$

The proofs of these equations proceed in essentially the same manner as the above lemmas and we omit them. By definition, the left hand side of (22) is the difference of the left hand sides of (23) and (24), proving the result. □

### *3.1. Illustration for $n = 2$*

We illustrate the ideas in the proof of (23) and (24), which are key to the proof of Theorem 5, for $n=k=2$. As in Section 2.1, we will look at $b=1$ and $e=2$ in detail and keep the labels $\ell_1=r$ and $\ell_2=s$ in addition to the permutation $\alpha$ arbitrary. We then have 4 2-pathmutations, which are shown in the left columns of Figs. 7, 8 and 9. Similarly, there are 4 such 2-pathmaps in $\mathcal{H}^2(1,2)$, which are shown in the right columns of Figs. 7, 8 and 9.

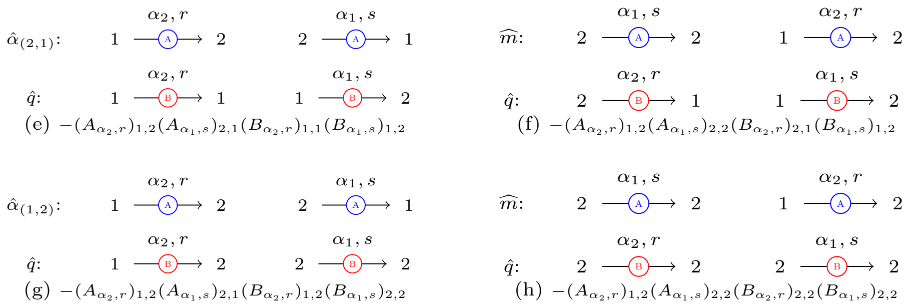



**Fig. 9.** The 2-pathmap terms proportional to $(A_{\alpha_2,r})_{1,2}$ along with their signed weights.

Let us compare the 2-pathmutation in Fig. 7(a) and the 2-pathmap in Fig. 7(b). One can check that the latter has positive sign. Now, the sum of weights of these are

$$\begin{aligned}
&\sum_{\alpha\in S_2}\sum_{r=1}^{2}\sum_{s=1}^{2}(A_{\alpha_2,s})_{2,2}(B_{\alpha_2,s})_{1,2}\Big((A_{\alpha_1,r})_{1,1}(B_{\alpha_1,r})_{1,1}+(A_{\alpha_1,r})_{1,2}(B_{\alpha_1,r})_{2,1}\Big)\\
&=\sum_{\alpha_1=1}^{2}\sum_{s=1}^{2}(A_{3-\alpha_1,s})_{2,2}(B_{3-\alpha_1,s})_{1,2}\sum_{r=1}^{2}(A_{\alpha_1,r}B_{\alpha_1,r})_{1,1},
\end{aligned}$$

which is zero by (19). A very similar computation goes through for the terms in Fig. 8(c) and (d) and gives

$$\sum_{\alpha_1=1}^{2}\sum_{s=1}^{2}(A_{3-\alpha_1,s})_{2,2}(B_{3-\alpha_1,s})_{2,2}\sum_{r=1}^{2}(A_{\alpha_1,r}B_{\alpha_1,r})_{1,2},$$

which is also zero for the same reason.

Complications arise in the remaining terms shown in Fig. 9(e), (f), (g) and (h). The sign for the terms in (g) and (h) are computed as described above. In this case, combining terms (e) and (g), we get

$$\begin{aligned}
&-\sum_{\alpha\in S_2}\sum_{r=1}^{2}\sum_{s=1}^{2}(A_{\alpha_2,r})_{1,2}(A_{\alpha_1,s})_{2,1}\Big((B_{\alpha_2,r})_{1,1}(B_{\alpha_1,s})_{1,2}+(B_{\alpha_2,r})_{1,2}(B_{\alpha_1,s})_{2,2}\Big)\\
&=-\sum_{\alpha\in S_2}\sum_{r=1}^{2}\sum_{s=1}^{2}(A_{\alpha_2,r})_{1,2}(A_{\alpha_1,s})_{2,1}\sum_{r=1}^{2}(B_{\alpha_2,r}B_{\alpha_1,s})_{1,2}\\
&=-\sum_{\alpha\in S_2}\sum_{r=1}^{2}\sum_{s=1}^{2}(A_{\alpha_2,r})_{1,2}(A_{\alpha_1,s})_{2,1}\sum_{r=1}^{2}(B_{\alpha_1,s}B_{\alpha_2,r})_{1,2}\\
&=-\sum_{\alpha\in S_2}\sum_{r=1}^{2}\sum_{s=1}^{2}(A_{\alpha_2,r})_{1,2}(A_{\alpha_1,s})_{2,1}\Big((B_{\alpha_1,s})_{1,1}(B_{\alpha_2,r})_{1,2}+(B_{\alpha_1,s})_{1,2}(B_{\alpha_2,r})_{2,2}\Big),
\end{aligned}$$

where we have used the commutativity of $B_{\alpha_2,r}$ and $B_{\alpha_1,s}$ in the third line. Similarly, combining terms (f) and (h), we get

$$\sum_{\alpha\in S_2}\sum_{r=1}^{2}\sum_{s=1}^{2}(A_{\alpha_2,r})_{1,2}(A_{\alpha_1,s})_{2,2}\Big((B_{\alpha_2,r})_{2,1}(B_{\alpha})_{1,2}+(B_{\alpha_1,s})_{2,2}(B_{\alpha_2,r})_{2,2}\Big).$$

Now, add the first summands in both the above equations to obtain

$$\begin{aligned}&-\sum_{\alpha\in S_2}\sum_{r=1}^{2}\sum_{s=1}^{2}(A_{\alpha_2,r})_{1,2}(B_{\alpha_2,r})_{1,2}\Big((A_{\alpha_1,s})_{2,1}(B_{\alpha_1,s})_{1,1}+(A_{\alpha_1,s})_{2,2}(B_{\beta})_{2,1}\Big)\\ =&-\sum_{\alpha_1=1}^{2}\sum_{r=1}^{2}(A_{3-\alpha_1,r})_{1,2}(B_{3-\alpha_1,r})_{1,2}\sum_{s=1}^{2}(A_{\alpha_1,s}B_{\alpha_1,s})_{2,1},\end{aligned}$$

which is now 0 by (19). A similar computation goes through for the second and fourth summands. This kind of computation is what needs to carried out to prove (23).

Now focus on the 2-pathmap terms in Figs. 7, 8 and 9, namely (b), (d), (f) and (h). Focus on the (b) and (f) figures. If we interchange $\alpha_1$ and $\alpha_2$ in the weight of the (b) figure, we obtain the negative of the weight of the (f) figure. Similarly, for (d) and (h) terms. Thus, an involution of the same kind used in the proof of Lemma 15 in addition to an appropriate involution on $\alpha$ will prove (24).

## Declaration of competing interest

The authors declare no competing interest.

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