Performance Evaluation 155–156 (2022) 102297

Contents lists available at ScienceDirect

Performance Evaluation

journal homepage: www.elsevier.com/locate/peva

Service scheduling for random requests with fixed waiting costs

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ARTICLE INFO

Article history:
Received 23 March 2021
Received in revised form 1 June 2022
Accepted 1 June 2022
Available online 16 June 2022

Keywords:
Service scheduling
Fixed waiting cost
Markov decision process

ABSTRACT

We study service scheduling problems in a slotted system in which agents arrive with service requests according to a Bernoulli process and have to leave within two slots after arrival, service costs are quadratic in service rates, and there is also a waiting cost. We consider fixed waiting costs. We frame the problem as an average cost Markov decision process. While the studied system is a linear system with quadratic costs, it has state dependent control. Moreover, it also possesses a non-standard cost function structure rendering the optimization problem complex. Here, we characterize the optimal policy. We also consider a system in which the agents make scheduling decisions for their respective service requests keeping their own cost in view. We frame this scheduling problem as a stochastic game. Here, we provide Nash equilibrium.

1. Introduction

Service scheduling problems have been widely studied in the literature. They apply to a wide range of applications like speed scaling in CPUs, scheduling of charging of electric vehicles (EVs), job scheduling in mobile edge computing (MEC), etc. In all these applications, service costs, measured in terms of energy consumption, increase with quantum of service. For instance, server energy consumption in cloud computing increases as a convex function of the quantum of service (see [1,2]). Similarly, in the context of EV charging, the energy cost can be modeled as a quadratic function of the service offered [3]. So, when quanta of services exceed certain thresholds, one may want to defer a part of service requests, saving energy cost in lieu of increased latency. However, large latencies must also be avoided.

We capture the above conflicting objectives through a model having soft and hard deadlines. It is desirable to complete service requests by their soft deadlines. The service requests can be deferred beyond their soft deadlines, but then they also incur waiting costs. The waiting cost behaves as a disincentive for deferring service to avoid excessive latencies. Of course, service requests *must* be completed before their hard deadlines. We aim at deriving service scheduling policies that optimize the time average sum of service and waiting costs.

Optimal scheduling that intends to minimize the service costs balances service over time. However, since deferring services also incur waiting costs, balancing the quanta of services is sub-optimal. We study service scheduling in slotted systems with Bernoulli service arrivals, quadratic service costs, and service delay guarantees. We consider fixed waiting costs. In particular, we consider the cases where the service requests can stay for two slots but incur fixed waiting costs in second slots. We see that this service scheduling problem is a special case of constrained linear quadratic control. We

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https://doi.org/10.1016/j.peva.2022.102297
0166-5316/

study optimal scheduling and Nash equilibria for selfish agents. These problems consider both service and waiting costs into account. We analyze optimal and equilibrium policies.

1.1. Related work

In [4], the authors propose a centralized algorithm to minimize the total charging cost of EVs. It determines the optimal amount of charging to be received at various charging stations en route. There is another line of work which intends to minimize waiting times at the charging stations. For instance, in [5] the authors propose a distributed scheduling algorithm that uses local information of traffic flows measured at the neighboring charging stations to uniformly utilize charging resources along the highway and minimize the total waiting time. In our work, we consider minimizing both charging and waiting costs simultaneously. More precisely we look fixed waiting cost. In the context of traffic routing and scheduling, the authors in [6] consider a scenario where agents compete for a common link to ship their demands to a destination. They obtain the optimal and equilibrium flows in the presence of polynomial congestion cost.

In [7], we consider routing on a ring network in the presence of quadratic congestion costs and also linear delay costs when traffic is redirected through the adjacent nodes. However, the problems in [7] are one-shot optimization problems as these do not have a temporal component. In [8], the authors consider a discrete time system in which jobs arrive according to a Poisson process and have exponential deadlines and job sizes.

Minimum energy scheduling has also been considered in the context of CPU power consumption [9], big data processing [10], production scheduling in plants [11]. In [12], the authors consider convex processing cost. They propose an optimal online algorithm for job arrivals with deadline uncertainty. They also derive competitive ratio for the proposed algorithm. Neither of the above discussed works accounts for waiting costs of jobs as considered in our work.

In an earlier work [13], we studied service scheduling for Bernoulli job arrivals, quadratic service costs and linear waiting costs. We obtained an optimal policy and a symmetric Nash equilibrium. We also extended the analysis to a scenario where job sizes can take distinct values, and job arrivals constitute a Markov chain. In [13] we discuss linear waiting costs. Analysis of fixed waiting cost is more complicated owing to discontinuity in the cost function. In contrast to [13] the optimal policy is not continuous in pending service (see Proposition 3.1, paragraphs above and below it for a detailed discussion on the optimal policy).

The authors in [14] consider a single server slotted system with impatient customers. Impatience of customers can be seen as their having stochastic deadlines. The authors assume that the customers have geometric sojourn times but fixed one-slot service time. They consider three costs, a fixed customer holding cost per slot, a fixed cost of losing a customer, a fixed service cost, for each customer. At the beginning of each slot, if the queue is nonempty, the server has to decide whether to serve a customer. The simple service discipline and cost structure allow the authors to derive a simple rule. The authors in [15] generalized the above model by considering exponential service times and γ-Cox distributed customer sojourn times. They consider two customer classes with different arrival rates and different linear customer waiting costs but no other costs. On each service completion, if customers of both the classes are waiting, the server has to decide which customer class to choose for service. However, the authors have only performed numerical value iteration and have obtained regions in which the first or the second customer classes are chosen. None of these works consider the case of rational customers.

Linear systems with quadratic cost have been widely studied in control theory. For instance, in infinite horizon unconstrained linear quadratic control, the optimal policy is found to be linear in system state and is given by the *Riccati equation* [16]. We have at our disposal control problems with state-dependent constraints. Moreover, in case of fixed waiting costs the problems do not conform to standard assumptions, e.g., positive definiteness of the control weighing matrix. In [17], the authors obtain a Nash equilibrium for a stochastic game where each arriving customer observes the current load and has to choose between a shared system whose service rate decreases with the number of customers or a constant service rate system. The optimal choice for each customer depends on the decisions of previous ones and the subsequent ones, through their effect on the current and future load in the shared server.

1.2. Applications and motivation

Our framework is general that can be applied to many contexts like scheduling charging of EVs, job scheduling in data centers, etc. In all these applications, both hard and soft deadlines arise naturally. In many cases, network (or, resource) managers schedule service requests to optimize time-average service and waiting costs while respecting their deadlines. For instance, in the examples of job scheduling in CPUs or in data centers, service schedulers may want to optimize average power and storage costs. These objectives are captured by the proposed optimal scheduling problem.

On the other hand, in some contexts the strategic agents who bring service requests to the system dictate their service schedules. Their scheduling decisions are aimed at minimizing their respective service and waiting costs. Such scenarios can naturally be modeled using non-cooperative stochastic games. For instance, if the EV owners in the EV charging example strive to minimizing their respective charging and waiting costs a stochastic game emerges.

In several systems of interest, agents can enter the system or leave only at slot boundaries, e.g., from [18], compute tasks derive utility only at slot boundaries. In such tasks that complete only at slot boundaries, the current operating job will be present in the system until its next slot boundary irrespective of the amount of pending service. Thus the waiting cost is fixed and does not depend on the amount of deferred service. Similarly, in data centers, the job in execution would hold a certain amount of fixed storage [19]. That storage is not released till the job exits the system. Thus we intend to capture the fixed storage costs in fixed waiting costs. In some other systems, service requests have soft deadlines; missing soft deadlines is tolerable but not desirable. The authors in [20] propose the notion of *tardiness* which is the difference between the service requests' actual service completion times and their soft deadlines. In our formulation, each request has a soft deadline of one slot and a hard deadline of two slots. The fixed waiting cost models the tardiness of a service request that is not completely served in its first slot. These scenarios motivate fixed waiting costs proposed in Section 2.2.

We also present a comparative numerical study to illustrate the impact of various waiting cost structures and performance criteria (optimal scheduling vs. strategic scheduling by selfish agents).

2. System model

We consider a time-slotted system where time is divided into discrete slots. The length of the slot depends on the application, e.g., in the case of CPU speed scaling the slots are of the order of ms where in the case of job scheduling the slots many of the order of several tens of minutes. Agents arrive over slots to a service facility. Every agent is characterized by its arrival time, deadline, and the amount of service it requires. Each service request has to be wholly served before its deadline. So service can be scheduled such that portions of the agents' required service are served in the future slots before their respective deadlines. Serving requests incur a cost, with the cost per unit service in a slot depending on the quantum of service delivered in that slot. Though the service facility has enough capacity to serve all the agents in the system, some of the service may be deferred to save on the service cost. We consider two scheduling problems: one where the service facility makes scheduling decisions to optimize the overall time-average cost and the other where the agents make scheduling decisions for their respective service requests to minimize their costs. Below we present the system model and both the problems formally.

2.1. Service request model

Agents with service requests arrive according to an i.i.d. Bernoulli(p) process; $p \in (0, 1)$. We assume that all the agents require equal amount of service, denoted as ψ. Further, each request can be met in at most two slots, i.e., a fraction of the service request arriving in a slot could be deferred to the next slot. As every agent leaves at the end of two slots, in any slot there can be a maximum of two agents. Hence the system remains stable. It is assumed that the service facility can serve up to 2ψ units in a slot.

2.2. Cost model

The cost consists of two components:

- *Service cost:* The service cost per unit service in a slot is a linear function of the total service offered in that slot. Thus the total service cost in a slot is square of the total offered service in that slot. For instance, in the context of EV charging, per unit electricity cost is modeled as a linear function of the load [3,21].
- *Waiting cost:* Each service incurs a fixed waiting cost $d > 0$ when a portion of the service is deferred to the next slot. This waiting cost can be interpreted as the penalty for not serving the service request in the same slot in which it has arrived. We introduce the waiting cost to strike a balance between service cost and latency. The constant d can be seen as relative weight of waiting cost vis-s-vis service cost for instance e.g., higher d indicates that the users are more sensitive to latency.

Let, for $k \geq 1$, x_k be the remaining demand from slot $k-1$ to slot k; $x_1 = 0$. This demand must be met in slot k. Also, for $k \geq 1$, let v_k be the extra service offered in slot k over x_k. Clearly, $v_k \in [0, \psi]$ and is 0 if there is no new request in slot k. A *scheduling policy* $\overline{\pi} = (\pi_k, k \geq 1)$ is a sequence of functions $\pi_k : [0, \psi] \to [0, \psi]$ such that if there is a service request in slot k then $\pi_k(x_k)$ gives the amount of service deferred from slot k to slot $k+1$. In other words,

$$x_{k+1} = \begin{cases} \pi_k(x_k) = \psi - v_k, & \text{if a request arrives in slot } k, \\ 0, & \text{otherwise.} \end{cases}$$

We consider the following two scheduling problems.

2.2.1. Optimal scheduling

We aim to minimize the time-averaged cost of the service facility. Here, waiting cost is imposed by the service facility to reduce the latency of the individual service requests. More precisely, we want to determine the scheduling policy $\overline{\pi}$ that minimizes

$$\lim_{T\to\infty}\frac{1}{T}\sum_{k=1}^{T}\mathbb{E}[(x_k+v_k)^2+d\mathbb{1}_{\{v_k\in(0,\psi)\}}]. \tag{1}$$

We obtain the optimal solution in Section 3.

At first glance, the optimization problem appears to be a special case of the well-studied *constrained linear quadratic control Markov decision problems*. In particular, if we define binary variables e_k, $k\geq 1$, as

$$e_k=\begin{cases}\psi, & \text{if slot } k \text{ has an arrival,}\\ 0, & \text{otherwise,}\end{cases}$$

then (x_k, e_k) can be considered to be the system state in slot k. The total service in slot k, $\bar{u}_k\in[x_k, x_k+e_k]$, and $w_k=e_{k+1}$ can be considered the action and the noise in slot k, respectively. Similar to [13, Section II A] the state evolution happens as $(x_{k+1}, e_{k+1})=(x_k+e_k-\bar{u}_k, w_k)$ and the single stage cost is $d\mathbb{1}_{x_k+e_k-\bar{u}_k>0}+\bar{u}_k^2$. *We see that the actions are subject to state dependent constraints and the single stage costs are not expressible in the form* $(x_k, e_k)^T Q(x_k, e_k)+\bar{u}_k^2$ *with* Q *a positive semidefinite matrix*. Thus the problem does not conform to the standard framework.

2.2.2. Equilibrium for selfish agents

Recall that, in our model each agent comes with a service request, all service requests being of the same size. Here, we consider rational agents, each determining how much of its request should be deferred. Further, each agent is aiming at minimizing his/her own service and waiting costs. We can frame this problem as a non-cooperative dynamic game among the agents. Here, the waiting cost is imposed by every individual agent in the system to minimize their respective waiting times. In this context, let us refer to π_k as a strategy of the agent who arrives in slot k (if there is one) and $\overline{\pi}=(\pi_k, k\geq 1)$ as a strategy profile.[1] If an agent k sees the system state as x, then the agent chooses the action $\pi_k(x)$. Then the total demand served in that slot is $x+\psi-\pi_k(x)$, which is per unit cost. Therefore, the total service cost levied on *the agent* is $(\psi-\pi_k(x))(x+\psi-\pi_k(x))$. The expected cost of an agent who arrives in slot k, if it sees a remaining demand x, is

$$\begin{aligned} c_k(x,\overline{\pi}) = &\ (\psi-\pi_k(x))(\psi-\pi_k(x)+x)+\pi_k(x)(\pi_k(x)+p(\psi-\pi_{k+1}(\pi_k(x))))\\ &+d\mathbb{1}_{\pi_k(x)>0}. \end{aligned} \tag{2}$$

A strategy profile $\overline{\pi}$ is called a *Nash equilibrium* if

$$c_k(x,\overline{\pi})\leq c_k(x,(\mu,\overline{\pi}_{-k}))$$

for all $k\geq 1, x\in[0,\psi]$ and strategies $\mu:[0,\psi]\to[0,\psi]$.[2] We focus on symmetric Nash equilibria of the form $(\pi,\pi,\dots)$ and obtain one such equilibrium in Section 4.

In the context of Job scheduling in data centers, the parameters introduced above could be mapped as follows.

1. x_k: CPU power pending in slot k for the job arrived in slot $k-1$.
2. v_k: CPU power offered in slot k to the job arrived in slot k.
3. e_k: Total CPU power requested in slot k by the job that arrived in slot k.

3. Optimal scheduling

As in [13], we first show that the optimal scheduling problem can be transformed into a stochastic shortest path problem. Let A_i, $i\geq 1$ be the successive slots that have service requests but do not have service requests in the preceding slots. More precisely,

$$A_i=\begin{cases}\min\{k: \text{slot } k \text{ has an arrival}\}, & \text{if } i=1,\\ \min\{k>A_{i-1}: \text{slot } k \text{ has an arrival but} & \\ \qquad k-1 \text{ does not}\}, & \text{if } i\geq 2.\end{cases}$$

Then A_i, $i\geq 1$ can be seen to be *renewal instants* of a delayed renewal process.

The following lemma gives the mean of renewal lifetimes, $A_{i+1}-A_i$, $i\geq 1$.

Lemma 3.1. $\mathbb{E}(A_{i+1}-A_i)=\frac{1}{p(1-p)}$.

[1] Notice that π consists of a strategy for each slot but there may not be any agent in a slot to use the corresponding strategy.

[2] $(\mu,\overline{\pi}_{-k})\triangleq(\pi_1,\dots,\pi_{k-1},\mu,\pi_{k+1},\dots)$.

Proof. See [22, Appendix A] ■

Hence, from the *Renewal Reward Theorem* [23],

$$\begin{aligned}
&\lim_{T\to\infty}\frac{1}{T}\sum_{k=1}^{T}\mathbb{E}[(x_k+v_k)^2+d\mathbb{1}_{\{v_k\in(0,\psi)\}}]\\
&=\frac{\mathbb{E}\left[\sum_{k=A_i}^{A_{i+1}-1}\left((x_k+v_k)^2+d\mathbb{1}_{\{v_k\in(0,\psi)\}}\right)\right]}{\mathbb{E}[A_{i+1}-A_i]}\\
&=p(1-p)\mathbb{E}\left[\sum_{k=A_i}^{A_{i+1}-1}\left((x_k+v_k)^2+d\mathbb{1}_{\{v_k\in(0,\psi)\}}\right)\right].
\end{aligned}$$

So, we can focus on minimizing the aggregate cost over a "renewal lifetime" $A_{i+1}-A_i$. But we do not incur any cost after service completion of the last customer in this lifetime. We can thus frame the problem as *stochastic shortest path problem* where terminal state corresponds to absence of request in a slot (see [24, Section 4.6] for a discussion on stochastic shortest path problem in infinite spaces).

Stochastic shortest path formulation. We follow the arguments in [13] and let x_k be the system state at any slot k and t be a special *terminal state* which is hit if there is no new request in a slot. Let x_{k+1} also denote the action in slot k. Clearly, the single stage cost before hitting the terminal state is $(x_k+\psi-x_{k+1})^2+d\mathbb{1}_{\{x_{k+1}>0\}}$. Given the state–action pair in slot k, (x_k,x_{k+1}), the next state is the terminal state with probability $1-p$ and the terminal cost is x_{k+1}^2. Note that [13] and this work differ in the single stage cost structure. The single stage cost function before hitting the terminal state in this article is not continuous in x_{k+1}, thus making the analysis much more complicated.

Let $J:[0,\psi]\to\mathbb{R}_+$ be the optimal cost function for the problem. It is the solution of the following Bellman's equation: For all $x\in[0,\psi]$,

$$J(x)=\min\{(\psi+x)^2+pJ(0),\ \min_{u\in(0,\psi]}\{(\psi-u+x)^2+d+pJ(u)+(1-p)u^2\}\}.$$

Notice that the term under the inner minimization at $u=0$ exceeds the first term by d. Hence we can change the constraint on u in the inner minimization to $[0,\psi]$ without altering the solution $J(\cdot)$. In other words, $J(\cdot)$ is also the solution to the following equation:

$$J(x)=\min\left\{(\psi+x)^2+pJ(0),\ \min_{u\in[0,\psi]}\left\{(\psi-u+x)^2+d+pJ(u)+(1-p)u^2\right\}\right\}. \tag{3}$$

The optimal cost is attained by a stationary policy of the form $(\pi^*,\pi^*,\dots)$ where $\pi^*(x)$ minimizes the right hand side in the above equation for all x. For brevity, we use π^* to refer to this policy. Let us define the "k-stage problem" as the one that allows at most $k+1$ service requests. More precisely, here the system is *forced to enter* the terminal state after $k+1$ service requests if it has not already done so. Let $J_k(\cdot)$ be the optimal cost function of the k-stage problem. Clearly,

$$J_0(x)=\min\{(\psi+x)^2,\ \min_{u\in[0,\psi]}\{(\psi+x-u)^2+d+u^2\}\} \tag{4}$$

and

$$J_k(x)=\min\{(\psi+x)^2+pJ_{k-1}(0),\ \min_{u\in[0,\psi]}\{(\psi+x-u)^2d+pJ_{k-1}(u)+(1-p)u^2\}\}. \tag{5}$$

Note that the maximum amount of service that can be offered in any slot is 2ψ. Hence the cost in any slot cannot be more than $4\psi^2+d$. Therefore $J_k(\cdot)$ is upper bounded by $\frac{4\psi^2+d}{1-p}$ for all $k\geq 1$. Observe that $J_0(x)>x^2$ from (4). The first and second terms on the right hand side of (5) are greater than the first and second terms, respectively, on the right hand side of (4). So, $J_1(x)>J_0(x)$. Inductively, we can see that $J_k(x)>J_{k-1}(x),\ \forall\, x$. So the sequence $J_k(\cdot)$s converges. We now outline the approach of determining the optimal policy. Let $\pi_k(\cdot)$ be the optimal controls of the k-stage problems (i.e., optimal controls in (4)–(5)). In the following we argue that $\pi_k(\cdot)$s are piece-wise linear discontinuous functions that are hard to fully characterize. We thus cannot follow the approach of deriving $\pi^*(\cdot)$ via taking limit of $\pi_k(\cdot)$, $k\geq 0$. We obtain optimal policy under certain conditions in Proposition 3.2. We also propose an approximate policy $\bar{\pi}(x)$ which forms an upper bound on the optimal policy (see Proposition 3.3). We then show that when the parameters does not satisfy the above mentioned conditions $\bar{\pi}(0)=0$, implying $\pi^*(0)=0$ (see Proposition 3.3). So in this region no service is deferred. This way we characterize the optimal policy for all the settings. The detailed analysis follows below.

Let us define $J_{-1}(x):=x^2$. We can then unify (4) and (5), i.e., we can use (5) to describe $J_k(\cdot)$, $k\geq 0$. We hypothesize that $J_k(\cdot)$s are quadratic functions and define, for all $k\geq 0$,

$$pJ_{k-1}(u)+(1-p)u^2=a_ku^2+b_ku+c_k. \tag{6}$$

where a_k, b_k and c_k are defined at appropriate places. Our hypothesis is clearly true for $k = 0$. In the following we see that it holds for all $k \geq 1$ as well. Also observe that for all $k \geq 0$,

$$\pi_k(x) = \arg\min_{u\in[0,\psi]}\{(\psi + x - u)^2 + d + pJ_{k-1}(u) + (1-p)u^2\}$$

if the minimum value is less than $(\psi + x)^2 + pJ_{k-1}(0)$ and $\pi_k(x) = 0$ otherwise. Let us define

$$\theta(a, b) := \sqrt{d(1 + a)} + \frac{b}{2} - \psi \text{ for } a, b \geq 0. \tag{7}$$

We begin with the following observation which we will repeatedly use. We use the following lemma later to show that the optimal policy does not defer any service up to certain value of pending service beyond which it defers strictly positive amount.

Lemma 3.2. *Let $\pi(x)$ be defined as follows*

$$\pi(x) = \begin{cases} \arg\min_{u\in[0,\psi]}\{(\psi + x - u)^2 + d + au^2 + bu + c\}, \\ \qquad \text{if } \min_{u\in[0,\psi]}\{(\psi + x - u)^2 + d + au^2 + bu + c\} \leq (\psi + x)^2 + c \\ 0, \ \text{otherwise}. \end{cases}$$

If $a\psi + \frac{b}{2} \geq \min\{\psi, \theta(a, b)\}$, then

$$\pi(x) = \begin{cases} 0, & \text{if } x \leq \theta(a, b) \\ \left[\frac{x+\psi-\frac{b}{2}}{1+a}\right]^{\psi}, & \text{otherwise} \end{cases}$$

else,

$$\pi(x) = \begin{cases} 0, & \text{if } x \leq \frac{(a-1)\psi+b}{2} + \frac{d}{2\psi} \\ \psi, & \text{otherwise}. \end{cases}$$

Proof. See [22, Appendix B]. ■

Remark 3.1. If $a\psi + \frac{b}{2} \geq \psi$, then $\frac{x+\psi-\frac{b}{2}}{1+a} \leq \psi$, $\forall x \in [0, \psi]$. Therefore,

$$\pi(x) = \begin{cases} 0, & \text{if } x \leq \theta(a, b) \\ \frac{x+\psi-\frac{b}{2}}{1+a}, & \text{otherwise}. \end{cases} \tag{8}$$

Let us define

$$\bar{a}_i = \begin{cases} 1, & \text{if } i = 0, \\ 1 - \frac{p}{1+\bar{a}_{i-1}}, & \text{otherwise}, \end{cases} \tag{9}$$

and

$$\bar{b}_i = \begin{cases} 2p\psi, & \text{if } i = 0, \\ \frac{p(2\bar{a}_{i-1}\psi+\bar{b}_{i-1})}{1+\bar{a}_{i-1}} & \text{otherwise}. \end{cases} \tag{10}$$

We show that the sequences $\bar{a}_k, \bar{b}_k, k \geq 0$ have the following monotonicity properties. We use these properties in deriving the optimal policy (e.g., see the proof of Proposition 3.2).

Lemma 3.3. *(a) $\bar{a}_k, k \geq 0$ is a decreasing sequence and converges to $\bar{a}_\infty := \sqrt{1-p}$.*

(b) $\bar{b}_k, k \geq 0$ is a decreasing sequence and converges to
$\bar{b}_\infty := \frac{2p\psi}{1+\sqrt{1-p}}$.

Proof. See [22, Appendix C]. ■

The following proposition shows that $\pi^*(\cdot)$ is in general a discontinuous piece-wise linear function with increasing slopes. We also know all the affine functions that constitute $\pi^*(\cdot)$, but do not know the jump epochs.

Proposition 3.1. *The optimal policy $\pi^*(\cdot)$ of (3) is of the form*

$$\pi^*(x) = \begin{cases} 0, & \text{if } 0 \le x \le \bar{x}_0 \\ \frac{x+\psi-\frac{\bar{b}_i}{2}}{1+\bar{a}_i}, & \text{if } \bar{x}_i < x \le \bar{x}_{i+1}, i \ge 0 \\ \frac{x+\psi-\frac{\bar{b}_\infty}{2}}{1+\bar{a}_\infty}, & \text{if } \bar{x}_\infty < x \le \psi \end{cases} \tag{11}$$

where $\bar{x}_i, i \ge 0$ are functions of $\bar{a}_i, i \ge 0$ and $\bar{b}_i, i \ge 0$.

Proof. See Appendix A ■

We now provide intuition behind the form of the optimal policies as given by the above proposition. Recall that the waiting cost d is fixed irrespective of the amount of deferred service whereas the marginal service cost [25] in a slot increases with the amount of service offered in the slot. Hence, for optimality, service is deferred only when the marginal service cost in the slot dominates the sum of d and expected marginal service cost in the subsequent slot. Further, given that some service has to be deferred from a slot to the next slot, amount of deferred service is chosen to optimize the service costs in the two slots causing a jump in the optimal policies. Subsequent jumps in the optimal policies can also be attributed to similar phenomenon. Finally, owing to increasing marginal service costs, the optimal policies tend to defer services more aggressively at higher values of pending services. This is why slopes of the successive line segments in the optimal policies increase monotonically.

We provide the exact optimal policies for a couple of special cases in Section 3.1. As we do not know the jump epochs in Proposition 3.1 we propose an approximate policy in Section 3.2. However, this approximate policy helps us characterize the optimal policy for all cases (see Section 3.3).

3.1. Optimal policy for special cases

Let us notice that (5) for $k \ge 0$ constitute value iteration starting with $J_{-1}(x) = x^2$. We can instead perform value iteration starting with a different function. From [26, Chapter 2, Proposition 1.2(b)], in any such iteration, $J_k(\cdot)$ will converge to the optimal cost function $J(\cdot)$ and $\pi_k(\cdot)$ will converge to $\pi^*(\cdot)$. The following proposition shows that, starting with certain initial functions, limits of $\pi_k(\cdot)$ can be obtained in certain special cases.

Proposition 3.2. (*a*) *If $\psi < \frac{\sqrt{2d}}{(2-p)}$, $\pi^*(x) = 0$ for all $x \in [0, \psi]$.*
(*b*) *If $\psi > \frac{\sqrt{d(1+\bar{a}_\infty)}}{\bar{a}_\infty}$,*

$$\pi^*(x) = \frac{x+\psi-\frac{\bar{b}_\infty}{2}}{1+\bar{a}_\infty}, \text{ for all } x \in [0, \psi].$$

Proof. See Appendix B. ■

3.2. Approximate policy

Let us consider a fictitious problem wherein an agent with demand ψ arrives with probability p and there is no arrival with probability $1-p$ but a fixed additional cost d is incurred for each service request whether or not a portion of the request is deferred to the subsequent slot. The optimal cost function for this fictitious problem, $J'(\cdot)$, is solution of the following Bellman's equation.

$$J'(x) = \min_{u\in[0,\psi]} \left\{(\psi-u+x)^2 + d + pJ'(\psi) + (1-p)u^2\right\}$$

This fictitious problem can be seen as a special case of the linear waiting cost problem in [13, Section III] with $d = 0$ but with a fixed additional cost d per request. Hence, following the analysis in [13, Appendix I.1] (also see [13, Section 3, Theorem 3.1(a)]), its optimal policy is

$$\pi'(x) = \frac{x+\psi-\frac{\bar{b}_\infty}{2}}{1+\bar{a}_\infty},$$

where $\bar{a}_\infty, \bar{b}_\infty$ are as in Lemma 3.3. Further, the optimal cost function satisfies

$$pJ'(x) + (1-p)x^2 = \bar{a}_\infty x^2 + \bar{b}_\infty x + \bar{c}_\infty,$$

where $\bar{c}_\infty$ is a certain constant. Let us now define the following cost function

$$\tilde{J}(x) = \min \Big\{(\psi+x)^2 + pJ'(0), \min_{u\in[0,\psi]} (\psi+x-u)^2 + d + pJ'(u) + (1-p)u^2\Big\} \tag{12}$$

Also note that

$$\begin{aligned}\bar{a}_\infty \psi + \frac{\bar{b}_\infty}{2} &= \sqrt{1-p}\psi + \frac{p\psi}{1+\sqrt{1-p}} \\ &= \psi. \end{aligned} \tag{13}$$

Hence, from (8), the optimal control of the cost function $\tilde{J}(x)$, say $\tilde{\pi}(\cdot)$, is given by

$$\tilde{\pi}(x) = \begin{cases} 0, & \text{if } x \le \theta(\bar{a}_\infty, \bar{b}_\infty) \\ \frac{x+\psi-\frac{\bar{b}_\infty}{2}}{1+\bar{a}_\infty}, & \text{otherwise.} \end{cases} \tag{14}$$

We propose to use the following policy for our fixed waiting cost problem.

$$\bar{\pi}(x) = \begin{cases} 0, & \text{if } \psi < \frac{\sqrt{2d}}{(2-p)}, \\ \frac{x+\psi-\frac{\bar{b}_\infty}{2}}{1+\bar{a}_\infty}, & \text{if } \psi > \frac{\sqrt{d(1+\bar{a}_\infty)}}{\bar{a}_\infty}, \\ \tilde{\pi}(x), & \text{otherwise.} \end{cases} \tag{15}$$

We do not have any performance bound for the proposed policy. However, we show below that, for any given backlog, we defer more under this policy than under the optimal policy.

Proposition 3.3. *$\bar{\pi}(x) \ge \pi^*(x)$ for all $x \in [0, \psi]$.*

Proof. See Appendix C. ■

Remark 3.2. Note that $\bar{\pi}(x) = 0$ implies $\pi^*(x) = 0$, i.e., the proposed approximate policy and the optimal policy agree when $\bar{\pi}(x) = 0$.

3.3. Optimal policy for the general case

The following theorem completely characterizes the optimal policy.

Theorem 3.1. *The optimal actions are given as follows*

1. *if $\psi > \frac{\sqrt{d(1+\bar{a}_\infty)}}{\bar{a}_\infty}$, then the optimal actions are taken in accordance with $\pi^*(x)$ as given by Proposition 3.2(b).*
2. *$\psi \le \frac{\sqrt{d(1+\bar{a}_\infty)}}{\bar{a}_\infty}$, $\pi^*(0) = 0$, and therefore none of the requests have their services deferred.*

Proof. Following the definitions of $\bar{a}_\infty, \bar{b}_\infty$ and $\theta(\bar{a}_\infty, \bar{b}_\infty)$ (see Lemma 3.3 and (7)) it can be easily checked that $\theta(\bar{a}_\infty, \bar{b}_\infty) \ge 0$ if and only if $\psi \le \frac{\sqrt{d(1+\bar{a}_\infty)}}{\bar{a}_\infty}$. Hence, if $\psi \le \frac{\sqrt{d(1+\bar{a}_\infty)}}{\bar{a}_\infty}$, $\bar{\pi}(0) = \tilde{\pi}(0) = 0$ from (14) and (15), and so, $\pi^*(0) = 0$ from Proposition 3.3. Notice that when $\pi^*(0) = 0$ none of the requests have their services deferred under the optimal policy.

We thus have complete characterization of the optimal scheduling in all the cases. ■

We illustrate the optimal and the approximate policies via a few examples in Fig. 1. We choose $\psi = 2, d = 1$ and $p = 0.5, 0.7$ and 0.85 for illustration. When $p = 0.5$, the parameters meet the hypothesis of Proposition 3.2(b), and hence, the optimal policy is provided by the proposition. For $p = 0.7$ and 0.85, the optimal policies have been computed by value iteration which involves discretization of the state and action spaces and hence is subject to quantization error. For both these cases the approximate policies are given by (15). When $p = 0.7$, $\bar{x}_i = \bar{x}_0 > \theta(\bar{a}_\infty, \bar{b}_\infty)$ for all $i \ge 1$ (see Proposition 3.1), and hence, the optimal and the approximate policies coincide for $x \ge \bar{x}_0$. For both, $p = 0.7$ and 0.85, the optimal policies exhibit jumps and are piece-wise linear with the slopes of successive line segments increasing as claimed in Proposition 3.1. For both these cases the approximate policies upper bound the optimal policies as shown in Proposition 3.3. *As expected, for the same pending service, the deferred service decreases as the expected quantum of service in the next slot increases, i.e., as p increases.*

3.4. More general models

We agree that the our model is quite simple and does not capture many attributes of real problems. However, evidently, analysis and optimization of this simple model also is very complex. Further, the optimal solution to this model can lead to heuristics for more general models. We briefly discuss here one such generalization allowing more general demand arrival processes. Assume that, in each slot, with probabilities p_i demands ψ_i arrive where $i = 1, 2, \ldots, N$, and with probability $1 - \sum_{i=1}^{N} p_i$ no demand arrives. We can formulate a fictitious problem with i.i.d. Bernoulli arrivals with arrival

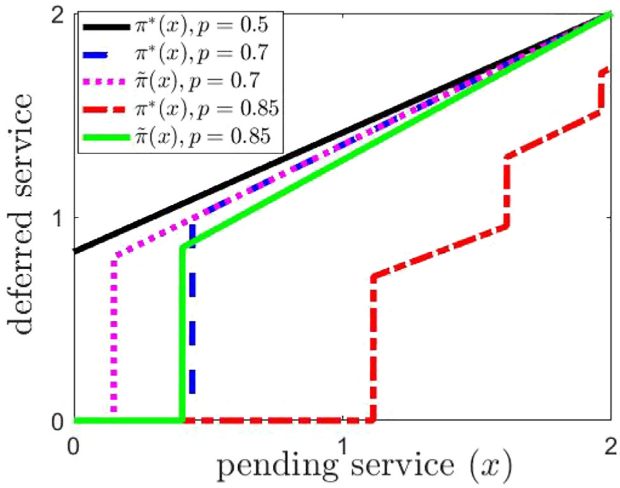

Fig. 1. The optimal and the approximate policies for $\psi = 2, d = 1, p \in \{0.5, 0.7, 0.85\}$. For $p = 0.5$ the optimal and the approximate policies are same.

constant demand $\bar{\psi} = (\sum_{j=1}^{N} p_j \psi_j)/\bar{p}$ and demand arrival probability $\bar{p} = \sum_{i=1}^{N} p_i$. We can then use the optimal policy associated with this fictitious problem for our original problem. Such heuristics are proposed and analyzed in [13] in the context of linear waiting costs.

4. Nash equilibrium

In this section we provide a Nash equilibrium for the non-cooperative game among the selfish agents (see Section 2). Specifically, we look at symmetric Nash equilibria where each agent's strategy is a piece-wise linear function of the remaining demand of the previous player.

Let $C : [0, \psi] \to \mathbb{R}_+$ give the optimal cost for a player as a function of the pending demand given that all other players use strategy $\pi' : [0, \psi] \to [0, \psi]$. Clearly, $C(x)$ is given by the following equation for all $x \in [0, \psi]$.

$$C(x) = \min\{(\psi + x)\psi, \min_{u\in[0,\psi]}\{(\psi - u)(\psi - u + x) + u(u + p(\psi - \pi'(u))) + d\}\}$$

We call $\bar{\pi}' = (\pi', \pi', ..)$ a symmetric Nash equilibrium if $\pi'(x)$ attains the optimal cost in the above optimization problem for all x, i.e., if

$$\pi'(x) = \arg\min_{u\in[0,\psi]}\{(\psi - u)(\psi - u + x) + u(u + p(\psi - \pi'(u))) + d\}$$

if the minimum value is less than $(\psi + x)^2 + c$ and $\pi'(x) = 0$ otherwise, for all $x \in [0, \psi]$. We characterize one such Nash equilibrium in the following. As in Section 3 we define k-stage problems, where the tagged player has atmost k service requests after it, before the terminal state is hit. Let $C_k(\cdot)$ be the tagged user's optimal cost in the k-stage problem and $\pi'_k(\cdot)$ be the corresponding optimal strategy. Then

$$C_0(x) = \min\{(\psi + x)\psi \min_{u\in[0,\psi]}\{(\psi - u)(\psi - u + x) + u^2 + d\}\} \tag{16}$$

and for all $k \geq 1$,

$$C_k(x) = \min\{(\psi + x)\psi, \min_{u\in[0,\psi]}\{(\psi - u)(\psi - u + x) + d + u(u + p(\psi - \pi'_{k-1}(u)))\}\}. \tag{17}$$

We can see $C(x)$ as the limit of $C_k(x)$ as k approaches infinity. Furthermore, the limit of the optimal strategy of k-stage problems yield a symmetric Nash equilibrium. We now outline the approach of determining the Nash equilibrium, policy. We obtain Nash equilibrium policy under certain conditions in Lemma 4.3 and Proposition 4.1. Later we characterize total Nash equilibrium policy in Theorem 4.1.

4.1. A symmetric Nash equilibrium for special case

We first focus on symmetric Nash equilibrium in a few special cases. We then use these results to obtain symmetric Nash equilibria for all the cases (see Section 4.2). We begin with defining sequences $\tilde{a}_k, \tilde{b}_k, k \geq -1$ as follows

$$\tilde{a}_k = \begin{cases} 0, & \text{if } k = -1 \\ \frac{1}{2(2 - p\tilde{a}_{k-1})}, & \text{otherwise} \end{cases} \tag{18}$$

$$\tilde{b}_k = \begin{cases} 0, & \text{if } k = -1 \\ \frac{(2-p)\psi + p\tilde{b}_{k-1}}{2(2-p\tilde{a}_{k-1})}, & \text{otherwise} \end{cases} \tag{19}$$

We state a few properties of the above sequences.

Lemma 4.1. *(a) The sequence $\tilde{a}_k, k \geq -1$ converges to*

$$\tilde{a}_\infty := \frac{1}{p} - \frac{\sqrt{4-2p}}{2p}.$$

Also, $\frac{1}{4} < \tilde{a}_\infty < \frac{1}{3}$.

(b) The sequence $\tilde{b}_k, k \geq -1$ converges to

$$\tilde{b}_\infty := \frac{\tilde{a}_\infty(2-p)\psi}{1 - \tilde{a}_\infty p}.$$

Proof. See [22, Appendix E]. ■

The following lemma states that $\tilde{a}_\infty x + \tilde{b}_\infty$ is strictly positive and strictly less than ψ for all $x \in [0, \psi]$. We use it later to show that under certain conditions, the symmetric Nash equilibria can be obtained via solving unconstrained optimization problems.

Lemma 4.2. $\tilde{a}_\infty x + \tilde{b}_\infty \in (0, \psi)$ *for all* $x \in [0, \psi]$.

Proof. See [22, Appendix F]. ■

Let us also define $x_\infty = \frac{\sqrt{2\tilde{a}_\infty d} - \tilde{b}_\infty}{\tilde{a}_\infty}$. The following lemma partially characterizes symmetric Nash equilibrium policies.

Lemma 4.3.

$$\pi'(x) = 0, \forall x \leq x_\infty$$

.

Proof. See [22, Appendix G]. ■

The following proposition gives a symmetric Nash equilibrium in a special case.

Proposition 4.1. *If $\frac{\tilde{b}_\infty}{1-\tilde{a}_\infty} \geq x_\infty$, then $\bar{\pi}' = (\pi', \pi', \ldots)$ is a symmetric Nash equilibrium where*

$$\pi'(x) = \begin{cases} 0, & \text{if } x \leq x_\infty \\ \tilde{a}_\infty x + \tilde{b}_\infty, & \text{otherwise.} \end{cases} \tag{20}$$

Proof. See Appendix D. ■

Notice that the Nash equilibrium as given by Proposition 4.1 can also have a discontinuity. This jump can be explained using a similar argument as for the jumps in optimal policies (see the paragraph following Proposition 3.1).

4.2. Nash Equilibrium for the general case

The following theorem completely characterizes Nash equilibrium policy.

Theorem 4.1. *The Nash equilibrium actions are given as follows*

1. *If $x_\infty \geq 0$, then $\pi'(0) = 0$, none of the requests have their services deferred.*
2. *If $x_\infty < 0$, then $\pi'(x)$, Nash equilibrium actions are taken in accordance with Proposition 4.1.*

Proof. If $x_\infty \geq 0$, $\pi'(0) = 0$ from Lemma 4.3. In this case, none of the agents defer any service as they do not see any pending service. On the other hand, if $x_\infty < 0$, Proposition 4.1 applies, giving the equilibrium scheduling decisions. We thus have complete characterization of the users' scheduling decisions in all the cases. ■

In Fig. 2, we illustrate symmetric Nash equilibria for the same parameters as used to illustrate the optimal policies in Section 3. In all these examples, it turns out that $x_\infty < 0$, and hence, the equilibria are given by Proposition 4.1. For the same reason the equilibria do not exhibit jumps.

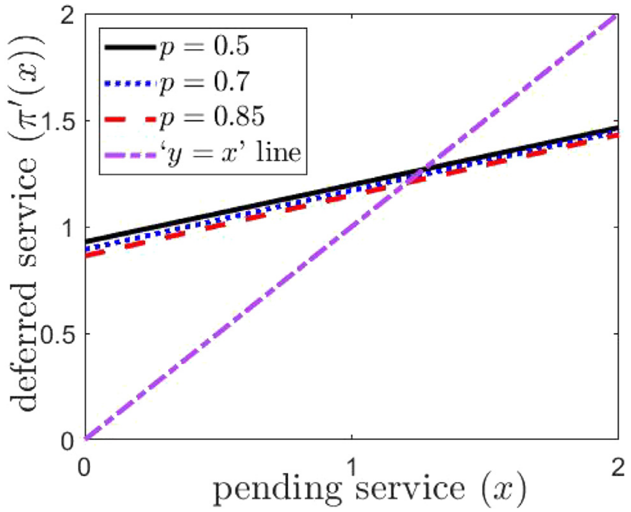

Fig. 2. The symmetric Nash equilibria for $\psi = 2, d = 1, p \in \{0.5, 0.7, 0.85\}$.

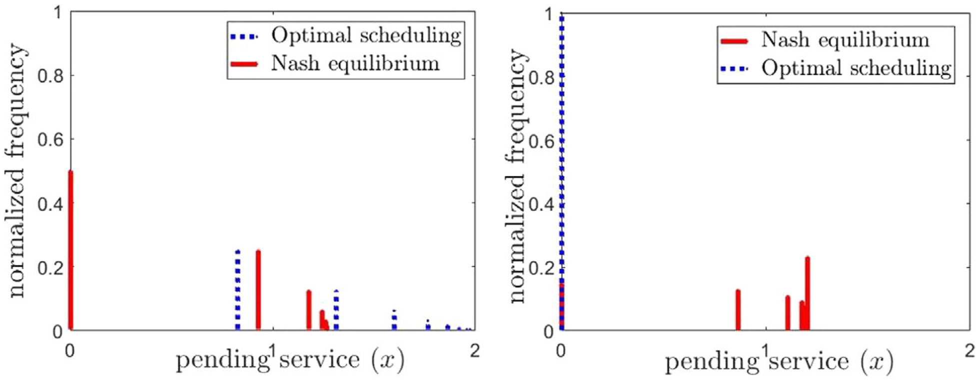

Fig. 3. Fixed waiting costs: histogram of the pending services seen by the jobs for $\psi = 2, d = 1, p = 0.5$ (left subfigure) and $p = 0.85$ (right subfigure).

5. Comparative numerical evaluation

We now discuss the effect of the fixed waiting cost structure, on the scheduling policies, deferred services and costs. For any given cost structure, we also compare the impact of performance criteria (optimal scheduling vs. strategic scheduling by selfish agents).

We begin with revisiting the optimal policies and Nash equilibria in Figs. 1 and 2. Recall that we had chosen $\psi = 2, d = 1$, and $p = 0.5, 0.7$ and 0.85. *Notice that for the same parameters and pending service, e.g., for $p = 0.85$ and $x = 1$, the optimal policy may not defer any service whereas the Nash equilibrium may differ substantial amount (larger than* 1*). Also, the equilibria are not as sensitive to p as the optimal policies.*

We show histograms of pending services seen by the jobs for both optimal policies and Nash equilibria in Fig. 3. We use $p = 0.5$ and $p = 0.85$ for left subfigure and right subfigure respectively. For $p = 0.85$, since $\pi^*(0) = 0$, all the jobs see zero pending service under the optimal scheduling policy. When $\pi(0) > 0$, $(1-p)$ fraction of jobs see $y_0 = 0$ pending service, and for $k \geq 1$, $p^k(1-p)$ fraction of jobs see $y_k = \pi(y_{k-1})$ pending service ($\pi \equiv \pi^*$ for an optimal policy whereas $\pi \equiv \pi'$ for a Nash equilibrium). Notice that, for all $k \geq 0$, y_k are upper bounded by, the fixed point of $\pi(x) = x$. *For $p = 0.85$, under Nash equilibrium the system attains a steady state wherein each user observes a pending service $= 1.2053$ (the fixed point of $\pi'(x) = x$ in Fig. 2) and defers the same amount of service*. Hence we see a mass $(1-p)\sum_{k=9}^{\infty} p^k = p^9$ at $y_9 = 1.2053$.

Next, in Fig. 4(a), we show variation of time-average cost under both optimal policy and Nash equilibrium as p is varied from 0 to 1. In Fig. 4(b), we show price of anarchy vs. p. We consider two sets of other parameters, $\psi = 2, d = 1$ and $\psi = 2.5, d = 1.5$. For $\psi = 2, d = 1$ and $p \geq 0.58$, no service is deferred in any slot under the optimal policy. Hence, the optimal average cost is $p\psi^2$ in this regime. Under the Nash equilibrium for $p = 1$, the system attains a steady state

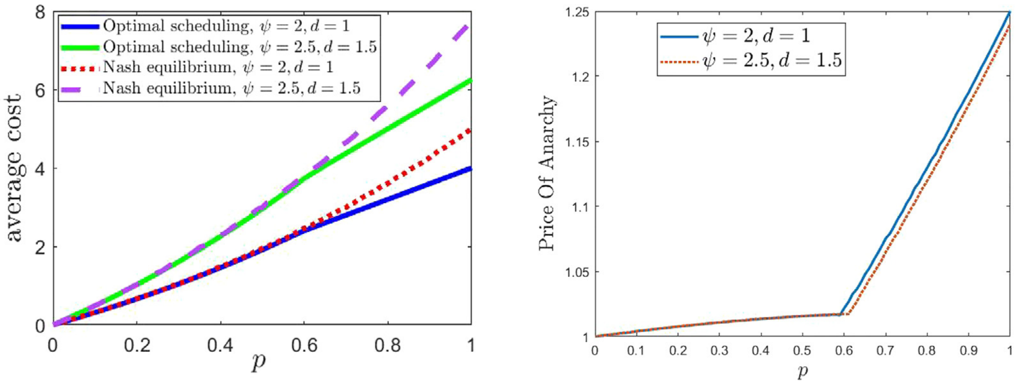

Fig. 4. (a) Average cost vs. p for $\psi = 2, d = 1$ and $\psi = 2.5, d = 1.5$. (b) Price of Anarchy vs. p for $\psi = 2, d = 1$ and $\psi = 2.5, d = 1.5$.

wherein each user observes a pending service given by the fixed point of $\pi'(x) = x$ and defers the same amount of service. Consequently, the amount of offered service in each slot equals ψ in the steady state, and the average cost equals $\psi^2 + d$. *The ratio of the average cost under Nash equilibrium and the optimal cost, often termed as efficiency loss, is 1 for p* $\gtrsim 0$ *and* $1 + \frac{d}{\psi^2}$ *for* $p = 1$. We observe same phenomena for $\psi = 2.5, d = 1.5$.

6. Conclusion

We studied service scheduling in slotted systems with Bernoulli request arrivals, quadratic service costs, fixed waiting costs and service delay guarantee of two slots. In the case of fixed waiting cost, we obtained optimal policy in special cases (Proposition 3.2). We proposed an approximate policy that is an upper bound on the optimal policy (Proposition 3.3). Finally, we characterize the optimal policy for all cases in Theorem 3.1. Subsequently, we also provided a symmetric Nash equilibrium when the parameters satisfy certain conditions (Proposition 4.1). And the total characterization of Nash equilibrium can be found in Theorem 4.1.

Our future work entails extending the results to the scenario where service delay guarantee is of three or more slots. We would also like to derive online algorithms for the cases where service request statistics are unknown.

Declaration of competing interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

Acknowledgments

The first and second authors acknowledge support from Research Fellowships of Visvesvaraya PhD Scheme and INSPIRE Faculty Research Grant (DSTO-1363).

Appendix A. Proof of Proposition 3.1

Recall that $J_{-1}(x) = x^2$. Substituting $k = 0$ in (6), $a_0 = 1$ and $b_0 = 0$. Observe that $a_0\psi + \frac{b_0}{2} = \psi$. Therefore, using (8),

$$\pi_0(x) = \begin{cases} 0, & \text{if } x \le \theta(a_0, b_0) \\ \frac{x+\psi-\frac{b_0}{2}}{1+a_0}, & \text{otherwise.} \end{cases}$$

Let us define

$$J_{00}(x) := (\psi + x)^2 \text{ and } J_{01}(x) := (\psi + x - \pi_0(x))^2 + d + \pi_0(x)^2.$$

So,

$$J_0(x) = \begin{cases} J_{00}(x), & \text{if } x \le \theta(a_0, b_0) \\ J_{01}(x), & \text{otherwise.} \end{cases}$$

Note that the function $J_0(x)$ can be one of the two quadratic functions $J_{00}(x), J_{01}(x)$ depending upon x. Observe that $\pi_0(x)$ is piece-wise linear but discontinuous with a jump at $\theta(a_0, b_0)$. However, by definition of $\theta(a_0, b_0)$

$$J_{00}(\theta(a_0, b_0)) = J_{01}(\theta(a_0, b_0)).$$

For all $x < \theta(a_0, b_0), J_{00}(\theta(a_0, b_0)) < J_{01}(\theta(a_0, b_0))$ and for all $x > \theta(a_0, b_0), J_{00}(x) > J_{01}(x)$. We define $a_{1,0}, b_{1,0}, c_{1,0}, a_{1,1}, b_{1,1}$ and $c_{1,1}$ as follows

$$pJ_{00}(u) + (1-p)u^2 = a_{1,0}u^2 + b_{1,0}u + c_{1,0}.$$

and

$$pJ_{01}(u) + (1-p)u^2 = a_{1,1}u^2 + b_{1,1}u + c_{1,1}.$$

Using (6) for $k = 1$, we obtain the following,

$$a_1 = \begin{cases} a_{1,0} = 1, & \text{if } u \le \theta(a_0, b_0) \\ a_{1,1} = 1 - \frac{p}{1+a_0}, & \text{otherwise,} \end{cases} \tag{A.1}$$

$$b_1 = \begin{cases} b_{1,0} = 2p\psi, & \text{if } u \le \theta(a_0, b_0) \\ b_{1,1} = \frac{p(2a_0\psi + b_0)}{1+a_0}, & \text{otherwise,} \end{cases} \tag{A.2}$$

$$\text{and } c_1 = \begin{cases} c_{1,0} = p\psi^2, & \text{if } u \le \theta(a_0, b_0) \\ c_{1,1} = p\Big(\frac{a_0\psi^2 + b_0\psi - \left(\frac{b_0}{2}\right)^2}{1+a_0} + d\Big), & \text{otherwise.} \end{cases}$$

Let us now define the following fictitious cost function.

$$J_1'(x) = \min_{u\in[0,\psi]} \{(\psi + x - u)^2 + d + a_1u^2 + b_1u + c_1\},$$

Let $\pi_1'(x)$ be the optimal action in the R.H.S. $J_1'(x)$ can be written as

$$J_1'(x) = \min\Bigg\{ \min_{u\in[0,\theta(a_0,b_0)]} \{(\psi + x - u)^2 + d + a_{1,0}u^2 + b_{1,0}u + c_{1,0}\},$$
$$\min_{u\in[\theta(a_0,b_0),\psi]} \{(\psi + x - u)^2 + d + a_{1,1}u^2 + b_{1,1}u + c_{1,1}\} \Bigg\}$$

Let us define

$$h_{a,b,c}(x,u) = (\psi + x - u)^2 + d + au^2 + b^2 + c \text{and } \pi_{a,b}(x) = \arg\min_{u\in[0,\psi]} h_{a,b,c}(x,u)$$

(a) If $\pi_{a_{10},b_{10}}(x) < \theta(a_0, b_0)$ and $\pi_{a_{11},b_{11}}(x) < \theta(a_0, b_0)$ then

$$\begin{aligned} \min_{u\in[0,\psi]} h_{a_{10},b_{10}}(x,u) &= \min_{u\in[0,\theta_{(a_0,b_0)}]} h_{a_{10},b_{10}}(x,u) \\ &< \min_{u\in[0,\psi]} h_{a_{11},b_{11}}(x,u) \\ &\le \min_{u\in[\theta_{(a_0,b_0)},\psi]} h_{a_{11},b_{11}}(x,u). \end{aligned}$$

Hence in this case $\pi_1'(x) = \pi_{a_{10},b_{10}}(x)$.

(b) If $\pi_{a_{10},b_{10}}(x) \ge \theta(a_0, b_0)$ and $\pi_{a_{11},b_{11}}(x) \ge \theta(a_0, b_0)$ then

$$\begin{aligned} \min_{u\in[0,\psi]} h_{a_{11},b_{11}}(x,u) &= \min_{u\in[\theta_{(a_0,b_0)},\psi]} h_{a_{11},b_{11}}(x,u) \\ &< \min_{u\in[0,\psi]} h_{a_{10},b_{10}}(x,u) \\ &\le \min_{u\in[0,\theta_{(a_0,b_0)}]} h_{a_{10},b_{10}}(x,u). \end{aligned}$$

Hence in this case $\pi_1'(x) = \pi_{a_{11},b_{11}}(x)$. Using (A.1) and (A.2), it can be easily verified that there exist $\tilde{x} < 0$ and $y < 0$ such that $\pi_{a_{10},b_{10}}(\tilde{x}) = \pi_{a_{11},b_{11}}(\tilde{x}) = y$. Let us define x' and x'' as follows

$$x' := \max\{x : \pi_{a_{10},b_{10}}(x) < \theta(a_0, b_0) \text{ and } \pi_{a_{11},b_{11}}(x) < \theta(a_0, b_0)\}$$

$$x'' := \max\{x : \pi_{a_{10},b_{10}}(x) \ge \theta(a_0, b_0) \text{ and } \pi_{a_{11},b_{11}}(x) \ge \theta(a_0, b_0)\}$$

Thus, using case(a) and case(b), we can see that $\pi'(x)$ can be written as

$$\pi_1'(x) = \begin{cases} \pi_{a_{10},b_{10}}(x), & \text{if } x < x' \\ \pi_{a_{11},b_{11}}(x), & \text{if } x > x'' \end{cases} \tag{A.3}$$

Using (A.3) we can write the following

$$J_1'(x) = \begin{cases} \frac{a_{1,0}}{1+a_{1,0}}x^2 + (\frac{2\psi a_{1,0}+b_{1,0}}{1+a_{1,0}})x + \frac{a_{1,0}\psi^2+b_{1,0}\psi-\frac{b_{1,0}^2}{4}}{(1+a_{1,0})} \\ +c_{1,0}+d := A_0x^2+B_0x+C_0, \text{ if } x < x' \\ \frac{a_{1,1}}{1+a_{1,1}}x^2 + (\frac{2\psi a_{1,1}+b_{1,1}}{1+a_{1,1}})x + \frac{a_{1,1}\psi^2+b_{1,1}\psi-\frac{b_{1,1}^2}{4}}{(1+a_{1,1})} \\ +c_{1,1}+d := A_1x^2+B_1x+C_1, \text{ if } x > x''. \end{cases} \tag{A.4}$$

Also,

$$\pi_1'(x) = \begin{cases} \pi_{a_{10},b_{10}}(x), & \text{if } x' \geq \psi \\ \pi_{a_{11},b_{11}}(x), & \text{if } x'' \leq 0 \end{cases} \tag{A.5}$$

Therefore, we only discuss the case $x' > 0$ and $x'' < \psi$.[3] Let us observe that solution to the following equation gives us $\bar{x}_1 \in [x', x'']$

$$A_0x^2 + B_0x + C_0 = A_1x^2 + B_1x + C_1 \tag{A.6}$$

As $x' > 0$, $C_0 < C_1$. Also, $A_0 > A_1$ as $a_{1,0} > a_{1,1}$. Thus the product of roots of (A.6) is negative. Hence there exists a $\bar{x}_1 \in [x', x'']$ such that

$$\pi_1'(x) = \begin{cases} \pi_{a_{10},b_{10}}(x), & \text{if } 0 < x \leq \bar{x}_1 \\ \pi_{a_{11},b_{11}}(x), & \text{if } \bar{x}_1 < x \leq \psi \end{cases}$$

It can be noted that $\bar{x}_1$ is a function of $a_{1,0}, a_{1,1}, b_{1,0}, b_{1,1}, c_{1,0}$ and $c_{1,1}$ but not easy to determine. Further, we study the optimal control for 1-stage problem.

$$J_1(x) = \min\{(\psi+x)^2 + pJ_0(0), \min_{u\in[0,\psi]}\{(\psi+x-u)^2 + d + a_1u^2 + b_1u^2 + c_1\}\}.$$

Also $pJ_0(0) = c_{1,0}$ if $0 \leq x''$ and $c_{1,1}$ otherwise. As we are discussing a case where $x' > 0$ and $x'' < \psi$, $pJ_0(0) = c_{1,0}$. Let us consider the following fictitious cost functions.

$$J_{1,0}(x) = \min\big\{(\psi+x)^2 + c_{1,0}, \min_{u\in[0,\theta(a_0,b_0)]}\{(\psi+x-u)^2 + d + a_{1,0}u^2 + b_{1,0}u^2 + c_{1,0}\}\big\}$$

and

$$J_{1,1}(x) = \min\big\{(\psi+x)^2 + c_{1,0}, \min_{u\in[\theta(a_0,b_0),\psi]}\{(\psi+x-u)^2 + d + a_{1,1}u^2 + b_{1,1}u^2 + c_{1,1}\}\big\}.$$

Let $\pi_{1,0}(x)$ and $\pi_{1,1}(x)$ be the optimal functions of $J_{1,0}(x)$ and $J_{1,1}(x)$ respectively. It can be noted that $a_{1,0}\psi + b_{1,0} = \psi$ and $a_{1,1}\psi + b_{1,1} = \psi$. Therefore using (8),

$$\pi_{1,0}(x) = \begin{cases} 0, & \text{if } 0 \leq x \leq \theta(a_{1,0}, b_{1,0}) \\ \pi_{a_{10},b_{10}}(x), & \text{if } \theta(a_{1,0}, b_{1,0}) < x \leq \psi \end{cases} \tag{A.7}$$

and

$$\pi_{1,1}(x) = \begin{cases} 0, & \text{if } 0 \leq x \leq \theta(a_{1,1}, b_{1,1}) \\ \pi_{a_{11},b_{11}}(x), & \text{if } \theta(a_{1,1}, b_{1,1}) < x \leq \psi \end{cases} \tag{A.8}$$

From (A.7), (A.8) we see that optimal function of $\pi_1(x)$ can be written as

$$\pi_1(x) = \begin{cases} 0, & \text{if } 0 \leq x \leq \bar{x}_0 \\ \frac{x+\psi-\frac{b_{1,0}}{2}}{1+a_{1,0}}, & \text{if } \bar{x}_0 < x \leq \bar{x}_1 \\ \frac{x+\psi-\frac{b_{1,1}}{2}}{1+a_{1,1}}, & \text{if } \bar{x}_1 < x \leq \psi \end{cases}$$

where $\bar{x}_0 = \min\{\theta(a_{1,0}, b_{1,0}), \theta(a_{1,1}, b_{1,1})\}$. We can similarly argue that the optimal policy $\pi^*(\cdot)$ is of the form (a few of the intervals $(\bar{x}_i, \bar{x}_{i+1}]$ can be empty sets)

$$\pi^*(x) = \begin{cases} 0, & \text{if } 0 \leq x \leq \bar{x}_0 \\ \frac{x+\psi-\frac{\bar{b}_i}{2}}{1+\bar{a}_i}, & \text{if } \bar{x}_i < x \leq \bar{x}_{i+1}, i \geq 0. \end{cases}$$

[3] When $x' \geq \psi$, $[\bar{x}_1, \psi]$ is an empty set. Similarly, when $x'' \leq 0$, then $[\bar{x}_0, \bar{x}_1]$ is an empty set.

Appendix B. Proof of Proposition 3.2

(a) Let us analyze value iteration starting with function $J_0(x) = (x+\psi)^2$. Substituting $k=1$ in (6), $a_1 = \bar{a}_0$, $b_1 = \bar{b}_0$ and $c_1 = \psi^2$. Following (5),

$$J_1(x) = \min\{(\psi + x)^2 + c_1, \min_{u\in[0,\psi]} (\psi + x - u)^2 + d + \bar{a}_0 u^2 + \bar{b}_0 u + c_1\}$$

Observe that $\bar{a}_0\psi + \frac{\bar{b}_0}{2} > \psi$. Hence, from (8), the optimal control in the 1-stage problem, $\pi_1(x)$, can be written as

$$\pi_1(x) = \begin{cases} 0, & \text{if } x \le \theta(\bar{a}_0, \bar{b}_0) = \sqrt{2d} + p\psi - \psi \\ \frac{x+\psi-p\psi}{2}, & \text{otherwise.} \end{cases}$$

Note that when $\sqrt{2d} \ge (2-p)\psi$, the second case does not arise, i.e., $\pi_1(x) = 0$ for all $x \in [0, \psi]$. It implies that

$$J_1(x) = (x+\psi)^2 + p\psi^2$$

for all $x \in [0, \psi]$. Again using (6) for $k=2$, we see that $a_2 = a_1$, $b_2 = b_1$. Hence, following similar arguments as before, $\pi_2(x) = 0$ for all $x \in [0, \psi]$. Continuing in this fashion we see that for all $k \ge 1$, $\pi_k(x) = 0$ for all $x \in [0, \psi]$. Therefore $\pi^*(x) = 0$ $x \in [0, \psi]$.

(b) Now we analyze value iteration starting with function $J_0(u)$ that satisfies

$$pJ_0(u) + (1-p)u^2 = \bar{a}_\infty u^2 + \bar{b}_\infty u + \bar{c}_\infty,$$

where $\bar{a}_\infty, \bar{b}_\infty$ are as defined in Lemma 3.3 and $\bar{c}_\infty$ is a certain constant. Substituting $k=1$ in (6), $a_1 = \bar{a}_\infty$ and $b_1 = \bar{b}_\infty$. Following (5),

$$J_1(x) = \min\{(\psi + x)^2 + \bar{c}_\infty, \min_{u\in[0,\psi]} (\psi + x - u)^2 + d + \bar{a}_\infty u^2 + \bar{b}_\infty u + \bar{c}_\infty\}.$$

Using definitions of $\bar{a}_\infty$ and $\bar{b}_\infty$, $\bar{a}_\infty\psi + \frac{\bar{b}_\infty}{2} = \psi$. Hence, from (8),

$$\pi_1(x) = \begin{cases} 0, & \text{if } x \le \theta(\bar{a}_\infty, \bar{b}_\infty) \\ \frac{x+\psi-\frac{\bar{b}_\infty}{2}}{1+\bar{a}_\infty}, & \text{otherwise.} \end{cases}$$

Further, when $\psi\bar{a}_\infty > \sqrt{d(1+a_\infty)}$,

$$\theta(\bar{a}_\infty, \bar{b}_\infty) = \sqrt{d(1+\bar{a}_\infty)} + \frac{\bar{b}_\infty}{2} - \psi < \frac{\bar{b}_\infty}{2} - \psi(1-\bar{a}_\infty) = 0,$$

implying that $\pi_1(x) = \frac{x+\psi-\frac{\bar{b}_\infty}{2}}{1+\bar{a}_\infty}$ for all $x \in [0, \psi]$. It further implies that

$$J_1(x) = (\psi + x - \pi_1(x))^2 + d + \bar{a}_\infty \pi_1(x)^2 + \bar{b}_\infty \pi_1(x) + \bar{c}_\infty$$

for all $x \in [0, \psi]$. Again using (6) for $k=2$, we see that

$$a_2 = 1 - \frac{p}{1+\bar{a}_\infty} \text{ and } b_2 = \frac{p(2\bar{a}_\infty\psi + \bar{b}_\infty)}{1+\bar{a}_\infty}.$$

Following Lemma 3.3, $a_2 = \bar{a}_\infty$ and $b_2 = \bar{b}_\infty$. Hence, following similar arguments as before, $\pi_2(x) = \pi_1(x)$ for all $x \in [0, \psi]$. Continuing in this fashion we see that for all $k \ge 1$, $\pi_k(x) = \pi_1(x)$ for all $x \in [0, \psi]$. Therefore $\pi^*(x) = \frac{x+\psi-\frac{\bar{b}_\infty}{2}}{1+\bar{a}_\infty}$ for all $x \in [0, \psi]$.

Appendix C. Proof of Proposition 3.3

Following Proposition 3.2 and (15) we see that $\bar{\pi}(x)$ is either $\pi^*(x)$ or $\tilde{\pi}(x)$ depending on the parameters. Therefore, it is enough to argue that

$$\tilde{\pi}(x) \ge \pi^*(x) \; \forall x \in [0, \psi]$$

irrespective of the parameters. We prove this by considering the following two cases separately.

Case (1) $x \le \theta(\bar{a}_\infty, \bar{b}_\infty)$: We assume $\theta(\bar{a}_\infty, \bar{b}_\infty) \ge 0$ else this case does not arise. In this case, $\tilde{\pi}(x) = 0$. We will argue that $\pi^*(x)$ also equals zero in this case. We will do this via iteratively showing that $\pi_k(x) = 0$ for all $k \ge 0$. First recall that $\bar{a}_\infty\psi + \frac{\bar{b}_\infty}{2} = \psi$ (see Section 3.2, (13)). From Lemma 3.3, $\bar{a}_k \ge \bar{a}_\infty$ and $\bar{b}_k \ge \bar{b}_\infty$ for all $k \ge 0$. Hence $\bar{a}_k\psi + \frac{\bar{b}_k}{2} > \psi$ for all $k \ge 0$ and also, $\theta(\bar{a}_k, \bar{b}_k) > \theta(\bar{a}_\infty, \bar{b}_\infty)$ for all $k \ge 0$.

Let us now consider value iteration starting with function $J_0(x) = (x + \psi)^2$ as in the proof of Proposition 3.2(a). Recall that $a_1 = 1 = \bar{a}_0$, $b_1 = 2p\psi = \bar{b}_0$ and

$$\pi_1(x) = \begin{cases} 0, & \text{if } x \le \theta(\bar{a}_0, \bar{b}_0) \\ \frac{x+\psi-p\psi}{2}, & \text{otherwise.} \end{cases}$$

Clearly, $\pi_1(x) = 0$ for all $x \le \theta(\bar{a}_\infty, \bar{b}_\infty)$. Next we analyze $\pi_2(x)$. Using (6) for $k = 2$, $pJ_1(u)+(1-p)u^2 = a_2u^2+b_2u+c_2$, where

$$a_2 = \begin{cases} a_{21} = \bar{a}_0, & \text{if } u \le \theta(\bar{a}_0, \bar{b}_0) \\ a_{22} = 1 - \frac{p}{1+\bar{a}_0}, & \text{otherwise} \end{cases}$$

$$b_2 = \begin{cases} b_{21} = \bar{b}_0, & \text{if } u \le \theta(\bar{a}_0, \bar{b}_0) \\ b_{22} = \frac{p(2\bar{a}_0\psi+\bar{b}_0)}{1+\bar{a}_0}, & \text{otherwise} \end{cases}$$

$$c_2 = \begin{cases} c_{21} = p(\psi^2 + \bar{c}_1), & \text{if } u \le \theta(\bar{a}_0, \bar{b}_0) \\ c_{22} = p(\frac{\bar{a}_0\psi^2+\bar{b}_0\psi-\frac{\bar{b}_0^2}{4}}{1+\bar{a}_0} + \bar{c}_1 + d), & \text{otherwise.} \end{cases}$$

Note that

$$a_{21}u^2 + b_{21}u + c_{21} < a_{22}u^2 + b_{22}u + c_{22}$$

for all $u \in [0, \theta(\bar{a}_0, \bar{b}_0))$, implying that $c_{21} < c_{22}$. Moreover,

$$J_2(x) = \min\Big\{(\psi + x)^2 + c_{21}, \min_{u\in[0,\theta(\bar{a}_0,\bar{b}_0)]} (\psi + x - u)^2 + d + a_{21}u^2 + b_{21}u + c_{21},$$
$$\min_{u\in[\theta(\bar{a}_0,\bar{b}_0),\psi]} (\psi + x - u)^2 + d + a_{22}u^2 + b_{22}u + c_{22}\Big\}.$$

Let us define functions

$$J_{21}(x) = \min\{(\psi + x)^2 + c_{21}, \min_{u\in[0,\psi]} (\psi + x - u)^2 + d + a_{21}u^2 + b_{21}u + c_{21}\}$$

and

$$J_{22}(x) = \min\{(\psi + x)^2 + c_{21}, \min_{u\in[0,\psi]} (\psi + x - u)^2 + d + a_{22}u^2 + b_{22}u + c_{22}\}.$$

The optimal controls in the above optimization problems are

$$\pi_{21}(x) = \begin{cases} 0, & \text{if } x \le \theta(a_{21}, b_{21}) \\ \frac{x+\psi-\frac{b_{21}}{2}}{1+a_{21}}, & \text{otherwise} \end{cases}$$

and

$$\pi_{22}(x) = \begin{cases} 0, & \text{if } x \le \sqrt{(d + c_{22} - c_{21})(1 + a_{22})} + \frac{b_{22}}{2} - \psi \\ \frac{x+\psi-\frac{b_{22}}{2}}{1+a_{22}}, & \text{otherwise} \end{cases}$$

respectively. Note that, since $c_{22} > c_{21}$, $\sqrt{(d + c_{22} - c_{21})(1 + a_{22})} + \frac{b_{22}}{2} - \psi > \theta(a_{22}, b_{22})$, and hence, $\pi_{22}(x) = 0$ for all $x \in [0, \theta(a_{22}, b_{22})]$. Finally, comparing J_2, J_{21} and J_{22}, we see that when both $\pi_{21}(x)$ and $\pi_{22}(x)$ equal zero, $\pi_2(x)$ also equals zero. In other words, $\pi_2(x) = 0$ for all $x \le \min\{\theta(a_{21}, b_{21}), \theta(a_{22}, b_{22})\}$. In particular, $\pi_2(x) = 0$ for all $x \le \theta(\bar{a}_\infty, \bar{b}_\infty)$.

We can similarly argue that, for all $k \ge 1$, $\pi_k(x) = 0$ for all $x \le \theta(\bar{a}_\infty, \bar{b}_\infty)$ as desired.

Case (2) $x > \theta(\bar{a}_\infty, \bar{b}_\infty)$*:* In this case

$$\tilde{\pi}(x) = \frac{x + \psi - \frac{\bar{b}_\infty}{2}}{1 + \bar{a}_\infty}.$$

From Lemma 3.3, $\bar{a}_k \ge \bar{a}_\infty$ and $\bar{b}_k \ge \bar{b}_\infty$ for all $k \ge 0$, and hence,

$$\tilde{\pi}(x) \ge \frac{x + \psi - \frac{\bar{b}_k}{2}}{1 + \bar{a}_k}$$

for all $x > \theta(\bar{a}_\infty, \bar{b}_\infty)$. Therefore, following (11), $\tilde{\pi}(x) \ge \pi^*(x)$ for all $x > \theta(\bar{a}_\infty, \bar{b}_\infty)$. Combining Cases (1) and (2) we see that $\tilde{\pi}(x) \ge \pi^*(x)$ for all $x \in [0, \psi]$ as desired.

Appendix D. Proof of Proposition 4.1

Let us define

$$h_{a,b}(x,u) = (\psi + x - u)(\psi - u) + d + u(u + p(\psi - au - b)), \tag{D.1}$$

$$\text{and} \qquad \pi_{a,b}(x) = \arg\min_{u\in[0,\psi]} h_{a,b}(x,u).$$

The following can be verified

$$\pi_{a,b}(x) = \frac{x + (2-p)\psi + pb}{2(2-pa)} \tag{D.2}$$

$$h_{a,b}(x,u) = \left(\frac{u^2}{2} - u\frac{(2-p)\psi + x + pb}{2(2-ap)} + \frac{d}{2(2-ap)}\right)2(2-ap) + \psi(\psi + x).$$

From [26, Chapter 2, Proposition 1.2(b)], $C_k(\cdot)$s converge to the optimal cost function $C(\cdot)$ and $\pi'_k(\cdot)$ converge to $\pi'(\cdot)$ irrespective of the initial function $C_0(x)$ in the value iteration. Now we analyze value iteration starting with a different function.

$$C_0(x) = \min\{(\psi + x)\psi, h_{\tilde{a}_\infty,\tilde{b}_\infty}(x, \pi_{\tilde{a}_\infty,\tilde{b}_\infty}(x))\}.$$

Recall that $\pi'_0(x)$ is the solution to $C_0(x)$. To determine $\pi'_0(x)$, we need to find $\arg\min h_{\tilde{a}_\infty,\tilde{b}_\infty}(x, \pi_{\tilde{a}_\infty,\tilde{b}_\infty}(x))$. Realize that $\pi_{\tilde{a}_\infty,\tilde{b}_\infty}(x)$ is $\arg\min h_{\tilde{a}_\infty,\tilde{b}_\infty}(x, \pi_{\tilde{a}_\infty,\tilde{b}_\infty}(x))$. Using (D.2), it can be seen that $\pi_{\tilde{a}_\infty,\tilde{b}_\infty}(x) = \frac{x+(2-p)\psi+p\tilde{b}_\infty}{2(2-p\tilde{a}_\infty)}$. As the sequences $\tilde{a}_k, \tilde{b}_k, k \geq -1$ converge to $\tilde{a}_\infty, \tilde{b}_\infty$ (see (18),(19)). From Lemma 4.2 we know that $\pi_{\tilde{a}_\infty,\tilde{b}_\infty}(x) \in (0,\psi)$, therefore we infer that

$$\pi_{\tilde{a}_\infty,\tilde{b}_\infty}(x) = \tilde{a}_\infty x + \tilde{b}_\infty. \tag{D.3}$$

Now to determine $\pi'_0(x)$, we need the following lemma which is proved at [22, Appendix II-C].

Lemma D.1. *The following inequality holds if and only if $x \leq x_\infty$.*

$$\psi(\psi + x) \leq h_{\tilde{a}_\infty,\tilde{b}_\infty}(x, \pi_{\tilde{a}_\infty,\tilde{b}_\infty}(x)). \tag{D.4}$$

Using Lemmas D.1 and 4.2, we infer

$$\pi'_0(x) = \begin{cases} 0, & \text{if } x \leq x_\infty \\ \tilde{a}_\infty x + \tilde{b}_\infty, & \text{otherwise} \end{cases} \tag{D.5}$$

Using (D.1), $\tilde{a}_\infty > 0, \tilde{b}_\infty > 0$ we infer the following

$$h_{\tilde{a}_\infty,\tilde{b}_\infty}(x,u) \leq h_{0,0}(x,u), \forall x, u \in [0,\psi] \tag{D.6}$$

Now from (17) and (D.5), the following can be written

$$C_1(x) = \min\{(\psi + x)\psi, \min_{u\in[0,x_\infty]} h_{0,0}(x,u), \min_{u\in[x_\infty,\psi]} h_{\tilde{a}_\infty,\tilde{b}_\infty}(x,u)\}.$$

We would now determine $\pi'_1(x)$. Let us study the following two cases separately.

Case 1. $x \leq x_\infty$: From Lemma D.1, we infer the following when $x \leq x_\infty$

$$\begin{aligned}\psi(\psi + x) &\leq \min_{u\in[0,\psi]} h_{\tilde{a}_\infty,\tilde{b}_\infty}(x,u), \\ &< \min\left\{\min_{u\in[0,x_\infty]} h_{0,0}(x,u), \min_{u\in[x_\infty,\psi]} h_{\tilde{a}_\infty,\tilde{b}_\infty}(x,u)\right\},\end{aligned}$$

where the second inequality follows from (D.6). Hence $\pi'_1(x) = 0, \forall x \leq x_\infty$.

Case 2. $x > x_\infty$: Note that $\frac{\tilde{b}_\infty}{1-\tilde{a}_\infty} \geq x_\infty$ implies $\tilde{a}_\infty x_\infty + \tilde{b}_\infty > x_\infty$. When $\tilde{a}_\infty x_\infty + \tilde{b}_\infty > x_\infty$ the following holds

$$\begin{aligned}\min_{u\in[x_\infty,\psi]} h_{\tilde{a}_\infty,\tilde{b}_\infty}(x,u) &= \min_{u\in[0,\psi]} h_{\tilde{a}_\infty,\tilde{b}_\infty}(x,u) \\ &< \min\left\{(\psi + x)\psi, \min_{u\in[0,x_\infty]} h_{0,0}(x,u)\right\}\end{aligned}$$

Last inequality follows from Lemma D.1 and (D.6). Hence, $\pi'_1(x) = \tilde{a}_\infty x + \tilde{b}_\infty, \forall x > x_\infty$.

Combining both the cases $\pi'_1(x) = \pi'_0(x), \forall x \in [0,\psi]$. We can iteratively show that $\pi'_k(x) = \pi'_0(x), \forall x \in [0,\psi]$. Hence $\pi'(x) = \pi'_0(x)$.

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