TOTALLY POSITIVE KERNELS, PÓLYA FREQUENCY FUNCTIONS, AND THEIR TRANSFORMS

By

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Abstract. The composition operators preserving total non-negativity and total positivity for various classes of kernels are classified, following three themes. Letting a function act by post composition on kernels with arbitrary domains, it is shown that such a composition operator maps the set of totally non-negative kernels to itself if and only if the function is constant or linear, or just linear if it preserves total positivity. Symmetric kernels are also discussed, with a similar outcome. These classification results are a byproduct of two matrix-completion results and the second theme: an extension of A. M. Whitney's density theorem from finite domains to subsets of the real line. This extension is derived via a discrete convolution with modulated Gaussian kernels. The third theme consists of analyzing, with tools from harmonic analysis, the preservers of several families of totally non-negative and totally positive kernels with additional structure: continuous Hankel kernels on an interval, Pólya frequency functions, and Pólya frequency sequences. The rigid structure of post-composition transforms of totally positive kernels acting on infinite sets is obtained by combining several specialized situations settled in our present and earlier works.

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1 Introduction and main results

1.1 Total positivity. Let X and Y be totally ordered sets. A kernel $K : X \times Y \to \mathbb{R}$ is said to be **totally positive** if the matrix $(K(x_i, y_j))_{i,j=1}^n$ is totally positive (that is, all of its minors are positive) for any choice of $x_1 < \cdots < x_n$ and $y_1 < \cdots < y_n$, where n is an arbitrary positive integer. Similarly, the kernel K is said to be **totally non-negative** if $(K(x_i, y_j))_{i,j=1}^n$ is totally non-negative (that is, all of its minors are non-negative).[1] For almost a century, these classes of kernels and matrices surfaced in the most unexpected circumstances, and this trend continues in full force today. The foundational work [26], the survey [3], the early monograph [27], and the more recent publications [40, 28, 48, 18] offer

[1]The monographs [40, 48] use the terms "strict total positivity" and "total positivity" instead of "total positivity" and "total non-negativity" respectively.

ample references to the fascinating history of total positivity, as well as accounts of its many surprising applications. Total positivity continues to make impacts in areas such as representation theory [45, 46, 51], network analysis [50], cluster algebras [11, 23, 24], Gabor analysis [29], statistics [17, 42], and combinatorics [15, 16]. A surprising link between positive Grassmannians, seen as the geometric impersonation of total positivity, and integrable systems [43, 44] is also currently developing at a fast pace.

A natural way to construct new kernels from existing ones is to compose them with a given function F. More precisely, every suitable function F induces a composition operator C_F mapping the kernel K to $C_F(K) := F \circ K$. The aim of the present work is to determine for which functions F the operator C_F leaves invariant the set of totally positive kernels defined on $X \times Y$, and to answer the analogous question for total non-negativity. When X and Y are finite, this is equivalent to determining when the entrywise calculus $(a_{ij}) \mapsto (F(a_{ij}))$ induced by F preserves the total positivity or total non-negativity of matrices.

The study of when entrywise transformations preserve notions of positivity has a long history. One of the first rigidity theorems for such transforms was proved by Schoenberg, who showed in the 1940s that entrywise transforms preserving positive semidefiniteness of matrices of all sizes must be given by convergent power series with non-negative coefficients [55]. That any such function preserves positive semidefiniteness when applied to matrices of arbitrary dimensions follows immediately from the Schur product theorem [60]. Reformulated in the language of kernels, Schoenberg's theorem shows that the composition operator C_F leaves invariant the set of positive-semidefinite kernels if and only if F admits a power series representation with non-negative coefficients. Schoenberg's discovery was part of a larger project of classifying the invariant distances on homogeneous spaces which are isometrically equivalent to a Hilbert-space distance; see Bochner's very informative article [14] for more details. This circle of ideas was further extended by the next generation of analysts, to operations which preserve Fourier coefficients of measures [33]. The analogous result to Schoenberg's theorem for matrices of a fixed size is more subtle. Roger Horn's doctoral dissertation contains the fundamental observation, attributed by Horn to Löwner, that the size of the positive matrices preserved by a smooth transform imposes non-negativity constraints on roughly the same number of its derivatives [36]. This observation left a significant mark on probability theory [37]. Determining the exact set of functions that preserve positivity when applied entrywise to positive-semidefinite matrices of a fixed dimension remains an open problem and the subject of active research [6, 9, 41].

More details about the evolution of matrix positivity transforms and their applications in areas such as data science and probability theory are contained in our recent surveys [7, 8]. The investigation of entrywise transforms preserving total positivity has recently revealed novel connections to type-A representation theory and to combinatorics. We refer the reader to the recent works [6, 30] and the recent paper [41] by Khare and Tao for more details.

1.2 Main results. Recall that a kernel is said to be totally non-negative of order p, denoted TN_p, if all of its minors of size $p \times p$ and smaller are non-negative. Similarly one defines TP_p kernels; see Definition 2.2 for the precise details.

An initial step in classifying total-positivity preservers over an arbitrary domain $X \times Y$ is to consider separately the cases where at least one of X and Y is finite and when both are infinite. The first of these cases leads to the following theorem.

Theorem 1.1. *Let X and Y be totally ordered sets, each of size at least 4, and let $F : [0, \infty) \to \mathbb{R}$. The operator $C_F : K \mapsto F \circ K$ maps the set of totally non-negative kernels of any fixed order at least 4 to itself if and only if F is constant, so that $F(x) = c$, or linear, so that $F(x) = cx$, with $c \geq 0$. The same holds if totally non-negative kernels are replaced by totally positive kernels of any fixed order at least 4, now with the requirement that $c > 0$.*

We develop the proof of Theorem 1.1 over several sections. In fact, we prove more; we provide a full characterization of entrywise transforms that preserve total non-negativity on $m \times n$ matrices or symmetric $n \times n$ matrices, for any fixed values of m and n, finite or infinite. We also prove the analogous classifications for preservers of total positivity on matrices of each size. See Tables 2.1 and 2.2 for further details, including variants involving test sets of symmetric matrices.

The proof strategy is broadly as follows. For preservers of total non-negativity, note that totally non-negative matrices of smaller size can be embedded into larger ones; this allows us to use, at each stage, properties of preservers for lower dimensions. Thus, we show the class of preservers to be increasingly restrictive as the dimension grows, and as soon as we reach 4×4 matrices (or 5×5 matrices if our test matrices are taken to be symmetric), we obtain the main result.

For total positivity, the problem is more subtle: as zero entries are not allowed, one can no longer use the previous technique. Instead, the key observation is that totally positive matrices are dense in totally non-negative matrices; this is an approximation theorem due to A. M. Whitney, which reduces the problem for continuous functions and finite sets X and Y to the previous case. The next step then is to prove the continuity of all total-positivity preservers; we achieve this by solving two totally positive matrix-completion problems. Finally, to go from finite X and Y to the case where one of X and Y is infinite, we extend Whitney's approximation theorem to totally positive kernels on arbitrary subsets of $\mathbb{R}$, as follows.

Theorem 1.2. *Given non-empty subsets X and Y of $\mathbb{R}$, and a positive integer p, any bounded TN_p kernel $K : X \times Y \to \mathbb{R}$ can be approximated locally uniformly at points of continuity in the interior of $X \times Y$ by a sequence of TP_p kernels on $X \times Y$. Furthermore, if $X = Y$ and K is symmetric, then the kernels in the approximating sequence may be taken to be symmetric.*

The proof of Theorem 1.2 is developed in Section 6, using discretized Gaussian convolution. The Gaussian function is found throughout mathematics, and this paper is no exception. It finds itself a crucial ingredient for several of the arguments below. As well as the discrete convolution, it allows regular Pólya frequency functions to be approximated by totally positive ones, and is employed in various places as a totally positive kernel which is particularly straightforward to manipulate.

The only remaining case is the classification of total positivity-preservers for kernels over $X \times Y$, with both X and Y infinite. In this situation, the test sets used to obtain the previous results are no longer sufficient, and new tools and test classes of kernels with additional structure are called for.

When considering other forms of structured kernels, the Hankel and Toeplitz classes stand out. The study of Hankel kernels with countable domains leads naturally to moment-preserving maps, and these form the main body of our previous investigation [9]. In the present article, we provide the classification of preservers for both Hankel and Toeplitz kernels with domains which are a continuum. A Hankel kernel has the form

$$X \times X \to \mathbb{R}; \quad (x, y) \mapsto f(x + y),$$

whereas a Toeplitz kernel has the form

$$X \times X \to \mathbb{R}; \quad (x, y) \mapsto f(x - y),$$

where $X \subseteq \mathbb{R}$.

The main results are summarized by the next five theorems; for more details, see Theorems 7.8 and 7.11 (the Hankel case), Theorems 8.5 and 8.10 (Pólya frequency functions), Theorems 8.7 and 8.10 (measurable Toeplitz kernels), Theorems 9.2 and 9.5 (Pólya frequency sequences), and Theorems 10.1 and 10.3 (one-sided Pólya frequency functions and sequences).

The class of TN or TP preservers for Hankel kernels consists essentially of absolutely monotonic functions. This is outlined in the following result, and our proof relies on prior work of Bernstein, Hamburger, Mercer, and Widder.

Theorem 1.3. *Let $X \subseteq \mathbb{R}$ be an open interval and let $F : [0, \infty) \to \mathbb{R}$. The composition map C_F preserves the set of continuous* TN *Hankel kernels on $X \times X$ if and only if $F(x) = \sum_{k=0}^{\infty} c_k x^k$ on $(0, \infty)$, with $c_k \geq 0$ for all k and $F(0) \geq 0$.*

A similar statement holds for preservers of TP *Hankel kernels on $X \times X$.*

In contrast, TN Toeplitz kernels possess a far more restricted class of preservers. Recall that a Pólya frequency function Λ is an integrable function on $\mathbb{R}$, non-zero at two or more points, such that the Toeplitz kernel $T_\Lambda : \mathbb{R} \times \mathbb{R} \to \mathbb{R};\ (x, y) \mapsto \Lambda(x-y)$ is TN.

Theorem 1.4. *Let $F : [0, \infty) \to [0, \infty)$. The composition map C_F preserves the set of Pólya frequency functions if and only if $F(x) = cx$ with $c > 0$.*

A similar statement holds for preservers of the class of TP *kernels of the form T_Λ, where Λ is a Pólya frequency function.*

If the integrability condition is removed, then the class of preservers of Toeplitz kernels is enlarged slightly. In the following theorem, measurability is required to hold in the sense of Lebesgue.

Theorem 1.5. *Let $F : [0, \infty) \to [0, \infty)$ be non-zero. The composition map C_F preserves* TN *measurable Toeplitz kernels on $\mathbb{R} \times \mathbb{R}$ if and only if $F(x) = c$ or $F(x) = cx$ or $F(x) = c\mathbf{1}_{x>0}$, for some $c > 0$.*

The only preservers of TP *Toeplitz kernels on $\mathbb{R} \times \mathbb{R}$, whether measurable or not, are the dilations $F(x) = cx$ with $c > 0$.*

The discrete analogue of a Pólya frequency function or a Toeplitz kernel is a Pólya frequency sequence, that is, a real sequence $\mathbf{a} = (a_n)_{n \in \mathbb{Z}}$ such that the Toeplitz kernel $T_{\mathbf{a}} : \mathbb{Z} \times \mathbb{Z} \to \mathbb{R};\ (i, j) \mapsto a_{i-j}$ is TN. These sequences have been widely studied in function theory, approximation theory, and combinatorics. It turns out their preservers display the same rigidity.

Theorem 1.6. *Let $F : [0, \infty) \to [0, \infty)$. The composition map C_F preserves Pólya frequency sequences if and only if $F(x) = c$ or $F(x) = cx$, with $c \geq 0$.*

A similar result holds for the preservers of TP *Pólya frequency sequences.*

In fact, we prove a more general version of Theorem 1.6 with the same rigidity, Theorem 9.5, where the common domain of the kernels is a pair of subsets that each contain arbitrarily long arithmetic progressions with equal increments.

As a final variation, we characterize the preservers of one-sided analogues of Pólya frequency functions and sequences, and of Toeplitz kernels, where a kernel is said to be one-sided if the associated function is: that is, it vanishes on an infinite semi-axis. The preservers of such kernels, when compared to the classifications obtained in the three previous theorems, turn out to be similarly restricted.

Theorem 1.7. *Let $F : [0, \infty) \to [0, \infty)$. The composition map C_F preserves the following classes:*

(1) *one-sided Pólya frequency functions,*
(2) *one-sided* TN *measurable Toeplitz kernels on $\mathbb{R} \times \mathbb{R}$,*
(3) *one-sided Pólya frequency sequences,*

if and only if the function F has the following form in each case, where $c > 0$:

(1) $F(x) = cx$*;*
(2) $F(x) = cx$, $F(x) = c\mathbf{1}_{x>0}$, *or* $F(x) = 0$*;*
(3) $F(x) = cx$, *or* $F(x) = 0$.

These results on the preservers of Toeplitz kernels rely on work of Schoenberg and his collaborators on Pólya frequency functions and sequences; a comprehensive exposition of this is found in Karlin's treatise [40].

As expected, these classes of invariant kernels and their preservers touch harmonic analysis, in particular the Fourier–Laplace transform. We elaborate a few details alongside our classification proofs in the main body of this article.

As part of our analysis, we provide an example of an even Pólya frequency function M with the property that M^n is not a Pólya frequency function for every integer $n \geq 2$, and also a one-sided version of such a function.

As a final consequence, we come full circle to classify TP preservers, by refining the class of test matrices used to prove Theorem 1.1 and its ramifications.

Theorem 1.8. *Let X and Y be infinite, totally ordered sets that admit a* TP *kernel on $X \times Y$. A function $F : (0, \infty) \to (0, \infty)$ is such that C_F preserves the set of* TP *kernels on $X \times Y$ if and only $F(x) = cx$ with $c > 0$.*

A similar result holds for the non-constant preservers of TN *kernels on $X \times Y$, and for the preservers of symmetric* TP *or* TN *kernels on $X \times X$.*

The proof for preservers of TP kernels on $X \times Y$ uses Theorem 9.5, together with the observation that any TP kernel on $X \times Y$ may be used to realize X and Y as subsets of $\mathbb{R}$. To classify the preservers of symmetric TP kernels on $X \times X$, we exploit and unify in a coherent proof most of the concepts arising in this paper: Vandermonde and Hankel kernels, Pólya frequency functions and sequences, order-preserving embeddings, discretization, and Whitney-type density theorems.

1.3 Contents. This work has three distinct themes: preservers of TN kernels and TP kernels, approximation of TN kernels by TP kernels, and preservers of structured kernels possessing various forms of positivity. In Sections 3 and 4, we provide characterizations for endomorphisms of TN and TP kernels, under various restrictions: symmetric or not, matricial (having finite domains) or not, and so on. See Section 2.1 for a tabulated compilation of these. Next, in Section 5, we strengthen a result of Vasudeva to show that TN preservation for symmetric 2×2 matrices with positive entries is equivalent to preservation of positive semidefiniteness on a much smaller set, and show that such functions must be continuous. Section 6 contains the results on discrete Gaussian convolution required to extend Whitney's approximation theorem and so establish some of the preceding characterizations. Section 7 is devoted to the classification of composition operators which preserve TN Hankel kernels defined on a continuum, and also the TP case. Section 8 examines those transforms which leave invariant the class of Pólya frequency functions and measurable Toeplitz kernels (for both the TN and TP cases), and Section 9 considers the analogous Pólya frequency sequences. The preservers of one-sided Pólya frequency functions and sequences are characterized in Section 10. We conclude in Section 11 by completing the classification problem for TP preservers in both the general setting and for the case of symmetric kernels. With the practitioner in mind, our final section collects some information about minimal test families that assure the rigidity of preservers for the larger class of kernels to which they belong. At the end of this last section, we record some of the ad hoc notation used in this paper.

The determination of post-composition transforms for totally positive kernels that is obtained in the following pages may seem rather discouraging at first sight: there are only trivial ones. However, many of the technical ingredients appearing in the proofs may be of independent interest, such as the Whitney-type approximation for kernels with infinite support, the TP completion of Hankel matrices, the structure of Loewner monotone maps, TN transforms of the Gaussian kernel, a family of even Pólya frequency functions whose higher integer powers cease to be Pólya frequency functions, and an order-preserving embedding of the supports of TP_2-kernels into the real line.

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2 Preliminaries and overview

Throughout this paper, the abbreviation "TN" stands either for the class of totally non-negative matrices, or the total-non-negativity property for a matrix or a kernel, and similarly for "TP". A kernel is a map $K : X \times Y \to \mathbb{R}$, where X and Y are sets; a kernel is **symmetric** if $X = Y$ and $K(x, y) = K(y, x)$ for all $x, y \in X$.

Notation 2.1. If X is a totally ordered set and $n \in \mathbb{N} := \{1, 2, 3, \ldots\}$, then

$$X^{n,\uparrow} := \{\mathbf{x} = (x_1, \ldots, x_n) \in X^n : x_1 < \cdots < x_n\}$$

and $[n] := \{1, \ldots, n\}$. If $K : X \times Y \to \mathbb{R}$, where X and Y are totally ordered sets, together with $\mathbf{x} \in X^{m,\uparrow}$ and $\mathbf{y} \in Y^{n,\uparrow}$, then $K[\mathbf{x}; \mathbf{y}]$ is defined to be the $m \times n$ matrix such that

$$K[\mathbf{x}; \mathbf{y}]_{ij} = K(x_i, y_j) \quad (i = 1, \ldots, m; j = 1, \ldots, n).$$

Definition 2.2. Let $p \in \mathbb{N}$. We say that K is

(i) TN_p if $\det K[\mathbf{x}; \mathbf{y}] \geq 0$ for all $n \in [p]$, $\mathbf{x} \in X^{n,\uparrow}$, and $\mathbf{y} \in Y^{n,\uparrow}$;

(ii) TP_p if $\det K[\mathbf{x}; \mathbf{y}] > 0$ for all $n \in [p]$, $\mathbf{x} \in X^{n,\uparrow}$, and $\mathbf{y} \in Y^{n,\uparrow}$.

If this holds, we say that the kernel is TN or TP of **order** p, denoted by TN_p and TP_p respectively. The kernel K is said to be TN if it is TN_p for all $p \in \mathbb{N}$, and similarly for TP, in which case the order is infinite.

Example 2.3. Given $n \in \mathbb{N}$, positive constants $u_1 < \cdots < u_n$ and real constants $\alpha_1 < \cdots < \alpha_n$, the generalized Vandermonde matrix $V = (u_i^{\alpha_j})_{i,j=1}^n$ is totally positive [25, Chapter XIII, §8, Example 1]. Reversing the order of both the rows and columns of V preserves TP (and TN), so the same is true if both sets of inequalities are reversed.

In particular, the kernels

$$K : I \times \mathbb{R} \to \mathbb{R}; \quad (x, y) \mapsto x^y \quad \text{and} \quad K' : I' \times \mathbb{R} \to \mathbb{R}; \quad (x, y) \mapsto e^{xy}$$

are TP for any sets $I \subseteq (0, \infty)$ and $I' \subseteq \mathbb{R}$.

Example 2.4. As a generalization of Example 2.3, let X and Y be totally ordered sets, let $g_X : X \to (0, \infty)$ and $g_Y : Y \to (0, \infty)$ be arbitrary, and let $h_X : X \to (0, \infty)$ and $h_Y : Y \to \mathbb{R}$ be increasing. Then the kernel

$$K : X \times Y \to (0, \infty);\ (x, y) \mapsto g_X(x) h_X(x)^{h_Y(y)} g_Y(y) \tag{2.1}$$

is TP. To see this, note that if $n \in \mathbb{N}$, $\mathbf{x} \in X^{n,\uparrow}$ and $\mathbf{y} \in Y^{n,\uparrow}$, then

$$K[\mathbf{x}; \mathbf{y}] = \operatorname{diag}(g_X[\mathbf{x}])(h_X(x_i)^{h_Y(y_j)})_{i,j=1}^n \operatorname{diag}(g_Y[\mathbf{y}]),$$

where $\operatorname{diag}(g_X[\mathbf{x}])$ is the matrix with $(g_X(x_1), \ldots, g_X(x_n))$ on the leading diagonal and zeros elsewhere, and similarly for $\operatorname{diag}(g_Y[\mathbf{y}])$. Example 2.3 now gives that $K[\mathbf{x}; \mathbf{y}]$ has positive determinant.

In general, for any kernel on $X \times Y$, the properties of being TN, TN_p, TP, or TP_p are each preserved after multiplying by functions $g_X : X \to (0, \infty)$ or $g_Y : Y \to (0, \infty)$.

2.1 Overview of classification results for TN and TP kernels.

Our primary focus in this paper is to classify the functions which, under composition, preserve classes of totally positive or totally non-negative kernels on $X \times Y$, where X and Y are totally ordered sets. In Section 3 and 4, we consider sixteen different classes of kernels, according to the following binary possibilities:

(1) totally non-negative or totally positive;
(2) matricial, so that X and Y are finite, or non-matricial, so that at least one of X and Y is infinite;
(3) order p with $p \geq \min\{|X|, |Y|\}$ or $p < \min\{|X|, |Y|\}$;
(4) symmetric, requiring that $X = Y$, or not.

Remark 2.5. If at least one of X and Y is finite, then the preservers of TP kernels on $X \times Y$ are precisely the preservers of TP_p kernels on $X \times Y$, for any $p \geq \min\{|X|, |Y|\}$. The same observation holds if TP is replaced by TN, and whether or not symmetry is imposed. Thus, the first alternative in (3) above may be replaced by $p = \min\{|X|, |Y|\}$ and we do this henceforth.

We now present tabulations of our classification results from the next two sections.

For the most part, the conclusions in the matricial and non-matricial situations are similar or even the same. However, and especially for TP preservers, the proofs are harder when at least one of the index sets is infinite. In addition to the results for the matricial cases, and the ideas behind their proofs, we require other, more

Table 2.1. Total non-negativity preservers

Characterization of endomorphisms	matricial	non-matricial	symmetric matricial	symmetric non-matricial
$p = \min\{\lvert X\rvert, \lvert Y\rvert\}$	Theorem 3.3	Corollary 3.4	Theorem 3.6	Theorem 3.6
$p < \min\{\lvert X\rvert, \lvert Y\rvert\}$	Theorem 3.3, Remark 3.5	Corollary 3.4, Remark 3.5	Theorem 3.7	Theorem 3.7

Table 2.2. Total-positivity preservers

Characterization of endomorphisms	matricial	non-matricial	symmetric matricial	symmetric non-matricial
$p = \min\{\lvert X\rvert, \lvert Y\rvert\}$	Theorem 4.1	Theorem 4.4	Theorem 4.9	Corollary 4.11
$p < \min\{\lvert X\rvert, \lvert Y\rvert\}$	Theorem 4.4	Theorem 4.4	Theorem 4.12	Theorem 4.12

involved techniques to extend these results to kernels. A particular issue is the lack of a tractable test set of TP kernels.

The preservers of symmetric TN_p or TP_p kernels differ depending on whether $p = \min\{|X|, |Y|\}$ or $p < \min\{|X|, |Y|\}$, and in the latter case these preservers coincide with the preservers of all $p \times p$ matrices which are TN or TP. The proofs rely on the careful analysis of preservers of totally non-negative kernels in each fixed dimension: we show that our test sets of $p \times p$ matrices which are TN occur already as minors of $(p + 1) \times (p + 1)$ symmetric TN matrices.

3 Total non-negativity preservers

We now begin to formulate and prove our characterization results for TN preservers. In this section, we are interested in understanding the following family of functions.

Definition 3.1. Given two totally ordered sets X and Y, let

$$\mathscr{F}^{\mathrm{TN}}_{X,Y} := \left\{ F\,[0,\infty) \to \mathbb{R} \,\middle|\, \begin{array}{l} \text{if } K : X \times Y \to \mathbb{R} \text{ is totally non-negative,} \\ \text{so is } F \circ K \end{array} \right\}. \tag{3.1}$$

We observe first that $\mathscr{F}^{\mathrm{TN}}_{X,Y}$ depends on only rather coarse features of X and Y. A totally ordered set has an **ascending chain** if it contains an infinite sequence of elements $x_1 < x_2 < \cdots$, and similarly for a **descending chain**. It is well known that an infinite totally ordered set must contain an ascending chain or a descending chain (or both).[2] We say that two totally ordered sets **have chains of the same type** if both contain an ascending chain or both contain a descending chain.

Recall that $[n] := \{1, \ldots, n\}$ whenever $n \in \mathbb{N}$.

Proposition 3.2. *Let X and Y be totally ordered sets. Then*

(1) $\mathscr{F}^{\mathrm{TN}}_{X,Y} = \mathscr{F}^{\mathrm{TN}}_{[n],[n]}$ *if at least one of X and Y is finite, and $n = \min\{|X|, |Y|\}$,*
(2) $\mathscr{F}^{\mathrm{TN}}_{X,Y} = \mathscr{F}^{\mathrm{TN}}_{\mathbb{N},\mathbb{N}}$ *if X and Y have chains of the same type,*
(3) $\mathscr{F}^{\mathrm{TN}}_{X,Y} = \mathscr{F}^{\mathrm{TN}}_{\mathbb{N},-\mathbb{N}}$ *if X and Y are infinite and do not have chains of the same type.*

Proof. The key observation is that, for any $X_0 \subseteq X$ and $Y_0 \subseteq Y$, a totally non-negative kernel $K : X_0 \times Y_0 \to \mathbb{R}$ trivially extends to a totally non-negative kernel $\tilde{K} : X \times Y \to \mathbb{R}$ by "padding by zeros", that is, by setting $K(x, y) = 0$ whenever $(x, y) \in (X \times Y) \setminus (X_0 \times Y_0)$. Conversely, it is immediate that any totally non-negative kernel on $X \times Y$ restricts to a totally non-negative kernel on $X_0 \times Y_0$.

If X and Y are both infinite, then each contains a copy of $\mathbb{N}$ or $-\mathbb{N}$. If they both contain copies of $\mathbb{N}$, then padding by zeros gives (2); similarly, if they both contain copies of $-\mathbb{N}$, noting that reversing the order of both rows and columns preserves TN, as observed in Example 2.3. If X and Y do not contain chains of the same type, then (3) holds, reversing rows and columns to swap the roles of $\mathbb{N}$ and $-\mathbb{N}$ if required.

Proposition 3.2 shows that characterizing $\mathscr{F}^{\mathrm{TN}}_{X,Y}$ is equivalent to determining which functions $F : \mathbb{R} \to \mathbb{R}$ preserve total non-negativity when applied entrywise to totally non-negative matrices of a fixed dimension (if X or Y is finite), or to totally non-negative matrices of all dimensions (if X and Y are infinite). The main result in this section answers this question.

Given a domain $I \subseteq \mathbb{R}$, a function $F : I \to \mathbb{R}$, and a matrix $A = (a_{ij}) \in I^{m \times n}$, we denote by $F[A] := (F(a_{ij}))$ the matrix obtained by applying F to the entries of A. We denote the Hadamard powers of A by $A^{\circ\alpha} := (a_{ij}^{\alpha})$. The convention $0^0 := 1$ is adopted throughout.

Theorem 3.3. *Let $F : [0, \infty) \to \mathbb{R}$ be a function and let $d := \min\{m, n\}$, where m and n are positive integers. The following are equivalent:*

(1) *F preserves total non-negativity entrywise on $m \times n$ matrices.*
(2) *F preserves total non-negativity entrywise on $d \times d$ matrices.*

[2]Here is a proof for completeness: let X be infinite and totally ordered, let $x_1 \in X$ and suppose $\{x \in X : x > x_1\}$ is infinite. (The proof is similar if $\{x \in X : x < x_1\}$ is infinite.) If there is no maximum element in X, then starting from x_1 one can inductively produce an ascending chain, as desired. Else, set $y_1 := \max X$. Inductively, given $y_1, \ldots, y_k$, either the infinite set $[x_1, y_k)$ has a maximum element y_{k+1}, or one can find an ascending chain $x_1 < x_2 < \cdots$ in $[x_1, y_k)$ as before. Thus, if there is no ascending chain starting from x_1, there is a descending chain $y_1 > y_2 > \cdots$.

(3) *F is either a non-negative constant or*
   (a) $(d = 1)$ $F(x) \geq 0$*;*
   (b) $(d = 2)$ $F(x) = c\mathbf{1}_{x>0}$ *or* cx^α *for some* $c > 0$ *and some* $\alpha > 0$*;*
   (c) $(d = 3)$ $F(x) = cx^\alpha$ *for some* $c > 0$ *and some* $\alpha \geq 1$*;*
   (d) $(d \geq 4)$ $F(x) = cx$ *for some* $c > 0$.

Proof. That (1) $\iff$ (2) is obvious, since the minors of a $m \times n$ matrix have dimension at most d. We will now prove that (2) $\iff$ (3) for each value of d.

The result is obvious when $d = 1$, since in this case a matrix is TN if and only if its entry is non-negative.

Suppose $F[-]$ preserves TN on 2×2 matrices and note that $F(x) \geq 0$ for all $x \geq 0$. Next, consider the following totally non-negative matrices:

$$A(x, y) := \begin{pmatrix} x & xy \\ 1 & y \end{pmatrix} \quad \text{and} \quad B(x, y) := \begin{pmatrix} xy & x \\ y & 1 \end{pmatrix} \quad (x, y \geq 0). \tag{3.2}$$

Considering the determinants of $F[A(x, y)]$ and $F[B(x, y)]$ gives that

$$F(xy)F(1) = F(x)F(y) \quad \text{for all } x, y \geq 0. \tag{3.3}$$

If $F(1) = 0$, then $F(x)F(y) = 0$, so $F(x) = 0$ for all $x \geq 0$. We will therefore assume that $F(1) > 0$. If $F(x) = 0$ for any $x > 0$, then Equation (3.3) implies that $F \equiv 0$, so we assume that $F(x) > 0$ for all $x > 0$. Applying F to the TN matrix

$$\begin{pmatrix} x & \sqrt{xy} \\ \sqrt{xy} & y \end{pmatrix} \quad (x, y \geq 0), \tag{3.4}$$

we conclude that $F(\sqrt{xy})^2 \leq F(x)F(y)$. As a result, the function $G(x) = \log F(e^x)$ is mid-point convex on $\mathbb{R}$. Also, applying F to the TN matrix

$$\begin{pmatrix} y & x \\ x & y \end{pmatrix} \quad (y \geq x \geq 0)$$

implies that F, so G, is non-decreasing. By [52, Theorem 71.C], we conclude that G is continuous on $\mathbb{R}$, and so F is continuous on $(0, \infty)$. Moreover, since $F(1) \neq 0$, Equation (3.3) implies

$$\frac{F(xy)}{F(1)} = \frac{F(x)}{F(1)} \frac{F(y)}{F(1)},$$

that is, the function $F/F(1)$ is multiplicative. From these facts, there exists $\alpha \geq 0$ such that $F(x) = F(1)x^\alpha$ for all $x > 0$. Finally, setting $y = 0$ in Equation (3.3), we see that

$$F(0)F(1) = F(x)F(0) \quad \text{for all } x \geq 0.$$

Thus either $F(0) = 0$ or $F \equiv F(1)$; in either case, the function F has the required form. The converse is immediate, and this proves the result in the case $d = 2$.

Next, suppose F preserves TN on 3×3 matrices and is non-constant. Since the matrix $A \oplus \mathbf{0}_{1\times 1}$ is totally non-negative if the 2×2 matrix A is, we conclude by part (b) that $F(x) = cx^\alpha$ for some $c > 0$ and $\alpha \geq 0$. The matrix

$$C := \begin{pmatrix} 1 & 1/\sqrt{2} & 0 \\ 1/\sqrt{2} & 1 & 1/\sqrt{2} \\ 0 & 1/\sqrt{2} & 1 \end{pmatrix} \tag{3.5}$$

is totally non-negative, and $\det F[C] = c^3(1 - 2^{1-\alpha})$. It follows that F does not preserve TN on 3×3 matrices when $\alpha < 1$. For higher powers, we use the following result [39, Theorem 4.2]; see [20, Theorem 5.2] for a shorter proof.

$$\alpha \geq 1 \quad \Longrightarrow \quad x^\alpha \text{ preserves TN and TP on } 3 \times 3 \text{ matrices.} \tag{3.6}$$

This concludes the proof of the case $d = 3$.

Finally, suppose F is non-constant and preserves TN on 4×4 matrices. Similarly to the above, considering matrices of the form $A \oplus \mathbf{0}_{1\times 1}$ gives, by part (c), that $F(x) = cx^\alpha$ for some $c > 0$ and some $\alpha \geq 1$. We now appeal to [20, Example 5.8], which examines Hadamard powers of the family of matrices $N(\varepsilon, x) := \mathbf{1}_{4\times 4} + xM(\varepsilon)$, where

$$\mathbf{1}_{4\times 4} := \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \end{pmatrix} \quad \text{and} \quad M(\varepsilon) := \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 2 & 3 \\ 0 & 2 & 4+\varepsilon & 6+\frac{5}{2}\varepsilon \\ 0 & 3 & 8 & 14+\varepsilon \end{pmatrix}. \tag{3.7}$$

As shown therein, the matrix $N(\varepsilon, x)$ is TN for all $\varepsilon \in (0, 1)$ and $x > 0$. Moreover, for small x and any $\alpha > 1$, the determinant of the Hadamard power

$$\det N(\varepsilon, x)^{\circ\alpha} = \varepsilon^2\alpha^3x^3 + \frac{1}{4}(8 - 70\varepsilon - 59\varepsilon^2 - 4\varepsilon^3)(\alpha^3 - \alpha^4)x^4 + O(x^5).$$

Thus $\det F[N(\varepsilon, x)] < 0$ for sufficiently small $\varepsilon = \varepsilon(\alpha) > 0$ and $x > 0$. We conclude that $F(x) = cx$ if $d = 4$. More generally, if F preserves TN on $d \times d$ matrices, where $d \geq 4$, then F also preserves TN on 4×4 matrices, and so $F(x) = cx$ for some $c > 0$, as desired. The converse is immediate. $\square$

Proposition 3.2 and Theorem 3.3 immediately combine to yield the following exact description of the set $\mathscr{F}^{\mathrm{TN}}_{X,Y}$.

Corollary 3.4. *Let X and Y be totally ordered sets. Then:*

(1) $\mathscr{F}^{\mathrm{TN}}_{X,Y} = \{F : \mathbb{R} \to \mathbb{R} \mid F(x) \geq 0$ *for all* $x \in \mathbb{R}\}$ *if* $\min\{|X|, |Y|\} = 1$.
(2) $\mathscr{F}^{\mathrm{TN}}_{X,Y} = \{c,\ c\mathbf{1}_{x>0},\ cx^{\alpha} : c \geq 0,\ \alpha > 0\}$ *if* $\min\{|X|, |Y|\} = 2$.
(3) $\mathscr{F}^{\mathrm{TN}}_{X,Y} = \{c,\ cx^{\alpha} : c \geq 0,\ \alpha \geq 1\}$ *if* $\min\{|X|, |Y|\} = 3$.
(4) $\mathscr{F}^{\mathrm{TN}}_{X,Y} = \{c,\ cx : c \geq 0\}$ *if* $\min\{|X|, |Y|\} \geq 4$ *or if X and Y are infinite.*

Remark 3.5. Given a positive integer $p \leq \min\{|X|, |Y|\}$, Corollary 3.4 immediately classifies the collection of all functions mapping the set of TN_p kernels on $X \times Y$ to itself. This is because any TN_p kernel on $[p] \times Y$ or $X \times [p]$ extends by "padding by zeros", as in the proof of Proposition 3.2, to a TN kernel on $X \times Y$.

3.1 Preservers of symmetric TN matrices and kernels.

Theorem 3.3 and Corollary 3.4 have a natural analogue for totally non-negative matrices and kernels which are symmetric. Note that any such matrix has non-negative principal minors and is therefore positive semidefinite.

Theorem 3.6. *Let $F : [0, \infty) \to \mathbb{R}$ and let d be a positive integer. The following are equivalent:*

(1) *F preserves total non-negativity entrywise on symmetric $d \times d$ matrices.*
(2) *F is either a non-negative constant or*
    (a) $(d = 1)$ $F(x) \geq 0$*;*
    (b) $(d = 2)$ *F is non-negative, non-decreasing, and multiplicatively mid-convex, that is,* $F(\sqrt{xy})^2 \leq F(x)F(y)$ *for all* $x, y \in [0, \infty)$*, so continuous on* $(0, \infty)$*;*
    (c) $(d = 3)$ $F(x) = cx^{\alpha}$ *for some* $c > 0$ *and some* $\alpha \geq 1$*;*
    (d) $(d = 4)$ $F(x) = cx^{\alpha}$ *for some* $c > 0$ *and some* $\alpha \in \{1\} \cup [2, \infty)$*;*
    (e) $(d \geq 5)$ $F(x) = cx$ *for some* $c > 0$.

The same characterizations hold for the preservers of symmetric TN *kernels on $X \times X$, where X is a totally ordered set of size d, which may now be infinite.*

Proof. The result is trivial when $d = 1$. When $d = 2$, a symmetric matrix is TN if and only if it is positive semidefinite, so part (b) follows immediately from [31, Theorem 2.5].

Now, suppose F preserves TN entrywise on symmetric 3×3 matrices and is non-constant. Considering matrices of the form $A \oplus \mathbf{0}_{1\times 1}$, it follows from part (b) that F is non-decreasing and continuous on $(0, \infty)$. Applying F entrywise to the matrix $x\,\mathrm{Id}_3$ for $x > 0$, where Id_3 is the 3×3 identity matrix, it follows easily that $F(0) = 0$. Next, let $L := \lim_{\varepsilon\to 0^+} F(\varepsilon)$, which exists since F is non-decreasing, and let C be the TN matrix in Equation (3.5). Then $0 \leq \lim_{\varepsilon\to 0^+} \det F[\varepsilon C] = -L^3$, whence $L = 0$. Thus F is continuous on $[0, \infty)$. Next, consider the symmetric totally non-negative matrices

$$A'(x,y) := \begin{pmatrix} x^2 & x & xy \\ x & 1 & y \\ xy & y & y^2 \end{pmatrix} \quad \text{and} \quad B'(x,y) := \begin{pmatrix} x^2y & xy & x \\ xy & y & 1 \\ x & 1 & 1/y \end{pmatrix} \qquad (x \geq 0, y > 0). \tag{3.8}$$

Note that $A'(x, y)$ contains the matrix $A(x, y)$ from Equation (3.2) as a submatrix, and the same is true for $B'(x, y)$ and $B(x, y)$. As in the proof of Theorem 3.3(a), it follows that

$$F(xy)F(1) = F(x)F(y) \quad \text{for all } x, y \geq 0.$$

Proceeding as there, and noting that the matrix C from Equation (3.5) is symmetric, we obtain $c > 0$ and $\alpha \geq 1$ such that $F(x) = cx^\alpha$. Moreover, each function of this form preserves TN entrywise, by (3.6). This concludes the proof of part (c).

To prove (d), we suppose the non-constant function F preserves TN on symmetric 4×4 matrices, and use part (c) with the usual embedding to obtain $c > 0$ and $\alpha \geq 1$ such that $F(x) = cx^\alpha$. To rule out $\alpha \in (1, 2)$, let $x \in (0, 1)$ and note that the infinite matrix $(1 + x^{i+j})_{i,j\geq 0}$ is the moment matrix of the two-point measure $\delta_1 + \delta_x$. Its leading principal 4×4 submatrix D is TN, by classical results in the theory of moments [26, 61], but if $\alpha \in (1, 2)$ then $D^{\circ\alpha}$ is not positive semidefinite, hence not TN, by [38, Theorem 1.1]. The converse follows from [20, Proposition 5.6]. This proves (d).

Finally, suppose F is non-constant and preserves TN on 5×5 symmetric matrices, and apply part (d) to obtain $c > 0$ and $\alpha \in \{1\} \cup [2, \infty)$ such that $F(x) = cx^\alpha$. To rule out the case $\alpha \geq 2$, we appeal to [20, Example 5.10], which studies the symmetric, totally non-negative matrices

$$T(x) := \mathbf{1}_{5\times 5} + x \begin{pmatrix} 2 & 3 & 6 & 14 & 36 \\ 3 & 6 & 14 & 36 & 98 \\ 6 & 14 & 36 & 98 & 276 \\ 14 & 36 & 98 & 284 & 842 \\ 36 & 98 & 276 & 842 & 2604 \end{pmatrix} \qquad (x > 0). \tag{3.9}$$

It is shown there that, for every $\alpha > 1$, there exists $\varepsilon = \varepsilon(\alpha) > 0$ such that the upper right 4×4 submatrix of $T(x)^{\circ\alpha}$ has negative determinant whenever $x \in (0, \varepsilon)$. It now follows that $F(x) = cx$ if $d = 5$. The general case, where $d \geq 5$, follows by the usual embedding trick, and the converse is once again immediate.

The final assertion is immediate, via padding by zeros. $\square$

We conclude this section with a characterization of symmetric TN_p preservers which is parallel to Remark 3.5.

Theorem 3.7. *Let $F : [0, \infty) \to \mathbb{R}$ and let d and p be positive integers, with $p < d$. The following are equivalent:*

(1) *F preserves TN_p entrywise on symmetric $d \times d$ matrices.*
(2) *F preserves TN_p entrywise on $d \times d$ matrices.*
(3) *F is either a non-negative constant or*
   (a) *($p = 1$) $F(x) \geq 0$;*
   (b) *($p = 2$) $F(x) = cx^\alpha$ for some $c > 0$ and some $\alpha \geq 0$;*
   (c) *($p = 3$) $F(x) = cx^\alpha$ for some $c > 0$ and some $\alpha \geq 1$;*
   (d) *($p \geq 4$) $F(x) = cx$ for some $c > 0$.*

The same functions characterize the preservers of symmetric TN_p kernels on $X \times X$, where X is a totally ordered set of size at least $p + 1$ (and possibly infinite).

In a sense, this result not immediately following from Theorem 3.6 is a manifestation of the fact that the definition of TN_p for a symmetric kernel differs from asking for every **principal** $r \times r$ minor being non-negative for $1 \leq r \leq p$.

Proof. That (2) $\implies$ (1) is immediate, while the equivalence (2) $\iff$ (3) follows from Theorem 3.3, since (2) is equivalent to preserving TN for $p \times p$ matrices. To see that (1) $\implies$ (3), it suffices to note that test matrices used to prove Theorems 3.3 and 3.6 occur as submatrices of $d \times d$ symmetric matrices which are TN_p.

This is immediate for $p = 1$, while for $p = 2$ the matrices in (3.2) and (3.4) embed as required, using (3.8) for the former and padding with zeros as necessary. Now working as in the proof of Theorem 3.3(b) gives that $F(x) = cx^\alpha$ with $c > 0$ and $\alpha \geq 0$.

Next, suppose $p = 3$. Then the $p = 2$ case, together with the matrix (3.5), implies as in the proof of Theorem 3.3(c) that $\alpha \geq 1$. Finally, if $p = 4$, then the matrices (3.9) imply as in the proof of Theorem 3.6(e) that $\alpha = 1$.

This concludes the proof for matrices, and the extension to kernels follows once again via padding by zeros. □

4 Total-positivity preservers. I. Semi-finite domains

We now turn to the more challenging problem of determining the functions which leave invariant the set of totally positive kernels,

$$\mathscr{F}^{\mathrm{TP}}_{X,Y} := \{F : (0, \infty) \to \mathbb{R} \mid \text{if } K : X \times Y \to \mathbb{R} \text{ is totally positive, so is } F \circ K\}.$$

There are two technical challenges one encounters once the underlying inequalities are strict. First, the embedding technique used to prove Theorem 3.3, which realizes totally non-negative $d \times d$ matrices as submatrices of totally non-negative $(d + 1) \times (d + 1)$ matrices, is lost. Second, the crucial property of multiplicative mid-point convexity is no longer available, since the matrices in (3.2) and (3.4) are not always totally positive. Following the approach of the previous section, we begin by indicating how these challenges can be addressed in the finite-dimensional case.

Theorem 4.1. *Let $F : (0, \infty) \to \mathbb{R}$ be a function and let $d := \min\{m, n\}$, where m and n are positive integers. The following are equivalent:*

(1) *F preserves total positivity entrywise on $m \times n$ matrices.*
(2) *F preserves total positivity entrywise on $d \times d$ matrices.*
(3) *The function F satisfies*
  (a) $(d = 1)$ $F(x) > 0$*;*
  (b) $(d = 2)$ $F(x) = cx^\alpha$ *for some* $c > 0$ *and some* $\alpha > 0$*;*
  (c) $(d = 3)$ $F(x) = cx^\alpha$ *for some* $c > 0$ *and some* $\alpha \geq 1$*;*
  (d) $(d \geq 4)$ $F(x) = cx$ *for some* $c > 0$.

In order to prove Theorem 4.1, we formulate two auxiliary results. We say that an $m \times n$ matrix (a_{ij}) occurs as a **submatrix** of a kernel K on $X \times Y$ if there exist $x_1 < \cdots < x_m$ and $y_1 < \cdots < y_n$ such that $a_{ij} = K(x_i, y_j)$ for all $i \in [m]$ and $j \in [n]$.

Lemma 4.2. *Fix integers $m \geq 2$ and $n \geq 2$. Every totally positive 2×2 matrix occurs as the leading principal submatrix of a positive multiple of a $m \times n$ generalized Vandermonde matrix, which is necessarily totally positive.*

In fact, any TP 2×2 matrix can be embedded at any specified location within a generalized Vandermonde matrix.[3] A stronger version of this result is given by Theorem 4.6 below.

Lemma 4.2 is an example of a **totally positive completion problem** [19]. Embedding results are known for arbitrary totally positive matrices, using, for example, the exterior-bordering technique discussed in [18, Chapter 9] or the parametrizations available in [11, 24]. Lemma 4.2 has the advantage of providing an explicit embedding into the well-known class of Vandermonde kernels, and is crucial to our final characterization results, found in the penultimate section of this paper.

The second result we require is a density theorem derived by A. M. Whitney in 1952, using generalized Vandermonde matrices and the Cauchy–Binet identity. The symmetric variant has the same proof as the version without this requirement.

[3] We thank Prakhar Gupta and Pranjal Warade for this observation.

Theorem 4.3 (Whitney, [66, Theorem 1]). *Given positive integers m, n, and p, the set of* TP_p $m \times n$ *matrices is dense in the set of* TN_p $m \times n$ *matrices. The same is true if both sets of matrices are taken to be symmetric.*

With these two observations to hand, we can now classify total-positivity preservers.

Proof of Theorem 4.1. That (3) $\implies$ (2) and (2) $\implies$ (1) are immediate, with the former using (3.6) when $d = 3$. We now prove that (1) $\implies$ (3). The case $d = 1$ is immediate, so we assume that $d \geq 2$. By Lemma 4.2, the map $F[-]$ preserves TP on 2×2 matrices. Considering the action of $F[-]$ on the matrices

$$\begin{pmatrix} y & x \\ x & x \end{pmatrix} \qquad (y > x > 0)$$

gives that F takes positive values and is increasing on $(0, \infty)$. Thus F is Borel measurable and continuous outside a countable set. Let $a > 0$ be a point of continuity and consider the totally positive matrices

$$A(x, y, \varepsilon) := \begin{pmatrix} ax & axy \\ a - \varepsilon & ay \end{pmatrix} \quad \text{and} \quad B(x, y, \varepsilon) := \begin{pmatrix} axy & ax \\ ay & a + \varepsilon \end{pmatrix}$$
$$(x, y > 0,\ 0 < \varepsilon < a).$$

Then

$$0 \leq \lim_{\varepsilon \to 0^+} \det F[A(x, y, \varepsilon)] = F(ax)F(ay) - F(axy)F(a)$$

and

$$0 \leq \lim_{\varepsilon \to 0^+} \det F[B(x, y, \varepsilon)] = F(a)F(axy) - F(ax)F(ay).$$

Hence, letting $G(x) := F(ax)/F(a)$, we have that

$$G(xy) = G(x)G(y) \quad \text{for all } x, y > 0.$$

Since G is measurable, classical results of Sierpińsky [62] and Banach [5] on the Cauchy functional equation imply there exists $\alpha \in \mathbb{R}$ such that $G(x) = x^\alpha$ for all $x > 0$. Thus if $c := F(a)a^{-\alpha} > 0$, then

$$F(x) = F(a)(x/a)^\alpha = cx^\alpha \quad \text{for all } x > 0.$$

As F is increasing, it holds that $\alpha > 0$. Hence $F(x) = cx^\alpha$ for some $c > 0$ and $\alpha > 0$. The result follows immediately if $d = 2$.

Finally, suppose $d \geq 3$. Since $F(x) = cx^\alpha$ for some $c > 0$ and $\alpha > 0$, it admits a continuous extension $\tilde{F}$ to $[0, \infty)$. By Theorem 4.3, we conclude that $\tilde{F}$ preserves TN entrywise on $m \times n$ matrices. Theorem 3.3 gives the form of $\tilde{F}$, and restricting to $(0, \infty)$ shows that F is as claimed. This proves that (1) $\implies$ (3), which completes the proof. □

The proof of Theorem 4.1 relies on Lemma 4.2; we will prove a stronger result presently. For now, we determine the set $\mathscr{F}^{\mathrm{TP}}_{X,Y}$ when at most one of X and Y is infinite.

Theorem 4.4. *Let X and Y be non-empty totally ordered sets. Then*

(a) $\mathscr{F}^{\mathrm{TP}}_{X,Y} = \{F : (0,\infty) \to (0,\infty)\}$ *if* $\min\{|X|, |Y|\} = 1$.
(b) $\mathscr{F}^{\mathrm{TP}}_{X,Y} = \{cx^\alpha : c > 0,\ \alpha > 0\}$ *if* $\min\{|X|, |Y|\} = 2$.
(c) $\mathscr{F}^{\mathrm{TP}}_{X,Y} = \{cx^\alpha : c > 0,\ \alpha \geq 1\}$ *if* $\min\{|X|, |Y|\} = 3$.
(d) $\mathscr{F}^{\mathrm{TP}}_{X,Y} = \{cx : c > 0\}$ *if* $4 \leq \min\{|X|, |Y|\} < \infty$.

Furthermore, if $p \in \mathbb{N}$ and both X and Y are of size at least p (and possibly infinite), then the functions preserving TP_p *kernels on $X \times Y$ are as above, with* $\min\{|X|, |Y|\}$ *replaced by p.*

When X and Y are both finite, Theorem 4.4 follows directly from Theorem 4.1. The case where one of X or Y is infinite is significantly more complicated. First, observe that for $X_0 \subseteq X$ and $Y_0 \subseteq Y$, a TP kernel on $X_0 \times Y_0$ cannot be extended to a TP kernel on $X \times Y$ simply by "padding by zeros", that is, by defining $K(x, y) = 0$ on the complement of $X_0 \times Y_0$. In the absence of a suitable extension result, we instead generalize Whitney's approximation theorem (Theorem 4.3) to arbitrary domains; see Theorem 1.2 above. The proof is obtained in Section 6 with the help of a form of discretized Gaussian convolution.

Moreover, contrary to the TN case of Proposition 3.2, the set $\mathscr{F}^{\mathrm{TP}}_{X,Y}$ does not only depend on whether X and Y are finite or not. For example, suppose X is of cardinality strictly larger than the continuum, and $|Y| \geq 2$. Choose distinct y_1 and y_2 in Y; since $|X| > |\mathbb{R}^2|$, by the pigeonhole principle it follows that $K|_{X \times \{y_1, y_2\}}$ contains a 2×2 submatrix with equal columns, which is therefore singular. This shows that K cannot be TP, and the "test set" in $\mathscr{F}^{\mathrm{TP}}_{X,Y}$ is, in fact, empty. Thus, the existence of a TP kernel on $X \times Y$ already imposes constraints on the sets X and Y.

The following result characterizes such sets, which must be order-isomorphic to subsets of the real line.

Lemma 4.5. *Suppose X and Y are non-empty totally ordered sets. The following are equivalent.*

(1) *There exists a totally positive kernel $K : X \times Y \to \mathbb{R}$.*
(2) *There exists a* TP_2 *kernel $K : X \times Y \to \mathbb{R}$.*
(3) *Either X or Y is a singleton, or there exist order-preserving injections from X and Y into* $(0, \infty)$.

The same equivalence holds if $X = Y$ and the kernels in (1) *and* (2) *are taken to be symmetric.*

Proof. If (3) holds and X or Y is a singleton, then the constant kernel $K \equiv 1$ shows that (1) holds. Otherwise, identify X and Y with subsets of $\mathbb{R}$ via order-preserving injections, and note that the restriction of K' from Example 2.3 is totally positive. Hence (3) $\implies$ (1). Clearly (1) $\implies$ (2), so it remains to show that (2) $\implies$ (3).

Suppose (2) holds, and neither X nor Y is a singleton. Fix $y_1 < y_2$ in Y; the TP_2 property of K implies that the ratio function

$$\varphi : X \to (0, \infty); \ x \mapsto K(x, y_2)/K(x, y_1)$$

is strictly increasing, so is an order-preserving injection. The same working applies with the roles of X and Y exchanged, and so (3) holds.

Finally, note that the same proof goes through verbatim if $X = Y$ and all kernels under consideration are required to be symmetric. □

Lemma 4.5 is useful not only in proving Theorem 4.4, but also for proving a stronger form of Lemma 4.2 that was promised above. A TP 2×2 matrix, which is necessarily proportional to one of generalized Vandermonde form, can be embedded in any position, not just in a TP matrix, but in a Vandermonde kernel on an essentially arbitrary domain.

Theorem 4.6. *Let A be a real 2×2 matrix. The following are equivalent:*

(1) *Given $\{i_1 < i_2\} \subseteq [m]$ and $\{j_1 < j_2\} \subseteq [n]$, where $m, n \geq 2$, there exists an $m \times n$ matrix $\widetilde{A}$, which is a positive multiple of a generalized Vandermonde matrix, such that $\widetilde{A}_{i_p, j_q} = a_{pq}$ for $p, q = 1, 2$.*
(2) *Given totally ordered sets X and Y, such that $X \times Y$ admits a* TP *kernel, and pairs $\{x_1 < x_2\} \subseteq X$ and $\{y_1 < y_2\} \subseteq Y$, there exists a* TP *kernel K on $X \times Y$ such that $K[(x_1, x_2); (y_1, y_2)] = A$.*
(3) *The matrix A is* TP.

This immediately implies Lemma 4.2, and so completes the proof of Theorem 4.1.

Proof. Clearly, (1) and (2) each imply (3). We will show that (3) $\implies$ (2); the construction used for this also shows that (3) $\implies$ (1). Furthermore, as X and Y both embed inside $\mathbb{R}$, by Lemma 4.5, henceforth we will consider X and Y to be subsets of $\mathbb{R}$.

We first show that an arbitrary TP 2×2 matrix A has the form $\lambda^{-1}(u_i^{\alpha_j})_{i,j=1}^2$, where the terms λ, u_1, and u_2 are positive, α_1 and α_2 are real, and either $u_1 < u_2$ and $\alpha_1 < \alpha_2$, or $u_1 > u_2$ and $\alpha_1 > \alpha_2$. The proof goes through various cases.

Suppose first that three entries of A are equal. Rescaling the matrix A, there are four cases to consider:

$$A_1 = \begin{pmatrix} x & 1 \\ 1 & 1 \end{pmatrix}, \quad A_2 = \begin{pmatrix} 1 & y \\ 1 & 1 \end{pmatrix}, \quad A_3 = \begin{pmatrix} 1 & 1 \\ y & 1 \end{pmatrix}, \quad \text{and} \quad A_4 = \begin{pmatrix} 1 & 1 \\ 1 & x \end{pmatrix},$$

where $x > 1$ and $0 < y < 1$. In the first case, the matrix A_1 equals $(u_i^{\alpha_j})$ where $u_1 = x$, $u_2 = 1$, $\alpha_1 = 1$, and $\alpha_2 = 0$. A similar construction can easily be obtained for A_2, A_3, and A_4.

Next, suppose two entries in a row or column of A are equal. There are again four cases:

$$A_5 = \begin{pmatrix} 1 & 1 \\ x & y \end{pmatrix}, \quad A_6 = \begin{pmatrix} y & x \\ 1 & 1 \end{pmatrix}, \quad A_7 = \begin{pmatrix} y & 1 \\ x & 1 \end{pmatrix}, \quad \text{and} \quad A_8 = \begin{pmatrix} 1 & x \\ 1 & y \end{pmatrix},$$

where $y > x > 0$ and $x, y \neq 1$. For A_5, we can take $u_1 = 1$, $u_2 = x$, $\alpha_1 = 1$, and $\alpha_2 = \log y / \log x$. If $u_1 < u_2$, then $\alpha_1 < \alpha_2$; similarly, when $u_1 > u_2$, we have that $\alpha_1 > \alpha_2$. Thus A_5 can be written as desired. The other cases are similar.

The remaining case is when

$$A := \begin{pmatrix} v & w \\ x & y \end{pmatrix} \qquad (v, w, x, y > 0, \ vy - wx > 0),$$

with $\{v, y\} \cap \{w, x\} = \emptyset$. We claim there exist $\lambda, u_1, u_2 > 0$, $\alpha_1 = 1$, and α_2 such that

$$\lambda \begin{pmatrix} v & w \\ x & y \end{pmatrix} = \begin{pmatrix} u_1 & u_1^{\alpha_2} \\ u_2 & u_2^{\alpha_2} \end{pmatrix},$$

and either $u_1 < u_2$ and $\alpha_1 < \alpha_2$, or $u_1 > u_2$ and $\alpha_1 > \alpha_2$. Applying the logarithm entrywise to both matrices and computing the determinants gives that

$$(L + V)(L + Y) = (L + W)(L + X),$$

where $L = \log \lambda$, $V = \log v$, $W = \log w$, $X = \log x$, and $Y = \log y$. This yields a linear equation in L, whence

$$\lambda = \exp\left(\frac{\log w \log x - \log v \log y}{\log(vy/wx)}\right).$$

Clearly, $u_1 = \lambda v$ and $u_2 = \lambda x$. Solving for α_2 explicitly, we obtain

$$\alpha_2 = \frac{\log(w/y)}{\log(v/x)}.$$

There are now two cases: if $u_1 < u_2$, then $v < x$, so $w/y < v/x < 1$ and $\alpha_2 > 1 = \alpha_1$. If, instead, $u_1 > u_2$, then $v/x > 1$ and $\alpha_2 < 1 = \alpha_1$.

Thus, for some $\lambda > 0$, the matrix λA is of the form $(\exp(\alpha_i\beta_j))_{i,j=1}^2$ with either $\alpha_1 < \alpha_2$ and $\beta_1 < \beta_2$, or $\alpha_1 > \alpha_2$ and $\beta_1 > \beta_2$. Furthermore, the latter case reduces to the former, since

$$\lambda A = (\exp(\alpha_i'\beta_j'))_{i,j=1}^2, \qquad \text{with } \alpha_i' = -\alpha_i \text{ and } \beta_j' = -\beta_j.$$

Thus A occurs as a submatrix of the scaled Vandermonde kernel

$$\mathbb{R} \times \mathbb{R} \to \mathbb{R};\ (x, y) \mapsto \lambda^{-1} \exp(xy).$$

To pass to a kernel on $X \times Y$, where X and Y are real sets, fix $x_1, x_2 \in X$ and $y_1, y_2 \in Y$, where $x_1 < x_2$ and $y_1 < y_2$, and let $\varphi_X : X \to \mathbb{R}$ and $\varphi_Y : Y \to \mathbb{R}$ be the unique linear maps such that $\varphi_X(x_i) := \alpha_i$ and $\varphi_Y(y_j) := \beta_j$ $(i, j = 1, 2)$. Then A occurs as the submatrix $K[(x_1, x_2); (y_1, y_2)]$ of the kernel

$$K : X \times Y \to \mathbb{R};\ (x, y) \mapsto \lambda^{-1} \exp(\varphi_X(x)\varphi_Y(y)). \qquad \square$$

Using these results, we can now classify the preservers of TP kernels on possibly infinite domains.

Proof of Theorem 4.4. We consider the two settings in a uniform manner: suppose $p \in \mathbb{N}$ and either (i) $p = \min\{|X|, |Y|\}$ and F preserves TP kernels on $X \times Y$, or (ii) X and Y both have size at least p and F preserves TP_p kernels on $X \times Y$.

If $p = 1$ then the result is immediate, so suppose $p \geq 2$. By Lemma 4.5, X and Y can be identified with subsets of $(0, \infty)$. Furthermore, by Lemma 4.2 and using suitable order-preserving maps, every TP 2×2 matrix can be realized as a submatrix of a TP kernel on $X \times Y$. Hence, by Theorem 4.1(3b), the function F has the form $F(x) = cx^\alpha$ for some $c > 0$ and $\alpha > 0$. Conversely, every such F is easily seen to satisfy (i) and (ii) above, which completes the case $p = 2$.

Otherwise, note first that F extends continuously to $[0, \infty)$. Let $d = \min\{p, 4\}$ and suppose $A = (a_{ij})_{i,j=1}^d$ is TN. Fix $\mathbf{x} \in X^{d,\uparrow}$ and $\mathbf{y} \in Y^{d,\uparrow}$, and let $\varepsilon > 0$ be such that $\min\{x_{i+1} - x_i,\ y_{i+1} - y_i : i \in [d-1]\} > 2\varepsilon$, where X and Y are identified with subsets of $\mathbb{R}$. Define

$$K : \mathbb{R} \times \mathbb{R} \to \mathbb{R};\ (x, y) \mapsto \begin{cases} a_{ij} & \text{if } |x - x_i| < \varepsilon \text{ and } |y - y_j| < \varepsilon \quad (i, j \in [d]), \\ 0 & \text{otherwise.} \end{cases}$$

Then K is TN_p and, by Theorem 1.2, there exists a sequence of TP_p kernels $(K_l)_{l\geq 1}$ converging to K at (x_i, y_j) for all $i, j \in [d]$. Hence $F \circ K_l$ is TP_p for all $l \geq 1$ and therefore $F[A]$ is TN. Since A was arbitrary, it follows that F preserves TN entrywise on $d \times d$ matrices. By Theorem 3.3, we see that F has the form claimed. The converse follows from Theorem 4.1(3c) and (3d). $\square$

The classification problems for preservers of TP kernels on $X \times Y$ is still to be resolved in the case when X and Y are both infinite, and the same is true when $X = Y$ and the kernels are required to be symmetric. As a first step in this direction, we show next that any such preserver must be a power function.

Proposition 4.7. *Suppose X and Y are totally ordered sets, each of size at least 2 and possibly infinite. If there exists a* TP *kernel on $X \times Y$, and $F : (0, \infty) \to (0, \infty)$ preserves all such kernels, or all* TP_2 *kernels on $X \times Y$, then $F(x) = cx^\alpha$ for some $c > 0$ and $\alpha > 0$. The same holds if $X = Y$ and the kernels are taken to be symmetric.*

Proof. By Theorem 4.6, any TP 2×2 matrix A has the form

$$\lambda^{-1}(\exp(\alpha_i \beta_j))_{i,j=1}^2,$$

where $\lambda > 0$, $\alpha_1 < \alpha_2$, and $\beta_1 < \beta_2$. Fix $x_1 < x_2$ in X and $y_1 < y_2$ in Y, considered as subsets of $\mathbb{R}$, and let $\alpha : X \to \mathbb{R}$ and $\beta : Y \to \mathbb{R}$ be order-preserving bijections such that $\alpha(x_i) = \alpha_i$ and $\beta(y_j) = \beta_j$ for $i, j = 1, 2$. Then the kernel

$$K : X \times Y \to \mathbb{R}; \quad (x, y) \mapsto \lambda^{-1} \exp(\alpha(x)\beta(y))$$

is totally positive and contains A as a submatrix; since $F \circ K$ is totally positive and A is arbitrary, it follows from Theorem 4.1 that F has the form claimed.

If $X = Y$ then we may arrange that $\alpha = \beta$, in which case K is symmetric. Thus, F has the same form as before. □

The full resolution of this classification question when X and Y are infinite is provided in Section 11. For now, we apply Proposition 4.7 to classify the TP preservers on Vandermonde matrices.

Corollary 4.8. *The functions which preserve the* TP *property of the scaled Vandermonde kernels*

$$\mathbb{R} \times \mathbb{R} \to \mathbb{R}; \quad (x, y) \mapsto \mu \exp(xy) \quad (\mu > 0)$$

*are precisely the power functions $F(x) = cx^\alpha$, where $c > 0$ and $\alpha > 0$. The same holds if "*TP*" is replaced by "*TP_2*".*

4.1 Preservers of symmetric TP matrices.

As in the totally non-negative case, Theorems 4.1 and 4.4 have analogues for symmetric matrices and kernels. The following result should be compared with Theorem 3.6.

Theorem 4.9. *Let $F : (0, \infty) \to \mathbb{R}$ and let d be a positive integer. The following are equivalent:*

1. *F preserves total positivity entrywise on symmetric $d \times d$ matrices.*
2. *The function F satisfies*
   (a) *($d = 1$) $F(x) > 0$;*
   (b) *($d = 2$) F is positive, increasing, and multiplicatively mid-convex, that is, $F(\sqrt{xy})^2 \leq F(x)F(y)$ for all $x, y \in (0, \infty)$, so continuous;*
   (c) *($d = 3$) $F(x) = cx^\alpha$ for some $c > 0$ and some $\alpha \geq 1$;*
   (d) *($d = 4$) $F(x) = cx^\alpha$ for some $c > 0$ and some $\alpha \in \{1\} \cup [2, \infty)$;*
   (e) *($d \geq 5$) $F(x) = cx$ for some $c > 0$.*

We now outline our proof strategy. Akin to Theorem 4.1, the idea is to derive the continuity of F from the 2×2 case, without the use of multiplicative mid-convexity, and then use the density of symmetric TP matrices in symmetric TN matrices. For the first step, we require the solution of a symmetric totally positive completion problem. The following result is analogous to Theorem 4.6.

Theorem 4.10. *Let the real 2×2 matrix A be symmetric. The following are equivalent.*

1. *For any $d \geq 2$ and any pair $\{k_1 < k_2\} \subseteq [d]$, there exists a* TP *Hankel $d \times d$ matrix $\widetilde{A}$ such that $\widetilde{A}_{k_p, k_q} = A_{pq}$ for $p, q = 1, 2$.*
2. *Given a totally ordered set X, such that $X \times X$ admits a* TP *kernel, and a pair $\{x_1 < x_2\} \subseteq X$, there exists a* TP *continuous Hankel kernel K on $X \times X$ such that $K[(x_1, x_2); (x_1, x_2)] = A$.*
3. *The matrix A is* TP.

Proof. Clearly (1) and (2) each imply (3), and (1) is a special case of (2). We will show that (3) implies (2).

The general form of a TP symmetric 2×2 matrix is

$$A = \begin{pmatrix} a & b \\ b & c \end{pmatrix} \qquad (a, b, c > 0,\ ac > b^2).$$

If $\alpha = \frac{1}{2}\log(ac/b^2)$ and $\beta = \frac{1}{2}\log(b^4/a^3c)$, then the continuous Hankel kernel

$$K : \mathbb{R} \times \mathbb{R} \to \mathbb{R};\ (x, y) \mapsto a\exp(\alpha(x + y)^2 + \beta(x + y)),$$

which is TP by Examples 2.3 and 2.4, contains A as the submatrix $K[(0, 1); (0, 1)]$. Working as in the final paragraph of the proof of Theorem 4.6 gives the general result. $\square$

With Theorem 4.10 at hand, we can classify the preservers of total positivity on the set of symmetric matrices.

Proof of Theorem 4.9. The result is clear when $d = 1$, so we assume $d \geq 2$ henceforth. First, suppose (1) holds. By Theorem 4.10, $F[-]$ must preserve total positivity for symmetric 2×2 matrices. As shown in the proof of Theorem 4.1, it follows that F is positive and increasing on $(0, \infty)$. In particular, F has countably many discontinuities, and each of these is a jump. Let $F^+(x) := \lim_{y \to x^+} F(y)$ for all $x > 0$. Then F^+ is increasing, coincides with F at every point where F is right continuous, and has the same jump as F at every point where F is not right continuous. Applying $F[-]$ to the totally positive matrices

$$M(x, y, \varepsilon) := \begin{pmatrix} x + \varepsilon & \sqrt{xy} + \varepsilon \\ \sqrt{xy} + \varepsilon & y + \varepsilon \end{pmatrix} \quad (x, y, \varepsilon > 0,\ x \neq y),$$

it follows that

$$F(\sqrt{xy} + \varepsilon)^2 < F(x + \varepsilon)F(y + \varepsilon).$$

Letting $\varepsilon \to 0^+$, we conclude that

$$F^+(\sqrt{xy})^2 \leq F^+(x)F^+(y) \quad \text{for all } x, y > 0;$$

this inequality holds trivially when $x = y$. Thus F^+ is multiplicatively mid-convex on $(0, \infty)$. As in the proof of Theorem 3.3, it follows by [52, Theorem 71.C] that F^+ is continuous. We conclude that F has no jumps and is therefore also continuous.

For $d = 2$, this completes the proof that (1) $\implies$ (2). If, instead, $d \geq 3$, note that F extends to a continuous function $\tilde{F}$ on $[0, \infty)$. As observed in [20, Theorem 2.6], the set of symmetric totally positive $r \times r$ matrices is dense in the set of symmetric totally non-negative $r \times r$ matrices. By continuity, it follows that $\tilde{F}$ preserves total non-negativity entrywise, and (2) now follows immediately from Theorem 3.6.

Conversely, suppose (2) holds for $d = 2$, and consider the totally positive matrix

$$A = \begin{pmatrix} a & b \\ b & c \end{pmatrix} \quad (a, b, c > 0,\ ac - b^2 > 0).$$

Since F is increasing, we have $F(\sqrt{b^2}) < F(\sqrt{ac})$. Using the multiplicative convexity of F, we conclude that

$$F(b)^2 = F(\sqrt{b^2})^2 < F(\sqrt{ac})^2 \leq F(a)F(c).$$

Thus $F[A]$ is totally positive and (1) holds. The implications for $d = 3$ and $d = 4$ follow from [20, Theorem 5.2 and Proposition 5.6], respectively, and the case of $d = 5$ is clear. $\square$

4.2 Preservers of symmetric TP kernels. The following result is the immediate reformulation of Theorem 4.9 to the setting of matricial kernels.

Corollary 4.11. *Let X be a totally ordered set of size $d \in \mathbb{N}$, and let $F : (0, \infty) \to \mathbb{R}$. Then $F \circ K$ is totally positive for any symmetric totally positive kernel $K : X \times X \to \mathbb{R}$ if and only if*

(a) *($d = 1$) $F(x) > 0$;*
(b) *($d = 2$) F is positive, increasing, and multiplicatively mid-convex, that is, $F(\sqrt{xy})^2 \leq F(x)F(y)$ for all $x, y \in (0, \infty)$, so continuous;*
(c) *($d = 3$) $F(x) = cx^\alpha$ for some $c > 0$ and some $\alpha \geq 1$;*
(d) *($d = 4$) $F(x) = cx^\alpha$ for some $c > 0$ and some $\alpha \in \{1\} \cup [2, \infty)$;*
(e) *($d \geq 5$) $F(x) = cx$ for some $c > 0$.*

Next, we formulate and prove a parallel result, in the spirit of the final assertion in Theorem 4.4.

Theorem 4.12. *Let $F : (0, \infty) \to \mathbb{R}$ and let p and d be positive integers, with $p < d$. The following are equivalent.*

(1) *F preserves TP_p entrywise on symmetric $d \times d$ matrices.*
(2) *F preserves TP_p entrywise on $d \times d$ matrices.*
(3) *F preserves TP entrywise on $p \times p$ matrices.*
(4) *The function F satisfies*
    (a) *($p = 1$) $F(x) > 0$;*
    (b) *($p = 2$) $F(x) = cx^\alpha$ for some $c > 0$ and some $\alpha > 0$;*
    (c) *($p = 3$) $F(x) = cx^\alpha$ for some $c > 0$ and some $\alpha \geq 1$;*
    (d) *($p \geq 4$) $F(x) = cx$ for some $c > 0$.*

The same functions characterize the preservers of symmetric TP_p kernels on $X \times X$, where X is a totally ordered set of size at least $p + 1$ (and possibly infinite).

Proof. By Theorem 4.4, the statements (2), (3) and (4) are equivalent, and clearly (2) $\implies$ (1). Now suppose (1) holds; we show (4) in several steps. If $p = 1$, then (4) is immediate, so we suppose henceforth that $p \geq 2$. Now, by Lemma 4.10, every symmetric 2×2 TP matrix admits an extension to a symmetric $d \times d$ TP matrix. Repeating the proof of Theorem 4.9, it follows that F is continuous. Hence F admits a continuous extension $\widetilde{F}$ to $[0, \infty)$, so, by the symmetric version of Whitney's Theorem 4.3, it follows that $\widetilde{F}$ preserves the class of symmetric TN_p $d \times d$ matrices. The claim now follows from Theorem 3.7 and the fact that F cannot be constant.

For the final assertion involving kernels, first note that if F is as in (4), then (2) holds, and so F preserves symmetric TP_p kernels on $X \times X$. Conversely, suppose F preserves the symmetric TP_p kernels on $X \times X$. Then the arguments used in the proof of Theorem 4.4 show that (1) $\implies$ (4), with Theorem 4.9 giving continuity of F when $p \geq 2$ and Theorem 3.7 used in place of Theorem 3.3. $\square$

5 Total-positivity preservers are continuous

As the vigilant reader will have noticed, we have shown above two similar assertions, that an entrywise map $F[-]$ preserves total non-negativity on the set of 2×2 symmetric matrices if and only if F is non-negative, non-decreasing, and multiplicatively mid-convex, with the corresponding changes if weak inequalities are replaced by strict ones. The variation lies in reducing the set of test matrices with which to work, while arriving at very similar conclusions.

Such a result was proved originally by Vasudeva [63], when classifying the entrywise preservers of positive semidefiniteness for 2×2 matrices with positive entries. To date, this remains the only known classification of positivity preservers in a fixed dimension greater than 1.

It is natural to seek a common strengthening of the results above, as well as of Vasudeva's result. We conclude by recording for completeness such a characterization, which uses a small test set of totally positive 2×2 matrices.

Notation 5.1. Let $\mathcal{P}$ denote the set of symmetric totally non-negative 2×2 matrices with positive entries, and let the subsets

$$\mathcal{P}' := \left\{ A(a,b) := \begin{pmatrix} a & b \\ b & a \end{pmatrix} : a > b > 0, \ a \text{ and } b \text{ not both irrational} \right\}$$

and

$$\mathcal{P}'' := \left\{ B(a,b,c) := \begin{pmatrix} a & b \\ b & c \end{pmatrix} : a, b, c > 0, \ ac > b^2, \ a, b \text{ and } c \text{ rational} \right\}.$$

Theorem 5.2. *Let $F : (0, \infty) \to \mathbb{R}$ be a function. The following are equivalent:*

(1) *The map $F[-]$ preserves total non-negativity on the set $\mathcal{P}$.*
(2) *The function F is non-negative, non-decreasing, and multiplicatively mid-convex on $(0, \infty)$.*
(3) *The map $F[-]$ preserves positive semidefiniteness on the set $\mathcal{P}' \cup \mathcal{P}''$.*

Moreover, every such function is continuous, and is either nowhere zero or identically zero.

The sets $\mathcal{P}'$ and $\mathcal{P}''$ are in bijection with the sets $(\mathbb{R} \times \mathbb{Q}) \cup (\mathbb{Q} \times \mathbb{R})$ and $\mathbb{Q}$, respectively, whereas $\mathcal{P}$ is a three-parameter family. The equivalence of (1) and (2) is Vasudeva's result.

Proof. To see that (2) $\implies$ (1), note that if $A = \left(\begin{smallmatrix} a & b \\ b & c \end{smallmatrix}\right) \in \mathcal{P}$, then $a, b, c > 0$ and $0 < b \leq \sqrt{ac}$. By (2), the matrix $F[A]$ has non-negative entries and

$$0 \leq F(b)^2 \leq F(\sqrt{ac})^2 \leq F(a)F(c),$$

so $F[A]$ is totally non-negative. Clearly, (1) $\implies$ (3). The main challenge in the proof is to show (3) $\implies$ (2). The first step is to observe that F is non-negative and non-decreasing on $(0, \infty)$. Let $y > x > 0$, choose rational a such that $x < a < y$, and consider the matrices $F[A(a, x)]$ and $F[A(y, a)]$, which are both positive semidefinite. From this, it follows that $F(y)$ is non-negative, and $F(y)^2 \geq F(a)^2 \geq F(x)^2$.

We now show that F is identically zero if it vanishes anywhere. Suppose $F(x) = 0$ for some $x > 0$. Then, as F is non-decreasing and non-negative, $F \equiv 0$ on $(0, x]$. Given $y > x > 0$, choose rational b and c such that $0 < c < x < y < b$. Considering $F[B(1 + (b^2/c), b, c)]$ and then $F[A(b, y)]$ shows that $F(b) = 0$ and then $F(y) = 0$. It follows that $F \equiv 0$.

Finally, we claim that F is multiplicatively mid-convex and continuous. Clearly this holds if $F \equiv 0$, so we assume that F is never zero. We first show that the function

$$F^+ : (0, \infty) \to [0, \infty); \quad F^+(x) := \lim_{y \to x^+} F(y)$$

is multiplicatively mid-convex and continuous. Note that F^+ is well defined because F is monotone. Given x, $y > 0$, we choose rational numbers $a_n \in (x, x + 1/n)$ and $c_n \in (y, y + 1/n)$ for each positive integer n. Since $a_n c_n > xy$, we may choose rational $b_n \in (\sqrt{xy}, \sqrt{a_n c_n})$. The matrix $B(a_n, b_n, c_n) \in \mathcal{P}''$ for each n, therefore

$$0 \leq \lim_{n \to \infty} \det F[B(a_n, b_n, c_n)] = F^+(x)F^+(y) - F^+(\sqrt{xy})^2.$$

Thus F^+ is multiplicatively mid-convex on $(0, \infty)$, and F^+ is non-decreasing since F is. Repeating the argument in the proof of Theorem 3.3, which requires the function to take positive values, it follows that F^+ is continuous. Hence F and F^+ are equal, and this gives the result. $\square$

Remark 5.3. The analogous version of Theorem 5.2 holds for any bounded domain, that is, for matrices with entries in $(0, \rho)$, with $\rho > 0$. The proof is a minimal modification of that given above, except for the argument to show that either $F \equiv 0$ or F vanishes nowhere. For this, see [31, Proposition 3.2(2)]. Also, it is clear that the set of rational numbers in the definitions of $\mathcal{P}'$ and $\mathcal{P}''$ may be replaced with any countable dense subset of the domain of F.

6 Extensions of Whitney's approximation theorem

The present section is devoted to constructive approximation schemes derived from discrete convolutions with the Gaussian kernel. The proof of Theorem 1.2 is obtained as an application.

6.1 Discretized Gaussian convolution.

Notation 6.1. For all $\kappa > 0$, let

$$G_\kappa : \mathbb{R} \times \mathbb{R} \to \mathbb{R}; \quad (x, y) \mapsto \exp(-\kappa(x - y)^2).$$

A key observation, going back at least to Pólya and Schoenberg, is the total positivity of this kernel. In our terminology, this means that G_κ is TP_p for all $p \in \mathbb{N}$, and this follows as a particular case of Example 2.4.

Proposition 6.2. *Let the kernel $K : A \times B \to \mathbb{R}$ be* TN_p*, where $A, B \subseteq \mathbb{R}$ and $p \in \mathbb{N}$. Suppose that $\kappa > 0$ and $n, N \in \mathbb{N}$ are greater than or equal to p. If $\mathbf{z} \in A^{n,\uparrow}$ and $\mathbf{w} \in B^{N,\uparrow}$, then*

$$T_{\kappa,\mathbf{z},\mathbf{w}}(K) : A \times B \to \mathbb{R}; \quad (x, y) \mapsto \sum_{j=1}^{n} \sum_{k=1}^{N} G_\kappa(x, z_j) K(z_j, w_k) G_\kappa(w_k, y)$$

is TN_p *and* $\mathrm{TP}_{\min\{p,r\}}$*, where r is the rank of* $K[\mathbf{z}; \mathbf{w}]$*, which is the same as the rank of* $T_{\kappa,\mathbf{z},\mathbf{w}}(K)[\mathbf{z}; \mathbf{w}]$.

Proof. Let $\mathbf{x} \in A^{m,\uparrow}$ and $\mathbf{y} \in B^{m,\uparrow}$, where $m \in [p]$. The Cauchy–Binet formula gives that

$$\det T_{\kappa,\mathbf{z},\mathbf{w}}(K)[\mathbf{x}; \mathbf{y}] = \sum_{\mathbf{j} \in [n]^{m,\uparrow}} \sum_{\mathbf{k} \in [N]^{m,\uparrow}} \det G_\kappa[\mathbf{x}; \mathbf{z_j}] \det K[\mathbf{z_j}; \mathbf{w_k}] \det G_\kappa[\mathbf{w_k}; \mathbf{y}], \tag{6.1}$$

where $\mathbf{z_j} := (z_{j_1}, \ldots, z_{j_m})$ if $\mathbf{j} = (j_1, \ldots, j_m)$ and similarly for $\mathbf{w_k}$.

Since $K[\mathbf{z}; \mathbf{w}]$ has rank r, it has a non-zero $r \times r$ minor, and so a non-zero minor of all smaller dimensions, but every strictly larger minor is zero. The result now follows. □

Notation 6.3. Given a vector $\mu \in \mathbb{R}^m$ and a positive-definite matrix $V \in \mathbb{R}^{m\times m}$, where m is a positive integer, the multivariate Gaussian probability density

$$f_{\mu,V} : \mathbb{R}^m \to [0,\infty);\ \mathbf{x} \mapsto \frac{(\det V)^{1/2}}{(2\pi)^{m/2}} e^{-\frac{1}{2}(\mathbf{x}-\mu)^T V(\mathbf{x}-\mu)}$$

has mean μ and inverse covariance matrix V. Note that

$$f_{\mu,V}(\mathbf{x}) = c_{\mu,V}\, g_{\mu,V}(\mathbf{x}) \quad \text{for all } \mathbf{x} \in \mathbb{R}^m,$$

where

$$c_{\mu,V} := (2\pi)^{-m/2}(\det V)^{1/2} e^{-\frac{1}{2}\mu^T V\mu} \quad \text{and} \quad g_{\mu,V}(\mathbf{x}) := e^{-\frac{1}{2}\mathbf{x}^T V\mathbf{x}+\mathbf{x}^T V\mu}.$$

For all $n \in \mathbb{N}$, let the $n \times n$ matrix Q be defined by setting $Q_1 = 1$ and

$$Q_{n+1} = Q_n \oplus 0_{1\times 1} + 0_{n-1\times n-1} \oplus \begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix},$$

so that

$$Q_n = \begin{pmatrix} 2 & -1 & 0 & \cdots & \\ -1 & 2 & -1 & & \\ 0 & -1 & 2 & -1 & \\ \vdots & & \ddots & 2 & -1 \\ & & & -1 & 1 \end{pmatrix}.$$

Lemma 6.4. *Let $\kappa > 0$. If $x_0, \dots, x_m \in \mathbb{R}$, then*

$$\prod_{j=1}^{m} G_\kappa(x_{j-1}, x_j) = e^{-\kappa x_0^2} g_{\mu,V}(x_1, \dots, x_m) = (\pi/\kappa)^{m/2} f_{\mu,V}(x_1, \dots, x_m), \tag{6.2}$$

where $\mu = x_0 \mathbf{1}_{m\times 1}$ and $V = 2\kappa Q_m$. Moreover, $\det V = (2\kappa)^m$ *and* $e^{-\kappa x_0^2} = e^{-\frac{1}{2}\mu^T V\mu}$.

Proof. Let $x_0, \dots, x_{m+1} \in \mathbb{R}$ be arbitrary. The first identity holds when $m = 1$, because

$$G_\kappa(x_0, x_1) = \exp(-\kappa x_0^2)\exp(-\kappa x_1^2 + 2\kappa x_0 x_1) = \exp(-\kappa x_0^2) g_{x_0, 2\kappa}(x_1).$$

Now suppose

$$G_\kappa(x_0, x_1) \cdots G_\kappa(x_{m-1}, x_m) = \exp(-\kappa x_0^2) g_{\mu,V}(x_1, \dots, x_m)$$

for some $\mu = (\mu_1, \dots, \mu_m) \in \mathbb{R}^m$ and $V \in \mathbb{R}^{m\times m}$. Then

$$\prod_{j=1}^{m+1} G_\kappa(x_{j-1}, x_j) = \exp(-\kappa x_0^2) g_{\mu,V}(x_1, \dots, x_m) G_\kappa(x_m, x_{m+1})$$

and, letting $\mathbf{x} := (x_1, \dots, x_{m+1})^T$ and $z \in \mathbb{R}$,

$$\begin{aligned} g_{\mu,V}(x_1,\dots,x_m)G_\kappa(x_m,x_{m+1}) \\ &= \exp\Big(-\frac{1}{2}\mathbf{x}^T(V\oplus 0)\mathbf{x} + \mathbf{x}^T(V\oplus 0)(\mu\oplus z) - \kappa(x_m - x_{m+1})^2\Big) \\ &= \exp\Big(-\frac{1}{2}\mathbf{x}^T V'\mathbf{x} + \mathbf{x}^T V'\mu'\Big), \end{aligned}$$

where

$$V' := V \oplus 0_{1\times 1} + 0_{m-1\times m-1} \oplus \begin{pmatrix} 2\kappa & -2\kappa \\ -2\kappa & 2\kappa \end{pmatrix} \quad \text{and} \quad \mu' = \mu \oplus \mu_m.$$

By induction, this gives the first identity. For the penultimate claim, note that adding the last row of V' to the penultimate row gives the matrix

$$V \oplus 0_{1\times 1} + 0_{m-1\times m-1} \oplus \begin{pmatrix} 0 & 0 \\ -2\kappa & 2\kappa \end{pmatrix},$$

which has determinant equal to 2κ times the determinant of V. The final identity is immediate, and the second identity in (6.2) now follows. $\square$

Notation 6.5. Given $z \in A \subseteq \mathbb{R}$ and $w \in B \subseteq \mathbb{R}$, let

$$\delta_{(z,w)} : A \times B \to \mathbb{R}; \quad (x,y) \mapsto \begin{cases} 1 & \text{if } x = z \text{ and } y = w, \\ 0 & \text{otherwise.} \end{cases}$$

Proposition 6.6. *With the notation and hypotheses of Proposition 6.2 and Notation 6.5, if $r < p$, then the kernel*

$$T_{\kappa,\mathbf{z},\mathbf{w}}(T_{\kappa,\mathbf{z},\mathbf{w}}(K) + e^{-\kappa}\delta_{(z_1,w_1)})$$

is TN_p *and* TP_{r+1}.

Proof. Let $K' := T_{\kappa,\mathbf{z},\mathbf{w}}(K) + e^{-\kappa}\delta_{(z_1,w_1)}$ and fix $m \in [p]$. If $\mathbf{j} \in [n]^{m,\uparrow}$ and $\mathbf{k} \in [N]^{m,\uparrow}$, then

$$K'[\mathbf{z_j};\mathbf{w_k}] = T_{\kappa,\mathbf{z},\mathbf{w}}(K)[\mathbf{z_j};\mathbf{w_k}] + e^{-\kappa}\delta_{(z_1,w_1)}(z_{j_1}, w_{k_1})E_{11},$$

where E_{11} is the $m \times m$ matrix with $(1,1)$ entry equal to 1 and 0 elsewhere. Hence

$$\begin{aligned} \det K'[\mathbf{z_j};\mathbf{w_k}] \\ &= \det T_{\kappa,\mathbf{z},\mathbf{w}}(K)[\mathbf{z_j};\mathbf{w_k}] + e^{-\kappa}\delta_{(z_1,w_1)}(z_{j_1}, w_{k_1}) \det T_{\kappa,\mathbf{z},\mathbf{w}}(K)[\mathbf{z_j}\setminus\{z_1\};\mathbf{w_k}\setminus\{w_1\}], \end{aligned}$$

where $\det T_{\kappa,\mathbf{z},\mathbf{w}}(K)[\emptyset,\emptyset] = 1$. This shows that K' is TN_p and that $K'[\mathbf{z_j};\mathbf{w_j}]$ has positive determinant if $\mathbf{j} = (1,\dots,r+1)$, since $T_{\kappa,\mathbf{z},\mathbf{w}}(K)$ is TP_r. Thus $K'[\mathbf{z};\mathbf{w}]$ has rank at least $r+1$, but since $K'[\mathbf{z};\mathbf{w}]$ is a rank-one perturbation of $T_{\kappa,\mathbf{z},\mathbf{w}}(K)[\mathbf{z};\mathbf{w}]$, its rank is exactly $r+1$. The result now follows from Proposition 6.2. $\square$

Corollary 6.7. *With the notation and hypotheses of Proposition 6.2 and Notation 6.5, if* $m = \max\{0, p-r\} + 1$*, then the kernel* $K^{(m)}_{\kappa,\mathbf{z},\mathbf{w}}$ *is* TP_p*, where*

$$K^{(1)}_{\kappa,\mathbf{z},\mathbf{w}} := T_{\kappa,\mathbf{z},\mathbf{w}}(K)$$
$$\text{and} \quad K^{(m)}_{\kappa,\mathbf{z},\mathbf{w}} := T^m_{\kappa,\mathbf{z},\mathbf{w}}(K) + e^{-\kappa} \sum_{j=1}^{m-1} T^j_{\kappa,\mathbf{z},\mathbf{w}}(\delta_{(z_1,w_1)}) \quad (m \geq 2).$$

6.2 Finite–continuum kernels. As a prélude to our main result, we establish the following. Recall that the **set of continuity** for a function is the set of points in its domain where it is continuous.

Theorem 6.8. *Let* $d,\ p \in \mathbb{N}$*, and suppose* $K : [d] \times \mathbb{R} \to \mathbb{R}$ *is bounded and* TN_p*. Then there exists a sequence of* TP_p *kernels* $(K_l)_{l\geq 1}$ *converging to* K *locally uniformly on its set of continuity.*

This theorem is an immediate consequence of the next result on discrete Gaussian convolution. For any $d, n \in \mathbb{N}$, let $\mathbf{d} := (1, \ldots, d)$ and

$$\mathbf{z}_n := (-n, -n + 2^{-n}, \ldots, n) \in [-n, n]^{N,\uparrow}, \quad \text{where } N = n2^{n+1} + 1.$$

Proposition 6.9. *Let* $K : [d] \times \mathbb{R} \to \mathbb{R}$ *be bounded and let* $K^{(m)}$ *be as in Corollary 6.7. Then*

$$2^{-mn}(n/\pi)^{m/2} K^{(m)}_{n,\mathbf{d},\mathbf{z}_n} \to K \quad \text{as } n \to \infty,$$

locally uniformly on the set of continuity for K*.*

Proof. Note first the elementary estimate

$$\|T_{\kappa,\mathbf{z},\mathbf{w}}(G)\|_\infty \leq dN \|G\|_\infty$$

for any kernel $G : A \times B \to \mathbb{R}$, where $\kappa > 0$, $\mathbf{z} \in A^{d,\uparrow}$, $\mathbf{w} \in B^{N,\uparrow}$, and $\|\cdot\|_\infty$ is the supremum norm on $A \times B$. Thus, if $j \in [m]$, then

$$\|T^j_{n,\mathbf{d},\mathbf{z}_n}(\delta_{(1,z_1)})\|_\infty \leq d^j N^j \leq d^m (4n)^m 2^{mn} = (4d)^m n^m 2^{mn},$$

hence

$$\begin{aligned}\left\| 2^{-mn}(n/\pi)^{m/2} e^{-n} \sum_{j=1}^{m-1} T^j_{n,\mathbf{d},\mathbf{z}_n}(\delta_{(1,z_1)}) \right\|_\infty &\leq 2^{-mn}(n/\pi)^{m/2} e^{-n} m (4d)^m n^m 2^{mn} \\ &= m(16d^2/\pi)^{m/2} n^{3m/2} e^{-n} \\ &\to 0 \qquad \text{as } n \to \infty.\end{aligned}$$

Next, let $(j_0, x_0) \in [d] \times \mathbb{R}$. Lemma 6.4 gives that

$$2^{-mn}(n/\pi)^{m/2} T^m_{n,\mathbf{d},\mathbf{z}_n}(K) = \sum_{j_1,\ldots,j_m \in [d]} I_{j_1,\ldots,j_m},$$

where

$$I_{j_1,\ldots,j_m}(j_0, x_0) := \exp(-n \sum_{k=1}^{m} (j_{k-1} - j_k)^2) \int_{[-n,n]^m} K(j_m, \varphi_n(x_m)) f_{x_0\mathbf{1},V}(\varphi_n(\mathbf{x}))\,\mathrm{d}\mathbf{x},$$

with $V = 2nQ_m$ and $\varphi_n(z) := \lfloor 2^n z \rfloor 2^{-n}$ if $z \in \mathbb{R}$ and $\varphi_n(\mathbf{z}) := (\varphi_n(z_1), \ldots, \varphi_n(z_m))$ if $\mathbf{z} \in \mathbb{R}^m$. Now let $\varepsilon_n(\mathbf{z}) := \mathbf{z} - \varphi_n(\mathbf{z}) \in [0, 2^{-n}]^m$ and note that

$$(\varphi_n(\mathbf{z}) - \mu)^T V (\varphi_n(\mathbf{z}) - \mu) = \mathbf{z}^T V \mathbf{z} - 2\varepsilon_n(\mathbf{z})^T V(\mathbf{z} - \mu) + \varepsilon_n(\mathbf{z})^T V \varepsilon_n(\mathbf{z}),$$

so

$$f_{\mu,V}(\varphi_n(\mathbf{z})) = f_{\mu,V}(\mathbf{z}) \exp(R_n(\mathbf{z}; \mu)),$$

where

$$R_n(\mathbf{z}; \mu) := \varepsilon_n(\mathbf{z})^T V(\mathbf{z} - \mu) - \frac{1}{2}\varepsilon_n(\mathbf{z})^T V \varepsilon_n(\mathbf{z}). \tag{6.3}$$

Note that, if $\mathbf{z} \in [-n, n]^m$, then

$$|R_n(\mathbf{z}; x_0\mathbf{1})| \leq C_n(x_0) := m2^{-n}(n + |x_0| + 2^{-n-1})(6m - 5)^{1/2}(2n)^m \to 0$$

as $n \to \infty$. Hence, if $(j_1, \ldots, j_m) \neq (j_0, \ldots, j_0)$, then

$$|I_{j_1,\ldots,j_m}| \leq e^{-n} \|K\|_\infty \exp(C_n(x_0)) \to 0 \quad \text{as } n \to \infty,$$

locally uniformly in x_0 on $\mathbb{R}$. Furthermore, if the random variable X has the probability density function $f_{x_0\mathbf{1},V}$, then $X \to x_0\mathbf{1}$ in distribution as $n \to \infty$, by Lévy's continuity theorem [13, p. 383] and the fact that $V^{-1} = (2n)^{-1}Q_m^{-1} \to 0$ as $n \to \infty$. Thus $X \to x_0\mathbf{1}$ in probability, and so

$$\begin{aligned}
&\left| I_{j_0,\ldots,j_0} - \int_{\mathbb{R}^m} K(j_0, \varphi_n(x_m)) f_{x_0\mathbf{1},V}(\mathbf{x})\,\mathrm{d}\mathbf{x} \right| \\
&\quad \leq \int_{[-n,n]^m} \|K\|_\infty f_{x_0\mathbf{1},V}(\mathbf{x})(\exp(C_n(x_0)) - 1)\,\mathrm{d}\mathbf{x} + \|K\|_\infty \mathbb{P}(\|X\|_\infty > n) \to 0
\end{aligned}$$

as $n \to \infty$, locally uniformly in x_0 on $\mathbb{R}$: if $n > |x_0| + 1/2$ then

$$\begin{aligned}
\mathbb{P}(\|X\|_\infty > n) &\leq \mathbb{P}(\|X\|_\infty > |x_0| + 1/2) \\
&\leq \mathbb{P}(\big| \|X\|_\infty - |x_0| \big| > 1/2) \\
&\leq \mathbb{P}(\|X - x_0\mathbf{1}\|_2 > 1/2),
\end{aligned}$$

since

$$\|X - x_0\mathbf{1}\|_2 \geq |X_j - x_0| \geq \big||X_j| - |x_0|\big| \quad \text{for any } j \in [m].$$

Finally, if $\varepsilon > 0$, then

$$\begin{aligned}
&\left| \int_{\mathbb{R}^m} K(j_0, \varphi_n(x_m)) f_{x_0\mathbf{1},V}(\mathbf{x}) \,\mathrm{d}\mathbf{x} - K(j_0, x_0) \right| \\
&\leq \int_{[x_0-\varepsilon, x_0+\varepsilon]^m} |K(j_0, \varphi_n(x_m) - K(j_0, x_0)| f_{x_0\mathbf{1},V}(\mathbf{x}) \,\mathrm{d}\mathbf{x} + 2\|K\|_\infty \\
&\qquad\qquad \times \mathbb{P}(\|X - x_0\mathbf{1}\|_\infty > \varepsilon) \\
&\leq \sup\{ |K(j_0, \varphi_n(x)) - K(j_0, x_0)| : |x - x_0| \leq \varepsilon\} + 2\|K\|_\infty \mathbb{P}(\|X - x_0\mathbf{1}\|_2 > \varepsilon).
\end{aligned}$$

This gives the result. □

6.3 Continuum–continuum kernels. Finally, we provide the technical heart of the proof of Theorem 1.2, which is the following modification of Proposition 6.9.

Theorem 6.10. *Let $K : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ be bounded and let $K^{(m)}$ be as in Corollary 6.7. Then*

$$4^{-mn}(n/\pi)^m K^{(m)}_{n,\mathbf{z}_n,\mathbf{z}_n} \to K \quad \text{as } n \to \infty,$$

locally uniformly on the set of continuity for K.

Proof. Working as in the proof of Proposition 6.9, note first that

$$\begin{aligned}
\left\| 4^{-mn}(n/\pi)^m e^{-n} \sum_{j=1}^{m-1} T^j_{n,\mathbf{z}_n,\mathbf{z}_n}(\delta_{(z_1,w_1)}) \right\|_\infty &\leq 4^{-mn}(n/\pi)^m e^{-n} m(4n)^{2m} 4^{mn} \\
&= m(16/\pi)^m n^{3m} e^{-n} \\
&\to 0 \quad \text{as } n \to \infty.
\end{aligned}$$

Next, fix $(x, y) \in \mathbb{R}^2$. Lemma 6.4 gives that

$$\begin{aligned}
J_n &:= 4^{-mn}(n/\pi)^m T^m_{n,\mathbf{z}_n,\mathbf{z}_n}(K)(x, y) \\
&= \int_{[-n,n]^m} \int_{[-n,n]^m} K(\varphi_n(x_m), \varphi_n(y_m)) f_{x\mathbf{1},V}(\varphi_n(\mathbf{x})) f_{y\mathbf{1},V}(\varphi_n(\mathbf{y})) \,\mathrm{d}\mathbf{x} \,\mathrm{d}\mathbf{y},
\end{aligned}$$

where $V = 2nQ_m$, $\varphi_n(z) := \lfloor 2^n z \rfloor 2^{-n}$ if $z \in \mathbb{R}$ and $\varphi_n(\mathbf{z}) := (\varphi_n(z_1), \ldots, \varphi_n(z_m))$ if $\mathbf{z} \in \mathbb{R}^m$. As above, let $\varepsilon_n(\mathbf{z}) := \mathbf{z} - \varphi_n(\mathbf{z}) \in [0, 2^{-n}]^m$ and note that

$$f_{\mu,V}(\varphi_n(\mathbf{z})) = f_{\mu,V}(\mathbf{z}) \exp(R_n(\mathbf{z}; \mu)),$$

where $R_n(\mathbf{z}; \mu)$ is as in (6.3). Hence

$$
\begin{aligned}
&\left| J_n - \int_{[-n,n]^m} \int_{[-n,n]^m} K(\varphi_n(x_m), \varphi_n(y_m)) f_{x\mathbf{1},V}(\mathbf{x}) f_{y\mathbf{1},V}(\mathbf{y}) \,\mathrm{d}\mathbf{x} \,\mathrm{d}\mathbf{y} \right| \\
&\quad \leq \|K\|_\infty \int_{[-n,n]^m} \int_{[-n,n]^m} |\exp(R_n(\mathbf{x}; x\mathbf{1}) + R_n(\mathbf{y}; y\mathbf{1})) - 1| f_{x\mathbf{1},V}(\mathbf{x}) f_{y\mathbf{1},V}(\mathbf{y}) \,\mathrm{d}\mathbf{x} \,\mathrm{d}\mathbf{y} \\
&\quad \leq \|K\|_\infty (\exp(m 2^{-n}(2n + |x| + |y| + 2^{-n})(6m-5)^{1/2}(2n)^m) - 1) \\
&\qquad \to 0 \quad \text{as } n \to \infty,
\end{aligned}
$$

locally uniformly in (x, y) on $\mathbb{R}^2$.

Next, let (X, Y) have probability density function $f_{x\mathbf{1},V} \times f_{y\mathbf{1},V}$ and note that $(X, Y) \to (x\mathbf{1}, y\mathbf{1})$ in probability. Thus,

$$
\begin{aligned}
&\int_{\mathbb{R}^m} \int_{\mathbb{R}^m} (1 - 1_{[-n,n]^m}(\mathbf{x}) 1_{[-n,n]^m}(\mathbf{y})) K(\varphi_n(x_m), \varphi_n(y_m)) f_{x\mathbf{1},V}(\mathbf{x}) f_{y\mathbf{1},V}(\mathbf{y}) \,\mathrm{d}\mathbf{x} \,\mathrm{d}\mathbf{y} \\
&\qquad\qquad \leq \|K\|_\infty (\mathbb{P}(\|X\|_\infty > n) + \mathbb{P}(\|Y\|_\infty > n)) \\
&\qquad\qquad\quad \to 0 \quad \text{as } n \to \infty,
\end{aligned}
$$

locally uniformly in (x, y) on $\mathbb{R}^2$. Finally, fix $\varepsilon > 0$ and note that

$$
\begin{aligned}
&\int_{\mathbb{R}^m} \int_{\mathbb{R}^m} |K(\varphi_n(x_m), \varphi_n(y_m)) - K(x, y)| f_{x\mathbf{1},V}(\mathbf{x}) f_{y\mathbf{1},V}(\mathbf{y}) \,\mathrm{d}\mathbf{x} \,\mathrm{d}\mathbf{y} \\
&\quad \leq \int_{[x-\varepsilon, x+\varepsilon]^m} \int_{[y-\varepsilon, y+\varepsilon]^m} |K(\varphi_n(x_m), \varphi_n(y_m)) - K(x, y)| f_{x\mathbf{1},V}(\mathbf{x}) f_{y\mathbf{1},V}(\mathbf{y}) \,\mathrm{d}\mathbf{x} \,\mathrm{d}\mathbf{y} \\
&\qquad + 2\|K\|_\infty (\mathbb{P}(\|X - x\mathbf{1}\|_\infty > \varepsilon) + \mathbb{P}(\|Y - y\mathbf{1}\|_\infty > \varepsilon)) \\
&\quad \leq \sup\{ |K(\varphi_n(\xi), \varphi_n(\eta)) - K(x, y)| : |\xi - x| \leq \varepsilon, \ |\eta - y| \leq \varepsilon \} \\
&\qquad + 2\|K\|_\infty (\mathbb{P}(\|X - x\mathbf{1}\|_2 > \varepsilon) + \mathbb{P}(\|Y - y\mathbf{1}\|_2 > \varepsilon)).
\end{aligned}
$$

The result follows. □

Remark 6.11 (Symmetric kernels)**.** If $K : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ is symmetric, then, since G_κ is symmetric as well,

$$
T_{\kappa,\mathbf{z},\mathbf{w}}(K)(x, y) = T_{\kappa,\mathbf{w},\mathbf{z}}(K)(y, x) \quad \text{for all } \kappa > 0, \ \mathbf{z} \in X^{n,\uparrow}, \ \mathbf{w} \in X^{N,\uparrow} \text{ and } x, y \in X.
$$

In particular, the map $T_{\kappa,\mathbf{z},\mathbf{z}}$ preserves symmetry. Thus, the kernels $K^{(m)}_{n,\mathbf{z}_n,\mathbf{z}_n}$ used in Proposition 6.10 are symmetric if K is.

We conclude as follows.

Proof of Theorem 1.2. We first extend K to a TN_p kernel $\widetilde{K}$ on $\mathbb{R} \times \mathbb{R}$ via padding by zeros. Proposition 6.10 now gives a sequence $(\widetilde{K}_l)_{l \geq 1}$ of TP_p kernels on $\mathbb{R} \times \mathbb{R}$ converging locally uniformly on the set of continuity of $\widetilde{K}$, which contains the points in the interior of $X \times Y$ where K is continuous. The result now follows by restricting each of these to $X \times Y$. The symmetric variant is proved in the same way, noting that if K is symmetric, then so are $\widetilde{K}$ and the kernels $\widetilde{K}_l$, by Remark 6.11.□

Remark 6.12. Propositions 7.4 and 8.9 provide further TP-density results, for TN Hankel kernels and Pólya frequency functions, respectively.

7 Totally non-negative and totally positive Hankel kernels

Having explored variations on our original theme, we now return to classification problems for total non-negativity and total positivity, now in the presence of additional structure. First, we consider Hankel matrices and kernels; in the following two sections, we examine the case of Toeplitz kernels.

7.1 Totally non-negative Hankel matrices. As noted in [9], the collection of TN Hankel matrices constitutes a test set that is closed under addition, multiplication by non-negative scalars, entrywise products, and pointwise limits. In particular, this test set, in each fixed dimension, is a closed convex cone. As the functions 1 and x preserve total non-negativity when applied entrywise, the same holds for any absolutely monotonic function $\sum_{k=0}^{\infty} c_k x^k$, where the Maclaurin coefficient $c_k \geq 0$ for all k. It is natural to ask if there are any other preservers. In [9], we show that, up to a possible discontinuity at the origin, there are no others.

Theorem 7.1 ([9])**.** *Given a function $F : [0, \infty) \to \mathbb{R}$, the following are equivalent.*

(1) *Applied entrywise, $F[-]$ preserves* TN *for Hankel matrices of all sizes.*
(2) *Applied entrywise, $F[-]$ preserves positivity for* TN *Hankel matrices of all sizes.*
(3) $F(x) = \sum_{k=0}^{\infty} c_k x^k$ *on* $(0, \infty)$ *with* $c_k \geq 0$ *for all k, and* $0 \leq F(0) \leq c_0$.

Theorem 7.1 thus completely resolves the problem of characterizing entrywise TN preservers on the set of Hankel matrices of all dimensions.

For the fixed-dimension context, we provide a brief summary of some recent progress. The following result provides a necessary condition, analogous to a result of Horn [36] for positivity preservers.

Theorem 7.2 ([9]). *Suppose $F : [0, \infty) \to \mathbb{R}$ is such that $F[-]$ preserves* TN *on the set of $d \times d$ Hankel matrices. Then F is $(d-3)$-times continuously differentiable, with $F, F', \dots, F^{(d-3)}$ non-negative on $(0, \infty)$, and $F^{(d-3)}$ is convex and non-decreasing. If, instead, F is analytic, then the first d non-zero Maclaurin coefficients of F are positive.*

Theorem 7.2 implies strong restrictions for the class of TN preservers of Hankel matrices. For instance, if one restricts to power functions x^α, the only such preservers in dimension d correspond to α being a non-negative integer or greater than $d-2$. The converse, that such functions preserve TN for $d \times d$ Hankel matrices, was shown in [20]. This is the same as the set of entrywise powers preserving positivity on $d \times d$ matrices, as proved by FitzGerald and Horn [22].

We conclude by noting that there exist power series which preserve total non-negativity on Hankel matrices of a fixed dimension and do not have all Maclaurin coefficients non-negative. The question of which of these coefficients can be negative was settled in [41]. Again, the characterization is the same as that for the class of positivity preservers, and this coincidence is explained by the following result of Khare and Tao.

Given $k, d \in \mathbb{N}$, with $k \leq d$ and a constant $\rho \in (0, \infty]$, we let $\mathcal{P}_d^k([0, \rho))$ denote the set of positive-semidefinite $d \times d$ matrices of rank at most k and with entries in $[0, \rho)$.

Theorem 7.3 ([41, Proposition 9.7]). *Suppose $F : [0, \rho) \to \mathbb{R}$ is such that the entrywise map $F[-]$ preserves positivity on $\mathcal{P}_d^k([0, \rho))$, where $k \leq d$ and $\rho \in (0, \infty]$. Then $F[-]$ preserves total non-negativity on the set of Hankel matrices in $\mathcal{P}_d^k([0, \rho))$.*

7.2 Hankel totally non-negative and totally positive kernels on infinite domains. We now consider the problem of classifying the preservers of TN and TP Hankel kernels on $X \times X$, where $X \subseteq \mathbb{R}$ is infinite. A Hankel kernel has the form

$$X \times X \to \mathbb{R}; \quad (x, y) \mapsto f(x+y)$$

for some function $f : X + X \to \mathbb{R}$, and so is automatically symmetric. Examples of such kernels abound; for example, given positive scalars $c_1, \dots, c_n$ and $u_1, \dots, u_n$, the kernel

$$\mathbb{R} \times \mathbb{R} \to \mathbb{R}; \quad (x, y) \mapsto \sum_{i=1}^{n} c_i u_i^{x+y}$$

is Hankel and TN on $\mathbb{R} \times \mathbb{R}$, as we will see below.

If X is an arbitrary subset of $\mathbb{R}$, then minors drawn from $X \times X$ may not embed in a larger Hankel matrix drawn from $X \times X$, since the arguments may be linearly independent over $\mathbb{Q}$. This issue is avoided by assuming that X is an interval and any kernel under consideration is a continuous function of its arguments.

Recall that the **Schur** or **pointwise** product of kernels K and K' with common domain $X \times X$ is the kernel

$$K \cdot K' : X \times X \to \mathbb{R}; \quad (x, y) \mapsto K(x, y)K'(x, y).$$

We equip the set of kernels on a given domain $X \times X$ with the topology of pointwise convergence. The following proposition summarizes some of the important properties of Hankel kernels. In particular, under appropriate assumptions, the sets of TN and TP kernels form convex cones that are closed under taking Schur products. See [20] for analogous results in the matrix case.

Proposition 7.4. *Suppose $X \subseteq \mathbb{R}$ is an interval.*

(1) *The space of* TN *continuous Hankel kernels on $X \times X$ is a closed convex cone, which is also closed under Schur products.*

(2) *Suppose X is an open interval and $K : X \times X \to \mathbb{R}$ is a continuous Hankel kernel. The following are equivalent.*
   (a) *K is* TN.
   (b) *K is positive semidefinite.*
   (c) *K is of the form*

$$X \times X \to \mathbb{R}; \quad (x, y) \mapsto \int_{\mathbb{R}} e^{(x+y)u} \, \mathrm{d}\sigma(u) \tag{7.1}$$

   for some non-decreasing function σ.

   Furthermore, K is TP *if and only if the measure corresponding to σ has infinite support.*

(3) *If X is an open interval, then the set of* TP *continuous Hankel kernels on $X \times X$ is dense in the set of* TN *continuous Hankel kernels on X.*

(4) *If X is an open interval, then the set of* TP *continuous Hankel kernels on $X \times X$ is a convex cone, which is closed under Schur products.*

The second part of Proposition 7.4 solves a Hamburger-type inverse problem for exponential moments of non-negative measures on $\mathbb{R}$. The third provides an extension of Whitney's theorem for Hankel kernels.

The proof of Proposition 7.4 uses several preliminary results. We begin with a well-known 1912 result of Fekete [21]. Recall that a minor is **contiguous** if it is formed from consecutive rows and columns.

Proposition 7.5. *Suppose m, $n \in \mathbb{N}$ and let A be an $m \times n$ matrix such that all its contiguous minors are positive. Then A is* TP.

From Proposition 7.5, we deduce the following corollary, which will be used below. Given a matrix A, we denote by $A^{(1)}$ the matrix obtained from A by deleting its first row and last column.

Corollary 7.6. *A square Hankel matrix A is* TP *if and only if A and $A^{(1)}$ are positive definite. A square Hankel matrix A is* TN *if and only if A and $A^{(1)}$ are positive semidefinite.*

Proof. The forward implication is immediate in both cases. For the converse, first suppose A and $A^{(1)}$ are positive definite. Note that any contiguous minor of A is a principal minor of either A or $A^{(1)}$, and so is positive, hence the claim follows by Proposition 7.5.

Finally, suppose A and $A^{(1)}$ are positive semidefinite. By the above observation, so is every contiguous square submatrix of A. Now let the matrix B be Hankel, TP and the same size as A; Example 2.4 provides the existence of such. Using the previous observation again, every contiguous square submatrix of B is positive definite. Hence for all $\varepsilon > 0$, every contiguous minor of $A + \varepsilon B$ is positive. It follows by Proposition 7.5 that $A + \varepsilon B$ is TP, whence A is TN, as desired. □

The final preliminary result is as follows.

Lemma 7.7. *Let $K : X \times X \to (0, \infty)$, where $X \subseteq \mathbb{R}$ is an interval. Each of the following statements implies the next.*

(1) *K is* TN.
(2) *All principal submatrices drawn from K are* TN.
(3) *All principal submatrices drawn from K with arguments in arithmetic progression are* TN.
(4) *All principal submatrices drawn from K with arguments in arithmetic progression are positive semidefinite.*

Conversely, (2) $\implies$ (1) for all K, (3) $\implies$ (2) if K is continuous, and (4) $\implies$ (3) *if K is continuous and Hankel.*

Proof. Clearly (1) $\implies$ (2) $\implies$ (3) $\implies$ (4).

If (2) holds, then, given $\mathbf{x}$, $\mathbf{z} \in X^{n,\uparrow}$ for some $n \in \mathbb{N}$, the matrix $K[\mathbf{x}; \mathbf{z}]$ is a submatrix of $K[\mathbf{x} \cup \mathbf{z}; \mathbf{x} \cup \mathbf{z}]$, where $\mathbf{x} \cup \mathbf{z}$ is obtained by taking the union of $\mathbf{x}$ and $\mathbf{z}$ in increasing order. Hence (1) holds.

Next, suppose (3) holds and K is continuous. Let $\mathbf{x} \in X^{n,\uparrow}$ for some $n \in \mathbb{N}$; by continuity, we may assume that each term $x_j \in \mathbf{x}$ is rational. Choose a positive integer N such that $N(x_j - x_1)$ is an integer for all j, and let

$$\mathbf{y} := (x_1, x_1 + N^{-1}, x_1 + 2N^{-1}, \dots, x_n).$$

By assumption, the matrix $K[\mathbf{y}; \mathbf{y}]$ is TN, thus so is the submatrix $K[\mathbf{x}; \mathbf{x}]$. This shows that (2) holds.

Finally, suppose (4) holds, and let a principal submatrix A be obtained by evaluating K at an arithmetic progression in X, say $x_1 < \cdots < x_n$. By assumption, A is positive semidefinite; furthermore, so is the $(n-1) \times (n-1)$ matrix B obtained by evaluating K at the arithmetic progression

$$\frac{x_1 + x_2}{2} < \frac{x_2 + x_3}{2} < \cdots < \frac{x_{n-1} + x_n}{2}.$$

But $B = A^{(1)}$, so (3) follows by Corollary 7.6. □

We now have the ingredients we require.

Proof of Proposition 7.4. Part (1) holds because property (4) of Lemma 7.7 is closed under addition, dilation, pointwise limits, and Schur products.

For part (2), note first that Lemma 7.7 gives the equivalence of (a) and (b). That positive semidefiniteness is necessary and sufficient for K to have the form (7.1) is a result of Bernstein [12] and Widder [67] which uses prior works of Hamburger and Mercer; see also [2, Theorem 5.5.4].

If the measure μ corresponding to σ has finite support, so may be written as $\sum_{k=1}^{r} c_k \delta_{u_k}$, and $\mathbf{x}, \mathbf{y} \in X^{n,\uparrow}$, then the submatrix

$$K[\mathbf{x}; \mathbf{y}] = \sum_{k=1}^{r} c_k (e^{(x_i+y_j)u_k})_{i,j=1}^{n} = \sum_{k=1}^{r} c_k \mathbf{z}_k \mathbf{w}_k^T, \tag{7.2}$$

where $\mathbf{z}_k := (e^{x_1 u_k}, \dots, e^{x_n u_k})^T$ and $\mathbf{w}_k := (e^{y_1 u_k}, \dots, e^{y_n u_k})^T$. Thus, submatrices of K have rank at most r, so K cannot be TP. Finally, if μ has infinite support, then the basic composition formula of Pólya and Szegő [40, p. 17] gives that

$$\det K[\mathbf{x}; \mathbf{y}] = \int_{\mathbb{R}^{m,\uparrow}} \det(\exp(x_i u_j))_{i,j=1}^{n} \det(\exp(u_j y_k))_{j,k=1}^{n} \, \mathrm{d}\sigma(u_1) \cdots \mathrm{d}\sigma(u_m)$$

for any $\mathbf{x}, \mathbf{y} \in X^{m,\uparrow}$, and so K is TP. This observation completes the proof of part (2).

For part (3), note that if K is a TN continuous Hankel kernel as in (2), then the continuous Hankel kernel

$$X \times X \to \mathbb{R}; \quad (x, y) \mapsto K(x, y) + \varepsilon \int_0^1 e^{(x+y)u} \, \mathrm{d}u$$

is TP for all $\varepsilon > 0$, since the measure corresponding to the representative function σ_ε has infinite support.

For the final part, note first that TP kernels are closed under positive rescaling. Furthermore, if the TP kernels K' and K'' have representative functions σ' and σ'', then the corresponding measures have infinite support, and therefore so does the measure corresponding to $\sigma' + \sigma''$. It follows that $K' + K''$ is TP.

Finally, to see that $K' \cdot K''$ is TP, note first that it is TN, so has a representative function τ. We assume the measure ν corresponding to τ has finite support, say of size r, and derive a contradiction. Suppose $\mathbf{x} \in X^{r+1,\uparrow}$ is an arithmetic progression, and consider the principal submatrices $M' = K'[\mathbf{x};\mathbf{x}]$ and $M'' = K''[\mathbf{x};\mathbf{x}]$. Both submatrices are TP by assumption, and Hankel by the choice of $\mathbf{x}$. Hence so is $M' \circ M''$, by Corollary 7.6 above and the Schur product theorem, so it must have rank $r+1$. But this contradicts the fact that ν has support of size r, by (7.2) with $K = K' \cdot K''$. ☐

Having gained a better understanding of our test set, we proceed to classify its preservers. As in the case of matrices of all sizes, the preservers of TN continuous Hankel kernels are absolutely monotonic functions.

Theorem 7.8. *Suppose $X \subseteq \mathbb{R}$ is an interval containing at least two points and let $F : [0,\infty) \to \mathbb{R}$. The following are equivalent.*

1. *The map C_F preserves* TN *for continuous Hankel kernels on $X \times X$.*
2. *The map C_F preserves positive semidefiniteness for* TN *continuous Hankel kernels on $X \times X$.*
3. *$F(x) = \sum_{k=0}^\infty c_k x^k$ on $(0,\infty)$, with $c_k \geq 0$ for all k, and $F(0) \geq 0$.*

The proof of this theorem uses the following observation about TN_2 Hankel kernels that vanish at a point. Recall that ∂X denotes the topological boundary of the set X; in particular, if $X \subseteq \mathbb{R}$ is an interval, then ∂X is the set of endpoints.

Lemma 7.9. *Suppose $X \subseteq \mathbb{R}$ is an interval and the kernel $K : X \times X \to \mathbb{R}$ is Hankel and* TN_2. *If $K(x,y) = 0$ for some point $(x,y) \in X \times X$, then K vanishes on $X \times X \setminus \{(x_0,x_0) : x_0 \in \partial X\}$. In particular, if K is also continuous, then $K \equiv 0$.*

Proof. Suppose K is as in the statement of the lemma, and X has interior (a,b) where $-\infty \leq a < b \leq \infty$. If $K(x,y) = 0$, then, since K is Hankel, $K(d_0,d_0) = 0$, where $d_0 := (x+y)/2$. By the Hankel property of K, it suffices to show $K(d,d) = 0$ for all $d \in X \setminus \partial X$. Now let $c \in (a,d_0)$; the positivity of $K[(c,d_0);(c,d_0)]$ gives that

$$0 \leq K(c,d_0)^2 \leq K(c,c)K(d_0,d_0) = 0,$$

so $K(c,d_0) = 0 = K((c+d_0)/2,(c+d_0)/2)$.

If $a = -\infty$, then this shows that $K(d, d) = 0$ for all $d \in (a, d_0)$. If, instead, $a > -\infty$, then this shows that $K(d, d) = 0$ for all $d \in ((a + d_0)/2, d_0)$.

We proceed inductively, assuming that $d_0 > a$ (otherwise there is nothing to prove). Let

$$d_n := (a + 3d_{n-1})/4 \in ((a + d_{n-1})/2, d_{n-1}) \quad (n \in \mathbb{N})$$

and note that $K(d_n, d_n) = 0$, so the previous working shows that $K(d, d) = 0$ for all $d \in ((a + d_n)/2, d_0)$. Since $d_n \to a$ as $n \to \infty$, we see that $K(d, d) = 0$ whenever $d \in (a, d_0)$.

A similar argument shows that $K(d, d)$ vanishes if $d \in (d_0, b)$. The extended result when K is continuous is immediate. □

Proof of Theorem 7.8. That (1) $\implies$ (2) is immediate. Next, we assume (3) and show (1), so suppose the continuous Hankel kernel $K : X \times X \to \mathbb{R}$ is TN. If K is never zero on $X \times X$, then $F \circ K$ is again TN, continuous, and Hankel, by Proposition 7.4(1). Otherwise K vanishes at a point, so Lemma 7.9 applies and $K \equiv 0$, but then $F \circ \mathbf{0}_{X \times X} = F(0)\mathbf{1}_{X \times X}$ is indeed TN, continuous, and Hankel.

Finally, to show (2) $\implies$ (3), we appeal to the following result.

Theorem 7.10 ([9, Theorem 4.2 and Remark 4.3])**.** *Fix $u_0 \in (0, 1)$ and suppose the function $F : (0, \infty) \to \mathbb{R}$ is such that $F[-]$ preserves positive semidefiniteness for 2×2 matrices of the form*

$$\begin{pmatrix} a & b \\ b & b \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} c^2 & cd \\ cd & d^2 \end{pmatrix} \quad (a, b, c, d > 0, \ a > b)$$

as well as for the matrices $(p + qu_0^{i+j})_{i,j=0}^n$ for all $p, q \geq 0$ with $p + q > 0$ and all $n \in \mathbb{N}$. Then F is smooth and $F^{(k)} \geq 0$ on $(0, \infty)$ for all $k \geq 0$.

A function F satisfying the hypotheses of this theorem is therefore absolutely monotonic on $(0, \infty)$, and so has a power-series representation there with non-negative Maclaurin coefficients.

Now suppose (2) holds. When $K = x\mathbf{1}_{X \times X}$, with $x \geq 0$, then $F \circ K$ being TN implies $F(x) \geq 0$. To apply Theorem 7.10, fix $n \in \mathbb{N}$ and choose points x_0, $x_n \in X$ with $x_0 < x_n$. Let $g : \mathbb{R} \to \mathbb{R}$ be the linear function such that $g(x_0) = 0$ and $g(x_n) = n$, and let $x_i = g^{-1}(i)$ for $i = 1, \ldots, n - 1$. Let $p, q \geq 0$ be such that $p + q > 0$. By assumption, the map C_F preserves positive semidefiniteness on the TN continuous Hankel kernel

$$K : X \times X \to \mathbb{R}; \quad (x, y) \mapsto p + qu_0^{g(x)+g(y)},$$

which contains $(p + qu_0^{i+j})_{i,j=0}^n$ as the principal submatrix

$$K[(x_0, \dots, x_n); (x_0, \dots, x_n)].$$

Similarly, given positive a, b, c, and d, with $a > b$, the TN continuous Hankel kernels

$$K' : X \times X \to \mathbb{R}; \quad (x, y) \mapsto \frac{(2a - b)^2}{4a - 3b}\Big(\frac{b}{2a - b}\Big)^{g(x)+g(y)} + \frac{b(a - b)}{4a - 3b} 2^{g(x)+g(y)}$$

and

$$K'' : X \times X \to \mathbb{R}; \; (x, y) \mapsto c^2(d/c)^{g(x)+g(y)}$$

have submatrices $K'[(x_0, x_1); (x_0, x_1)]$ and $K''[(x_0, x_1); (x_0, x_1)]$ which appear in the statement of Theorem 7.10. Thus $F[-]$ preserves TN on these matrices, so the hypotheses of Theorem 7.10 are satisfied. It follows that F is as claimed. □

To conclude this part, we classify the preservers of TP Hankel kernels.

Theorem 7.11. *Suppose $X \subseteq \mathbb{R}$ is an open interval and let $F : (0, \infty) \to \mathbb{R}$. The following are equivalent:*

(1) *The map C_F preserves* TP *continuous Hankel kernels on $X \times X$.*
(2) *The map C_F preserves positive definiteness for* TP *continuous Hankel kernels on $X \times X$.*
(3) *$F(x) = \sum_{k=0}^{\infty} c_k x^k$ on $(0, \infty)$, where $c_k \geq 0$ for all k, and F is non-constant.*

Proof. Clearly (1) $\implies$ (2). We now assume (3) and show (1). Suppose F is as specified and let $n_0 \in \mathbb{N}$ be such that $c_{n_0} > 0$. If $K : X \times X \to \mathbb{R}$ is a TP continuous Hankel kernel, then so is K^{n_0}, by Proposition 7.4(4). Let $G(x) := F(x) - c_{n_0} x^{n_0}$ and note that $G \circ K$ is TN, by Theorem 7.8. Now K^{n_0} and $G \circ K$ have integral representations as in Proposition 7.4(2), with corresponding measures μ and ν, respectively. Furthermore, the measure μ has infinite support, and therefore so does $c_{n_0}\mu + \nu$. Thus $F \circ K$ is TP.

Finally, suppose (2) holds. By Theorem 4.10, any TP symmetric 2×2 matrix occurs as a submatrix of a continuous Hankel TP kernel on $X \times X$. It follows from Theorem 4.9 that F is continuous on $(0, \infty)$. But then, by the density assertion in Proposition 7.4(3), the map C_F preserves the set of TN continuous Hankel kernels on $X \times X$. It now follows from Theorem 7.8 that F is a power series with non-negative Maclaurin coefficients, and F cannot be constant as then it cannot preserve positive definiteness. This shows (3). □

8 Pólya frequency functions and Toeplitz kernels

As the analysis in the previous sections shows, only small test sets of matrices and kernels are required to assure the rigidity of TN and TP endomorphisms, that is, to obtain Theorem 1.1. In this section, we explore another classical family of distinguished kernels, those associated to Pólya frequency functions. Such kernels are of central importance for time-frequency analysis and the theory of splines. The landmark contributions of Schoenberg, starting with his first full article on the subject [58], are highly recommended to the uninitiated reader. See also the monographs of Karlin [40] and Hirschman and Widder [35].

Definition 8.1. A **Pólya frequency function** is a function $\Lambda : \mathbb{R} \to [0, \infty)$ which is Lebesgue integrable, non-zero at two or more points and such that the Toeplitz kernel

$$T_\Lambda : \mathbb{R} \times \mathbb{R} \to \mathbb{R}; \quad (x, y) \mapsto \Lambda(x - y)$$

is totally non-negative.

This is a Toeplitz counterpart of the Hankel kernels encountered in Section 7. Even the condition that the kernel T_Λ is TN_2 is very restrictive. Indeed, this, measurability and the non-vanishing condition imply that

$$\Lambda(x) = \exp(-\phi(x)) \quad (x \in \mathbb{R}), \tag{8.1}$$

where the function ϕ is convex on an open interval, so continuous there, with possible discontinuities at the boundary and infinite values outside: see [58, Definition 3 and Lemma 1]. It also implies [58, Lemma 2] that Λ either decays exponentially at infinity, and so is integrable, or is monotone.

Schoenberg proved [58, Corollary 2] that the only discontinuous Pólya frequency functions are affine transforms $x \mapsto \lambda(ax + b)$, where $a, b \in \mathbb{R}$ and $a \neq 0$, of the map

$$\lambda : \mathbb{R} \to [0, \infty); \quad x \mapsto \begin{cases} 0 & (x < 0), \\ e^{-x} & (x \geq 0), \end{cases}$$

except possibly at the origin. In fact, one can alter the function λ to obtain

$$\lambda_d : \mathbb{R} \to [0, \infty); \quad x \mapsto \begin{cases} 0 & (x < 0), \\ d & (x = 0), \\ e^{-x} & (x > 0), \end{cases} \tag{8.2}$$

where $d \in [0, 1]$, without affecting the total non-negativity property. This may be proved directly in the same way as [35, Chapter IV, Lemma 7.1a]; see also [40, p. 16]. The requirement (8.1) ensures that this is the only possible variation on λ.

This class of kernels was linked by Pólya and Schur [49] to earlier studies pursued by Laguerre and devoted to coefficient operations which preserve polynomials with purely real roots. More specifically, convolution with such kernels maps polynomials to polynomials of the same degree and such a map possesses a series of striking root-location and root-counting properties. It should be no surprise, then, that the Fourier–Laplace transform of a Pólya frequency function is very special; see, for instance, [32]. The main theorems of Schoenberg [58] are the culmination of half a century of discoveries on this theme. To be precise, the bilateral Laplace transform $\mathcal{B}\{\Lambda\}$ of a Pólya frequency function Λ, given by

$$\mathcal{B}\{\Lambda\}(z) := \int_{\mathbb{R}} e^{-xz} \Lambda(x) \, \mathrm{d}x,$$

is an analytic function in the vertical strip $\{z \in \mathbb{C} : \alpha < \Re z < \beta\}$, where the bounds α and β can have infinite values and are such that

$$\alpha = \lim_{x \to \infty} \frac{\log \Lambda(x)}{x} < 0 \quad \text{and} \quad \beta = \lim_{x \to -\infty} \frac{\log \Lambda(x)}{x} > 0;$$

see [58, Lemma 10]. The characteristic feature of the bilateral Laplace transform of a Pólya frequency function is the structure of its reciprocal.

Theorem 8.2 ([58, Theorems 1 and 2]). *If Λ is a Pólya frequency function then the map $z \mapsto 1/\mathcal{B}\{\Lambda\}(z)$ is, up to an exponential factor, the restriction of an entire function of genus zero or one, with purely real zeros.*

Examples abound, and in general they are related to Hadamard factorizations of elementary transcendental functions [32, 58, 40]. For instance, e^{-x^2}, $e^{-|x|}$, $1/\cosh x$ and $e^{-x-e^{-x}}$ are all Pólya frequency functions.

8.1 Preservers of Pólya frequency functions.

In this subsection, we classify all composition transforms $\Lambda \mapsto F \circ \Lambda$ which leave invariant the class of Pólya frequency functions. The Gaussian kernel stands out, as the sole generator via affine changes of coordinates of a prominent family of test functions. We start by investigating this particular situation.

An immediate inspection of such transforms applied to the TP kernels

$$cG_\kappa : \mathbb{R} \times \mathbb{R} \to \mathbb{R}; \quad (x, y) \mapsto c \exp(-\kappa(x-y)^2) \quad (c > 0, \ \kappa > 0)$$

shows that we may expect a larger class of preservers than found in the rigid conclusions contained in our general theorems. Indeed, all maps of the form $c_0 x^\alpha$ for positive c_0 and α preserve TP on these kernels. However, more exotic preservers exist in this setting. As

$$K_\alpha : \mathbb{R} \times \mathbb{R} \to \mathbb{R}; \quad (x, y) \mapsto \exp(-\alpha|x-y|)$$

is also a Pólya frequency function for any $\alpha > 0$, it follows that

$$F : (0, \infty) \to \mathbb{R}; \quad t \mapsto \exp(-\sqrt{-\log \max\{t, 1\}})$$

is an admissible transformer of G_κ for all $\kappa > 0$. Such an analysis can be refined to consider the TN_p property, but we do not pursue this path here. For a recent characterization of Pólya frequency functions of order at most 3, see [65].

As an initial step, we obtain the following proposition.

Proposition 8.3. *Given a function $F : (0, \infty) \to \mathbb{R}$, each of the following statements implies the next.*

(1) *$F(x) = c_0 x^\alpha$ for some $c_0 > 0$ and $\alpha > 0$.*
(2) *$F \circ cG_\kappa$ is totally positive on $\mathbb{R} \times \mathbb{R}$ for all $c > 0$ and $\kappa > 0$.*
(3) *$F \circ cG_1$ is TP_3 on $\mathbb{R} \times \mathbb{R}$ for all $c > 0$.*
(4) *F is positive, increasing, and continuous on $(0, \infty)$.*

If (3′) $F \circ cG_1$ is TN_3 on $\mathbb{R} \times \mathbb{R}$ for all $c > 0$, then (4′) F is non-negative, non-decreasing, and continuous on $(0, \infty)$.

Proof. That (1) $\implies$ (2) and (2) $\implies$ (3) is immediate. We next assume (3) and show (4). Given $p, q > 0$ with $p < q$, let $x := \sqrt{\log(q/p)}$ and $\mathbf{y} := (0, x)$. Then the 2×2 matrix $F[qG_1[\mathbf{y}; \mathbf{y}]]$ has positive determinant and positive entries. This shows that F must be positive and increasing on $(0, \infty)$. In particular, the function F has at most countably many discontinuities.

Now we set $F^\pm(x) := \lim_{y \to x^\pm} F(y)$ for all $x > 0$. To complete the proof, we fix $p > 0$ and show that $F^+(p) = F(p) = F^-(p)$. To see this, choose $q > p$ such that F is continuous at q, and let $x := \sqrt{\log(q/p)}$ as before. Let $\mathbf{z} := (0, y, x)$ and $\mathbf{w} := (0, x, z)$ for $y, z > 0$ such that $y < x < z$, and consider the positive-definite matrices

$$A_y := F[qG_1[\mathbf{z}; \mathbf{z}]] = \begin{pmatrix} F(q) & F(qe^{-y^2}) & F(p) \\ F(qe^{-y^2}) & F(q) & F(qe^{-(x-y)^2}) \\ F(p) & F(qe^{-(x-y)^2}) & F(q) \end{pmatrix}$$

and

$$B_z := F[qG_1[\mathbf{w}; \mathbf{w}]] = \begin{pmatrix} F(q) & F(p) & F(qe^{-z^2}) \\ F(p) & F(q) & F(qe^{-(x-z)^2}) \\ F(qe^{-z^2}) & F(qe^{-(x-z)^2}) & F(q) \end{pmatrix}.$$

Note that

$$\lim_{y \to x^-} F(qe^{-y^2}) = F^-(p), \qquad \lim_{z \to x^+} F(qe^{-z^2}) = F^+(p),$$

and F is continuous at q. Hence

$$\lim_{y \to x^-} \det A_y = -F(q)(F^-(p) - F(p))^2, \qquad \lim_{z \to x^-} \det B_z = -F(q)(F^+(p) - F(p))^2.$$

Since both limits are non-negative, and $F(q) > 0$ from the previous working, it follows that $F^{+}(p) = F(p) = F^{-}(p)$, as required. Hence (3) $\implies$ (4).

To show (3′) $\implies$ (4′), we may repeat the argument above, assuming without loss of generality that F is non-constant, so that given $p > 0$, we may choose a continuity point $q > p$ with $F(q) > 0$. □

The next result shows that the square or higher integer powers do not preserve Pólya frequency functions.

Lemma 8.4. *There exists a Pólya frequency function M such that*

(1) *M is even, continuous and vanishes nowhere,*
(2) *M is increasing on $(-\infty, 0]$ and decreasing on $[0, \infty)$, and*
(3) *$M^n : x \mapsto M(x)^n$ is not a Pólya frequency function for any integer $n \geq 2$.*

Proof. We claim that the Laplace-type function $M(x) := 2e^{-|x|} - e^{-2|x|}$ has the desired properties. More generally, we provide a one-parameter family of functions, each of which is as required. Given a real number $\alpha > 0$, let

$$M_\alpha : \mathbb{R} \to (0, \infty); \ x \mapsto (\alpha + 1) \exp(-\alpha|x|) - \alpha \exp(-(\alpha + 1)|x|). \tag{8.3}$$

It is readily verified that $M = M_\alpha$ has properties (1) and (2). Furthermore, a short calculation shows that

$$\mathcal{B}\{M\}(s) = \frac{2\alpha(\alpha + 1)(2\alpha + 1)}{(s^2 - \alpha^2)(s^2 - (\alpha + 1)^2)}$$

on a neighborhood of 0. Hence $1/\mathcal{B}\{M\}(s)$ is a polynomial function with non-zero real roots and positive at the origin, and so M is a Pólya frequency function [58, Theorem 1].

We now analyze the Laplace transform of the higher integer powers of M. A second calculation reveals that

$$\mathcal{B}\{M^n\}(s) = 2\sum_{k=0}^{n} (-1)^{k+1} \binom{n}{k} \frac{\alpha^k(\alpha + 1)^{n-k}(n\alpha + k)}{s^2 - (n\alpha + k)^2} = \frac{p_n(s)}{q_n(s)} \quad (n \in \mathbb{N}),$$

where the polynomial $q_n(s) := \prod_{k=0}^{n}(s^2 - (n\alpha + k)^2)$ has simple roots and degree $2n + 2$, and the polynomial $p_n(s)$ has degree no more than $2n$.

We claim that, if $n > 1$, then p_n is non-constant and coprime to q_n. This implies that q_n/p_n is not an entire function, whence M^n is not a Pólya frequency function. To see this claim, note that

$$p_n(\pm(n\alpha + k)) = 2(-1)^{k+1} \binom{n}{k} \alpha^k(\alpha + 1)^{n-k}(n\alpha + k) \prod_{j \neq k}((n\alpha + k)^2 - (n\alpha + j)^2) \neq 0$$

for $k = 0, \dots, n$, and so

$$\frac{p_n(n\alpha)}{p_n(n\alpha+n)} = \prod_{j=1}^{n-1} \frac{(2n\alpha+j)(\alpha+1)}{(2n(\alpha+1)-j)\alpha}.$$

When $n > 1$, each factor in the final product is greater than 1. This shows the claim, and concludes the proof. $\square$

We now use this result to obtain the very small class of maps that preserve all Pólya frequency functions.

Theorem 8.5. *Let $F : [0,\infty) \to [0,\infty)$. If $F \circ \Lambda$ is a Pólya frequency function for every Pólya frequency function Λ, then $F(x) = cx$ for some $c > 0$.*

The converse is, of course, immediate.

Proof. As cG_1 is a Pólya frequency function for all $c > 0$, Proposition 8.3 implies that F is non-decreasing and continuous on $(0,\infty)$. Furthermore, since $F \circ \lambda$ is a Pólya frequency function, the integrability condition gives that $F(0) = 0$.

Since $F \circ \lambda$ is non-zero at least at two points, there exists $t_0 > 0$ with $F(t_0) > 0$. Thus $F \circ (t_0\lambda)$ has a point of discontinuity, as it has distinct left and right limits at the origin, and therefore

$$F(t_0\lambda(x)) = c_0 e^{-b_0 x} \quad \text{for all } x > 0,$$

where c_0 and b_0 are positive constants. Therefore

$$F(t) = c_0 t^{b_0} \quad \text{for all } t \in [0, t_0);$$

if $t_1 > t_0$, then, as F is non-decreasing, repeating this working shows the existence of positive constants c_1 and b_1 such that

$$F(t) = c_1 t^{b_1} \quad \text{for all } t \in [0, t_1).$$

It is readily seen that $b_1 = b_0$ and $c_1 = c_0$, and therefore $F(t) = c_0 t^{b_0}$ for all $t \geq 0$.

Next, since $\phi(x) = x\lambda(x)$ is also a Pólya frequency function [58, p. 343], it follows that $x^{b_0}\lambda(b_0 x)$ is a Pólya frequency function. The bilateral Laplace transform of this function is

$$\int_0^\infty e^{-xs} x^{b_0} e^{-b_0 x} \, \mathrm{d}x = \int_0^\infty e^{-x(s+b_0)} x^{b_0} \, \mathrm{d}x = \frac{\Gamma(b_0+1)}{(s+b_0)^{b_0+1}} \quad (s > -b_0).$$

The reciprocal $(s+b_0)^{b_0+1}$ admits an analytic continuation to an entire function, as required by [58, Theorem 1], only for integer values of b_0. Lemma 8.4 now gives the result. $\square$

To conclude, we provide a result that will be useful presently, as well as being notable in its own right: the classification of preservers of TN Toeplitz kernels. The following definition is a slight variation on [58, Definition 1] that is more convenient for our purposes.

Definition 8.6. A function $f : \mathbb{R} \to \mathbb{R}$ is **totally non-negative** (or TN) if it is Lebesgue measurable and the Toeplitz kernel

$$T_f : \mathbb{R} \times \mathbb{R} \to \mathbb{R}; \ (x, y) \mapsto f(x - y)$$

is TN.

Thus a Pólya frequency function is a TN function which is integrable and non-zero at two or more points.

Theorem 8.7. *Let $F : [0, \infty) \to [0, \infty)$ be non-zero. The following are equivalent:*

(1) *Given any* TN *function f that is non-zero at two or more points, the composition $F \circ f$ is* TN.
(2) *Given any Pólya frequency function Λ, the composition $F \circ \Lambda$ is* TN.
(3) *The function F is of the form $F(x) = c$, $F(x) = cx$, $F(x) = c\mathbf{1}_{x>0}$ or $F(x) = c\mathbf{1}_{x=0}$, for some $c > 0$.*

Similarly, the function F preserves TN *functions if and only if $F(x) = c$, $F(x) = cx$, or $F(x) = c\mathbf{1}_{x>0}$, for some $c > 0$.*

Proof. We recall first a result of Schoenberg [58, Lemma 4], that if f is a TN function which is non-zero at two or more points and not of the form $f(x) = \exp(ax + b)$, where $a, b \in \mathbb{R}$, then there exists $\gamma \in \mathbb{R}$ such that $x \mapsto e^{\gamma x} f(x)$ is a Pólya frequency function. (See also [40, Chapter 7, Proposition 1.3].)

Clearly (1) $\implies$ (2), and that (3) $\implies$ (1) is immediate for constant or linear maps of the form under consideration, so suppose $F(x) = c\mathbf{1}_{x=0}$ or $F(x) = c\mathbf{1}_{x>0}$ for some $c > 0$. By (8.1), the zero set of a TN function that is non-zero at two or more points can have one of the following forms:

$$\emptyset, \quad (-\infty, a\rangle, \quad (\infty, a\rangle \cup \langle b, \infty), \quad \text{or} \quad \langle b, \infty) \quad (a, b \in \mathbb{R}, \ a < b),$$

where the angle bracket indicates the intervals may be either open or closed. However, the third possibility is ruled out by the fact that Pólya frequency functions cannot have compact support [58, Corollary 1].

Thus $F \circ f$ is a non-negative constant or, up to positive scaling and translation of the argument, one of the following functions:

$$\mathbf{1}_{x\geq 0}, \quad \mathbf{1}_{x>0}, \quad \mathbf{1}_{x\leq 0}, \quad \text{or} \quad \mathbf{1}_{x<0}.$$

Since λ_1 and λ_0 are TN, so are the first two of these; the remaining two follow from this, because a kernel K on $\mathbb{R} \times \mathbb{R}$ is TN if and only if the "order-reversed" kernel $K' : (x, y) \mapsto K(-x, -y)$ is TN. Hence (3) $\implies$ (1).

Next, suppose F satisfies (2); we wish to show that (3) holds. It follows from Proposition 8.3 that F is non-negative, non-decreasing, and continuous on $(0, \infty)$. If F has the form $c\mathbf{1}_{x=0}$ or $c\mathbf{1}_{x>0}$ for some $c > 0$, then we are done, so we assume otherwise.

If F is constant on $(0, \infty)$, then the only remaining possibility is that it is non-zero there and also non-zero at 0. Thus, applying F to the Pólya frequency function λ_d, where $d \in [0, 1]$ is fixed for the remainder of the proof, we see that

$$0 \leq \det(F \circ T_{\lambda_d})[(-1, 1); (0, 2)] = \begin{vmatrix} F(0) & F(0) \\ F(1) & F(0) \end{vmatrix} = F(0)(F(0) - F(1))$$

whereas

$$0 \leq \det(F \circ T_{\lambda_d})[(0, 2); (-1, 1)] = \begin{vmatrix} F(1) & F(0) \\ F(1) & F(1) \end{vmatrix} = F(1)(F(1) - F(0)).$$

Hence F is a positive constant and (3) holds.

We may now suppose F is not constant on $(0, \infty)$. It follows by continuity that F is positive and not constant on an open interval (r, s), where $s > r > 0$. Now there are two cases to consider.

First, suppose $F(0) < F(t_0)$ for some $t_0 > 0$ and fix $t > \max\{s, t_0\}$. By assumption, there exists $\gamma_t \in \mathbb{R}$ such that

$$\Lambda_t(x) := e^{\gamma_t x} F(t\lambda_d(x))$$

is a Pólya frequency function or of the form e^{ax+b}. As $F(t) \geq F(t_0) > F(0)$, so Λ_t is discontinuous at 0, and therefore it cannot have the latter form. Moreover, $F(t\lambda_d(x))$ is positive on an open sub-interval of $(0, \infty)$. It follows that $\Lambda_t(x) = p_t \lambda_{d_t}(q_t x)$ for suitable constants $p_t, q_t > 0$ and $d_t \in [0, 1]$, so

$$F(0) = e^{\gamma_t} p_t \lambda_{d_t}(-q_t) = 0 \quad \text{and} \quad F(te^{-x}) = p_t e^{-(\gamma_t + q_t)x} \quad \text{for all } x > 0.$$

Since t can be taken to be arbitrarily large, a simple argument shows that $F(y) = cy^\alpha$ for all $y > 0$, where $c > 0$ and $\alpha > 0$ because F is non-constant and non-decreasing on $(0, \infty)$. Applying F to $\phi(x) = x\lambda(x)$ gives a Pólya frequency function, since $x^\alpha e^{-\alpha x}$ is positive and integrable on $(0, \infty)$. The proof of Theorem 8.5 now shows that $\alpha \in \mathbb{N}$. Furthermore, if M is as in Lemma 8.4, then $F(M) = c^\alpha M^\alpha$ is integrable and positive, so a Pólya frequency function. Thus $\alpha = 1$, as required.

The second and final case is when $F(0) \geq F(t)$ for all $t > 0$. Choose $t \in (r, s)$ such that $F(0) > F(t) > 0$ and F is positive and not constant on (r, t). As before, note that

$$\Lambda(x) := e^{\gamma x} F(t\lambda_d(x))$$

is a Pólya frequency function for some choice of γ; it cannot be of the form e^{ax+b}, since Λ is discontinuous at 0. This discontinuity, and the positivity of Λ on some sub-interval of $(0, \infty)$, means that $\Lambda(x) = p\lambda_d(qx)$ for some $d \in [0, 1]$ and constants $p, q > 0$. Then

$$F(0) = F(t\lambda_d(-q)) = e^{\gamma}\Lambda(-1) = e^{\gamma} p\lambda_d(-q) = 0 < F(t) < F(0),$$

a contradiction.

This shows the first set of equivalences. We now turn to the final assertion, beginning with the "only if" part. As (1) $\implies$ (3), we see that F is from one of four families, and it remains to rule out the function $F(x) = c\mathbf{1}_{x=0}$, where $c > 0$. This follows by applying F to itself, as F is readily seen to be TN, but $F \circ F = c - F$, which is not even TN_2.

Conversely, to show the "if" part, since (3) $\implies$ (1), it suffices to verify that $F \circ f$ is TN when $F(x) = \mathbf{1}_{x>0}$ and $f(x) = f_a(x) = a\mathbf{1}_{x=b}$ for any $a \geq 0$ and $b \in \mathbb{R}$. In this case, either $F \circ f_a = f_1$, when $a > 0$, or $F \circ f_0 = f_0$. This completes the proof. $\square$

Remark 8.8. The preceding proof shows that a non-zero function

$$F : [0, \infty) \to [0, \infty)$$

belongs to the classes of functions in (3) if it preserves TN for the following restricted set of test functions:

(i) the Gaussian functions $cG_1(x)$ for all $c > 0$,
(ii) the Pólya frequency functions $t\lambda_d(x)$ for one $d \in [0, 1]$ and all $t > 0$,
(iii) the Pólya frequency function $\phi(x) = x\lambda(x)$, and
(iv) the Pólya frequency function M from Lemma 8.4.

If F also preserves TN for a positive multiple of the function $\mathbf{1}_{x=0}$, then F cannot have this form itself.

8.2 Totally positive Pólya frequency functions. The rigidity of the above class of endomorphisms carries over to other, related problems. We begin by showing the same rigidity for the class of TP Pólya frequency functions, where we say that a Pólya frequency function Λ is TP whenever the associated

Toeplitz kernel T_Λ has that property. The precise description of conditions on the data $x_1 < x_2 < \cdots < x_n$ and $y_1 < y_2 < \cdots < y_n$ to that ensure that

$$\det(\Lambda(x_i - y_j))_{i,j=1}^n > 0$$

are contained in Schoenberg's third article [59]. As much as convolution with the Gaussian kernel was essential in the previous sections, it is also very useful in this new framework.

For any $\gamma > 0$, let

$$g_\gamma : \mathbb{R} \to \mathbb{R}; \quad x \mapsto \frac{1}{2\sqrt{\pi\gamma}} \exp(-x^2/4\gamma)$$

be the normalized Gaussian function, so that $\int_{\mathbb{R}} g_\gamma(t)\,\mathrm{d}t = 1$, and let Λ be an arbitrary Pólya frequency function. As the class of Pólya frequency functions is closed under convolution [58, Lemma 5], the convolution $g_\gamma * \Lambda$ is also a Pólya frequency function, with bilateral Laplace transform equal to the product of the transforms of g_γ and Λ. In view of [59, Theorem 1], the kernel

$$\mathbb{R} \times \mathbb{R} \to \mathbb{R}; \quad (x, y) \mapsto (g_\gamma * \Lambda)(x - y)$$

is TP.

A Pólya frequency function Λ is bounded and has left and right limits everywhere, so, for any $x_0 \in \mathbb{R}$,

$$\begin{aligned}\lim_{\gamma\to 0^+} (g_\gamma * \Lambda)(x_0) &= \int_{\mathbb{R}} g_1(t)\Lambda(x_0 - t\sqrt{\gamma})\,\mathrm{d}t \\ &= \frac{1}{2}\Big(\lim_{y\to x_0^+} \Lambda(y) + \lim_{y\to x_0^-} \Lambda(y)\Big).\end{aligned}$$

We say that a Pólya frequency function is **regular** if it is equal to the arithmetic mean of its left and right limits at every point.

Putting together these observations, we obtain the following result.

Proposition 8.9. *Let Λ be a regular Pólya frequency function. There exists a sequence of* TP *Pólya frequency functions $(\Lambda_n)_{n\geq 1}$ such that*

$$\lim_{n\to\infty} \Lambda_n(x) = \Lambda(x)$$

for every $x \in \mathbb{R}$.

As an application, we complement Theorems 8.5 and 8.7 above, by considering TP kernels. We say that a kernel K on $\mathbb{R} \times \mathbb{R}$ is **measurable** if K is a Lebesgue-measurable function.

Theorem 8.10. *Given a function $F : (0,\infty) \to (0,\infty)$, the following are equivalent:*

1. *$F \circ K$ is a* TP *Toeplitz kernel on $\mathbb{R} \times \mathbb{R}$ whenever K is.*
2. *$F \circ K$ is a* TP *measurable Toeplitz kernel on $\mathbb{R} \times \mathbb{R}$ whenever K is.*
3. *$F \circ \Lambda$ is a* TP *Pólya frequency function whenever Λ is.*
4. *$F \circ \Lambda$ is a* TP *Pólya frequency function whenever Λ is a regular* TP *Pólya frequency function.*
5. *$F(x) = cx$, where $c > 0$.*

Proof. It is clear that (1), (2) and (3) both imply (4) and are implied by (5). Thus it remains to show (4) $\implies$ (5).

Let F satisfy (4). Then F is positive, increasing, and continuous on $(0,\infty)$, by Proposition 8.3, so F extends to a continuous, increasing function

$$\widetilde{F} : [0,\infty) \to [0,\infty).$$

Now suppose Λ is one of the regular Pólya frequency functions listed in Remark 8.8, and note that this includes $\lambda_{1/2}$. By Proposition 8.9, there exists a sequence $(\Lambda_n)_{n\geq 1}$ of TP Pólya frequency functions such that $\Lambda_n \to \Lambda$ pointwise. Hence $F \circ \Lambda_n$ gives rise to a TP kernel for each $n \geq 1$, and so $\widetilde{F} \circ \Lambda$ is TN. By Remark 8.8, the restriction F of $\widetilde{F}$ is constant or linear, and the former is impossible. This completes the proof. $\square$

9 Pólya frequency sequences

We continue our study of total non-negativity preservers with an exploration of the class of Pólya frequency sequences. A **Pólya frequency sequence** is a bi-infinite sequence of real numbers $\mathbf{a} = (a_n)_{n\in\mathbb{Z}}$ such that the Toeplitz kernel

$$T_{\mathbf{a}} : \mathbb{Z} \times \mathbb{Z} \to \mathbb{R};\ (i,j) \mapsto a_{i-j}$$

is TN. Recall in this context the groundbreaking body of work by Aissen, Edrei, Schoenberg, and Whitney (see [1] and the monograph by Karlin [40]). These sequences are characterized in terms of negative real-rootedness of the associated generating polynomial when most terms a_n are zero, or a product expansion when all negatively indexed terms a_n vanish. More recently, Pólya frequency sequences have found numerous applications in combinatorics, owing to their connections to log concavity. See the works of Brenti [15, 16] and subsequent papers.

Pólya frequency sequences turn out to be as rigid as Pólya frequency functions are, as far as their endomorphisms are concerned, with their preservers being dilations or constants. In order to demonstrate this fact, we first introduce Toeplitz kernels on a more general class of domains than $\mathbb{R} \times \mathbb{R}$.

Definition 9.1. We say that a pair of subsets $X \subseteq \mathbb{R}$ and $Y \subseteq \mathbb{R}$ is **admissible** if, for each integer $n \geq 2$, the sets contain n-step arithmetic progressions $\mathbf{x} \in X^{n,\uparrow}$ and $\mathbf{y} \in Y^{n,\uparrow}$ that are **equi-spaced**, so that their terms have the same increments:

$$x_{j+1} - x_j = y_{j+1} - y_j = x_2 - x_1 \quad \text{for all } j \in [n-1].$$

We let

$$X - Y := \{x - y : x \in X,\ y \in Y\}$$

and say that a kernel $K : X \times Y \to \mathbb{R}$ is *Toeplitz* if there exists a function $f : X - Y \to \mathbb{R}$ such that $K(x, y) = f(x - y)$ for all $x \in X$ and $y \in Y$.

The following theorem is a variant on Theorem 8.7 for this new setting.

Theorem 9.2. *Suppose that X and Y are a pair of admissible sets. If $F : [0, \infty) \to [0, \infty)$ is non-zero and preserves the* TN *Toeplitz kernels on $X \times Y$, then either $F(x) = c$, $F(x) = cx$, or $F(x) = c\mathbf{1}_{x>0}$, for some $c > 0$.*

The converse to this theorem does not necessarily hold for a given admissible pair. For example, if $X = Y = \mathbb{R}$, then Theorem 8.7 shows the converse, at least for the case of measurable kernels, but for $X = Y = \mathbb{Z}$ we will see below that $c\mathbf{1}_{x>0}$ is not a preserver of Pólya frequency sequences.

Proof. For ease of exposition, we split the proof into several steps.

Step 1: The function F is non-decreasing on $(0, \infty)$. Fix equi-spaced arithmetic progressions $\mathbf{x} \in X^{2,\uparrow}$ and $\mathbf{y} \in Y^{2,\uparrow}$, so that $x_2 - x_1 = y_2 - y_1$. Let p and q be positive real numbers, with $p < q$, and consider the kernel

$$K : X \times Y \to \mathbb{R};\ (x, y) \mapsto qG_1\Big(\sqrt{\log(q/p)}\frac{(x - x_1) - (y - y_1)}{x_2 - x_1}\Big).$$

This is a Toeplitz kernel which is TP since Gaussian kernels are, so $F \circ K$ is TN by assumption. Thus

$$0 \leq \det(F \circ K)[(x_1, x_2); (y_1, y_2)] = \begin{vmatrix} F(q) & F(p) \\ F(p) & F(q) \end{vmatrix} = F(q)^2 - F(p)^2.$$

As p and q are arbitrary, the claim follows. Furthermore, the function F has at most countably many discontinuities, so is Borel measurable, and the left-limit and right-limit functions

$$F^+ : (0, \infty) \to [0, \infty); \quad x \mapsto \lim_{y \to x^+} F(y),$$
$$F^- : (0, \infty) \to [0, \infty); \quad x \mapsto \lim_{y \to x^-} F(y)$$

are well defined.

Step 2: The function F is continuous on $(0, \infty)$. We will show that, for any $p > 0$, $F^+(p) = F(p) = F^-(p)$. This is trivial if $F \equiv 0$ on $(0, \infty)$, so we assume otherwise. Fix a point $q > p$ where F is continuous and $F(q) > 0$, choose an integer $n \geq 3$, and let $\mathbf{x} \in X^{n,\uparrow}$ and $\mathbf{y} \in Y^{n,\uparrow}$ be equi-spaced arithmetic progressions. As before, the kernel

$$K : X \times Y \to \mathbb{R}; \quad (x, y) \mapsto qG_1\Big(\sqrt{\log(q/p)}\frac{(x - x_1) - (y - y_1)}{(x_2 - x_1)(n - 2)}\Big)$$

is Toeplitz and TP. Furthermore, a straightforward computation shows that

$$K(x_i, y_j) = q(p/q)^{(i-j)^2/(n-2)^2} \quad \text{for all } i, j \in [n].$$

Since $F \circ K$ is TN, we have that

$$0 \leq \lim_{n\to\infty} \det(F \circ K)[(x_1, x_{n-2}, x_{n-1}); (y_1, y_{n-2}, y_{n-1})] \to \begin{vmatrix} F(q) & F^-(p) & F(p) \\ F^-(p) & F(q) & F(q) \\ F(p) & F(q) & F(q) \end{vmatrix}$$

$$= -F(q)(F(p) - F^-(p))^2,$$

and so $F(p) = F^-(p)$. Similarly,

$$0 \leq \lim_{n\to\infty} \det(F \circ K)[(x_1, x_{n-1}, x_n); (y_1, y_{n-1}, y_n)] = -F(q)(F(p) - F^+(p))^2.$$

This establishes the second claim.

Step 3: The function F belongs to one of the four families of functions in Theorem 8.7. Suppose not, and note that Remark 8.8 gives a Pólya frequency function Λ such that $F \circ \Lambda$ is not TN. As F is Borel measurable, the function $F \circ \Lambda$ is Lebesgue measurable and therefore $F \circ T_\Lambda$ is not TN; furthermore, Λ is continuous except possibly at the origin, and is either positive everywhere, or zero on $(-\infty, 0)$ and positive on $(0, \infty)$. It follows that $F \circ \Lambda$ is continuous except possibly at the origin. By Lemma 7.7, there exists $\mathbf{z} \in \mathbb{R}^{n,\uparrow}$, where $n \geq 2$, such that the principal submatrix $(F \circ T_\Lambda)[\mathbf{z}; \mathbf{z}]$ has at least one minor which is negative. As $F \circ \Lambda$ is continuous except at possibly the origin, we may assume that $\mathbf{z} = (z_1, \ldots, z_n) \in \mathbb{Q}^{n,\uparrow}$.

Choose $N \in \mathbb{N}$ sufficiently large so that Nz_i is an integer for every $i \in [n]$, and let $m = N(z_n - z_1) + 1$. Let $\mathbf{x} \in X^{m,\uparrow}$ and $\mathbf{y} \in Y^{m,\uparrow}$ be equi-spaced arithmetic progressions, and let

$$K : X \times Y \to \mathbb{R}; \quad (x, y) \mapsto \Lambda\Big(\frac{(x - x_1) - (y - y_1)}{(x_2 - x_1)N}\Big).$$

This Toeplitz kernel is TN, since Λ is, and therefore so is $F \circ K$. However, the submatrix

$$(F \circ K)[(1, \ldots, m); (1, \ldots, m)] = (F(\Lambda((i - j)/N)))_{i,j=1,\ldots,m}$$

contains $(F \circ T_\Lambda)[\mathbf{z}; \mathbf{z}]$, and this is the desired contradiction.

Step 4: The function F cannot have the form $c\mathbf{1}_{x=0}$, where $c > 0$. To see this, fix equi-spaced sequences $\mathbf{x} \in X^{2,\uparrow}$ and $\mathbf{y} \in Y^{2,\uparrow}$, and let

$$K : X \times Y \to \mathbb{R} : \ (x, y) \mapsto \begin{cases} 1 & \text{if } x - y = x_1 - y_1, \\ 0 & \text{otherwise.} \end{cases}$$

Then K is Toeplitz and TN, since each row of any submatrix of K contains 1 at most once, and similarly for each column. However, $c\mathbf{1}_{x=0} \circ K = c(1 - K)$ and

$$\det(1 - K)[(x_1, x_2); (y_1, y_2)] = \begin{vmatrix} 0 & 1 \\ 1 & 0 \end{vmatrix} = -1. \qquad \square$$

As a consequence, we now classify the preservers of Pólya frequency sequences.

Corollary 9.3. *Let $F : [0, \infty) \to [0, \infty)$ be non-constant. The sequence $F \circ \mathbf{a}$ is a Pólya frequency sequence for every Pólya frequency sequence $\mathbf{a}$ if and only if $F(x) = cx$ for some $c > 0$.*

Proof. One implication is immediate. For the other, an application of Theorem 9.2 with $X = Y = \mathbb{Z}$ means we need only show that F is continuous at the origin to obtain the desired result.

Define sequences $\mathbf{b} = (b_m)_{m\in\mathbb{Z}}$ and $\mathbf{c} = (c_m)_{m\in\mathbb{Z}}$ by setting

$$b_m = \begin{cases} 1 & \text{if } m = 0, \\ 0 & \text{otherwise} \end{cases} \quad \text{and} \quad c_m = \begin{cases} 1 & \text{if } m = 0 \text{ or } 2, \\ 2 & \text{if } m = 1, \\ 0 & \text{otherwise.} \end{cases}$$

These are Pólya frequency sequences, by [1, Theorem 6], since the only zeros of their polynomial generating functions 1 and $1 + 2z + z^2$ are negative. Hence

$$\begin{aligned} \lim_{t\to 0^+} \det(F \circ tT_{\mathbf{b}})[(1, 3, 4); (1, 2, 3)] &= \begin{vmatrix} F^+(0) & F(0) & F(0) \\ F(0) & F(0) & F^+(0) \\ F(0) & F(0) & F(0) \end{vmatrix} \\ &= -F(0)(F^+(0) - F(0))^2 \end{aligned}$$

is non-negative, as is

$$\lim_{t\to 0^+} \det(F \circ tT_{\mathbf{c}})[(2, 3, 4); (1, 2, 3)] = -F^+(0)(F^+(0) - F(0))^2.$$

Now either $F(0) = F^+(0) = 0$, in which case we are done, or at least one of $F(0)$ and $F^+(0)$ is positive, in which case they are equal. $\square$

Remark 9.4. The test families of Pólya frequency functions used to classify the preservers of TN functions listed in Remark 8.8 can be used to obtain test sets of Pólya frequency sequences. To see this, suppose Λ is a Pólya frequency function that is continuous except possibly at the origin, and let $F : [0, \infty) \to [0, \infty)$ be continuous. Then $F \circ \Lambda$ is TN if and only if $F \circ \Lambda_{(N)}$ is TN for every Pólya frequency sequence $\Lambda_{(N)} := (\Lambda(n/N))_{n \in \mathbb{Z}}$, where $N \in \mathbb{N}$. One implication is immediate, and the converse follows using similar reasoning to Lemma 7.7 and Step 3 in the proof of Theorem 9.2. In particular, for any integer $n \geq 2$, there exists some $N \in \mathbb{N}$ such that $M^n_{(N)}$ is not a Pólya frequency sequence, where M is as in Lemma 8.4.

We now turn to classifying TP preservers for Pólya frequency sequences. The next result is a version of Theorem 8.10 in the same setting as that of Theorem 9.2.

Theorem 9.5. *Let X and Y be a pair of admissible sets and let the function $F : (0, \infty) \to (0, \infty)$. The following are equivalent:*

(1) *The composition operator C_F preserves total positivity for all Toeplitz kernels on $X \times Y$.*
(2) *$F(x) = cx$ for some $c > 0$.*

As an immediate consequence, we have the following result.

Corollary 9.6. *The preservers of total positivity for Pólya frequency sequences are precisely the dilations $F(x) = cx$, where $c > 0$.*

Proof of Theorem 9.5. That (2) $\implies$ (1) is immediate. For the converse, suppose (1) holds and note that F must be positive, increasing, and continuous on $(0, \infty)$; the argument is essentially that of Step 1 in the proof of Theorem 9.2.

Next, we extend F to a continuous, increasing function $\widetilde{F} : [0, \infty) \to [0, \infty)$ and suppose for contradiction that F is not a dilation. Then, by Remark 8.8, there exists a regular Pólya frequency function Λ that is continuous except perhaps at the origin and such that $\widetilde{F} \circ \Lambda$ is not TN.

By Lemma 7.7, there exists $\mathbf{z} \in \mathbb{R}^{n,\uparrow}$, where $n \geq 2$, such that $A = (\widetilde{F} \circ T_\Lambda)[\mathbf{z}; \mathbf{z}]$ is not TN, that is, A has at least one negative minor. By continuity, we may assume that $\mathbf{z} = (z_1, \ldots, z_n) \in \mathbb{Q}^{n,\uparrow}$.

By Proposition 8.9, there exists a sequence $(\Lambda_k)_{k \geq 1}$ of TP Pólya frequency functions such that $\Lambda_k \to \Lambda$ pointwise as $k \to \infty$. Since A depends on the value of Λ only at the finite set of values $\{z_i - z_j : i, j \in [n]\}$, there exists $k \in \mathbb{N}$ such that $A' = (\widetilde{F} \circ T_{\Lambda_k})[\mathbf{z}; \mathbf{z}]$ has a negative minor.

We now follow the last part of Step 3 in the proof of Theorem 9.2. Let $N \in \mathbb{N}$ be sufficiently large so that $N z_i \in \mathbb{Z}$ for all $i \in [n]$, let $m = N(z_n - z_1) + 1$ and choose $\mathbf{x} \in X^{m,\uparrow}$ and $\mathbf{y} \in Y^{m,\uparrow}$ to define the kernel

$$\Lambda' : X \times Y \to \mathbb{R}; \quad (x, y) \mapsto \Lambda_k\Big(\frac{(x - x_1) - (y - y_1)}{(x_2 - x_1)N}\Big).$$

Then Λ' is TP, since Λ_k is and therefore so is $F \circ \Lambda'$. However, A' occurs as a submatrix of $F \circ \Lambda'$ and this is the desired contradiction. $\square$

The final theorem in this section classifies the powers that preserve symmetric Toeplitz matrices.

Theorem 9.7. *The only power functions preserving* TN *symmetric Toeplitz matrices are $F(x) = 1$ and $F(x) = x$.*

As a first step in the proof of this theorem, we obtain the following result.

Proposition 9.8. *Let $A := (\cos((i - j)\theta))_{i,j=1}^n$, where $n \geq 2$ and the angle θ is such that $0 < \theta < \pi/(2n - 2)$. Then $A^{\circ\alpha}$ is positive semidefinite if and only if $\alpha \in \mathbb{N}_0$ or $\alpha \in [n - 2, \infty)$.*

Proof. We appeal to a result of Jain [38, Theorem 1.1], which states that, for distinct positive real numbers $x_1, \ldots, x_n$, the entrywise αth power of $X := (1 + x_i x_j)_{i,j=1}^n$ is positive semidefinite if and only if $\alpha \in \mathbb{N}_0$ or $\alpha \in [n - 2, \infty)$. Now let $x_j := \tan(j\theta)$, and let D be the diagonal $n \times n$ matrix with (j, j) entry $\cos(j\theta)$. Then $X^{\circ\alpha}$ is positive semidefinite if and only if $D^\alpha X^{\circ\alpha} D^\alpha = (DXD)^{\circ\alpha}$ is, but $DXD = A$. $\square$

Proof of Theorem 9.7. We begin with the observation that the cosine Toeplitz kernel

$$K : (-\pi/4, \pi/4) \times (-\pi/4, \pi/4) \to \mathbb{R}; \quad (x, y) \mapsto \cos(x - y)$$

is TN. Note first that this kernel has rank 2, by the cosine identity for differences: given $\mathbf{x}, \mathbf{y} \in (-\pi/4, \pi/4)^{n,\uparrow}$ for some $n \in \mathbb{N}$, we have that

$$K[\mathbf{x}; \mathbf{y}] = \mathbf{c}_x \mathbf{c}_y^T + \mathbf{s}_x \mathbf{s}_y^T,$$

where

$$\mathbf{c}_x := (\cos x_1, \ldots, \cos x_n)^T \quad \text{and} \quad \mathbf{s}_x := (\sin x_1, \ldots, \sin x_n)^T,$$

and similarly for $\mathbf{c}_y$ and $\mathbf{s}_y$. Thus, every minor of size at least 3×3 vanishes. If $(x_1, x_2), (y_1, y_2) \in (-\pi/4, \pi/4)^{2,\uparrow}$, then a direct computation shows that

$$\begin{vmatrix} \cos(x_1 - y_1) & \cos(x_1 - y_2) \\ \cos(x_2 - y_1) & \cos(x_2 - y_2) \end{vmatrix} = \sin(x_2 - x_1)\sin(y_2 - y_1) > 0,$$

and that $\cos(x - y) \geq 0$ whenever $|x| < \pi/4$ and $|y| < \pi/4$ is immediate.

Given this observation, we proceed to eliminate possibilities for α. The test matrix

$$\begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}$$

shows that α cannot be negative. Next, suppose α is positive and not an integer, and let n be an integer greater than $\alpha+2$. If A is the $n \times n$ matrix of Proposition 9.8, with $\theta = \pi/2n$, then A is TN, since it occurs as a submatrix of K, but $A^{\circ\alpha}$ is not positive semidefinite, so not TN.

The final case is if $F(x) = x^k$ for some integer $k \geq 2$. Let M be the even function of Lemma 8.4, and let $\mathbf{z} \in \mathbb{R}^{n,\uparrow}$ be such that $B := T_{M^k}[\mathbf{z}; \mathbf{z}]$ has a negative minor. By continuity, we may assume that $\mathbf{z} = (z_1, \ldots, z_n) \in \mathbb{Q}^{n,\uparrow}$. We may also translate each coordinate by $-z_1$, since this leaves B unchanged, thus $z_1 = 0 < z_n$. Choose $N \in \mathbb{N}$ such that $z_i N \in \mathbb{Z}$ for every $i \in [n]$, and consider the symmetric Toeplitz matrix $C := (M((i-j)/N))_{i,j=0}^{z_n N}$. This is TN, since it occurs as a submatrix of M, and therefore so is $F[C] = C^{\circ k}$, but this matrix contains B as a principal submatrix, and so has a negative minor. This contradiction completes the proof. $\square$

We conclude with two observations. First, the above classifications of preservers of Pólya frequency functions and sequences, including Lemma 8.4, sit in marked contrast to [65, Theorem 3]. There, Weinberger shows that the set PF_3 of functions defined analogously to Pólya frequency functions, but with the TN condition replaced by TN_3, is closed under taking any real power greater than or equal to 1.

Second, a result in the parallel paradigm of the holomorphic functional calculus, not the Schur–Hadamard calculus considered here, can be found in [40, pp. 451–452]. There, it is proved that a smooth function preserves TN matrices via the functional calculus if and only if it is a non-negative integer power. A close look at the proof reveals that the same conclusion can be deduced by using the smaller test set of TN upper-triangular Toeplitz matrices. This stands in contrast to the results of the next section.

10 One-sided Pólya frequency functions and sequences

As a final variation on the Pólya-frequency theme, we turn to the class of one-sided Pólya frequency sequences, where the terms vanish for negative indices. As discussed at the start of the previous section, Pólya frequency sequences, including the one-sided variant, are well studied, with a representation theorem [1] and applications in analysis, combinatorics, and other areas.

We prove here that the only preservers of this class are homotheties and Heaviside functions. In the spirit of the previous results, we begin by showing the analogous result for Pólya frequency functions and TN functions, akin to Theorem 8.7.

We say that a function $f : \mathbb{R} \to \mathbb{R}$ is **one sided** if there exists $x_0 \in \mathbb{R}$ such that either $f(x) = 0$ for all $x < x_0$, or $f(x) = 0$ for all $x > x_0$.

Theorem 10.1. *Let $F : [0, \infty) \to [0, \infty)$.*

(1) *The function $F \circ \Lambda$ is a one-sided Pólya frequency function whenever Λ is, if and only if $F(x) = cx$ for some $c > 0$.*
(2) *The function $F \circ f$ is a one-sided* TN *function that is non-zero at two or more points whenever f is, if and only if $F(x) = cx$, $F(x) = c\mathbf{1}_{x>0}$, or $F(x) = c\mathbf{1}_{x=0}$, for some $c > 0$.*
(3) *Suppose F is non-zero. Then $F \circ f$ is a one-sided* TN *function whenever f is, if and only if $F(x) = cx$ or $F(x) = c\mathbf{1}_{x>0}$, for some $c > 0$.*

The proof uses the following one-sided variant of Lemma 8.4.

Lemma 10.2. *Let a_1, a_2, and a_3 be positive real numbers, with $a_1 < a_2 < a_3$, such that the set $\{ a_1, a_2, a_3 \}$ is linearly independent over the rational numbers, and let the non-zero real numbers c_1, c_2, and c_3 be such that*

$$c_1 > 0, \quad c_1 + c_2 + c_3 = 0, \quad \textit{and} \quad a_1c_1 + a_2c_2 + a_3c_3 = 0. \tag{10.1}$$

Then

$$N : \mathbb{R} \to \mathbb{R};\ x \mapsto \begin{cases} c_1 e^{-a_1 x} + c_2 e^{-a_2 x} + c_3 e^{-a_3 x} & \textit{if } x \geq 0, \\ 0 & \textit{if } x < 0 \end{cases}$$

is a continuous Pólya frequency function such that $N^n : x \mapsto N(x)^n$ is not a Pólya frequency function for any integer $n \geq 2$.

The function N of Lemma 10.2 is, up to scaling, a member of the class of one-sided Pólya frequency functions which we call Hirschman–Widder densities. These were studied by Hirschman and Widder in their 1949 paper [34], and their 1955 monograph [35] contains a detailed analysis of such exponential polynomials and their Laplace transforms.

The work of Hirschman and Widder is closely intertwined with that of Schoenberg. In 1947, Schoenberg [56] announced the notion of a Pólya frequency function. In their 1949 work, Hirschman and Widder studied these maps and their order of smoothness, via their Laplace transforms. This was followed by Schoenberg's first full paper on Pólya frequency functions [58] in 1951.

In a forthcoming piece of work [10], we show that Hirschman–Widder densities satisfy the conclusions of Lemma 10.2 generically, as long as they involve at least three distinct exponential terms: their higher integer powers are not TN, and so are not Pólya frequency functions. Indeed, a stronger result is established, with powers replaced by non-homethetic polynomial functions.

Proof of Lemma 10.2. Throughout this proof, sums and products are taken over non-negative integers satisfying the given conditions. For any $n \in \mathbb{N}$, let

$$F_n(s) := \mathcal{B}\{N^n\}(s) = \sum_{i+j+k=n} \binom{n}{i,j,k} \frac{c_1^i c_2^j c_3^k}{s + ia_1 + ja_2 + ka_3} = \frac{p_n(s)}{q_n(s)}$$

be the bilateral Laplace transform of N, where the monic polynomial

$$q_n(s) := \prod_{i+j+k=n} (s + ia_1 + ja_2 + ka_3)$$

has degree $(n+1)(n+2)/2$ and

$$p_n(s) := \sum_{i+j+k=n} \binom{n}{i,j,k} c_1^i c_2^j c_3^k \prod_{\substack{i'+j'+k'=n \\ (i',j',k') \neq (i,j,k)}} (s + i'a_1 + j'a_2 + k'a_3)$$

has degree no more than $n(n+3)/2$. In fact, the leading coefficient of $p_n(s)$ is

$$\sum_{i+j+k=n} \binom{n}{i,j,k} c_1^i c_2^j c_3^k = (c_1 + c_2 + c_3)^n = 0,$$

whereas the next-highest-order coefficient of $p_n(s)$ is

$$\begin{aligned}
\sum_{i+j+k=n} \binom{n}{i,j,k} c_1^i c_2^j c_3^k \Big(\Big(\sum_{i'+j'+k'=n} i'a_1 + j'a_2 + k'a_3 \Big) - ia_1 - ja_2 - ka_3 \Big) \\
&= \frac{1}{6} n(n+1)(n+2)(a_1 + a_2 + a_3)(c_1 + c_2 + c_3)^n \\
&\quad - n(c_1 + c_2 + c_3)^{n-1}(a_1c_1 + a_2c_2 + a_3c_3) \\
&= 0,
\end{aligned}$$

since

$$\sum_{i'+j'+k'=n} i' = \frac{1}{6} n(n+1)(n+2)$$

and

$$\sum_{i+j+k=n} \binom{n}{i,j,k} i x^i y^j z^k = x \frac{\partial}{\partial x}(x+y+z)^n = nx(x+y+z)^{n-1}.$$

The constraints (10.1), together with the ordering of a_1, a_2, and a_3, imply that

$$c_2 < 0, \quad c_1 + c_2 < 0 \quad \text{and} \quad p_1 \equiv \frac{c_1 c_2 (a_1 - a_2)^2}{c_1 + c_2} > 0,$$

whence $1/F_1(s) = q_1(s)/p_1(s)$ is a polynomial with real roots $-a_1$, $-a_2$ and $-a_3$. Furthermore, as

$$\frac{1}{F_1(0)} = \frac{q_1(0)}{p_1(0)} = \frac{a_1 a_2 a_3 (c_1 + c_2)}{c_1 c_2 (a_1 - a_2)^2} > 0,$$

it follows that N is a Pólya frequency function, by [58, Theorem 1].

For the second part, we will show that $1/F_n(s) = q_n(s)/p_n(s)$ is the restriction of a rational function with simple poles whenever $n \geq 2$, and so it cannot be the restriction of an entire function; it then follows from [58, Theorem 1] that N^n is not a Pólya frequency function for such n.

Note first that none of the roots of $q_n(s)$ are roots of $p_n(s)$, since if i, j and k are non-negative integers such that $i + j + k = n$, then

$$\begin{aligned} &p_n(-ia_1 - ja_2 - ka_3) \\ &\qquad = \binom{n}{i,j,k} c_1^i c_2^j c_3^k \prod_{\substack{i'+j'+k'=n \\ (i',j',k') \neq (i,j,k)}} ((i' - i)a_1 + (j' - j)a_2 + (k' - k)a_3) \neq 0. \end{aligned}$$

We will now show that $p_n(s)$ is non-constant, which establishes that $1/F_n(s)$ is as claimed. We have that

$$\begin{aligned} \gamma_n := \frac{p_n(-na_3)}{p_n(-na_1)} &= \left(\frac{c_3}{c_1}\right)^n \frac{na_1 - na_3}{na_3 - na_1} \prod_{\substack{i+j+k=n \\ i,k \neq n}} \frac{ia_1 + ja_2 + ka_3 - na_3}{ia_1 + ja_2 + ka_3 - na_1} \\ &= -\left(\frac{c_3}{c_1}\right)^n \prod_{\substack{i+j+k=n \\ i,k \neq n}} \frac{(a_1 - a_3)i + (a_2 - a_3)j}{(a_2 - a_1)j + (a_3 - a_1)k} \\ &= (-1)^{n+1} \left(\frac{c_1 + c_2}{c_1}\right)^n \prod_{\substack{i+j+k=n \\ i,k \neq n}} \frac{c_1 j - c_2 i}{(c_1 + c_2)j + c_2 i}, \end{aligned}$$

where the last step uses the fact that the placeholder variables i and k are symmetric in the product for the denominator. Since

$$\prod_{\substack{i+j+k=n \\ i,k \neq n}} \frac{c_1 j - c_2 i}{(c_1 + c_2)j + c_2 i} = (-1)^{n-1} \left(\frac{c_1}{c_1 + c_2}\right)^n \prod_{2 \leq i+j \leq n} \frac{c_1 j - c_2 i}{(c_1 + c_2)j + c_2 i},$$

it follows that

$$\frac{\gamma_{n+1}}{\gamma_n} = \prod_{\substack{i+j=n+1 \\ i,j \geq 1}} \frac{c_1 j - c_2 i}{(c_1 + c_2)j + c_2 i} = (-1)^n \prod_{j=1}^{n} \frac{n + 1 - (\alpha + 1)j}{n + 1 + \alpha j},$$

where

$$\alpha = \frac{c_1}{c_2} = \frac{-(a_3 - a_2)}{a_3 - a_1} \in (-1, 0).$$

If $j = 1, \dots, n$, then

$$\frac{n+1-(\alpha+1)j}{n+1+\alpha j} < 1 \iff n+1-(\alpha+1)j < n+1+\alpha j \iff 0 < 2\alpha+1,$$

and

$$0 < 2\alpha + 1 = \frac{a_3 - a_1 - 2a_3 + 2a_2}{a_3 - a_1} \iff a_3 - a_2 < a_2 - a_1,$$

whereas

$$\frac{n+1-(\alpha+1)j}{n+1+\alpha j} > 1 \iff 0 > 2\alpha+1 \iff a_3 - a_2 > a_2 - a_1.$$

Since equality is impossible, by linear independence, it follows that either $|\gamma_{n+1}/\gamma_n| < 1$ for all $n \in \mathbb{N}$ or $|\gamma_{n+1}/\gamma_n| > 1$ for all $n \in \mathbb{N}$; in each case, since $\gamma_1 = 1$, the polynomial p_n is non-constant for all $n \geq 2$. This completes the proof. □

With the preliminary result established, we can now obtain the promised characterizations.

Proof of Theorem 10.1. Let $F : [0, \infty) \to [0, \infty)$. We show first that, if $G = F \circ t\lambda_d$ is a TN_2 function for all $t > 0$ and all $d \in (0, 1)$, in that $\det T_G[\mathbf{x}; \mathbf{y}] \geq 0$ for all $\mathbf{x}, \mathbf{y} \in \mathbb{R}^{2,\uparrow}$, then F is non-decreasing on $(0, \infty)$. This follows because, if $t > 0$, $\varepsilon > 0$, and $d = e^{-2\varepsilon}$, then

$$0 \leq \det(F \circ te^{\varepsilon} T_{\lambda_d})[(\varepsilon, 2\varepsilon); (0, \varepsilon)] = \begin{vmatrix} F(t) & F(te^{-\varepsilon}) \\ F(te^{-\varepsilon}) & F(t) \end{vmatrix}.$$

Next, letting $t_1 := \sup\{t > 0 : F(t) = 0\} \cup \{0\} \in [0, \infty]$, we see that F vanishes on $(0, t_1)$ and is positive on (t_1, ∞).

We now show that F is continuous on (t_1, ∞). If $t_1 = \infty$, then this is immediate. Otherwise, we first show that F is multiplicatively mid-concave on (t_1, ∞). Given any $q > p > t_1$, choose $t > q$ and let a and b be such that $q = te^{-a}$ and $p = te^{-b}$. Then, for any $d \in (0, 1)$,

$$0 \leq \det(F \circ tT_{\lambda_d})[((b+a)/2, b); (0, (b-a)/2)] = F(\sqrt{pq})^2 - F(p)F(q).$$

The continuity of F on (t_1, ∞) now follows by [52, Theorem 71.C]. We can now establish each of the three assertations of the theorem.

(1) Every homothety preserves the class of one-sided Pólya frequency functions. Conversely, since $F \circ \lambda$ is a Pólya frequency function, it is integrable, so $F(0) = 0$, and has unbounded support, so $t_1 = 0$: if $t_1 > 0$ then $F(e^{-x})$ will vanish for all sufficiently large x. We now follow the relevant parts of the proof of Theorem 8.5. By considering $F \circ p\lambda$ for any $p > 0$, we obtain the existence of positive constants b_0 and c_0 such that $F(t) = c_0 t^{b_0}$ for all $t \in (0, p)$; since p is arbitrary, this identity must hold everywhere on $(0, \infty)$. Moreover, the exponent b_0 must be an integer, by [58, Theorem 1] and the form of the bilateral Laplace transform of $F \circ \phi$, where $\phi(x) = x\lambda(x)$. That this integer equals 1 now follows from Lemma 10.2.

(2) That the given functions are preservers follows from the proof that (3) $\implies$ (1) in Theorem 8.7. Conversely, let t_1 be as above and note that $F(t\mathbf{1}_{x>0}) \equiv 0$ whenever $t \in (0, t_1)$. Hence $t_1 = 0$ and we may now follow the proof that (2) $\implies$ (3) in Theorem 8.7, replacing the use of Lemma 8.4 with Lemma 10.2.

(3) As $\mathbf{1}_{x=0}$ is now in our test set, the function $F(x) = c\mathbf{1}_{x=0}$ is no longer a preserver of total non-negativity; the other functions in the preceding part do preserve $a\mathbf{1}_{x=b}$ for any $a \geq 0$ and $b \in \mathbb{R}$. For the converse, if $t_1 = 0$, then the working for (2) and the previous observation gives the result. Otherwise, if $t_1 > 0$ and $F(0) \neq 0$, then $F(t\mathbf{1}_{x=0}) = F(0)(1 - \mathbf{1}_{x=0})$ is not TN whenever $t \in (0, t_1)$. It remains to consider the case $F(0) = 0$ and $t_1 \in (0, \infty)$, but then $F \circ 2t_1\lambda$ would be a TN function which is non-zero at two or more points and has compact support, an impossibility. This completes the proof. □

With Theorem 10.1 to hand, we show a similar result for the preservers of one-sided Pólya frequency sequences, but in slightly greater generality. See Definition 9.1 for the definition of an admissible pair; we say that a Toeplitz kernel $T_f : X \times Y \to \mathbb{R}$ is **one-sided** if the associated function $f : X - Y \to \mathbb{R}$ is, that is, there exists $x - y \in X - Y$ such that f vanishes on $\{z \in X - Y : z < x - y\}$ or on $\{z \in X - Y : z > x - y\}$.

Theorem 10.3. *Let X and Y be a pair of admissible sets such that $X - Y$ does not have a maximum or minimum element. If $F : [0, \infty) \to [0, \infty)$ is non-zero and preserves the one-sided* TN *Toeplitz kernels on $X \times Y$, then $F(x) = cx$ or $F(x) = c\mathbf{1}_{x>0}$ for some $c > 0$.*

Proof. We suppose that F is non-zero and preserves one-sided TN Toeplitz kernels on $X \times Y$ and proceed in a number of steps.

Step 1: $F(0) = 0$ and F is non-decreasing. Since $F \not\equiv 0$ on $[0, \infty)$, there exists $t > 0$ such that $F(t) \neq F(0)$. For any $x - y \in X - Y$, the one-sided function

$$\Lambda_0(z) = t\mathbf{1}_{z=x-y}$$

is a Pólya frequency function, so the kernel

$$(F \circ T_{\Lambda_0})(z) = F(0) + (F(t) - F(0))\mathbf{1}_{z=x-y}$$

is one-sided. As F equals the constant $F(0)$ on $X - Y$, except at $x - y$, and $X - Y$ has no extremal element, it follows that $F(0) = 0$. In particular, the operator C_F preserves one-sided kernels, and henceforth we focus on preserving TN.

We next show that F is non-decreasing, by following the proof of Theorem 10.1. From the previous working, we have that $F(0) = 0 \leq F(t)$ for all $t > 0$. Next, given any $t > 0$ and $\varepsilon > 0$, we see that $F(t) \geq F(te^{-\varepsilon})$, by considering $\det(F \circ T_{\Lambda_1})[\mathbf{x}; \mathbf{y}]$, where the arithmetic progressions $\mathbf{x} = (x_1, x_2) \in X^{2,\uparrow}$ and $\mathbf{y} = (y_1, y_2) \in Y^{2,\uparrow}$ are equi-spaced, and Λ_1 is the one-sided TN function

$$\Lambda_1(z) := te^{\varepsilon} \lambda_{e^{-2\varepsilon}}\left(\varepsilon \frac{z - x_1 + y_2}{x_2 - x_1}\right).$$

This shows the monotonicity of F, and so the existence of $t_1 \in [0, \infty]$ such that F vanishes on $(0, t_1)$ and is positive on (t_1, ∞).

Step 2: F is continuous on (t_1, ∞). It suffices to show that F is multiplicatively mid-concave on (t_1, ∞), by [52, Theorem 71.C]. Given $p, q \in (t_1, \infty)$ with $p < q$, choose any $t > q$ and note that

$$0 < a := \log(t/q) < b := \log(t/p).$$

With equi-spaced arithmetic progressions $\mathbf{x} \in X^{2,\uparrow}$ and $\mathbf{y} \in Y^{2,\uparrow}$ as in the previous step, and any $d \in [0, 1]$, we consider the one-sided TN function

$$\Lambda_2(z) := t\lambda_d\Big(a + \frac{z - x_1 + y_2}{2(x_2 - x_1)}(b - a)\Big).$$

The determinant

$$\det(F \circ T_{\Lambda_2})[\mathbf{x}; \mathbf{y}] = \begin{vmatrix} F(te^{-(a+b)/2}) & F(te^{-a}) \\ F(te^{-b}) & F(te^{-(a+b)/2}) \end{vmatrix} = F(\sqrt{pq})^2 - F(p)F(q)$$

is non-negative, as required. Hence F is continuous on (t_1, ∞).

Step 3: $t_1 = 0$. We suppose otherwise, and show first that $F(t_1) = 0$. For this, it suffices to prove that $F^+(t_1) := \lim_{t \to t_1^+} F(t) = 0$. We fix equi-spaced arithmetic progressions $\mathbf{x} = (x_1, x_2, x_3) \in X^{3,\uparrow}$ and $\mathbf{y} = (y_1, y_2, y_3) \in Y^{3,\uparrow}$. For arbitrary $\varepsilon > 0$, consider the one-sided TN function

$$\Lambda_{3,\varepsilon}(z) := t_1 e^{5\varepsilon} \lambda_1\Big(2\varepsilon \frac{z - x_1 + y_2}{x_2 - x_1}\Big).$$

A straightforward computation, using the fact that $F(t) = 0$ for all $t < t_1$, gives that

$$(F \circ T_{\Lambda_{3,\varepsilon}})[\mathbf{x}; \mathbf{y}] = \begin{pmatrix} F(t_1 e^{3\varepsilon}) & F(t_1 e^{5\varepsilon}) & 0 \\ F(t_1 e^{\varepsilon}) & F(t_1 e^{3\varepsilon}) & F(t_1 e^{5\varepsilon}) \\ 0 & F(t_1 e^{\varepsilon}) & F(t_1 e^{3\varepsilon}) \end{pmatrix}.$$

As this matrix is TN by assumption, we have that

$$0 \leq \lim_{\varepsilon \to 0^+} \det(F \circ T_{\Lambda_{3,\varepsilon}})[\mathbf{x}; \mathbf{y}] = -F^+(t_1)^3.$$

Thus $F^+(t_1) = 0$.

We now fix $t_2 > t_1$. Given any $k \in \mathbb{N}$, we choose equi-spaced arithmetic progressions $\mathbf{x} = (x_1, \dots, x_{k+2}) \in X^{k+2,\uparrow}$ and $\mathbf{y} = (y_1, \dots, y_{k+2}) \in Y^{k+2,\uparrow}$, let

$$\delta = \delta(k) := \frac{1}{k} \log(t_2/t_1) > 0,$$

and consider the one-sided TN function

$$\Lambda_{4,k}(z) := t_1 e^{(2k+1)\delta} \lambda_{e^{-\delta}} \Big(\delta \frac{z - x_1 + y_{k+1}}{x_2 - x_1} \Big).$$

As $F(0) = 0 = F(t_1)$, it follows that

$$(F \circ T_{\Lambda_{4,k}})[(x_1, x_2, x_{k+2}); (y_1, y_{k+1}, y_{k+2})] = \begin{pmatrix} F(t_2 e^{\delta}) & F(t_2^2/t_1) & 0 \\ F(t_2) & F(t_2^2/t_1) & F(t_2^2/t_1) \\ 0 & F(t_2) & F(t_2 e^{\delta}) \end{pmatrix}.$$

Since this matrix is TN, and F is positive on (t_1, ∞), taking determinants gives that

$$2F(t_2) \leq F(t_2 e^{\delta}).$$

Letting $k \to \infty$ yields $0 < 2F(t_2) \leq F(t_2)$, as F is continuous at t_2. Thus $F(t_2) \leq 0$, a contradiction since $t_2 > t_1$.

Step 4: F has the form claimed. The previous steps give that F is continuous, positive, and non-decreasing on $(0, \infty)$. We first assume that F does not have the form $c\mathbf{1}_{x>0}$ for any $c > 0$, so that F is non-constant on $(0, \infty)$. If $F \circ f$ is a TN function whenever f has the form $t\lambda_d$ for $t > 0$ and $d \in [0, 1]$, $\phi(x) = x\lambda(x)$ or N as in Lemma 10.2, then working as in the proof of Theorem 8.7 shows that $F(x) = cx$ for some $c > 0$. Hence we assume otherwise: suppose $G := F \circ f$ is not TN for one of these functions. Then, by Lemma 7.7, there exists $\mathbf{z} \in \mathbb{R}^{n,\uparrow}$, where $n \geq 2$, such that $A := T_G[\mathbf{z}; \mathbf{z}]$ is not TN. Since G is continuous except possibly at the origin, we may assume that $\mathbf{z} \in \mathbb{Q}^{n,\uparrow}$. Taking $N \in \mathbb{N}$ such that Nz_i is an integer for all $i \in [n]$, we set $m = N(z_n - z_1) + 1$, choose equi-spaced arithmetic progressions $\mathbf{x} \in X^{m,\uparrow}$ and $\mathbf{y} \in Y^{m,\uparrow}$, and let

$$\Lambda_5 : X \times Y \to \mathbb{R};\ (x, y) \mapsto G\Big(\frac{(x - x_1) - (y - y_1)}{(x_1 - y_1)N}\Big).$$

Then $\Lambda_5 = F \circ T_f|_{X\times Y}$ is TN, since T_f is a one-sided TN Toeplitz kernel on $X \times Y$, but Λ_5 contains A as a principal submatrix. This contradiction completes the proof. □

We conclude with the case of Pólya frequency sequences, where $X = Y = \mathbb{Z}$. Note that a shift of origin $(a_n)_{n\in\mathbb{Z}} \mapsto (a_{n+k})_{n\in\mathbb{Z}}$ preserves the TN property for any $k \in \mathbb{Z}$, as does the reflection $(a_n)_{n\in\mathbb{Z}} \mapsto (a_{-n})_{n\in\mathbb{Z}}$, so we may consider only one-sided sequences that vanish for negative indices.

Corollary 10.4. *Let $F : [0, \infty) \to [0, \infty)$. Then F preserves the set of one-sided Pólya frequency sequences if and only if $F(x) = cx$ for some $c \geq 0$.*

Proof. One implication is immediate. For the converse, by Theorem 10.3, we have that $F(x) = cx$ or $F(x) = c\mathbf{1}_{x>0}$; to rule out the latter possibility, we show that F is continuous at the origin. Given the one-sided Pólya frequency sequence $\mathbf{a}$ such that $a_0 = a_2 = 1$, $a_1 = 2$, and $a_n = 0$ otherwise, we note that

$$C_\varepsilon := \det(F \circ \varepsilon T_{\mathbf{a}})[(0, 1, 2); (-1, 0, 1)] = \begin{vmatrix} F(2\varepsilon) & F(\varepsilon) & 0 \\ F(\varepsilon) & F(2\varepsilon) & F(\varepsilon) \\ 0 & F(\varepsilon) & F(2\varepsilon) \end{vmatrix} \geq 0$$

for any $\varepsilon > 0$. Hence

$$0 \leq \lim_{\varepsilon\to 0^+} C_\varepsilon = \lim_{\varepsilon\to 0^+} -F(\varepsilon)^3,$$

which gives continuity at the origin as claimed. □

We conclude with a corollary on lower-triangular matrices, in the spirit of the final observation in the previous section.

Corollary 10.5. *Let $F : [0, \infty) \to [0, \infty)$. Then F preserves the set of* TN *lower-triangular Toeplitz matrices if and only if $F(x) = cx$ for some $c \geq 0$.*

Proof. One implication is immediate. For the converse, suppose $F(0) = 0$ but F is not of the form $F(x) = cx$ for any $c \geq 0$. By Corollary 10.4, there exists a one-sided Pólya frequency sequence $\mathbf{a} = (a_n)_{n\in\mathbb{Z}}$ such that $F \circ \mathbf{a}$ is not a one-sided Pólya frequency sequence, so there exist $\mathbf{x} = (x_1, \dots, x_n) \in \mathbb{Z}^{n,\uparrow}$ and $\mathbf{y} = (y_1, \dots, y_n) \in \mathbb{Z}^{n,\uparrow}$, where $n \in \mathbb{N}$, such that $A := (F \circ T_{\mathbf{a}})[\mathbf{x}; \mathbf{y}]$ has negative determinant. Let $m := \min\{x_1, y_1\}$, $M := \max\{x_n, y_n\}$, and let $\mathbf{z} := (m, \dots, M)$.

Then the Toeplitz matrices $T_{\mathbf{a}}[\mathbf{z};\mathbf{z}]$ and $(F \circ T_{\mathbf{a}})[\mathbf{z};\mathbf{z}]$ are lower triangular, since $a_n = 0$ whenever $n < 0$, but the latter contains A as a submatrix, so cannot be TN. This contradiction gives the result. $\square$

The observation prior to Corollary 10.4 means that the previous result applies equally to upper-triangular matrices.

11 Total-positivity preservers. II: General domains

It is time to return to our original problem, of classifying the preservers of TP kernels on $X \times Y$ for totally ordered sets X and Y. So far, we have resolved this when at least one of X and Y is finite. The picture is completed by the following result.

Theorem 11.1. *Suppose X and Y are infinite totally ordered sets such that there exists a* TP *kernel on $X \times Y$. A function $F : (0, \infty) \to (0, \infty)$ preserves the set of* TP *kernels on $X \times Y$ if and only if $F(x) = cx$ for some $c > 0$.*

Proof. Without loss of generality, we assume X and Y are infinite subsets of $\mathbb{R}$, by Lemma 4.5. Next, we establish a stronger version of the chain property used to prove Proposition 3.2, that is, the existence of an order-preserving bijection $\varphi_X : X \to X'$, where $X' \subseteq \mathbb{R}$ contains an arithmetic progression of length 2^n and increment 4^{-n} for each integer $n \geq 2$. To show this, let $(a_n)_{n\geq 1}$ be an increasing sequence of positive real numbers, containing an arithmetic progression of length 2^n and increment 4^{-n} for each integer $n \geq 2$, and converging to 1. Such a sequence can be constructed, by taking each arithmetic progression of the form $j4^{-n}$, where $j \in [2^n]$, and concatenating these, at each stage adding the last term of the existing sequence to each term of the next progression:

$$\frac{1}{4}, \quad \frac{2}{4}, \quad \frac{2}{4} + \frac{1}{16}, \quad \ldots, \quad \frac{2}{4} + \frac{4}{16}, \quad \frac{2}{4} + \frac{4}{16} + \frac{1}{64}, \quad \ldots, \quad \frac{2}{4} + \frac{4}{16} + \frac{8}{64}, \ldots.$$

As observed in Section 3, the set X has either an infinite ascending chain or an infinite descending chain. Without loss of generality, we assume the former; the argument for a descending chain is similar. If X is unbounded above, let $(x_n)_{n\geq 1} \subseteq X$ be an increasing sequence such that $x_n \to \infty$. For all $n \in \mathbb{N}$, let $\varphi_X(x_n) = a_n$ and extend φ_X piecewise linearly on $\{x \in X : x_n < x < x_{n+1}\}$. This provides an order-preserving embedding of $\{x \in X : x \geq x_1\}$ into $(0, 1)$ containing the desired arithmetic progressions. If, instead, X is bounded above, with $\sup X = m$, let $(x_n)_{n\geq 1} \subseteq X$ be any increasing sequence in X, and let $x \in \mathbb{R}$ be its limit. Set $\varphi_X(x_n) = a_n$, $\varphi_X(x) = 1$, and $\varphi_X(m) = 2$ if $m \neq x$, and extend φ_X piecewise linearly between these points. Then φ_X is again an order-preserving embedding of $\{x \in X : x \geq x_1\}$ into $(0, 2]$ and containing the desired arithmetic progressions. A similar argument can be used to extend φ_X to the whole of X by mapping $\{x \in X : x < x_1\}$ into $[-2, 0)$. This gives φ_X as claimed.

To complete the proof, let $\varphi_X : X \to X'$ and $\varphi_Y : Y \to Y'$ be order-preserving bijections as constructed above, and note that X' and Y' are an admissible pair in the sense of Definition 9.1. Thus, if F preserves TP for kernels on $X \times Y$ but is not a dilation, Theorem 9.5 gives a TP Toeplitz kernel K' on $X' \times Y' \to \mathbb{R}$ such that $F \circ K'$ is not TP. However, the kernel

$$K : X \times Y \to \mathbb{R};\ (x, y) \mapsto K'(\varphi_X(x), \varphi_Y(y))$$

is TP, since K' is, and therefore so is $F \circ K$, by the assumption on F. As any submatrix of $F \circ K'$ occurs as a submatrix of $F \circ K$, we have a contradiction. As dilations clearly preserve TP, the proof is complete. □

For our final result, we consider the symmetric version of the previous theorem.

Theorem 11.2. *Suppose X is an infinite totally ordered set such that there exists a symmetric* TP *kernel on $X \times X$. A function $F : (0, \infty) \to (0, \infty)$ preserves* TP *for symmetric kernels on $X \times X$ if and only if $F(x) = cx$ for some $c > 0$.*

Proof. In the usual fashion, we first identity X with a subset of $\mathbb{R}$ using Lemma 4.5. Proposition 4.7 now implies that any such preserver must be of the form $F(x) = cx^\alpha$ for positive constants c and α.

The construction in the proof of Theorem 11.1 gives an order-preserving bijection $\varphi : X \to X'$, where $X' \subseteq \mathbb{R}$ contains arithmetic progressions of arbitrary length. This bijection provides a correspondence between TP symmetric kernels on $X \times X$ and those on $X' \times X'$, so we may assume, without loss of generality, that X contains arithmetic progressions of arbitrary length.

Given $u_0 \in [0, 1)$, p, $q \geq 0$ such that $p + q > 0$ and $\varepsilon > 0$, note first that the Hankel kernel

$$K : \mathbb{R} \times \mathbb{R} \to \mathbb{R};\ (x, y) \mapsto p + qu_0^{x+y} + \varepsilon \int_0^1 e^{(x+y)u} \, \mathrm{d}u$$

is TP, by Proposition 7.4. Let $\mathbf{x} = (x_1, \ldots, x_n) \in X^{n,\uparrow}$ be an arithmetic progression, where $n \geq 2$, and note that the kernel

$$K' : X \times X \to \mathbb{R};\ (x, y) \mapsto K\left(\frac{x - x_1}{x_2 - x_1}, \frac{y - x_1}{x_2 - x_1}\right)$$

is symmetric and TP, hence so is

$$(F \circ K')[\mathbf{x}; \mathbf{x}] = (F \circ K)[(0, 1, \ldots, n - 1); (0, 1, \ldots, n - 1)].$$

By the continuity of F, the matrix

$$\lim_{\varepsilon\to 0^+}(F\circ K)[(0,1,\dots,n-1);(0,1,\dots,n-1)] = (F(p+qu_0^{i+j}))_{i,j=0}^{n-1}$$

is positive semidefinite. As n can be arbitrarily large, it follows from [9, Theorem 4.1] that F is the restriction of an entire function $\sum_{k=0}^{\infty} c_k x^k$ with $c_k \geq 0$ for all k. Since $F(x) = cx^{\alpha}$, it must be that α is a positive integer.

To conclude, we suppose for contradiction that $\alpha \geq 2$ and let M be a Pólya frequency function as in Lemma 8.4. Then M is non-vanishing and, by Lemma 7.7, there exists $\mathbf{z} \in \mathbb{R}^{n,\uparrow}$ such that $(F\circ T_M)[\mathbf{z};\mathbf{z}]$ has a negative minor; by continuity, we may assume that $\mathbf{z} \in \mathbb{Q}^{n,\uparrow}$. Working as in the final two paragraphs of the proof of Theorem 9.5, with $X = Y$ and $\mathbf{x} = \mathbf{y}$, now gives the result. $\square$

12 Concluding remarks: Minimal test families

In the first part of this section, we identify a few directions for future exploration. The second part contains an enumeration of minimal test criteria to demonstrate the rigidity of preservers.

12.1 Open questions. By specializing the families of maps that the post-composition transform leaves invariant, we may obtain a plethora of classification questions. Some of these questions, which appear artificial at first sight, might gain weight due to future applications. Here, we simply touch the surface.

Question 12.1. Which functions preserve the class of one-sided Pólya frequency sequences with finitely many non-zero terms, or those generated by evaluating a polynomial?

This question has more positive answers than just the homotheties. For example, the power maps x^n are preservers of both sub-classes of one-sided Pólya frequency sequences for all $n \in \mathbb{N}$, by results of Maló [47] and Wagner [64], respectively.

Question 12.2. Are the totally positive Pólya frequency sequences dense in the set of all Pólya frequency sequences?

Question 12.3. Given a TN kernel $K : X \times Y \to \mathbb{R}$, where X and Y are infinite subsets of $\mathbb{R}$, can K be approximated by a sequence of TP kernels, at least at points of continuity?

For the latter question, recall that Section 6 contains such an approximation by TP_p kernels, for every $p \in \mathbb{N}$.

While positive solutions to the preceding questions could help provide alternate proofs of the classifications of the classes of TP preservers in these settings, we have already achieved these classifications via different methods.

12.2 Minimal testing families. Many of the proofs above isolate some minimal classes of kernels against which putative TN or TP preservers must be tested. For the reader interested solely in the dimension-free setting of Theorem 1.1, we end with some toolkit observations.

If a function $F : [0, \infty) \to \mathbb{R}$ preserves TN for (a) all TN 2×2 matrices, (b) the 3×3 matrix C from (3.5), and (c) the two-parameter family of 4×4 matrices $N(\varepsilon, x)$ defined above (3.7), then F is either constant or linear. Specifically, as the proof of Theorem 3.3 shows, preserving TN for the 2×2 test set implies that F is either a non-negative constant or $F(x) = cx^{\alpha}$ for some $c > 0$ and $\alpha \geq 0$. Using the matrix C, we see that $\alpha \geq 1$. Finally, using the test set $\{N(\varepsilon, x) : \varepsilon \in (0, 1),\ x > 0\}$, we obtain $\alpha = 1$.

As noted in Remark 8.8, a non-zero function $F : [0, \infty) \to [0, \infty)$ preserves Pólya frequency functions if and only if the transforms by F of $t\lambda(x)$ and $x\lambda(x)$ are TN for all $t > 0$, as are the transforms of Gaussians cG_1 for all $c > 0$, as well as that of a single function M as in Lemma 8.4.

We also note that Theorem 1.1 for TN preservers was proved by different means, in the context of Hankel positivity preservers, in [9, Section 5]. By comparison, the proof given here has clear benefits, including completing the classification in every fixed size and isolating a small set of matrices on which the preservation of the TN property can be tested. Our present approach also leads to the classification of preservers of total positivity for matrices of a prescribed size, as well as classifications of the preservers when restricted to symmetric matrices.

List of symbols. For the convenience of the reader, we list some of the symbols used in this paper.

- $\mathbb{N}$ and $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$ denote the sets of positive integers and non-negative integers, respectively.
- For any $n \in \mathbb{N}$, $[n]$ denotes the set $\{1, \ldots, n\}$.
- Given a totally ordered set X and $n \in \mathbb{N}$, the set $X^{n,\uparrow}$ comprises all increasing n-tuples $(x_1, \ldots, x_n)$ in X, so that $x_1 < \cdots < x_n$.
- Let X and Y be totally ordered sets and suppose $K : X \times Y \to \mathbb{R}$. Given tuples $\mathbf{x} \in X^{m,\uparrow}$ and $\mathbf{y} \in Y^{n,\uparrow}$, $K[\mathbf{x}; \mathbf{y}]$ denotes the $m \times n$ matrix with (i, j) entry $K(x_i, y_j)$.

- For totally ordered sets X and Y,

$$\mathscr{F}_{X,Y}^{\mathrm{TN}} := \left\{ F : [0,\infty) \to \mathbb{R} \middle| \begin{array}{l} \text{if } K : X \times Y \to \mathbb{R} \text{ is totally non-negative,} \\ \text{so is } F \circ K \end{array} \right\}$$

and

$$\mathscr{F}_{X,Y}^{\mathrm{TP}} := \left\{ F : (0,\infty) \to \mathbb{R} \middle| \begin{array}{l} \text{if } K : X \times Y \to \mathbb{R} \text{ is totally positive,} \\ \text{so is } F \circ K \end{array} \right\}.$$

- Given $p \in \mathbb{N}$, TN_p and TP_p are the sets of matrices or kernels whose submatrices of order $d \times d$ have non-negative or positive determinants, respectively, for all $d \in [p]$.
- Given a set X and a domain $I \subseteq \mathbb{R}$,

$$\mathscr{F}_X^{\mathrm{psd}}(I) := \left\{ f : I \to \mathbb{R} \middle| \begin{array}{l} \text{if } K : X \times X \to I \text{ is positive semidefinite,} \\ \text{so is } f \circ K \end{array} \right\}$$

and

$$\mathscr{F}_X^{\mathrm{pd}}(I) := \{ f : I \to \mathbb{R} \mid \text{if } K : X \times X \to I \text{ is positive definite, so is } f \circ K \}.$$

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