# Discrete-time zero-sum games for Markov chains with risk-sensitive average cost criterion 

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#### Abstract

We study zero-sum stochastic games for controlled discrete time Markov chains with risk-sensitive average cost criterion with countable/compact state space and Borel action spaces. The payoff function is nonnegative and possibly unbounded for countable state space case and for compact state space case it is a real-valued and bounded function. For countable state space case, under a certain Lyapunov type stability assumption on the dynamics we establish the existence of the value and a saddle point equilibrium. For compact state space case we establish these results without any Lyapunov type stability assumptions. Using the stochastic representation of the principal eigenfunction of the associated optimality equation, we completely characterize all possible saddle point strategies in the class of stationary Markov strategies. Also, we present and analyze an illustrative example. © 2022 Elsevier B.V. All rights reserved.


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## 1. Introduction

We address a risk-sensitive discrete-time zero-sum game with long-run (or ergodic) cost criterion where the underlying state dynamics is given by a controlled Markov processes determined by a prescribed transition kernel. The state space is a denumerable/compact set,

[^0]actions spaces are Borel spaces and the cost function is possibly unbounded for countable state space case and for compact state space case it is a real-valued and bounded function. In [7] this problem is studied with bounded cost under a uniform ergodicity condition. Here we have extended the results of [7] to the case with unbounded cost. This is carried out under a certain Lyapunov type stability condition. Also, we have extended the results of [7] to a compact state space case.

In the risk-neutral criterion, players consider the expected value of the total cost, but in the risk-sensitive criterion, they consider the expected value of the exponential of the total costs. As a result, the risk-sensitive criterion provides comprehensive protection from the risk since it captures the effects of the higher order moments of the cost as well as its expectation; for more details see [54]. We refer to [30,56] for risk-neutral Markov decision processes (MDP), and $[28,29,52]$ for stochastic games with risk-neutral criterion.

The analysis of stochastic systems with the risk-sensitive average criteria can be traced back to the seminal papers by Jacobson in [37] and Howard and Matheson in [36]. The literature on risk-sensitive MDP under different cost criteria is quite extensive, e.g., [1,9,13-16,20,21,27,31$33,36,41,42,51,54]$. The corresponding literature on discrete-time ergodic risk-sensitive games can be found in $[7,8,10,53]$. In this respect we mention some interesting works, $[6,38,39]$ studying multiplicative ergodic theorem for geometrically stable Markov processes. In [39, p. 77 , sec. 2.4] authors made a strong connection between ergodic theory and Perron-Frobenius eigenvalue theory. For the classical approach to study risk-sensitive ergodic control problem based on equivalent game formulation, one can see [24]. In [15], the authors studied risksensitive ergodic cost criterion for discrete-time MDP with bounded cost using a simultaneous Doeblin condition on a countable state space. Also, see [1,14] and the references therein for multiplicative ergodic theory. These papers used eigenvalue approach to study risk-sensitive ergodic control problem. Ergodic problem for controlled Markov processes refers to the problem of minimizing a time average cost over an infinite time horizon. Hence the cost over any finite initial time segment does not affect the ergodic cost. This makes the analysis of ergodic problem analytically more difficult. The authors in [27,49] used the results of $[38,39]$ to study their risk-sensitive ergodic control problems. Also, in the context of controlled diffusions, eigenvalue approach is used in [3-5,12] to study the risk-sensitive ergodic control problems. The articles [7,10] address zero-sum risk-sensitive stochastic games for discrete-time Markov chains with discounted as well as ergodic cost criteria. The analysis of the ergodic cost criterion in [7] is carried out using vanishing discount asymptotics. The results of the article [7] are extended to the general state space case in [10]. In [10], the ergodic cost criterion is studied under a local minorization property and a Lyapunov condition. The analogous results in continuous time setup are carried out in [26]. The corresponding nonzero-sum risksensitive ergodic stochastic games for discrete-time Markov chains are studied in [8,53]. The papers $[7,10,26]$ studied the game problems under the assumption that the running cost is a bounded function, but in many real-life situations the cost functions may be unbounded, for example in inventory control, queuing control etc.

In this article, we study the stochastic game problems for ergodic cost criterion by analyzing the principal eigenpair of the associated Shapley equation. The analysis of our ergodic game problems is inspired from the work of [1,13]. In [13], the authors studied risk-sensitive discrete/continuous-time ergodic control problem for controlled Markov processes with countable state space. They established the existence of a principal eigenpair of the associated ergodic HJB equation. For this, they first studied the corresponding Dirichlet eigenvalue problems on finite set and then pass to the limit by increasing the finite sets to countable state space.

In [1], authors used a novel technique to provide a variational formula for infinite horizon risk-sensitive reward on a compact state and action spaces. They build a nonlinear version of Krein-Rutman theorem to study the corresponding ergodic HJB equation which leads to the existence of optimal ergodic control.

In the literature, the Krein-Rutman theorem has been studied extensively, see [1,2,40, $43,44,46-48,55$ ] and the references therein. In the pioneering works of Perron [50] and Frobenius [25], it was proved that the spectral radius of a nonnegative square matrix is an eigenvalue with a nonnegative eigenvector. In [40], Krěin-Rutman extended the results of Perron and Frobenius's theory to a positive compact linear operator, which is the celebrated Krein-Rutman theorem. For Krein-Rutman theorem of a linear/nonlinear operator on ordered Banach space (under different set of conditions), see [1,2,44,46-48,55] and the references therein.

In this manuscript, using a nonlinear version of the Krein-Rutman theorem, we establish the existence of a principal eigenpair to the associated Shapley equations for both countable/compact state space cases. Under a certain condition, for both countable/compact state space, we show that the principal eigenvalues are the values of the corresponding games. Also, we establish the existence of a saddle-point equilibrium via the outer maximizing/minimizing selectors of the associated Shapley equations. Additionally, we give a complete characterization of all possible saddle-point strategies in the space of stationary Markov strategies.

The rest of this article is arranged as follows. Section 2 deals with problem description and preliminaries. In Section 3, we study Dirichlet eigenvalue problems. In Section 4, we show that the risk-sensitive optimality equation (i.e., Shapley equation) has a solution, obtain the value of the game and saddle-point equilibrium in the class of stationary Markov strategies. We also completely characterize all possible saddle point strategies in the class of stationary strategies in this section. In Section 5, we present an illustrative example. In the next section, we study the same problem on compact state space. Section 7 concludes the paper with some concluding remarks.

## 2. The game model

In this section we introduce a discrete-time zero-sum stochastic game model which consists of the following elements

$$
\begin{equation*}
\{S, U, V,(U(i) \subset U, V(i) \subset V, i \in S), P(\cdot \mid i, u, v), c(i, u, v)\} \tag{2.1}
\end{equation*}
$$

Here $S$ is the state space which is assumed to be the set of all nonnegative integers endowed with the discrete topology of our controlled Markov processes $\boldsymbol{X}:=\left\{X_{0}, X_{1}, \ldots\right\} ; U$ and $V$ are action spaces for players 1 and 2 , respectively. The action spaces $U$ and $V$ are assumed to be Borel spaces with the Borel $\sigma$-algebras $\mathcal{B}(U)$ and $\mathcal{B}(V)$, respectively. For each $i \in S$, $U(i) \in \mathcal{B}(U)$ and $V(i) \in \mathcal{B}(V)$ denote the sets of admissible actions for players 1 and 2 , respectively, when the system is at state $i$. For any metric space $Y$, let $\mathcal{P}(Y)$ denote the space of probability measures on $\mathcal{B}(Y)$ with Prohorov topology. Next $P: \mathcal{K} \rightarrow \mathcal{P}(S)$ is a transition (stochastic) kernel, where $\mathcal{K}:=\{(i, u, v) \mid i \in S, u \in U(i), v \in V(i)\}$, a Borel subset of $S \times U \times V$. We assume that the function $P(j \mid i, u, v)$ is continuous in $(u, v) \in U(i) \times V(i)$ for any fixed $i, j \in S$. Finally, the function $c: \mathcal{K} \rightarrow \mathbb{R}_{+}$denotes the cost function which is assumed to be continuous in $(u, v) \in U(i) \times V(i)$ for any fixed $i \in S$.

The game evolves as follows. When the state $i \in S$ at time $t \in \mathbb{N}_{0}:=\{0,1, \ldots\}$, players independently choose actions $u_{t} \in U(i)$ and $v_{t} \in V(i)$ according to some strategies, respectively. As a consequence of this, the following happens:

- player 1 incurs an immediate cost $c\left(i, u_{t}, v_{t}\right)$ and player 2 receives a reward $c\left(i, u_{t}, v_{t}\right)$;
- the system moves to a new state $j \neq i$ with the probability determined by $P\left(j \mid i, u_{t}, v_{t}\right)$.

When the state of the system transits to a new state $j$, the above procedure repeats. Both the players have full information of past and present states and past actions of both players. The goal of player 1 is to minimize his/her accumulated costs, whereas that of player 2 is to maximize the same with respect to some performance criterion $\mathcal{J} \cdot(\cdot, \cdot)$, which in our present case is defined by (2.3). At each stage, the players choose their actions on the basis of accumulated information. The available information for decision making at time $t \in \mathbb{N}_{0}$, i.e., the history of the process up to time $t$ is given by

$$
h_{t}:=\left(i_{0}^{\prime},\left(u_{0}, v_{0}\right), i_{1}^{\prime},\left(u_{1}, v_{1}\right), \ldots, i_{t-1}^{\prime},\left(u_{t-1}, v_{t-1}\right), i_{t}^{\prime}\right),
$$

where $H_{0}=S, H_{t}=H_{t-1} \times(U \times V \times S), \ldots, H_{\infty}=(U \times V \times S)^{\infty}$ are the history spaces. An admissible strategy for player 1 is a sequence $\pi^{1}:=\left\{\pi_{t}^{1}: H_{t} \rightarrow \mathcal{P}(U)\right\}_{t \in \mathbb{N}_{0}}$ of stochastic kernels satisfying $\pi_{t}^{1}\left(U\left(X_{t}\right) \mid h_{t}\right)=1$, for all $h_{t} \in H_{t} ; t \geq 0$, where $\left\{X_{t}\right\}$ is the state process. The set of all such strategies for player 1 is denoted by $\Pi_{a d}^{1}$. A strategy for player 1 is called a Markov strategy i if

$$
\pi_{t}^{1}\left(\cdot \mid h_{t-1}, u, v, i\right)=\pi_{t}^{1}\left(\cdot \mid h_{t-1}^{\prime}, u^{\prime}, v^{\prime}, i\right)
$$

for all $h_{t-1}, h_{t-1}^{\prime} \in H_{t-1}, u, u^{\prime} \in U, v, v^{\prime} \in V, i \in S, t \in \mathbb{N}_{0}$. Thus a Markov strategy for player 1 can be identified with a sequence of maps, denoted by $\pi^{1} \equiv\left\{\pi_{t}^{1}: S \rightarrow \mathcal{P}(U)\right\}_{t \in \mathbb{N}_{0}}$. A Markov strategy $\left\{\pi_{t}^{1}\right\}$ is called stationary Markov for player 1, if it does not have any explicit time dependence, i.e., $\pi_{t}^{1}\left(\cdot \mid h_{t}\right)=\tilde{\phi}\left(\cdot \mid i_{t}^{\prime}\right)$ for all $h_{t} \in H_{t}$ for some mapping $\tilde{\phi}$ satisfying $\tilde{\phi}(U(i) \mid i)=1$ for all $i \in S$. The set of all Markov strategies and all stationary Markov strategies for player 1, are denoted by $\Pi_{M}^{1}$ and $\Pi_{S M}^{1}$, respectively. Similarly, the set of all admissible strategies, Markov strategies and stationary Markov strategies for player 2 are defined similarly and denoted by $\Pi_{a d}^{2}, \Pi_{M}^{2}$, and $\Pi_{S M}^{2}$, respectively. For each $i, j \in S, \mu \in \mathcal{P}(U(i))$ and $v \in \mathcal{P}(V(i))$, the cost function $c$ and the transition kernel $P$ are extended as follows:

$$
\begin{aligned}
& c(i, \mu, v):=\int_{V(i)} \int_{U(i)} c(i, u, v) \mu(d u) v(d v) \\
& P(j \mid i, \mu, v):=\int_{V(i)} \int_{U(i)} P(j \mid i, u, v) \mu(d u) \nu(d v)
\end{aligned}
$$

(by an abuse of notation we use the same notation $c$ and $P$ ). For a given initial distribution $\tilde{\pi}_{0} \in \mathcal{P}(S)$ and a pair of strategies $\left(\pi^{1}, \pi^{2}\right) \in \Pi_{a d}^{1} \times \Pi_{a d}^{2}$, by Tulcea's Theorem (see Proposition 7.28 of [11]), there exists unique probability measure $P_{\tilde{\pi}_{0}}^{\pi^{1}, \pi^{2}}$ on $(\Omega, \mathcal{B}(\Omega))$, where $\Omega=(S \times U \times V)^{\infty}$. When $\tilde{\pi}_{0}=\delta_{i}, i \in S$ this probability measure is simply written by $P_{i}^{\pi^{1}, \pi^{2}}$ satisfying

$$
\begin{equation*}
P_{i}^{\pi^{1}, \pi^{2}}\left(X_{0}=i\right)=1 \text { and } P_{i}^{\pi^{1}, \pi^{2}}\left(X_{t+1} \in A \mid H_{t}, \pi_{t}^{1}, \pi_{t}^{2}\right)=P\left(A \mid X_{t}, \pi_{t}^{1}, \pi_{t}^{2}\right) \forall A \in \mathcal{B}(S) \tag{2.2}
\end{equation*}
$$

Let $E_{i}^{\pi^{1}, \pi^{2}}$ denote the expectation with respect to the probability measure $P_{i}^{\pi^{1}, \pi^{2}}$. Now from [34, p. 6], we know that under any $\left(\pi^{1}, \pi^{2}\right) \in \Pi_{M}^{1} \times \Pi_{M}^{2}$, the corresponding stochastic process $X$ is strong Markov.

We now introduce some useful notations.

## Notations:

For any finite set $\tilde{\mathcal{D}} \subset S$, we define $\mathcal{B}_{\tilde{D}}=\{f: S \rightarrow \mathbb{R} \mid f$ is Borel measurable and $f(i)=$ $\left.0 \forall i \in \tilde{\mathcal{D}}^{c}\right\}, \mathcal{B}_{\tilde{\mathcal{D}}}^{+} \subset \mathcal{B}_{\tilde{\mathcal{D}}}$ denotes the cone of all nonnegative functions vanishing outside $\tilde{\mathcal{D}}$. Given any real-valued function $\mathcal{V} \geq 1$ on $S$, we define a Banach space $\left(L_{\mathcal{V}}^{\infty},\|\cdot\|_{\mathcal{V}}^{\infty}\right)$ of $\mathcal{V}$-weighted functions by

$$
L_{\mathcal{V}}^{\infty}=\left\{f: S \rightarrow \mathbb{R} \mid\|f\|_{\mathcal{V}}^{\infty}:=\sup _{i \in S} \frac{|f(i)|}{\mathcal{V}(i)}<\infty\right\}
$$

For any ordered Banach space $\tilde{\mathcal{X}}$, a subset $\tilde{\mathcal{C}} \subset \tilde{\mathcal{X}}$ and $x, y \in \tilde{\mathcal{X}}$, we define $\succeq$ as $x \succeq y \Leftrightarrow$ $x-y \in \tilde{\mathcal{C}}$, i.e., the partial ordering in $\tilde{\mathcal{X}}$ with respect to the cone $\tilde{\mathcal{C}}$. For any subset $\underset{\hat{\mathcal{B}}}{\overline{\mathcal{D}}} \subset S$, $\check{\tau}(\hat{\mathcal{B}})=\inf \left\{t: X_{t} \in \hat{\mathcal{B}}\right\}$, i.e., the first entry time of $X_{t}$ to $\hat{\mathcal{B}}$. Also, for any subset $\tilde{\mathcal{D}} \subset S$, $\tau(\tilde{\mathcal{D}}):=\inf \left\{t>0: X_{t} \notin \tilde{\mathcal{D}}\right\}$ denotes the first exit time from $\tilde{\mathcal{D}}$.

We now introduce the cost evaluation criterion.
Ergodic cost criterion: Now we define the risk-sensitive average cost criterion for zerosum discrete-time games. Let $\theta>0$ be the risk-sensitive parameter. For each $i \in S$ and any $\left(\pi^{1}, \pi^{2}\right) \in \Pi_{a d}^{1} \times \Pi_{a d}^{2}$, the risk-sensitive ergodic cost criterion is given by

$$
\begin{equation*}
\mathcal{J}^{\pi^{1}, \pi^{2}}(i, c):=\limsup _{T \rightarrow \infty} \frac{1}{T} \ln E_{i}^{\pi^{1}, \pi^{2}}\left[e^{\theta \sum_{t=0}^{T-1} c\left(X_{t}, \pi_{t}^{1}, \pi_{t}^{2}\right)}\right] \tag{2.3}
\end{equation*}
$$

Since the risk-sensitive parameter remains the same throughout, we assume without loss of generality that $\theta=1$. The lower value and upper value of the game, are functions on $S$, defined as
$\mathcal{L}(i):=\sup _{\pi^{2} \in \Pi_{a d}^{2}} \inf _{\pi^{1} \in \Pi_{a d}^{1}} \partial^{\pi^{1}, \pi^{2}}(i, c)$ and $\mathcal{U}(i):=\inf _{\pi^{1} \in \Pi_{a d}^{1}} \sup _{\pi^{2} \in \Pi_{a d}^{2}} \partial^{\pi^{1}, \pi^{2}}(i, c)$ respectively. It is easy to see that

$$
\mathcal{L}(i) \leq \mathcal{U}(i) \text { for all } i \in S .
$$

If $\mathcal{L}(i)=\mathcal{U}(i)$ for all $i \in S$, then the common function is called the value of the game and is denoted by $\mathcal{J}^{*}(i)$. A strategy $\pi^{* 1}$ in $\Pi_{a d}^{1}$ is said to be optimal for player 1 if

$$
\mathcal{J}^{\pi^{* 1}, \pi^{2}}(i, c) \leq \sup _{\pi^{2} \in \Pi_{a d}^{2}} \inf _{\pi^{1} \in \Pi_{a d}^{1}} \mathcal{J}^{1}, \pi^{2}(i)=\mathcal{L}(i) \forall i \in S, \forall \pi^{2} \in \Pi_{a d}^{2}
$$

Similarly, $\pi^{* 2} \in \Pi_{a d}^{2}$ is optimal for player 2 if

$$
\partial^{\pi^{1}, \pi^{* 2}}(i, c) \geq \inf _{\pi^{1} \in \Pi_{a d}^{1}} \sup _{\pi^{2} \in \Pi_{a d}^{2}} \partial^{\pi^{1}, \pi^{2}}(i)=\mathcal{U}(i) \forall i \in S, \forall \pi^{1} \in \Pi_{a d}^{1}
$$

If $\pi^{* k} \in \Pi_{a d}^{k}$ is optimal for player $\mathrm{k}(\mathrm{k}=1,2)$, then $\left(\pi^{* 1}, \pi^{* 2}\right)$ is called a pair of optimal strategies. The pair of strategies $\left(\pi^{* 1}, \pi^{* 2}\right)$ at which this value is attained i.e., if

$$
\mathcal{J}^{\pi^{* 1}, \pi^{2}}(i, c) \leq \mathcal{J}^{\pi^{* 1}, \pi^{* 2}}(i, c) \leq \mathcal{J}^{\pi^{1}, \pi^{* 2}}(i, c), \quad \forall \pi^{1} \in \Pi_{a d}^{1}, \forall \pi^{2} \in \Pi_{a d}^{2},
$$

then the pair $\left(\pi^{* 1}, \pi^{* 2}\right)$ is called a saddle-point equilibrium, and then $\pi^{* 1}$ and $\pi^{* 2}$ are optimal for player 1 and player 2 , respectively.

Following [7], the Shapley equation for the above problem is given by

$$
\begin{aligned}
e^{\rho} \psi(i) & =\sup _{v \in \mathcal{P}(V(i))} \inf _{\mu \in \mathcal{P}(U(i))}\left[e^{c(i, \mu, v)} \sum_{j \in S} \psi(j) P(j \mid i, \mu, v)\right] \\
& =\inf _{\mu \in \mathcal{P}(U(i)))} \sup _{v \in \mathcal{P}(V(i))}\left[e^{c(i, \mu, v)} \sum_{j \in S} \psi(j) P(j \mid i, \mu, \nu)\right], i \in S .
\end{aligned}
$$

In the above equation, $\rho$ is a scalar and $\psi$ is an appropriate function.
Our goal is to establish the existence of a saddle-point equilibrium among the class of admissible history-dependent strategies and provide its complete characterization. We now describe briefly our technique for establishing the existence of a saddle-point equilibrium. We first construct an increasing sequence of bounded subsets of the state space $S$. Then we apply Kre i n-Rutman theorem [2] on each bounded subset to obtain a bounded solution of the corresponding Dirichlet eigenvalue problem, i.e., a solution to the above equation on each finite subset with the condition that the solution is zero in the complement of that subset. Using a suitable Lyapunov stability condition (to be stated shortly), we pass to the limit and show that risk-sensitive zero sum ergodic optimality equation admits a principal eigenpair. Subsequently we establish a stochastic representation of the principal eigenfunction. This enables us to characterize all possible saddle point equilibria in the space of stationary Markov strategies. To this end we make certain assumptions. First we define a norm-like function which is used in our assumptions.

Definition 2.1. A function $f: S \rightarrow \mathbb{R}$ is said to be norm-like if for every $k \in \mathbb{R}$, the set $\{i \in S: f(i) \leq k\}$ is either empty or finite.

Since the cost function (i.e., $c(i, u, v)$ ) may be unbounded, to guarantee the finiteness of $g^{1}, \pi^{2}(i, c)$, we use the following assumption.

Assumption 2.1. We assume that the Markov chain $\left\{X_{t}\right\}_{t \geq 0}$ is irreducible under every pair of stationary Markov strategies $\left(\pi^{1}, \pi^{2}\right) \in \Pi_{S M}^{1} \times \Pi_{S M}^{2}$. Also, assume that there exist a constant $\tilde{C}>0$, a real-valued function $\mathcal{W} \geq 1$ on $S$ and, a finite set $\tilde{\mathcal{K}}$ such that one of the following hold.
(a) If the running cost is bounded: For some positive constant $\tilde{\gamma}>\|c\|_{\infty}$, we have the following blanket stability condition

$$
\begin{equation*}
\sup _{(u, v) \in U(i) \times V(i)} \sum_{j \in S} \mathcal{W}(j) P(j \mid i, u, v) \leq \tilde{C} I_{\tilde{\mathcal{K}}}(i)+e^{-\tilde{\gamma} \mathcal{W}(i) \forall i \in S, ~, ~, ~} \tag{2.4}
\end{equation*}
$$

where $\|c\|_{\infty}:=\sup _{(i, u, v) \in \mathcal{K}} c(i, u, v)$.
(b) If the running cost is unbounded: For some real-valued nonnegative norm-like function $\tilde{\ell}$ on $S$ it holds that

$$
\begin{equation*}
\sup _{(u, v) \in U(i) \times V(i)} \sum_{j \in S} \mathcal{W}(j) P(j \mid i, u, v) \leq \tilde{C} I_{\tilde{\mathcal{K}}}(i)+e^{-\tilde{\ell}(i)} \mathcal{W}(i) \forall i \in S, \tag{2.5}
\end{equation*}
$$

where the function $\tilde{\ell}(\cdot)-\max _{(u, v) \in U(\cdot) \times V(\cdot)} c(\cdot, u, v)$ is norm-like.
Assumption 2.1 and its variants are key conditions of standard ergodicity hypothesis, see [13,30,45]. In this context, [15] used Doeblin condition, a stronger assumption than a variant of Assumption 2.1(a) to study ergodic control problems. The condition (2.5) plays important role
in studying the ergodic optimal control problems with unbounded running cost. We show that, (2.5) implies (2.3) is finite. Similar condition is also used in [6, Theorem 1.2], [14, Theorem $2.2]$ in the study of multiplicative ergodicity. Also, we refer [17-19,53] to see the importance of Lyapunov stability assumption in studying stochastic control problem.

Let $i_{0} \in S$ be a fixed state, we call it as the reference state. Consider an increasing sequence of finite subsets $\tilde{\mathcal{D}}_{n} \subset S$ such that $\cup_{n=1}^{\infty} \tilde{\mathcal{D}}_{n}=S$ and $i_{0} \in \tilde{\mathcal{D}}_{n}$ for all $n \in \mathbb{N}$. Recall that $\tau\left(\tilde{\mathcal{D}}_{n}\right):=\inf \left\{t>0: X_{t} \notin \tilde{\mathcal{D}}_{n}\right\}$, is the first exit time from $\tilde{\mathcal{D}}_{n}$. For our game problem, we wish to establish the existence of a saddle-point equilibrium in the space of stationary Markov strategies. To ensure the existence of saddle-point equilibrium, we make the following assumptions.

## Assumption 2.2.

(i) The admissible action spaces $U(i)(\subset U)$ and $V(i)(\subset V)$ are compact for each $i \in S$.
(ii) We assume that for any $n$ and any pair $i, j \in \tilde{\mathcal{D}}_{n}$, the probability of hitting $j$ from $i$ before exiting $\tilde{\mathcal{D}}_{n}$ is bounded from below by some $\delta_{i j, n}>0$ under all stationary Markov strategies i.e.,

$$
\begin{equation*}
\inf _{\left(\pi^{1}, \pi^{2}\right) \in \Pi_{S M}^{1} \times \Pi_{S M}^{2}} P_{i}^{\pi^{1}, \pi^{2}}\left(\check{\tau}_{j}<\tau\left(\tilde{\mathcal{D}}_{n}\right)\right) \geq \delta_{i j, n} \tag{2.6}
\end{equation*}
$$

where $\check{\tau}_{j}$ denotes the hitting time to $j$ i.e., for any pair $i, j \in \tilde{\mathcal{D}}_{n}$, under any pair of strategies $\left(\pi^{* 1}, \pi^{* 2}\right) \in \Pi_{S M}^{1} \times \Pi_{S M}^{2}$, there exists $i_{1}, i_{2}, \ldots, i_{m} \in \tilde{\mathcal{D}}_{n}$ satisfying

$$
\begin{equation*}
P\left(j \mid i_{m}, \pi^{* 1}\left(i_{m}\right), \pi^{* 2}\left(i_{m}\right)\right) P\left(i_{m} \mid i_{m-1}, \pi^{* 1}\left(i_{m-1}\right), \pi^{* 2}\left(i_{m-1}\right)\right) \cdots P\left(i_{1} \mid i, \pi^{* 1}(i), \pi^{* 2}(i)\right)>0 . \tag{2.7}
\end{equation*}
$$

(iii) $(i, u, v) \rightarrow \sum_{j \in S} \mathcal{W}(j) P(j \mid i, u, v)$ is continuous in $(u, v) \in U(i) \times V(i)$, where $\mathcal{W}$ is the Lyapunov function defined in Assumption 2.1.

## Remark 2.1.

(1) Assumption 2.2(i) and (iii) are standard continuity-compactness assumption.
(2) Under Assumption 2.2(i), for each $i \in S$, by in [11, Proposition 7.22, p. 130], we know that $\mathcal{P}(U(i))$ and $\mathcal{P}(V(i))$ are compact and metrizable. Note that $\pi^{1} \in \Pi_{S M}^{1}$ can be identified with a map $\pi^{1}: S \rightarrow \mathcal{P}(U)$ such that $\pi^{1}(\cdot \mid i) \in \mathcal{P}(U(i))$ for each $i \in S$. Thus, we have $\Pi_{S M}^{1}=\Pi_{i \in S} \mathcal{P}(U(i))$. Similarly, $\Pi_{S M}^{2}=\Pi_{i \in S} \mathcal{P}(V(i))$. Therefore by Tychonoff theorem, the sets $\Pi_{S M}^{1}$ and $\Pi_{S M}^{2}$ are compact metric spaces endowed with the product topology. Also, it is clear that these sets are convex.
(3) Instead of using (2.6), we can assume $\inf _{\left(\pi^{1}, \pi^{2}\right) \in \Pi_{S M}^{1} \times \Pi_{S M}^{2}} P_{i}^{\pi^{1}, \pi^{2}}\left(\check{\tau}_{j}<\tau\left(\tilde{\mathcal{D}}_{n}\right)\right)>0$. Then this weaker condition also implies that $\psi_{n}>0$, (see Lemma 3.3).

Using generalized Fatou's lemma as in [23], [35, Lemma 8.3.7], from Assumption 2.2 one can easily get the following result, which will be used in subsequent sections; we omit the details.

Lemma 2.1. Under Assumptions 2.1 and 2.2, the functions $\sum_{j \in S} P(j \mid i, \mu, v) f(j)$ and $c(i, \mu, \nu)$ are continuous at $(\mu, \nu)$ on $\mathcal{P}(U(i)) \times \mathcal{P}(V(i))$ for each fixed $f \in L_{\mathcal{W}}^{\infty}$ and $i \in S$.

## 3. Dirichlet eigenvalue problems

We begin this section by stating a version of the nonlinear Krein-Rutman theorem from [2, Section 3.1], (cf. [40]) which plays a crucial role in our analysis of the Dirichlet eigenvalue problems.

Theorem 3.1. Let $\tilde{\mathcal{X}}$ be an ordered Banach space and $\tilde{\mathcal{C}}$ a nonempty closed subset of $\tilde{\mathcal{X}}$ satisfying $\tilde{\mathcal{X}}=\tilde{\mathcal{C}}-\tilde{\mathcal{C}}$. Let $\tilde{T}: \tilde{\mathcal{X}} \rightarrow \tilde{\mathcal{X}}$ be a 1-homogeneous, order-preserving, continuous, and compact map satisfying the property that for some nonzero $\zeta \in \tilde{\mathcal{C}}$ and $\hat{N}>0$, we have $\hat{N} \tilde{T}(\zeta) \succeq \zeta$. Then there exists a nontrivial $\hat{f} \in \tilde{\mathcal{C}}$ and a scalar $\tilde{\lambda}>0$, such that $\tilde{T} \hat{f}=\tilde{\lambda} \hat{f}$.

In the following lemma we establish a few important estimates which will play crucial role in our analysis.

Lemma 3.1. Suppose that Assumption 2.1 holds. Let $\tilde{\mathcal{B}} \supset_{\tilde{\mathcal{B}}} \tilde{\mathcal{K}}$ be a finite subset of $S$ and let $\check{\tau}(\tilde{\mathcal{B}})=\inf \left\{t: X_{t} \in \tilde{\mathcal{B}}\right\}$, be the first entry time of $X_{t}$ to $\tilde{\mathcal{B}}$. Then for any pair of strategies $\left(\pi^{1}, \pi^{2}\right) \in \Pi_{a d}^{1} \times \Pi_{a d}^{2}$ we have the following:
(i) If Assumption 2.1(a) holds: Then

$$
\begin{equation*}
E_{i}^{\pi^{1}, \pi^{2}}\left[e^{\tilde{\gamma} \check{\imath}(\tilde{\mathcal{B}})} \mathcal{W}\left(X_{\tilde{\tau}(\tilde{\mathcal{B}})}\right)\right] \leq \mathcal{W}(i) \forall i \in \tilde{\mathcal{B}}^{c} . \tag{3.1}
\end{equation*}
$$

(ii) If Assumption 2.1(b) holds:

$$
\begin{equation*}
E_{i}^{\pi^{1}, \pi^{2}}\left[e^{\sum_{s=0}^{\check{\tau}(\tilde{\mathcal{B}})-1} \tilde{\ell}\left(X_{s}\right)} \mathcal{W}\left(X_{\check{\tau}(\tilde{\mathcal{B}})}\right)\right] \leq \mathcal{W}(i) \forall i \in \tilde{\mathcal{B}}^{c} . \tag{3.2}
\end{equation*}
$$

Proof. This result is proved in [13, Lemma 2.3] for one controller case. The proof for two controller case is analogous.

Now we prove the following existence result which is useful in establishing the existence of a Dirichlet eigenpair.

Proposition 3.1. Suppose Assumption 2.2 holds. Take any function $\bar{c}: \mathcal{K} \rightarrow \mathbb{R}$ which is continuous in $(u, v) \in U(i) \times V(i)$ for each fixed $i \in S$, satisfying the relation $\bar{c}<-\delta$ in $\tilde{\mathcal{D}}_{n}$, where $\delta>0$ is a constant and $\tilde{\mathcal{D}}_{n}$ is a finite set as described previously. Then for any $g \in \mathcal{B}_{\tilde{D}_{n}}$, there exits a unique solution $\varphi \in \mathcal{B}_{\tilde{D}_{n}}$ to the following nonlinear equation

$$
\begin{align*}
\varphi(i) & =\inf _{\mu \in \mathcal{P}(U(i))} \sup _{v \in \mathcal{P}(V(i))}\left[e^{\bar{c}(i, \mu, \nu)} \sum_{j \in S} \varphi(j) P(j \mid i, \mu, \nu)+g(i)\right] \\
& =\sup _{v \in \mathcal{P}(V(i))} \inf _{\mu \in \mathcal{P}(U(i))}\left[e^{\bar{c}(i, \mu, \nu)} \sum_{j \in S} \varphi(j) P(j \mid i, \mu, \nu)+g(i)\right] \forall i \in \tilde{\mathcal{D}}_{n} . \tag{3.3}
\end{align*}
$$

Moreover, we have

$$
\begin{align*}
\varphi(i) & =\inf _{\pi^{1} \in \Pi_{a d}^{1}} \sup _{\pi^{2} \in \Pi_{a d}^{2}} E_{i}^{\pi^{1}, \pi^{2}}\left[\sum_{t=0}^{\tau\left(\tilde{\mathcal{D}}_{n}\right)-1} e^{\sum_{s=0}^{t-1} \bar{c}\left(X_{s}, \pi_{s}^{1}, \pi_{s}^{2}\right)} g\left(X_{t}\right)\right] \\
& =\sup _{\pi^{2} \in \Pi_{a d}^{2}} \inf _{\pi^{1} \in \Pi_{a d}^{1}} E_{i}^{\pi^{1}, \pi^{2}}\left[\sum_{t=0}^{\tau\left(\tilde{\mathcal{D}}_{n}\right)-1} e^{\sum_{s=0}^{t-1} \bar{c}\left(X_{s}, \pi_{s}^{1}, \pi_{s}^{2}\right)} g\left(X_{t}\right)\right] \forall i \in S, \tag{3.4}
\end{align*}
$$

where $\tau\left(\tilde{\mathcal{D}}_{n}\right):=\inf \left\{t>0: X_{t} \notin \tilde{\mathcal{D}}_{n}\right\}$, first exit time from $\tilde{\mathcal{D}}_{n}$.

Proof. Let $g \in \mathcal{B}_{\tilde{D}_{n}}$. Define a map $\hat{T}: \mathcal{B}_{\tilde{D}_{n}} \rightarrow \mathcal{B}_{\tilde{D}_{n}}$ by

$$
\begin{align*}
& \sup _{v \in \mathcal{P}(V(i))} \inf _{\mu \in \mathcal{P}(U(i))}\left[e^{\bar{c}(i, \mu, v)} \sum_{j \in S} \tilde{\phi}(j) P(j \mid i, \mu, v)+g(i)\right]=\hat{T} \tilde{\phi}(i), i \in \tilde{\mathcal{D}}_{n}, \tilde{\phi} \in \mathcal{B}_{\tilde{\mathcal{D}}}^{n} \\
& \quad \text { and } \hat{T} \tilde{\phi}(i)=0 \text { for } i \in \tilde{\mathcal{D}}_{n}^{c} . \tag{3.5}
\end{align*}
$$

Now, let $\tilde{\phi}_{1}, \tilde{\phi}_{2} \in \mathcal{B}_{\tilde{\mathcal{D}}_{n}}$. Then

$$
\left(\hat{T} \tilde{\phi}_{2}(i)-\hat{T} \tilde{\phi}_{1}(i)\right) \leq \max _{i \in \tilde{\mathcal{D}}_{n}} \sup _{\nu \in \mathcal{P}(V(i))} \sup _{\mu \in \mathcal{P}(U(i))} e^{\bar{c}(i, \mu, \nu)}\left\|\tilde{\phi}_{2}-\tilde{\phi}_{1}\right\|_{\tilde{\mathcal{D}}_{n}}
$$

Similarly, we have

$$
\left(\hat{T} \tilde{\phi}_{1}(i)-\hat{T} \tilde{\phi}_{2}(i)\right) \leq \max _{i \in \tilde{\mathcal{D}}_{n}} \sup _{\nu \in \mathcal{P}(V(i))} \sup _{\mu \in \mathcal{P}(U(i))} e^{\bar{c}(i, \mu, \nu)}\left\|\tilde{\phi}_{2}-\tilde{\phi}_{1}\right\|_{\tilde{\mathcal{D}}_{n}}
$$

Hence

$$
\left\|\hat{T} \tilde{\phi}_{1}(i)-\hat{T} \tilde{\phi}_{2}(i)\right\|_{\tilde{\mathcal{D}}_{n}} \leq \max _{i \in \tilde{\mathcal{D}}_{n}} \sup _{\nu \in \mathcal{P}(V(i))} \sup _{\mu \in \mathcal{P}(U(i))} e^{\bar{c}(i, \mu, \nu)}\left\|\tilde{\phi}_{2}-\tilde{\phi}_{1}\right\|_{\tilde{\mathcal{D}}_{n}}
$$

where for any function $f \in \mathcal{B}_{\tilde{\mathcal{D}}_{n}},\|f\|_{\tilde{\mathcal{D}}_{n}}=\max \left\{|f(i)|: i \in \tilde{\mathcal{D}}_{n}\right\}$. Since $\bar{c}<0$, it is easy to see that $\max _{i \in \tilde{\mathcal{D}}_{n}} \sup _{v \in \mathcal{P}(V(i))} \sup _{\mu \in \mathcal{P}(U(i))} e^{\bar{c}(i, \mu, v)}<1$. Hence $\hat{T}$ is a contraction map. Thus by Banach fixed point theorem, there exists a unique $\varphi \in \mathcal{B}_{\tilde{D}_{n}}$ such that $\hat{T}(\varphi)=\varphi$. Now by applying Fan's minimax theorem in [22, Theorem 3], we get

$$
\begin{aligned}
& \sup _{v \in P(V(i))} \inf _{\mu \in P(U(i))}\left[e^{\bar{c}(i, \mu, \nu)} \sum_{j \in S} \varphi(j) P(j \mid i, \mu, \nu)\right] \\
& \quad=\inf _{\mu \in P(U(i))} \sup _{v \in P(V(i))}\left[e^{\bar{c}(i, \mu, \nu)} \sum_{j \in S} \varphi(j) P(j \mid i, \mu, \nu)\right] .
\end{aligned}
$$

Hence we conclude that (3.3) has unique solution. Now let $\left(\pi_{n}^{* 1}, \pi_{n}^{* 2}\right) \in \Pi_{S M}^{1} \times \Pi_{S M}^{2}$ be a mini-max selector of (3.3), i.e.,

$$
\begin{align*}
\varphi(i) & =\inf _{\mu \in \mathcal{P}(U(i))}\left[e^{\bar{c}\left(i, \mu, \pi_{n}^{* 2}(i)\right)} \sum_{j \in S} \varphi(j) P\left(j \mid i, \mu, \pi_{n}^{* 2}(i)\right)+g(i)\right] \\
& =\sup _{v \in \mathcal{P}(V(i))}\left[e^{\bar{c}\left(i, \pi_{n}^{* 1}(i), v\right)} \sum_{j \in S} \varphi(j) P\left(j \mid i, \pi_{n}^{* 1}(i), v\right)+g(i)\right] . \tag{3.6}
\end{align*}
$$

Now by Dynkin's formula [53, Lemma 3.1], for any $\left(\pi^{1}, \pi^{2}\right) \in \Pi_{a d}^{1} \times \Pi_{a d}^{2}$ and $N \in \mathbb{N}$, we have

$$
\begin{align*}
& E_{i}^{\pi^{1}, \pi^{2}}\left[e^{\sum_{t=0}^{N \wedge \tau\left(\tilde{\mathcal{D}}_{n}\right)-1} \bar{c}\left(X_{t}, \pi_{t}^{1}, \pi_{t}^{2}\right)} \varphi\left(X_{N \wedge \tau\left(\tilde{\mathcal{D}}_{n}\right)}\right)\right]-\varphi(i) \\
& =E_{i}^{\pi^{1}, \pi^{2}}\left[\sum_{t=1}^{N \wedge \tau\left(\tilde{\mathcal{D}}_{n}\right)} e^{\sum_{r=0}^{t-1} \bar{c}\left(X_{r}, \pi_{r}^{1}, \pi_{r}^{2}\right)}\right. \\
& \left.\quad \times\left(\sum_{j \in S} \varphi(j) P\left(j \mid X_{t-1}, \pi_{t-1}^{1}, \pi_{t-1}^{2}\right)-e^{-\bar{c}\left(X_{t-1}, \pi_{t-1}^{1}, \pi_{t-1}^{2}\right)} \varphi\left(X_{t-1}\right)\right)\right] . \tag{3.7}
\end{align*}
$$

Then, using (3.6) and (3.7), we obtain

$$
\begin{aligned}
& E_{i}^{\pi_{n}^{* 1}, \pi^{2}}\left[\sum_{t=0}^{N \wedge \tau\left(\tilde{\mathcal{D}}_{n}\right)-1} e^{\sum_{s=0}^{t-1} \bar{c}\left(X_{s}, \pi_{n}^{* 1}\left(X_{s}\right), \pi_{s}^{2}\right)} g\left(X_{t}\right)\right] \\
& \leq-E_{i}^{\pi_{n}^{* 1}, \pi^{2}}\left[e^{\sum_{s=0}^{N \wedge \tau\left(\tilde{\mathcal{D}}_{n}\right)-1} \bar{c}\left(X_{s}, \pi_{n}^{* 1}\left(X_{s}\right), \pi_{s}^{2}\right)} \varphi\left(X_{N \wedge \tau\left(\tilde{\mathcal{D}}_{n}\right)}\right)\right]+\varphi(i) .
\end{aligned}
$$

Since $\bar{c}<0$ and $\varphi \in \mathcal{B}_{\tilde{D}_{n}}$, taking $N \rightarrow \infty$ in the above equation and using the dominated convergence theorem, we deduce that

$$
\begin{aligned}
& E_{i}^{\pi_{n}^{* 1}, \pi^{2}}\left[\sum_{t=0}^{\tau\left(\tilde{\mathcal{D}}_{n}\right)-1} e^{\sum_{s=0}^{t-1} \bar{c}\left(X_{s}, \pi_{n}^{* 1}\left(X_{s}\right), \pi_{s}^{2}\right)} g\left(X_{t}\right)\right] \\
& \leq-E_{i}^{\pi_{n}^{* 1}, \pi^{2}}\left[e^{\sum_{s=0}^{\tau\left(\tilde{\mathcal{D}}_{n}\right)-1} \bar{c}\left(X_{s}, \pi_{n}^{* 1}\left(X_{s}\right), \pi_{s}^{2}\right)} \varphi\left(X_{\tau\left(\tilde{\mathcal{D}}_{n}\right)}\right)\right]+\varphi(i) .
\end{aligned}
$$

Hence

$$
\varphi(i) \geq E_{i}^{\pi_{n}^{* 1}, \pi^{2}}\left[\sum_{t=0}^{\tau\left(\tilde{\mathcal{D}}_{n}\right)-1} e^{\sum_{s=0}^{t-1} \bar{c}\left(X_{s}, \pi_{n}^{* 1}\left(X_{s}\right), \pi_{s}^{2}\right)} g\left(X_{t}\right)\right] .
$$

Since $\pi^{2} \in \Pi^{2}$ is arbitrary,

$$
\begin{align*}
\varphi(i) & \geq \sup _{\pi^{2} \in \Pi_{a d}^{2}} E_{i}^{\pi_{n}^{* 1}, \pi^{2}}\left[\sum_{t=0}^{\tau\left(\tilde{\mathcal{D}}_{n}\right)-1} e^{\sum_{s=0}^{t-1} \bar{c}\left(X_{s}, \pi_{n}^{* 1}\left(X_{s}\right), \pi_{s}^{2}\right)} g\left(X_{t}\right)\right] \\
& \geq \inf _{\pi^{1} \in \Pi_{a d}^{1} \pi^{2} \in \Pi^{2}} \sup _{i}^{\pi^{1}, \pi^{2}}\left[\sum_{t=0}^{\tau\left(\tilde{\mathcal{D}}_{n}\right)-1} e^{\sum_{s=0}^{t-1} \bar{c}\left(X_{s}, \pi_{s}^{1}, \pi_{s}^{2}\right)} g\left(X_{t}\right)\right] . \tag{3.8}
\end{align*}
$$

By similar arguments, using (3.6), (3.7) and the dominated convergence theorem, we obtain

$$
\begin{align*}
\varphi(i) & \leq \inf _{\pi^{1} \in \Pi_{a d}^{1}} E_{i}^{\pi^{1}, \pi_{n}^{* 2}}\left[\sum_{t=0}^{\tau\left(\tilde{\mathcal{D}}_{n}\right)-1} e^{\sum_{s=0}^{t-1} \bar{c}\left(X_{s}, \pi_{s}^{1}, \pi_{n}^{* 2}\left(X_{s}\right)\right)} g\left(X_{t}\right)\right] \\
& \leq \sup _{\pi^{2} \in \Pi_{a d}^{2}} \inf _{\pi^{1} \in \Pi_{a d}^{1}} E_{i}^{\pi^{1}, \pi^{2}}\left[\sum_{t=0}^{\tau\left(\tilde{\mathcal{D}}_{n}\right)-1} e^{\sum_{s=0}^{t-1} \bar{c}\left(X_{s}, \pi_{s}^{1}, \pi_{s}^{2}\right)} g\left(X_{t}\right)\right] . \tag{3.9}
\end{align*}
$$

Now combining (3.8) and (3.9), we obtain (3.4).
Next using Theorem 3.1, we show that for each $n \in \mathbb{N}$, Dirichlet eigenpair exists in $\tilde{\mathcal{D}}_{n}$. That is we establish the following result.

Lemma 3.2. Suppose Assumptions 2.1 and 2.2 hold. Then there exists an eigenpair $\left(\rho_{n}, \psi_{n}\right) \in$ $\mathbb{R} \times \mathcal{B}_{\tilde{\mathcal{D}}_{n}}^{+}, \psi_{n} \ngtr 0$ on $\tilde{\mathcal{D}}_{n}$, for the following Dirichlet nonlinear eigenequation

$$
\begin{align*}
e^{\rho_{n}} \psi_{n}(i) & =\inf _{\mu \in \mathcal{P}(U(i))} \sup _{v \in \mathcal{P}(V(i))}\left[e^{c(i, \mu, \nu)} \sum_{j \in S} \psi_{n}(j) P(j \mid i, \mu, \nu)\right] \\
& =\sup _{v \in \mathcal{P}(V(i))} \inf _{\mu \in \mathcal{P}(U(i))}\left[e^{c(i, \mu, \nu)} \sum_{j \in S} \psi_{n}(j) P(j \mid i, \mu, v)\right] . \tag{3.10}
\end{align*}
$$

The eigenvalue of the above equation satisfies

$$
\begin{equation*}
\rho_{n} \leq \sup _{\pi^{2} \in \Pi_{a d}^{2}} \inf _{\pi^{1} \in \Pi_{a d}^{1}} g^{\pi^{1}, \pi^{2}}(i, c), \tag{3.11}
\end{equation*}
$$

for all $i \in S$ such that $\psi_{n}(i)>0$.
Proof. For some constant $\delta>0$, let us define $c^{\prime}(i, \mu, \nu)=c(i, \mu, \nu)-k_{n}-\delta$ in $\tilde{\mathcal{D}}_{n}$, where $k_{n}=\sup _{(i, \mu, v) \in \tilde{\mathcal{D}}_{n} \times \mathcal{P}(U(i)) \times \mathcal{P}(V(i))}|c(i, \mu, \nu)|$. Then it is easy to see that $c^{\prime}(i, \mu, \nu)<-\delta$, $\forall(i, \mu, \nu) \in \tilde{\mathcal{D}}_{n}, \times \mathcal{P}(U(i)) \times \mathcal{P}(V(i))$. Now consider a mapping $\bar{T}_{n}: \mathcal{B}_{\tilde{\mathcal{D}}_{n}} \rightarrow \mathcal{B}_{\tilde{\mathcal{D}}_{n}}$ defined by

$$
\begin{equation*}
\bar{T}_{n}(g)(i):=\sup _{\pi^{2} \in \Pi_{a d}^{2}} \inf _{\pi^{1} \in \Pi_{a d}^{1}} E_{i}^{\pi^{1}, \pi^{2}}\left[\sum_{t=0}^{\tau\left(\tilde{\mathcal{D}}_{n}\right)-1} e^{\sum_{s=0}^{t-1} c^{\prime}\left(X_{s}, \pi_{s}^{1}, \pi_{s}^{2}\right)} g\left(X_{t}\right)\right], i \in \tilde{\mathcal{D}}_{n} \tag{3.12}
\end{equation*}
$$

with $\bar{T}_{n}(g)(i)=0$ for $i \in \tilde{\mathcal{D}}_{n}^{c}$, where $g \in \mathcal{B}_{\tilde{\mathcal{D}}_{n}}$.
From Proposition 3.1 it is clear that $\bar{T}_{n}$ is well defined. Since $c^{\prime}<-\delta$, for $g_{1}, g_{2} \in \hat{\mathcal{B}}_{\tilde{D}_{n}}$, it follows that

$$
\left\|\bar{T}_{n}\left(g_{1}\right)-\bar{T}_{n}\left(g_{2}\right)\right\|_{\tilde{\mathcal{D}}_{n}} \leq \alpha_{1}\left\|g_{1}-g_{2}\right\|_{\tilde{\mathcal{D}}_{n}},
$$

for some constant $\alpha_{1}>0$. Hence the map $\bar{T}_{n}$ is continuous.
Let $g_{1}, g_{2} \in \mathcal{B}_{\tilde{D}_{n}}$ with $g_{1} \succeq g_{2}$. Also, let $\bar{T}_{n}\left(g_{k}\right)=\varphi_{k}, k=1$, 2. Thus $\varphi_{2}$ is a solution of

$$
\begin{aligned}
\varphi_{2}(i) & =\sup _{v \in \mathcal{P}(V(i))} \inf _{\mu \in \mathcal{P}(U(i))}\left[e^{c^{\prime}(i, \mu, v)} \sum_{j \in \tilde{\mathcal{D}}_{n}} \varphi_{2}(j) P(j \mid i, \mu, v)+g_{2}(i)\right] \\
& =\inf _{\mu \in \mathcal{P}(U(i))}\left[e^{c^{\prime}\left(i, \mu, \pi_{n}^{* 2}(i)\right)} \sum_{j \in \tilde{\mathcal{D}}_{n}} \varphi_{2}(j) P\left(j \mid i, \mu, \pi_{n}^{* 2}(i)\right)+g_{2}(i)\right] \forall i \in \tilde{\mathcal{D}}_{n},
\end{aligned}
$$

where $\pi_{n}^{* 2} \in \Pi_{S M}^{2}$ is an outer maximizing selector. Therefore

$$
\begin{aligned}
\bar{T}_{n}\left(g_{1}\right)(i)- & \bar{T}_{n}\left(g_{2}\right)(i) \\
= & \sup _{\pi^{2} \in \Pi_{a d}^{2}} \inf _{\pi^{1} \in \Pi_{a d}^{1}} E_{i}^{\pi^{1}, \pi^{2}}\left[\sum_{t=0}^{\tau\left(\tilde{\mathcal{D}}_{n}\right)-1} e^{\sum_{s=0}^{t-1} c^{\prime}\left(X_{s}, \pi_{s}^{1}, \pi_{s}^{2}\right)} g_{1}\left(X_{t}\right)\right] \\
& -\sup _{\pi^{2} \in \Pi_{a d}^{2}} \inf _{\pi^{1} \in \Pi_{a d}^{1}} E_{i}^{\pi^{1}, \pi^{2}}\left[\sum_{t=0}^{\tau\left(\tilde{\mathcal{D}}_{n}\right)-1} e^{\sum_{s=0}^{t-1} c^{\prime}\left(X_{s}, \pi_{s}^{1}, \pi_{s}^{2}\right)} g_{2}\left(X_{t}\right)\right] \\
= & \sup _{\pi^{2} \in \Pi_{a d}^{2} \pi^{1} \in \Pi_{a d}^{1}} \inf _{i}^{\pi^{1}, \pi^{2}}\left[\sum_{t=0}^{\tau\left(\tilde{\mathcal{D}}_{n}\right)-1} e^{\sum_{s=0}^{t-1} c^{\prime}\left(X_{s}, \pi_{s}^{1}, \pi_{s}^{2}\right)} g_{1}\left(X_{t}\right)\right] \\
& -\inf _{\pi^{1} \in \Pi_{a d}^{1}} E_{i}^{\pi^{1}, \pi_{n}^{* 2}}\left[\sum_{t=0}^{\tau\left(\tilde{\mathcal{D}}_{n}\right)-1} e^{\sum_{s=0}^{t-1} c^{\prime}\left(X_{s}, \pi_{s}^{1}, \pi_{n}^{* 2}\left(X_{s}\right)\right)} g_{2}\left(X_{t}\right)\right] \\
\geq & \inf _{\pi^{1} \in \Pi_{a d}^{1}} E_{i}^{\pi^{1}, \pi_{n}^{* 2}}\left[\sum_{t=0}^{\tau\left(\tilde{\mathcal{D}}_{n}\right)-1} e^{\sum_{s=0}^{t-1} c^{\prime}\left(X_{s}, \pi_{s}^{1}, \pi_{n}^{* 2}\left(X_{s}\right)\right) d s} g_{1}\left(X_{t}\right)\right]
\end{aligned}
$$

$$
\begin{aligned}
& -\inf _{\pi^{1} \in \Pi_{a d}^{1}} E_{i}^{\pi^{1}, \pi_{n}^{* 2}}\left[\sum_{t=0}^{\tau\left(\tilde{\mathcal{D}}_{n}\right)-1} e^{\sum_{s=0}^{t-1} c^{\prime}\left(X_{s}, \pi_{s}^{1}, \pi_{n}^{* 2}\left(X_{s}\right)\right)} g_{2}\left(X_{t}\right)\right] \\
\geq & \inf _{\pi^{1} \in \Pi_{a d}^{1}} E_{i}^{\pi^{1}, \pi_{n}^{* 2}}\left[\sum_{t=0}^{\tau\left(\tilde{\mathcal{D}}_{n}\right)-1} e^{\sum_{s=0}^{t-1} c^{\prime}\left(X_{s}, \pi_{s}^{1}, \pi_{n}^{* 2}\left(X_{s}\right)\right)}\left(g_{1}\left(X_{t}\right)-g_{2}\left(X_{t}\right)\right)\right] .
\end{aligned}
$$

Hence $\bar{T}_{n}\left(g_{1}\right)(i)-\bar{T}_{n}\left(g_{2}\right)(i) \geq 0$ for all $i \in S$. This implies that $\bar{T}_{n}\left(g_{1}\right) \succeq \bar{T}_{n}\left(g_{2}\right)$. Choose a function $g \in \mathcal{B}_{\tilde{D}_{n}}$ such that $g\left(i_{0}\right)=1$ and $g(j)=0$ for all $j \neq i_{0}$, where $i_{0}$ is a fixed state (see p. 7). Thus by (3.12), we have

$$
\bar{T}_{n}(g)\left(i_{0}\right) \geq g\left(i_{0}\right)>0 .
$$

Thus we have $\bar{T}_{n}(g) \succeq g$. Let $\left\{g_{m}\right\} \subset \mathcal{B}_{\tilde{\mathcal{D}}_{n}}$ be a bounded sequence. Then since $c^{\prime}<0$, from (3.12), we get $\left\|\bar{T}_{n} g_{m}\right\|_{\infty} \leq \alpha_{2}$, for some constant $\alpha_{2}>0$. So, by a diagonalization argument, there exists a subsequence $m_{k}$ of $m$ and a function $\phi \in \hat{\mathcal{B}}_{\tilde{D}_{n}}$ such that $\left\|\bar{T}_{n} g_{m_{k}}-\phi\right\|_{\tilde{\mathcal{D}}_{n}} \rightarrow 0$ as $k \rightarrow \infty$. Thus the map $\bar{T}_{n}$ is completely continuous. By the definition of the map $\bar{T}_{n}$, it is easy to see that $\bar{T}_{n}(\lambda g)=\lambda \bar{T}_{n}(g)$ for all $\lambda \geq 0$. Hence by Theorem 3.1, there exists a nontrival $\psi_{n} \in \mathcal{B}_{\tilde{\mathcal{D}}_{n}}^{+}$and a constant $\lambda_{\tilde{\mathcal{D}}_{n}}^{\prime}>0$ such that
$\bar{T}_{n}\left(\psi_{n}\right)=\lambda_{\tilde{\mathcal{D}}_{n}}^{\prime} \psi_{n}$ i.e.,

$$
\begin{equation*}
\lambda_{\tilde{\mathcal{D}}_{n}^{\prime}}^{\prime} \psi_{n}(i)=\sup _{\nu \in \mathcal{P}(V(i))} \inf _{\mu \in \mathcal{P}(U(i))}\left[e^{c^{\prime}(i, \mu, \nu)} \sum_{j \in \tilde{\mathcal{D}}_{n}} \lambda_{\tilde{\mathcal{D}}_{n}}^{\prime} \psi_{n}(j) P(j \mid i, \mu, \nu)+\psi_{n}(i)\right] \forall i \in \tilde{\mathcal{D}}_{n} . \tag{3.13}
\end{equation*}
$$

Since $\psi_{n} \geq 0$ and $\psi_{n}(i)>0$, for some $i \in \tilde{\mathcal{D}}_{n}$, it follows from (3.13) that $\left[\frac{\lambda_{\tilde{\mathcal{D}}_{n}}^{\prime}-1}{\lambda_{\tilde{D}_{n}}^{\prime}}\right] \geq 0$. Next we prove (3.11). Now if $\left[\frac{\lambda_{\tilde{D}_{n}}^{\prime}-1}{\lambda_{\tilde{D}_{n}}^{\prime}}\right]=0$, it is easy to show that (3.11) holds. Assume that $\left[\frac{\lambda_{\tilde{\mathcal{D}}_{n}}^{\prime}-1}{\lambda_{\tilde{D}_{n}}^{\prime}}\right]>0$. Let $\rho_{n}^{\prime}=\log \left[\frac{\lambda_{\tilde{\mathcal{D}}_{n}}^{\prime}-1}{\lambda_{\tilde{\mathcal{D}}_{n}}^{\prime}}\right]$. Then from, (3.13), we get

$$
\begin{equation*}
e^{\rho_{n}^{\prime}} \psi_{n}(i)=\sup _{\nu \in \mathcal{P}(V(i))} \inf _{\mu \in \mathcal{P}(U(i))}\left[e^{c^{\prime}(i, \mu, \nu)} \sum_{j \in \tilde{\mathcal{D}}_{n}} \psi_{n}(j) P(j \mid i, \mu, \nu)\right] \forall i \in \tilde{\mathcal{D}}_{n} \tag{3.14}
\end{equation*}
$$

Now multiplying both sides of (3.14) by $e^{k_{n}+\delta}$ and applying Fan's minimax theorem, (see [22, Theorem 3]), we obtain

$$
\begin{align*}
e^{\rho_{n}} \psi_{n}(i) & =\sup _{v \in \mathcal{P}(V(i))} \inf _{\mu \in \mathcal{P}(U(i))}\left[e^{c(i, \mu, \nu)} \sum_{j \in S} \psi_{n}(j) P(j \mid i, \mu, \nu)\right] \\
& =\inf _{\mu \in \mathcal{P}(U(i)))} \sup _{v \in \mathcal{P}(V(i))}\left[e^{c(i, \mu, \nu)} \sum_{j \in S} \psi_{n}(j) P(j \mid i, \mu, \nu)\right] \forall i \in \tilde{\mathcal{D}}_{n}, \tag{3.15}
\end{align*}
$$

where $\rho_{n}=\rho_{n}^{\prime}+k_{n}+\delta$, (where $k_{n}$ is defined on p .12 ).
Let $\pi_{n}^{* 2} \in \Pi_{S M}^{2}$ be an outer maximizing selector of (3.10). Then we have

$$
\begin{equation*}
e^{\rho_{n}} \psi_{n}(i)=\inf _{\mu \in \mathcal{P}(U(i))}\left[e^{c\left(i, \mu, \pi_{n}^{* 2}(i)\right)} \sum_{j \in S} \psi_{n}(j) P\left(j \mid i, \mu, \pi_{n}^{* 2}(i)\right)\right] \forall i \in \tilde{\mathcal{D}}_{n} \tag{3.16}
\end{equation*}
$$

Therefore by using Dynkin's formula and (3.16), we obtain

$$
\begin{align*}
\psi_{n}(i) & \leq E_{i}^{\pi^{1}, \pi_{n}^{* 2}}\left[e^{\sum_{s=0}^{T-1}\left(c\left(X_{s}, \pi_{s}^{1}, \pi_{n}^{* 2}\left(X_{s}\right)\right)-\rho_{n}\right)} \psi_{n}\left(X_{T}\right) I_{\left\{T<\tau\left(\tilde{\mathcal{D}}_{n}\right)\right\}}\right] \\
& \leq\left(\sup _{\tilde{D}_{n}} \psi_{n}\right) E_{i}^{\pi^{1}, \pi_{n}^{* 2}}\left[e^{\sum_{s=0}^{T-1}\left(c\left(X_{s}, \pi_{s}^{1}, \pi_{n}^{* 2}\left(X_{s}\right)\right)-\rho_{n}\right)}\right] . \tag{3.17}
\end{align*}
$$

Now, taking logarithm on the both sides of (3.17), dividing by $T$ and letting $T \rightarrow \infty$, for each $i \in S$ for which $\psi_{n}>0$, we deduce that

$$
\rho_{n} \leq \mathcal{J}^{\pi^{1}, \pi_{n}^{* 2}}(i, c) .
$$

Since $\pi^{1} \in \Pi_{a d}^{1}$ is arbitrary, we get

$$
\rho_{n} \leq \inf _{\pi^{1} \in \Pi_{a d}^{1}} \mathcal{J}^{\pi^{1}, \pi_{n}^{* 2}}(i, c) \leq \sup _{\pi^{2} \in \Pi_{a d}^{2}} \inf _{\pi^{1} \in \Pi_{a d}^{1}} \mathcal{J}^{\pi^{1}, \pi^{2}}(i, c)
$$

Now, we show that the sequence $\{\rho\}_{n}$ is bounded, and for each $n, \psi_{n}>0$ on $\tilde{\mathcal{D}}_{n}$ and $\liminf _{n \rightarrow \infty} \rho_{n} \geq 0$.

Lemma 3.3. Suppose Assumptions 2.1 and 2.2 hold. Then for each $n, \psi_{n}>0$ on $\tilde{\mathcal{D}}_{n}$ and the sequence of eigenvalues $\left\{\rho_{n}\right\}_{n}$ of Eq. (3.10) is bounded. Moreover, we have

$$
\begin{equation*}
\liminf _{n \rightarrow \infty} \rho_{n} \geq 0 \tag{3.18}
\end{equation*}
$$

Proof. We first prove that $\left\{\rho_{n}\right\}_{n}$ is bounded. Under Assumption 2.1(a) since $\|c\|_{\infty}<\tilde{\gamma}$, it is easy to see that $\mathcal{J}^{1}, \pi^{2}(i, c) \leq \tilde{\gamma}$. Under Assumption 2.1(b) since $\tilde{\mathcal{K}}$ is finite, there exists a constant $k_{1}$ such that (2.5) can be written as

$$
\begin{equation*}
\sup _{(u, v) \in U(i) \times V(i)} \sum_{j \in S} \mathcal{W}(j) P(j \mid i, u, v) \leq e^{\left(k_{1}-\tilde{\ell}(i)\right)} \mathcal{W}(i) \forall i \in S \tag{3.19}
\end{equation*}
$$

Then by using (2.2) and successive conditioning, we get

$$
\begin{equation*}
E_{i}^{\pi^{1}, \pi^{2}}\left[e^{\sum_{t=0}^{T-1}\left(\tilde{\ell}\left(X_{t}\right)-k_{1}\right)} \mathcal{W}\left(X_{T}\right)\right] \leq \mathcal{W}(i) \forall i \in S \tag{3.20}
\end{equation*}
$$

Since, $\mathcal{W} \geq 1$, from (3.20), we get

$$
\partial^{\pi^{1}, \pi^{2}}(i, \tilde{\ell}) \leq k_{1} \text { for all } i \in S
$$

Now since $\tilde{\ell}-\sup _{(u, v) \in U(i) \times V(i)} c(\cdot, u, v)$ is norm-like, there exists a constant $k_{2}$ such that for all $i \in S$, we have $\sup _{(u, v) \in U(i) \times V(i)} c(i, u, v) \leq \tilde{\ell}(i)+k_{2}$. Hence we get

$$
\begin{equation*}
\partial^{\pi^{1}, \pi^{2}}(i, c) \leq k_{1}+k_{2} \quad \forall\left(\pi^{1}, \pi^{2}\right) \in \Pi_{a d}^{1} \times \Pi_{a d}^{2}, \forall i \in S \tag{3.21}
\end{equation*}
$$

Therefore using (3.11), it is clear that $\rho_{n}$ has an upper bound.
Next we want to show that $\rho_{n}$ is bounded below. To this end, first we claim that $\psi_{n}>0$ on $\tilde{\mathcal{D}}_{n}$ for each $n$. Let $n \in \mathbb{N}$ be fixed. Suppose that the claim is not true, then there exists $\tilde{i} \in \tilde{\mathcal{D}}_{n}$ such that $\psi_{n}(\tilde{i})=0$. Also, since $\psi_{n} \geq 0$ on $\tilde{\mathcal{D}}_{n}$, there exists $\hat{i} \in \tilde{\mathcal{D}}_{n}$ such that $\psi_{n}(\hat{i})>0$. Now, for any outer minimizing selector $\pi_{n}^{* 1} \in \Pi_{S M}^{1}$ of (3.15), Eq. (3.16) can be rewritten as

Again, in view of Assumption 2.2(ii), under any pair of strategies $\left(\pi_{n}^{* 1}, \pi_{n}^{* 2}\right) \in \Pi_{S M}^{1} \times \Pi_{S M}^{2}$, there exists $i_{1}, i_{2}, \ldots, i_{m} \in \tilde{\mathcal{D}}_{n}$ satisfying

$$
\begin{equation*}
P\left(\hat{i} \mid i_{m}, \pi_{n}^{* 1}\left(i_{m}\right), \pi_{n}^{* 2}\left(i_{m}\right)\right) P\left(i_{m} \mid i_{m-1}, \pi_{n}^{* 1}\left(i_{m-1}\right), \pi_{n}^{* 2}\left(i_{m-1}\right)\right) \cdots P\left(i_{1} \mid \tilde{i}, \pi_{n}^{* 1}(\tilde{i}), \pi_{n}^{* 2}(\tilde{i})\right)>0 . \tag{3.23}
\end{equation*}
$$

Thus, from (3.22) and (3.23), we deduce that $\psi_{n}(\hat{i})=\psi_{n}\left(i_{1}\right)=\cdots=\psi_{n}\left(i_{m}\right)=\psi_{n}(\tilde{i})=0$. But this contradicts to the fact that $\psi_{n}$ is nontrivial. Since, $n$ is arbitrary, this establishes our claim. So, for all $n$ we can pin $\psi_{n}$ such that $\psi_{n}\left(i_{0}\right)=1$, where $i_{0}$ is a reference state (defined as in p. 7).

Now, suppose that the sequence $\left\{\rho_{n}\right\}_{n}$ is not bounded below. Hence, along a subsequence $\rho_{n} \rightarrow-\infty$ as $n \rightarrow \infty$. So, $\rho_{n}<0$ for all large enough $n$. Let $\left(\pi_{n}^{* 1}, \pi_{n}^{* 2}\right) \in \Pi_{S M}^{1} \times \Pi_{S M}^{2}$ be a mini-max selector of (3.10), thus we have

$$
\begin{align*}
1=\psi_{n}\left(i_{0}\right) & =e^{-\rho_{n}} \sup _{v \in \mathcal{P}\left(V\left(i_{0}\right)\right)}\left[e^{c\left(i_{0}, \pi_{n}^{* 1}\left(i_{0}\right), \nu\right)} \sum_{j \in S} \psi_{n}(j) P\left(j \mid i_{0}, \pi_{n}^{* 1}\left(i_{0}\right), \nu\right)\right] \\
& =e^{-\rho_{n}}\left[e^{c\left(i_{0}, \pi_{n}^{* 1}\left(i_{0}\right), \pi_{n}^{* 2}\left(i_{0}\right)\right)} \sum_{j \in S} \psi_{n}(j) P\left(j \mid i_{0}, \pi_{n}^{* 1}\left(i_{0}\right), \pi_{n}^{* 2}\left(i_{0}\right)\right)\right] . \tag{3.24}
\end{align*}
$$

Since $\rho_{n}<0$ for all large enough $n$, and our cost function $c$ is nonnegative, it is easy to see that $c\left(i_{0}, \pi_{n}^{* 1}\left(i_{0}\right), \pi_{n}^{* 2}\left(i_{0}\right)\right)-\rho_{n}>0$, for all large enough $n$. Assumption 2.2(ii), implies that for any $j \in \tilde{\mathcal{D}}_{n}$, under any pair of strategies $\left(\pi^{1}, \pi^{2}\right) \in \Pi_{S M}^{1} \times \Pi_{S M}^{2}$, there exists $i_{1}, i_{2}, \ldots, i_{m} \in \tilde{\mathcal{D}}_{n}$ satisfying

$$
\begin{equation*}
P\left(j \mid i_{m}, \pi^{1}\left(i_{m}\right), \pi^{2}\left(i_{m}\right)\right) P\left(i_{m} \mid i_{m-1}, \pi^{1}\left(i_{m-1}\right), \pi^{2}\left(i_{m-1}\right)\right) \cdots P\left(i_{1} \mid i_{0}, \pi^{1}\left(i_{0}\right), \pi^{2}\left(i_{0}\right)\right)>0 . \tag{3.25}
\end{equation*}
$$

We claim that if $j \in \tilde{\mathcal{D}}_{n}$, then

$$
\begin{equation*}
\inf _{\left(\pi^{1}, \pi^{2}\right) \in \Pi_{S M}^{1} \times \Pi_{S M}^{2}} P_{i_{0}}^{\pi^{1}, \pi^{2}}\left(\check{\tau}_{j} \leq n \wedge \tau\left(\tilde{\mathcal{D}}_{n}\right)\right) \geq k(j, n), \text { for some constant } k(j, n)>0 . \tag{3.26}
\end{equation*}
$$

If not, suppose there exists a pair $\left(\tilde{\pi}_{k}^{1}, \tilde{\pi}_{k}^{2}\right) \in \Pi_{S M}^{1} \times \Pi_{S M}^{2}$ such that $P_{i_{0}}^{\tilde{\pi}_{k}, \tilde{\pi}_{k}^{2}}\left(\check{\tau}_{j} \leq n \wedge \tau\left(\tilde{\mathcal{D}}_{n}\right)\right) \rightarrow 0$ as $k \rightarrow \infty$. Now, since $\Pi_{S M}^{1}$ and $\Pi_{S M}^{2}$ are compact, there exist a further subsequence and $\tilde{\pi}^{1} \in \Pi_{S M}^{1}$ and $\tilde{\pi}^{2} \in \Pi_{S M}^{2}$, such that $\tilde{\pi}_{k}^{1} \rightarrow \tilde{\pi}^{1}$ and $\tilde{\pi}_{k}^{2} \rightarrow \tilde{\pi}^{2}$ as $k \rightarrow \infty$. By Assumption 2.2, we know that the law of $\boldsymbol{X}_{k}$ converges to $\boldsymbol{X}$, where $\boldsymbol{X}_{k}(\boldsymbol{X})$ is the DTCMC governed by $\left(\tilde{\pi}_{k}^{1}, \tilde{\pi}_{k}^{2}\right)$ ( $\left(\tilde{\pi}^{1}, \tilde{\pi}^{2}\right)$ respectively). So, for every $p \leq n$,

$$
\begin{aligned}
& P_{i_{0}}^{\tilde{\pi}^{1}, \tilde{\pi}^{2}}\left(X_{i} \in \tilde{\mathcal{D}}_{n} \backslash\left\{i_{0}, j\right\}, X_{p}=j \text { for all } i \leq p-1\right) \\
& \quad=\lim _{k \rightarrow \infty} P_{i_{0}}^{\tilde{\pi}_{k}^{1}, \tilde{\pi}_{k}^{2}}\left(X_{k, i} \in \tilde{\mathcal{D}}_{n} \backslash\left\{i_{0}, j\right\}, X_{k, p}=j \text { for all } i \leq p-1\right) \\
& \quad \leq \lim _{k \rightarrow \infty} P_{i_{0}}^{\tilde{\pi}_{k}^{1}, \tilde{\pi}_{k}^{2}}\left(\check{\tau}_{j} \leq n \wedge \tau\left(\tilde{\mathcal{D}}_{n}\right)\right)=0 .
\end{aligned}
$$

So, this contradicts (3.25). Hence, we must have (3.26).

From the monotonicity of $\tau\left(\tilde{\mathcal{D}}_{n}\right)$, it then follows that for $\tilde{\mathcal{D}}_{n} \supset \tilde{\mathcal{D}}_{m} \ni j$, we have

$$
\begin{align*}
\inf _{\left(\pi^{1}, \pi^{2}\right) \in \Pi_{S M}^{1} \times \Pi_{S M}^{2}} P_{i_{0}}^{\pi^{1}, \pi^{2}}\left(\check{\tau}_{j} \leq m \wedge \tau\left(\tilde{\mathcal{D}}_{n}\right)\right) & \geq \inf _{\left(\pi^{1}, \pi^{2}\right) \in \Pi_{S M}^{1} \times \Pi_{S M}^{2}} P_{i_{0}}^{\pi^{1}, \pi^{2}}\left(\check{\tau}_{j} \leq m \wedge \tau\left(\tilde{\mathcal{D}}_{m}\right)\right) \\
& \geq k(j, m) . \tag{3.27}
\end{align*}
$$

Since for large enough $n, c\left(i_{0}, \pi_{n}^{* 1}\left(i_{0}\right), \pi_{n}^{* 2}\left(i_{0}\right)\right)-\rho_{n}>0$, from (3.24), we have

$$
\begin{aligned}
1=\psi_{n}\left(i_{0}\right) & =E_{i_{0}}^{\pi_{n}^{* 1}, \pi_{n}^{* 2}}\left[e^{\sum_{t=0}^{m \wedge \tau\left(\tilde{\mathcal{D}}_{n}\right) \wedge \check{\tau}_{j}-1}\left(c\left(X_{t}, \pi_{n}^{* 1}\left(X_{t}\right), \pi_{n}^{* 2}\left(X_{t}\right)\right)-\rho_{n}\right)} \psi_{n}\left(X_{m \wedge \tau\left(\tilde{\mathcal{D}}_{n}\right) \wedge \check{\tau}_{j}}\right)\right] \\
& \geq E_{i_{0}}^{\pi_{n}^{* 1}, \pi_{n}^{* 2}}\left[\psi_{n}\left(X_{\left.m \wedge \tau\left(\tilde{\mathcal{D}}_{n}\right) \wedge \check{\tau}_{j}\right)}\right)\right] \\
& \geq \psi_{n}(j) \quad \inf _{\left(\pi^{1}, \pi^{2}\right) \in \Pi_{S M}^{1} \times \Pi_{S M}^{2}} P_{i_{0}}^{\pi^{1}, \pi^{2}}\left(\check{\tau}_{j} \leq m \wedge \tau\left(\tilde{\mathcal{D}}_{n}\right)\right) \\
& \geq k(j, m) \psi_{n}(j)(\operatorname{using}(3.27)) .
\end{aligned}
$$

Choose $m=j+1$. Then for all $n>j$, we have $\psi_{n}(j) \leq \frac{1}{k(j, j+1)}, \forall j \in S$. This implies that, $\left\{\psi_{n}\right\}$ has an upper bound. Thus by a standard diagonalization argument, there exists a subsequence (by an abuse of notation denoting by the same sequence) and a bounded function $\psi \geq 0$ with $\psi\left(i_{0}\right)=1$ such that $\psi_{n}(i) \rightarrow \psi(i)$, as $n \rightarrow \infty$ for all $i \in S$. Now, since $\Pi_{S M}^{1}$ and $\Pi_{S M}^{2}$ are compact, there exist a further subsequence and $\pi^{* 1} \in \Pi_{S M}^{1}$ and $\pi^{* 2} \in \Pi_{S M}^{2}$, such that $\pi_{n}^{* 1} \rightarrow \pi^{* 1}$ and $\pi_{n}^{* 2} \rightarrow \pi^{* 2}$ as $n \rightarrow \infty$. Since $c \geq 0$, (3.10) gives us

$$
\begin{equation*}
e^{\rho_{n}} \psi_{n}(i) \geq\left[\sum_{j \in S} \psi_{n}(j) P\left(j \mid i, \pi_{n}^{* 1}(i), \pi_{n}^{* 2}(i)\right)\right] . \tag{3.28}
\end{equation*}
$$

Hence, by taking $n \rightarrow \infty$, it follows that

$$
\begin{equation*}
\sum_{j \in S} \psi(j) P\left(j \mid i, \pi^{* 1}(i), \pi^{* 2}(i)\right) \leq 0, i \in S \tag{3.29}
\end{equation*}
$$

In view of (3.29), we claim that $\psi \equiv 0$. If not then there exists $\hat{i} \in S$ such that $\psi^{*}(\hat{i})>0$. Also, since $\psi \geq 0$ from (3.29), it is easy to see that there exists a point $\tilde{i} \in S$ for which $\psi(\tilde{i})=0$. Now, since $\left\{X_{t}\right\}$ is irreducible under any pair of strategies $\left(\pi^{* 1}, \pi^{* 2}\right) \in \Pi_{S M}^{1} \times \Pi_{S M}^{2}$, there exists $i_{1}, i_{2}, \ldots, i_{m} \in S$ satisfying

$$
P\left(\hat{i} \mid i_{m}, \pi^{* 1}\left(i_{m}\right), \pi^{* 2}\left(i_{m}\right)\right) P\left(i_{m} \mid i_{m-1}, \pi^{* 1}\left(i_{m-1}\right), \pi^{* 2}\left(i_{m-1}\right)\right) \cdots P\left(i_{1} \mid \tilde{i}, \pi^{* 1}(\tilde{i}), \pi^{* 2}(\tilde{i})\right)>0 .
$$

Thus, from (3.29) we deduce that $\psi(\hat{i})=\psi\left(i_{1}\right)=\cdots=\psi\left(i_{m}\right)=\psi(\tilde{i})=0$. But this contradicts to the fact that $\psi(\hat{i})>0$. This proves the claim. But since $\psi\left(i_{0}\right)=1$, this is a contradiction. Therefore, we obtain that, $\left\{\rho_{n}\right\}$ is bounded below.

Now we show that $\rho^{*}=\liminf _{n \rightarrow \infty} \rho_{n} \geq 0$. If not, then on contrary, $\rho^{*}<0$. So, for large enough $n, \rho_{n}<0$. Since, our cost function $c$ is nonnegative, for large enough $n$, $c(i, \mu, \nu)-\rho_{n}>0$ for all $(\mu, \nu) \in \mathcal{P}(U(i)) \times \mathcal{P}(V(i))$. So, by repeating the above arguments, there exists a subsequence (by an abuse of notation denoting by the same sequence) and a bounded function $\phi \geq 0$ with $\phi\left(i_{0}\right)=1$ such that $\psi_{n}(i) \rightarrow \phi(i)$, as $n \rightarrow \infty$ for all $i \in S$. From (3.10), we have

$$
\begin{equation*}
\psi_{n}(i) \geq\left[\sum_{j \in S} \psi_{n}(j) P\left(j \mid i, \pi_{n}^{* 1}(i), \pi_{n}^{* 2}(i)\right)\right], \tag{3.30}
\end{equation*}
$$

where $\left(\pi_{n}^{* 1}, \pi_{n}^{* 2}\right)$ is a mini-max selector of (3.10). By Fatou's lemma, taking $n \rightarrow \infty$, we deduce that

$$
\phi(i) \geq E_{i}^{\pi^{* 1}, \pi^{* 2}}\left[\phi\left(X_{1}\right)\right] \forall i \in S,
$$

for some pair of stationary strategies $\left(\pi^{* 1}, \pi^{* 2}\right) \in \Pi_{S M}^{1} \times \Pi_{S M}^{2}$. Hence, $\left\{\phi\left(X_{m}\right), \mathcal{F}_{m}\right\}$ is supermartingale where $\left\{X_{t}\right\}$ is the Markov process under the pair of stationary strategies $\left(\pi^{* 1}, \pi^{* 2}\right) \in \Pi_{S M}^{1} \times \Pi_{S M}^{2}$. So, by Doob's martingale convergence theorem $\phi\left(X_{m}\right) \rightarrow \hat{Y}$ almost surely, as $m \rightarrow \infty$. On the other hand by Assumption 2.1, we have $\left\{X_{t}\right\}$ is recurrent. Hence $\left\{X_{t}\right\}$ visits every state (in particular $i_{0}$ ) of $S$ infinitely often. Since, $\phi\left(i_{0}\right)=1,\left\{\phi\left(X_{m}\right)\right\}$ converges only if $\phi \equiv 1$. Now, taking limit $n \rightarrow \infty$ in (3.10), we obtain

$$
1=\phi(i) \geq e^{c\left(i, \pi^{* 1}(i), \pi^{* 2}(i)\right)-\rho^{*}}>1
$$

But this is a contradiction. Thus, ${\lim \inf _{n \rightarrow \infty} \rho_{n} \geq 0 \text {. }}_{\text {. }}$

## 4. Existence of risk-sensitive average optimal strategies

In this section we prove the existence of a risk-sensitive average optimal stationary strategy using the Shapley equation. Now we state and prove our main result of this section.

Theorem 4.1. Suppose Assumptions 2.1 and 2.2 hold. Then there exists a unique (up to a scalar multiplication) eigenpair $\left(\rho^{*}, \psi^{*}\right) \in \mathbb{R}_{+} \times L_{\mathcal{W}}^{\infty}$ with $\psi^{*}>0$, such that

$$
\begin{align*}
e^{\rho^{*}} \psi^{*}(i) & =\sup _{v \in \mathcal{P}(V(i))} \inf _{\mu \in \mathcal{P}(U(i))}\left[e^{c(i, \mu, \nu)} \sum_{j \in S} \psi^{*}(j) P(j \mid i, \mu, \nu)\right] \\
& =\inf _{\mu \in \mathcal{P}(U(i)))} \sup _{v \in \mathcal{P}(V(i))}\left[e^{c(i, \mu, \nu)} \sum_{j \in S} \psi^{*}(j) P(j \mid i, \mu, \nu)\right], i \in S . \tag{4.1}
\end{align*}
$$

Moreover, we have the following
(i)

$$
\begin{equation*}
\rho^{*}=\inf _{i \in S} \sup _{\pi^{2} \in \Pi_{a d}^{2}} \inf _{\pi^{1} \in \Pi_{a d}^{1}} g^{\pi^{1}, \pi^{2}}(i, c)=\inf _{i \in S} \inf _{\pi^{1} \in \Pi_{a d}^{1}} \sup _{\pi^{2} \in \Pi_{a d}^{2}} \mathcal{J}^{1}, \pi^{2}(i, c) . \tag{4.2}
\end{equation*}
$$

(ii) If $\left(\pi^{* 1}, \pi^{* 2}\right) \in \Pi_{S M}^{1} \times \Pi_{S M}^{2}$ be a mini-max selector of (4.1), then $\left(\pi^{* 1}, \pi^{* 2}\right) \in \Pi_{S M}^{1} \times \Pi_{S M}^{2}$ is a saddle point equilibrium, i.e.,

$$
\begin{equation*}
\mathcal{J}^{\pi^{* 1}, \pi^{2}}(i, c) \leq \mathcal{J}^{\pi^{* 1}, \pi^{* 2}}(i, c)=\rho^{*} \leq \mathcal{J}^{\pi^{1}, \pi^{* 2}}(i, c), \quad \forall \pi^{1} \in \Pi_{a d}^{1}, \quad \forall \pi^{2} \in \Pi_{a d}^{2} \tag{4.3}
\end{equation*}
$$

Thus the value of the game is independent of the initial state.
(iii) Let $\left(\pi^{* 1}, \pi^{* 2}\right) \in \Pi_{S M}^{1} \times \Pi_{S M}^{2}$ is a saddle point equilibrium, then this pair is a mini-max selector of (4.1).

Rest of this section is dedicated to the proof of Theorem 4.1.
Since $c \geq 0$, using Assumption 2.1, there exists a finite set $\hat{\mathcal{B}}$ containing $\tilde{\mathcal{K}}$ such that we have the following:

- Under Assumption 2.1(a): since $\tilde{\gamma}>\|c\|_{\infty}$, from (3.11) we have $\rho_{n} \leq \tilde{\gamma}$. Thus, for all large enough $n$ it holds that

$$
\begin{equation*}
\left(\sup _{(u, v) \in U(i) \times V(i)} c(i, u, v)-\rho_{n}\right)<\tilde{\gamma} \forall i \in \hat{\mathcal{B}}^{c} . \tag{4.4}
\end{equation*}
$$

- Under Assumption 2.1(b): since the function $\ell(\cdot)-\max _{(u, v) \in U(\cdot) \times V(\cdot)} c(\cdot, u, v)$ is norm-like, for all large enough $n$ it holds that

$$
\begin{equation*}
\left(\sup _{(u, v) \in U(i) \times V(i)} c(i, u, v)-\rho_{n}\right)<\tilde{\ell}(i) \forall i \in \hat{\mathcal{B}}^{c} . \tag{4.5}
\end{equation*}
$$

Now letting $n \rightarrow \infty$ from (3.10) we show that the limiting equation admits a positive eigenpair.
Lemma 4.1. Suppose Assumptions 2.1 and 2.2 hold. Then there exists an eigenpair $\left(\rho^{*}, \psi^{*}\right) \in$ $\mathbb{R}_{+} \times L_{\mathcal{W}}^{\infty}$ with $\psi^{*}>0$, such that

$$
\begin{align*}
e^{\rho^{*}} \psi^{*}(i) & =\sup _{v \in \mathcal{P}(V(i))} \inf _{\mu \in \mathcal{P}(U(i))}\left[e^{c(i, \mu, v)} \sum_{j \in S} \psi^{*}(j) P(j \mid i, \mu, \nu)\right] \\
& =\inf _{\mu \in \mathcal{P}(U(i))} \sup _{v \in \mathcal{P}(V(i))}\left[e^{c(i, \mu, v)} \sum_{j \in S} \psi^{*}(j) P(j \mid i, \mu, v)\right], i \in S . \tag{4.6}
\end{align*}
$$

Furthermore, for any mini-max selector $\left(\pi^{* 1}, \pi^{* 2}\right) \in \Pi_{S M}^{1} \times \Pi_{S M}^{2}$ of (4.6) we have the following:
(i)

$$
\begin{equation*}
\rho^{*} \leq \inf _{i \in S} \sup _{\pi^{2} \in \Pi_{a d}^{2}} \inf _{\pi^{1} \in \Pi_{a d}^{1}} g^{\pi^{1}, \pi^{2}}(i, c) \tag{4.7}
\end{equation*}
$$

(ii) For any finite set $\hat{\mathcal{B}}_{1} \supset \hat{\mathcal{B}}$, we have the following stochastic representation of the eigenfunction

$$
\begin{align*}
\psi^{*}(i) & =\inf _{\pi^{1} \in \Pi_{a d}^{1}} E_{i}^{\pi^{1}, \pi^{* 2}}\left[e^{\sum_{t=0}^{\tau\left(\hat{\mathcal{B}}_{1}\right)-1}\left(c\left(X_{t}, \pi_{t}^{1}, \pi^{* 2}\left(X_{t}\right)\right)-\rho^{*}\right)} \psi^{*}\left(X_{\check{\tau}\left(\hat{\mathcal{B}}_{1}\right)}\right)\right] \\
& =\sup _{\pi^{2} \in \Pi_{a d}^{2}} E_{i}^{\pi^{* 1}, \pi^{2}}\left[e^{\sum_{t=0}^{\tau \tau\left(\hat{\mathcal{B}}_{1}\right)-1}\left(c\left(X_{t}, \pi^{* 1}\left(X_{t}\right), \pi_{t}^{2}\right)-\rho^{*}\right)} \psi^{*}\left(X_{\check{\tau}\left(\hat{\mathcal{B}}_{1}\right)}\right)\right] \forall i \in \hat{\mathcal{B}}_{1}^{c} . \tag{4.8}
\end{align*}
$$

Proof. First we scale $\psi_{n}$ in such a way that we obtain $\psi_{n}(i) \leq \mathcal{W}(i)$ for all $i \in S$. Set

$$
\tilde{\theta}_{n}=\sup \left\{\alpha>0:\left(\mathcal{W}-\alpha \psi_{n}\right)>0 \text { in } S\right\} .
$$

Since $\psi_{n}$ vanishes in $\tilde{D}_{n}^{c}$ and $\psi_{n}>0$ on $\tilde{D}_{n}$, it follows that $\tilde{\theta}_{n}$ is finite. We claim that if we replace $\psi_{n}$ by $\tilde{\theta}_{n} \psi_{n}$, then $\psi_{n}$ touches $\mathcal{W}$ inside $\hat{\mathcal{B}}$. If this is not true, then on the contrary, we assume that for some state $\hat{i} \in \hat{\mathcal{B}}^{c} \cap \tilde{\mathcal{D}}_{n},\left(\mathcal{W}-\psi_{n}\right)(\hat{i})=0$ and $\mathcal{W}-\psi_{n}>0$ in $\hat{\mathcal{B}} \cup \tilde{\mathcal{D}}_{n}^{c}$. Let $\pi_{n}^{* 2}$ be an outer maximizing selector of (3.10). Then under Assumption 2.1(b), applying Dynkin's formula (as in [53, Lemma 3.1]), we obtain

$$
\begin{aligned}
\psi_{n}(\hat{i}) & \leq E_{\hat{i}}^{\pi^{1}, \pi_{n}^{* 2}}\left[e ^ { \sum _ { s = 0 } ^ { N \wedge \tau ̌ } ( \hat { \mathcal { B } } ) - 1 } \left(c\left(X_{s}, \pi_{s}^{1}, \pi_{n}^{\left.\left.* 2\left(X_{s}\right)\right)-\rho_{n}\right)} \psi_{n}\left(X_{N \wedge \check{\tau}(\hat{\mathcal{B}})}\right) I_{\left\{N \wedge \check{\tau}(\hat{\mathcal{B}})<\tau\left(\tilde{\mathcal{D}}_{n}\right)\right\}}\right]\right.\right. \\
& \leq E_{\hat{i}}^{\pi^{1}, \pi_{n}^{* 2}}\left[e^{\sum_{s=0}^{N \wedge \check{\tau}(\hat{\mathcal{B}})-1} \tilde{\ell}\left(X_{s}\right)} \psi_{n}\left(X_{N \wedge \check{\tau}(\hat{\mathcal{B}})}\right) I_{\left\{N \wedge \check{\tau}(\hat{\mathcal{B}})<\tau\left(\tilde{\mathcal{D}}_{n}\right)\right\}}\right] .
\end{aligned}
$$

Since $\psi_{n} \leq \mathcal{W}$ (by our scaling), in view of Lemma 3.1, by the dominated convergence theorem taking $N \rightarrow \infty$, we get

$$
\psi_{n}(\hat{i}) \leq E_{\hat{i}}^{\pi^{1}, \pi_{n}^{* 2}}\left[e^{\sum_{s=0}^{\check{( }(\hat{\mathcal{B}})-1} \tilde{\ell}\left(X_{s}\right) d s} \psi_{n}\left(X_{\check{\tau}(\hat{\mathcal{B}})}\right)\right] .
$$

Combining this and (3.2), we get

$$
0=\left(\mathcal{W}-\psi_{n}\right)(\hat{i}) \geq E_{\hat{i}}^{\pi^{1}, \pi_{n}^{* 2}}\left[e^{\sum_{s=0}^{\tau(\hat{\mathcal{B}})-1} \tilde{\ell}\left(X_{s}\right) d s}\left(\mathcal{W}-\psi_{n}\right)\left(X_{\check{\tau}(\hat{\mathcal{B}})}\right)\right]>0
$$

But this is a contradiction. Thus $\psi_{n}$ touches $\mathcal{W}$ inside $\hat{\mathcal{B}}$. Using estimate as in (3.1), one can show that similar conclusion holds under Assumption 2.1(a).

So, there exists a point $i^{*} \in \hat{\mathcal{B}}$ such that $\left(\mathcal{W}-\psi_{n}\right)\left(i^{*}\right)=0$, for all large $n$. Since $\psi_{n} \leq \mathcal{W}$, by diagonalization arguments, there exist a subsequence (here we use the same sequence by an abuse of notation), and a function $\psi^{*} \leq \mathcal{W}$ such that $\psi_{n} \rightarrow \psi^{*}$ as $n \rightarrow \infty$. Again, from Lemma 3.3, we know that the sequence $\left\{\rho_{n}\right\}$ is bounded and $\liminf _{n \rightarrow \infty} \rho_{n} \geq 0$, thus along a further subsequence we have $\rho_{n} \rightarrow \rho^{*}$ as $n \rightarrow \infty$ for some $\rho^{*} \geq 0$.

Also, we have $\left(\mathcal{W}-\psi^{*}\right)\left(\hat{i}^{*}\right)=0$ for some $\hat{i}^{*} \in \hat{\mathcal{B}}$. By the continuity-compactness assumptions, for any mini-max selector $\left(\pi_{n}^{* 1}, \pi_{n}^{* 2}\right) \in \Pi_{S M}^{1} \times \Pi_{S M}^{2}$ of (3.10), we get

$$
\begin{align*}
e^{\rho_{n}} \psi_{n}(i) & =\sup _{v \in \mathcal{P}(V(i))}\left[e^{c\left(i, \pi_{n}^{* 1}(i), v\right)} \sum_{j \in S} \psi_{n}(j) P\left(j \mid i, \pi_{n}^{* 1}(i), \nu\right)\right] \\
& =\inf _{\mu \in \mathcal{P}(U(i))}\left[e^{c\left(i, \mu, \pi_{n}^{* 2}(i)\right)} \sum_{j \in S} \psi_{n}(j) P\left(j \mid i, \mu, \pi_{n}^{* 2}(i)\right)\right] . \tag{4.9}
\end{align*}
$$

Note that since $\psi_{n} \in L_{\mathcal{W}}^{\infty}$, we have

$$
\begin{equation*}
\sum_{j \in S} \psi_{n}(j) P(j \mid i, u, v) \leq \sum_{j \in S} \mathcal{W}(j) P(j \mid i, u, v) \forall(i, u, v) \in \mathcal{K} . \tag{4.10}
\end{equation*}
$$

Since $\Pi_{S M}^{1}$ and $\Pi_{S M}^{2}$ are compact there exists $\left(\pi^{* 1}, \pi^{* 2}\right) \in \Pi_{S M}^{1} \times \Pi_{S M}^{2}$ such that $\pi_{n}^{* 1} \rightarrow \pi^{* 1}$ and $\pi_{n}^{* 2} \rightarrow \pi^{* 2}$ as $n \rightarrow \infty$. Now from (4.9) we obtain,

$$
\begin{equation*}
e^{\rho_{n}} \psi_{n}(i) \geq\left[e^{c\left(i, \pi_{n}^{* 1}(i), \nu\right)} \sum_{j \in S} \psi_{n}(j) P\left(j \mid i, \pi_{n}^{* 1}(i), \nu\right)\right] \tag{4.11}
\end{equation*}
$$

Then, using Lemma 2.1, taking $n \rightarrow \infty$ from (4.11), by the extended Fatou's lemma [23], [35, Lemma 8.3.7], we obtain

$$
e^{\rho^{*}} \psi^{*}(i) \geq e^{c\left(i, \pi^{* 1}(i), \nu\right)} \sum_{j \in S} \psi^{*}(j) P\left(j \mid i, \pi^{* 1}(i), \nu\right) .
$$

Thus

$$
\begin{align*}
e^{\rho^{*}} \psi^{*}(i) & \geq \sup _{v \in \mathcal{P}(V(i))}\left[e^{c\left(i, \pi^{* 1}(i), v\right)} \sum_{j \in S} \psi^{*}(j) P\left(j \mid i, \pi^{* 1}(i), v\right)\right] \\
& \geq \inf _{\mu \in \mathcal{P}(U(i))} \sup _{\mu \in \mathcal{P}(V(i))}\left[e^{c(i, \mu, \nu)} \sum_{j \in S} \psi^{*}(j) P(j \mid i, \mu, v)\right] . \tag{4.12}
\end{align*}
$$

Also, from (4.9), we get

$$
e^{\rho_{n}} \psi_{n}(i) \leq\left[e^{c\left(i, \mu, \pi_{n}^{* 2}(i)\right)} \sum_{j \in S} \psi_{n}(j) P\left(j \mid i, \mu, \pi_{n}^{* 2}(i)\right)\right] .
$$

Using (4.10), by the dominated convergence theorem, taking limit $n \rightarrow \infty$ in above equation, we deduce that

$$
\begin{align*}
e^{\rho^{*}} \psi^{*}(i) & \leq \inf _{\mu \in \mathcal{P}(U(i))}\left[e^{c\left(i, \mu, \pi^{* 2}(i)\right)} \sum_{j \in S} \psi^{*}(j) P\left(j \mid i, \mu, \pi^{* 2}(i)\right)\right] \\
& \leq \sup _{v \in \mathcal{P}(V(i))} \inf _{\mu \in \mathcal{P}(U(i))}\left[e^{c(i, \mu, v)} \sum_{j \in S} \psi^{*}(j) P(j \mid i, \mu, \nu)\right] . \tag{4.13}
\end{align*}
$$

Hence by (4.12) and (4.13), we get (4.6). Since we have $\left(\mathcal{W}-\psi^{*}\right)\left(\hat{i}^{*}\right)=0$ and $\mathcal{W} \geq 1$, it follows that $\psi^{*}$ is nontrivial.

Now, we claim that $\psi^{*}>0$. If not, then on contrary there exists a point $\tilde{i} \in S$ for which $\psi^{*}(\tilde{i})=0$. Again by continuity-compactness assumptions, there exists a mini-max selector $\left(\pi^{* 1}, \pi^{* 2}\right)$ such that (4.6) can be rewritten as

$$
e^{\rho^{*}} \psi^{*}(i)=\left[e^{c\left(i, \pi^{* 1}(i), \pi^{* 2}(i)\right)} \sum_{j \in S} \psi^{*}(j) P\left(j \mid i, \pi^{* 1}(i), \pi^{* 2}(i)\right)\right] \quad \forall i \in S
$$

So, we get

$$
\begin{equation*}
0=e^{\rho^{*}} \psi^{*}(\tilde{i})=\left[e^{c\left(\tilde{i}, \pi^{* 1}(\tilde{i}), \pi^{* 2}(\tilde{i})\right)} \sum_{j \in S} \psi^{*}(j) P\left(j \mid \tilde{i}, \pi^{* 1}(\tilde{i}), \pi^{* 2}(\tilde{i})\right)\right] \tag{4.14}
\end{equation*}
$$

Since $\psi^{*}$ is nontrivial, there exists $\hat{i} \in S$ such that $\psi^{*}(\hat{i})>0$. Again, since $X$ is irreducible under any pair of strategies $\left(\pi^{* 1}, \pi^{* 2}\right) \in \Pi_{S M}^{1} \times \Pi_{S M}^{2}$, there exists $i_{1}, i_{2}, \ldots, i_{n} \in S$ satisfying

$$
P\left(\hat{i} \mid i_{n}, \pi^{* 1}\left(i_{n}\right), \pi^{* 2}\left(i_{n}\right)\right) P\left(i_{n} \mid i_{n-1}, \pi^{* 1}\left(i_{n-1}\right), \pi^{* 2}\left(i_{n-1}\right)\right) \cdots P\left(i_{1} \mid \tilde{i}, \pi^{* 1}(\tilde{i}), \pi^{* 2}(\tilde{i})\right)>0 .
$$

Thus, from (4.14) we deduce that $\psi^{*}(\hat{i})=\psi^{*}\left(i_{1}\right)=\cdots=\psi^{*}\left(i_{n}\right)=\psi^{*}(\tilde{i})=0$. But this contradicts to the fact that $\psi^{*}$ is nontrivial. This establishes our claim.

Next we prove (4.7). Since $\psi_{n}>0$ on $\tilde{\mathcal{D}}_{n}$ for all $n$, using (3.11), we have $\rho^{*}=\lim _{n \rightarrow \infty} \rho_{n} \leq$ $\sup _{\pi^{2} \in \Pi_{a d}^{2}} \inf _{\pi^{1} \in \Pi_{a d}^{1}} \mathcal{J}^{\pi^{1}, \pi^{2}}(i, c)$ for all $i \in S$.

Finally we prove the stochastic representation (4.8) of $\psi^{*}$. As before there exists a pair of strategies $\left(\pi^{* 1}, \pi^{* 2}\right) \in \Pi_{S M}^{1} \times \Pi_{S M}^{2}$ satisfying

$$
\begin{align*}
e^{\rho^{*}} \psi^{*}(i) & =\sup _{v \in \mathcal{P}(V(i))}\left[e^{c\left(i, \pi^{* 1}(i), v\right)} \sum_{j \in S} \psi^{*}(j) P\left(j \mid i, \pi^{* 1}(i), v\right)\right] \\
& =\inf _{\mu \in \mathcal{P}(U(i))}\left[e^{c\left(i, \mu, \pi^{* 2}(i)\right)} \sum_{j \in S} \psi^{*}(j) P\left(j \mid i, \mu, \pi^{* 2}(i)\right)\right] . \tag{4.15}
\end{align*}
$$

Now for any finite set $\hat{\mathcal{B}}_{1} \supset \hat{\mathcal{B}}$, applying Dynkin's formula (as in [53, Lemma 3.1]) from (4.15), we get

$$
\psi^{*}(i) \leq E_{i}^{\pi^{1}, \pi^{* 2}}\left[e^{\sum_{t=0}^{\check{\tau}\left(\hat{\mathcal{B}}_{1}\right) \wedge N-1}\left(c\left(X_{t}, \pi_{t}^{1}, \pi^{* 2}\left(X_{t}\right)\right)-\rho^{*}\right)} \psi^{*}\left(X_{\check{\tau}\left(\hat{\mathcal{B}}_{1}\right) \wedge N}\right)\right] \forall i \in \hat{\mathcal{B}}_{1}^{c} .
$$

Since $\psi^{*} \leq \mathcal{W}$, using estimates of Lemma 3.1, by the dominated convergence theorem taking $N \rightarrow \infty$, it follows that

$$
\begin{equation*}
\psi^{*}(i) \leq E_{i}^{\pi^{1}, \pi^{* 2}}\left[e^{\sum_{t=0}^{\tau\left(\hat{\mathcal{B}}_{1}\right)-1}\left(c\left(X_{t}, \pi_{t}^{1}, \pi^{* 2}\left(X_{t}\right)\right)-\rho^{*}\right)} \psi^{*}\left(X_{\check{\tau}\left(\hat{\mathcal{B}}_{1}\right)}\right)\right] \forall i \in \hat{\mathcal{B}}_{1}^{c} . \tag{4.16}
\end{equation*}
$$

Hence

$$
\begin{align*}
\psi^{*}(i) & \leq \inf _{\pi^{1} \in \Pi_{a d}^{1}} E_{i}^{\pi^{1}, \pi^{* 2}}\left[e^{\sum_{t=0}^{\check{( }\left(\hat{\mathcal{B}}_{1}\right)-1}\left(c\left(X_{t}, \pi_{t}^{1}, \pi^{* 2}\left(X_{t}\right)\right)-\rho^{*}\right)} \psi^{*}\left(X_{\check{\tau}\left(\hat{\mathcal{B}}_{1}\right)}\right)\right] \\
& \leq \sup _{\pi^{2} \in \Pi_{a d}^{2}} \inf _{\pi^{1} \in \Pi_{a d}^{1}} E_{i}^{\pi^{1}, \pi^{2}}\left[e^{\sum_{t=0}^{\check{\tau}\left(\hat{\mathcal{S}}_{1}\right)-1}\left(c\left(X_{t}, \pi_{t}^{1}, \pi_{t}^{2}\right)-\rho^{*}\right)} \psi^{*}\left(X_{\check{\tau}\left(\hat{\mathcal{B}}_{1}\right)}\right)\right], \forall i \in \hat{\mathcal{B}}_{1}^{c} . \tag{4.17}
\end{align*}
$$

Now using (4.15) and Dynkin's formula

$$
\psi^{*}(i) \geq E_{i}^{\pi^{* 1}, \pi^{2}}\left[e^{\sum_{t=0}^{\check{\tau}\left(\hat{\mathcal{B}}_{1}\right) \wedge N-1}\left(c\left(X_{t}, \pi^{* 1}\left(X_{t}\right), \pi_{t}^{2}\right)-\rho^{*}\right)} \psi^{*}\left(X_{\check{\tau}\left(\hat{\mathcal{B}}_{1}\right) \wedge N}\right)\right] \forall i \in \hat{\mathcal{B}}_{1}^{c} .
$$

In view of Lemma 3.1 by Fatou's lemma taking $N \rightarrow \infty$, we get

$$
\begin{equation*}
\psi^{*}(i) \geq E_{i}^{\pi^{* 1}, \pi^{2}}\left[e^{\sum_{t=0}^{\tau\left(\hat{\mathcal{B}}_{1}\right)-1}\left(c\left(X_{t}, \pi^{* 1}\left(X_{t}\right), \pi_{t}^{2}\right)-\rho^{*}\right)} \psi^{*}\left(X_{\check{\tau}\left(\hat{\mathcal{B}}_{1}\right)}\right)\right], \forall i \in \hat{\mathcal{B}}_{1}^{c} . \tag{4.18}
\end{equation*}
$$

Hence,

$$
\begin{align*}
\psi^{*}(i) & \geq \sup _{\pi^{2} \in \Pi_{a d}^{2}} E_{i}^{\pi^{* 1}, \pi^{2}}\left[e^{\sum_{t=0}^{\check{( }\left(\hat{\mathcal{B}}_{1}\right)-1}\left(c\left(X_{t}, \pi^{* 1}\left(X_{t}\right), \pi_{t}^{2}\right)-\rho^{*}\right)} \psi^{*}\left(X_{\check{\tau}\left(\hat{\mathcal{B}}_{1}\right)}\right)\right] \\
& \geq \inf _{\pi^{1} \in \Pi_{a d}^{1}} \sup _{\pi^{1} \in \Pi_{a d}^{1}} E_{i}^{\pi^{1}, \pi^{2}}\left[e^{\sum_{t=0}^{\check{\tau}\left(\hat{\mathcal{S}}_{1}\right)-1}\left(c\left(X_{t}, \pi_{t}^{1}, \pi_{t}^{2}\right)-\rho^{*}\right)} \psi^{*}\left(X_{\check{\tau}\left(\hat{\mathcal{B}}_{1}\right)}\right)\right], \forall i \in \hat{\mathcal{B}}_{1}^{c} . \tag{4.19}
\end{align*}
$$

From (4.17) and (4.19), we get Eq. (4.8).
Next we prove the existence of the value of the game. To this end we first perturb the cost function as follows:

- When Assumption 2.1(a) holds: Let $\alpha_{3}>0$, be a small number satisfying $\|c\|_{\infty}+\alpha_{3}<\tilde{\gamma}$. Now we define $\tilde{c}_{n}(i, u, v)=c(i, u, v) I_{\tilde{\mathcal{D}}_{n}}(i)+\left(\|c\|_{\infty}+\alpha_{3}\right) I_{\tilde{\mathcal{D}}_{n}^{c}} \forall(u, v) \in U(i) \times V(i)$, $i \in S$. Note $\left\|\tilde{c}_{n}\right\|_{\infty}<\tilde{\gamma}$, where $\left\|\tilde{c}_{n}\right\|_{\infty}=\sup _{(i, u, v) \in \mathcal{K}} \tilde{c}_{n}(i, u, v)$.
- When Assumption 2.1(b) holds: Define

$$
\tilde{c}_{n}(i, u, v)=c(i, u, v)+\frac{1}{n}\left[\tilde{\ell}(i)-\sup _{(u, v) \in U(i) \times V(i)} c(i, u, v)\right]_{+} \forall(u, v) \in U(i) \times V(i), i \in S .
$$

Since the function $\left[\tilde{\ell}(\cdot)-\sup _{(u, v) \in U(\cdot) \times V(\cdot)} c(\cdot, u, v)\right]_{+}$is norm-like function, we have $\tilde{\ell}-\sup _{(u, v) \in U(\cdot) \times V(\cdot)} \tilde{c}_{n}(\cdot, u, v)$ is norm-like for large enough $n$.

Theorem 4.2. Suppose that Assumptions 2.1 and 2.2 hold. Let $\left(\pi^{* 1}, \pi^{* 2}\right) \in \Pi_{S M}^{1} \times \Pi_{S M}^{2}$ be any mini-max selector of (4.6), i.e. $\left(\pi^{* 1}, \pi^{* 2}\right) \in \Pi_{S M}^{1} \times \Pi_{S M}^{2}$ satisfies

$$
\begin{align*}
e^{\rho^{*}} \psi^{*}(i) & =\sup _{v \in \mathcal{P}(V(i))} \inf _{\mu \in \mathcal{P}(U(i))}\left[e^{c(i, \mu, \nu)} \sum_{j \in S} \psi^{*}(j) P(j \mid i, \mu, v)\right] \\
& =\inf _{\mu \in \mathcal{P}(U(i))} \sup _{v \in \mathcal{P}(V(i))}\left[e^{c(i, \mu, \nu)} \sum_{j \in S} \psi^{*}(j) P(j \mid i, \mu, v)\right] \\
& =\inf _{\mu \in \mathcal{P}(U(i))}\left[e^{c\left(i, \mu, \pi^{* 2}(i)\right)} \sum_{j \in S} \psi^{*}(j) P\left(j \mid i, \mu, \pi^{* 2}(i)\right)\right] \\
& =\sup _{\nu \in \mathcal{P}(V(i))}\left[e^{c\left(i, \pi^{* 1}(i), v\right)} \sum_{j \in S} \psi^{*}(j) P\left(j \mid i, \pi^{* 1}(i), v\right)\right], i \in S . \tag{4.20}
\end{align*}
$$

Then we have

$$
\begin{align*}
\rho^{*} & =\inf _{i \in S} \sup _{\pi^{2} \in \Pi_{a d}^{2}} \inf _{\pi^{1} \in \Pi_{a d}^{1}} \mathcal{J}^{\pi^{1}, \pi^{2}}(i, c)=\inf _{i \in S} \inf _{\pi^{1} \in \Pi_{a d}^{1}} \sup _{\pi^{2} \in \Pi_{a d}^{2}} \mathcal{J}^{1}, \pi^{2}(i, c) \\
& =\inf _{i \in S_{\pi^{1} \in \Pi_{a d}^{1}} \inf ^{\pi^{1}, \pi^{* 2}}(i, c)=\inf _{i \in S} \sup _{\pi^{2} \in \Pi_{a d}^{2}} \mathcal{J}^{\pi^{* 1}, \pi^{2}}(i, c)=\mathcal{J}^{\pi^{* 1}, \pi^{* 2}}(i, c) .} . \tag{4.21}
\end{align*}
$$

Proof. Arguing as Lemma 4.1, for the stationary strategy $\pi^{* 1} \in \Pi_{S M}^{1}$, there exists an eigenpair $\left(\hat{\rho}_{n}, \hat{\psi}_{n}\right) \in \mathbb{R}_{+} \times L_{W}^{\infty}$ with $\hat{\psi}_{n}>0$ satisfying

$$
\begin{equation*}
e^{\hat{\rho}_{n}} \hat{\psi}_{n}(i)=\sup _{\nu \in \mathcal{P}(B(i))}\left[e^{\tilde{c}_{n}\left(i, \pi^{* 1}(i), v\right)} \sum_{j \in S} \hat{\psi}_{n}(j) P\left(j \mid i, \pi^{* 1}(i), \nu\right)\right], i \in S \tag{4.22}
\end{equation*}
$$

such that

$$
\begin{equation*}
0 \leq \hat{\rho}_{n} \leq \sup _{\pi^{2} \in \Pi_{a d}^{2}} \mathcal{J}^{\pi^{* 1}, \pi^{2}}\left(i, \tilde{c}_{n}\right) \tag{4.23}
\end{equation*}
$$

Also,

$$
\begin{equation*}
\hat{\psi}_{n}(i)=\sup _{\pi^{2} \in \Pi_{a d}^{2}} E_{i}^{\pi^{* 1}, \pi^{2}}\left[e^{\sum_{t=0}^{\check{\Sigma_{\mathcal{B}}}\left(\hat{\mathcal{D}}_{1}\right)-1}\left(\tilde{c}_{n}\left(X_{t}, \pi^{* 1}\left(X_{t}\right), \pi_{t}^{2}\right)-\hat{\rho}_{n}\right)} \hat{\psi}_{n}\left(X_{\check{\tau}\left(\hat{\mathcal{B}}_{1}\right)}\right)\right], i \in \hat{\mathcal{B}}_{1}^{c} \tag{4.24}
\end{equation*}
$$

for some finite set $\hat{\mathcal{B}}_{1}$ containing $\hat{\mathcal{B}}$.
Now as in Lemma 4.1, we have a finite set $\tilde{\mathcal{B}}_{1}$, depending on $n$, containing $\tilde{\mathcal{K}}$ such that:

- Under Assumption 2.1(a): From (4.23), we have $\hat{\rho}_{n} \leq\left\|\tilde{c}_{n}\right\|_{\infty}$. So, from the above definition of $\tilde{c}_{n}$, for $i \in \hat{\mathcal{D}}_{n}^{c}$, we have $\tilde{c}_{n}(i, u, v)-\hat{\rho}_{n} \geq 0$ for all $(u, v) \in U(i) \times V(i)$. Consequently, we may take $\tilde{\mathcal{B}}_{1}=\hat{\mathcal{D}}_{n}$ such that $\tilde{c}_{n}(i, u, v)-\hat{\rho}_{n} \geq 0$ in $\tilde{\mathcal{B}}_{1}^{c}$ for all $(u, v) \in U(i) \times V(i)$.
- Under Assumption 2.1(b): since $\tilde{c}_{n}$ is norm-like function, we can choose suitable finite set $\tilde{\mathcal{B}}_{1}$ such that $\left(\tilde{c}_{n}(i, u, v)-\hat{\rho}_{n}\right) \geq 0$ in $\tilde{\mathcal{B}}_{1}^{c}$ for all $(u, v) \in U(i) \times V(i)$.

From (4.22), we obtain

$$
\begin{equation*}
\hat{\psi}_{n}(i) \geq\left[e^{\left(\tilde{c}_{n}\left(i, \pi^{* 1}(i), v\right)-\hat{\rho}_{n}\right)} \sum_{j \in S} \hat{\psi}_{n}(j) P\left(j \mid i, \pi^{* 1}(i), v\right)\right] . \tag{4.25}
\end{equation*}
$$

By Dynkin's formula from (4.25), we deduce that

$$
\hat{\psi}_{n}(i) \geq E_{i}^{\pi^{* 1}, \pi^{2}}\left[e^{\sum_{t=0}^{\check{\tau}\left(\tilde{\mathcal{B}}_{1}\right) \wedge N-1}\left(\tilde{c}_{n}\left(X_{t}, \pi^{* 1}\left(X_{t}\right), \pi_{t}^{2}\right)-\hat{\rho}_{n}\right)} \hat{\psi}_{n}\left(X_{\check{\tau}\left(\tilde{\mathcal{B}}_{1}\right) \wedge N}\right)\right] .
$$

Since $\tilde{c}_{n}(i, u, v)-\hat{\rho}_{n} \geq 0$, in $\tilde{\mathcal{B}}_{1}^{c}$, for all $(u, v) \in U(i) \times V(i)$, by Fatou lemma taking $N \rightarrow \infty$, we obtain

$$
\hat{\psi}_{n}(i) \geq E_{i}^{\pi^{* 1}, \pi^{2}}\left[e^{\sum_{t=0}^{\check{\tau}\left(\tilde{\mathcal{B}}_{1}\right)-1}\left(\tilde{c}_{n}\left(X_{t}, \pi^{* 1}\left(X_{t}\right), \pi_{t}^{2}\right)-\hat{\rho}_{n}\right)} \hat{\psi}_{n}\left(X_{\check{\tau}\left(\tilde{\mathcal{B}}_{1}\right)}\right)\right] \geq \min _{\tilde{\mathcal{B}}_{1}} \hat{\psi}_{n} \forall i \in \tilde{\mathcal{B}}_{1}^{c} .
$$

So, $\hat{\psi}_{n}$ has a lower bound. Again by Dynkin's formula from (4.22), we get

$$
\hat{\psi}_{n}(i) \geq E_{i}^{\pi^{* 1}, \pi^{2}}\left[e^{\sum_{t=0}^{T \wedge \tau\left(\tilde{\mathcal{D}}_{m}\right)-1}\left(\tilde{c}_{n}\left(X_{t}, \pi^{* 1}\left(X_{t}\right), \pi_{t}^{2}\right)-\hat{\rho}_{n}\right)} \hat{\psi}_{n}\left(X_{T \wedge \tau\left(\tilde{\mathcal{D}}_{m}\right)}\right)\right] .
$$

By Fatou's lemma, taking $m \rightarrow \infty$, we obtain

$$
\begin{aligned}
\hat{\psi}_{n}(i) & \geq E_{i}^{\pi^{* 1}, \pi^{2}}\left[e^{\sum_{t=0}^{T-1}\left(\tilde{c}_{n}\left(X_{t}, \pi^{* 1}\left(X_{t}\right), \pi_{t}^{2}\right)-\hat{\rho}_{n}\right)} \hat{\psi}_{n}\left(X_{T}\right)\right] \\
& \geq\left(\min _{\tilde{\mathcal{B}}_{1}} \hat{\psi}_{n}\right) E_{i}^{\pi^{* 1}, \pi^{2}}\left[e^{\sum_{t=0}^{T-1}\left(\tilde{c}_{n}\left(X_{t}, \pi^{* 1}\left(X_{t}\right), \pi_{t}^{2}\right)-\hat{\rho}_{n}\right)}\right] .
\end{aligned}
$$

So, taking logarithm both sides, dividing by $T$ and letting $T \rightarrow \infty$, we deduce that

$$
\hat{\rho}_{n} \geq \mathcal{J}^{\pi^{* 1}, \pi^{2}}\left(i, \tilde{c}_{n}\right) .
$$

Since $\pi^{2} \in \Pi_{a d}^{2}$ is arbitrary,

$$
\hat{\rho}_{n} \geq \sup _{\pi^{2} \in \Pi_{a d}^{2}} g^{\pi^{* 1}, \pi^{2}}\left(i, \tilde{c}_{n}\right) \geq \sup _{\pi^{2} \in \Pi_{a d}^{2}} g^{\pi^{* 1}, \pi^{2}}(i, c)
$$

Using this and (4.23), we get $\sup _{\pi^{2} \in \Pi_{a d}^{2}} \mathcal{J}^{\pi^{* 1}, \pi^{2}}(i, c) \leq \sup _{\pi^{2} \in \Pi_{a d}^{2}} \partial^{\pi^{* 1}, \pi^{2}}\left(i, \tilde{c}_{n}\right)=\hat{\rho}_{n}$ for all $n$. Now, by suitable scaling as in the proof of Lemma 4.1, it is easy to see that $\hat{\psi}_{n} \leq \mathcal{W}$ and it touches $\mathcal{W}$. Also, we note from the definition of $\tilde{c}_{n}$ that $\hat{\rho}_{n}$ is a monotone decreasing sequence bounded below. Thus, using diagonalization arguments, there exists subsequence (denoting the same sequence) and a pair ( $\hat{\rho}, \hat{\psi}$ ), $\hat{\psi}>0$ such that $\hat{\rho}_{n} \rightarrow \hat{\rho}$ and $\hat{\psi}_{n} \rightarrow \hat{\psi}$ as $n \rightarrow \infty$. Now arguing as in the proof of Lemma 4.1, taking $n \rightarrow \infty$ in (4.22), we get

$$
\begin{equation*}
e^{\hat{\rho}} \hat{\psi}(i)=\sup _{v \in \mathcal{P}(V(i))}\left[e^{c\left(i, \pi^{* 1}(i), v\right)} \sum_{j \in S} \hat{\psi}(j) P\left(j \mid i, \pi^{* 1}(i), v\right)\right] . \tag{4.26}
\end{equation*}
$$

Also, we have $\lim _{n \rightarrow \infty} \hat{\rho}_{n}=\hat{\rho} \geq \sup _{\pi^{2} \in \Pi^{2}} \partial^{\pi^{* 1}, \pi^{2}}(i, c) \geq \inf _{\pi^{1} \in \Pi^{1}} \sup _{\pi^{2} \in \Pi^{2}} \partial^{\pi^{1}, \pi^{2}}(i, c) \geq \rho^{*}$.
We want to show that $\hat{\rho}=\rho^{*}$. By continuity-compactness assumptions, there exists $\hat{\pi}^{* 2}$ such that (4.26) can be rewritten as

$$
\begin{equation*}
e^{\hat{\rho}} \hat{\psi}(i)=\left[e^{c\left(i, \pi^{* 1}(i), \hat{\pi}^{* 2}(i)\right)} \sum_{j \in S} \hat{\psi}(j) P\left(j \mid i, \pi^{* 1}(i), \hat{\pi}^{* 2}(i)\right)\right] . \tag{4.27}
\end{equation*}
$$

By Dynkin's formula, for some $\hat{\mathcal{B}}_{2}$ containing $\hat{\mathcal{B}}$, we have

$$
\begin{equation*}
\hat{\psi}(i)=E_{i}^{\pi^{* 1}, \hat{\pi}^{* 2}}\left[e^{\sum_{t=0}^{\check{\tau}\left(\hat{\mathcal{B}}_{2}\right) \wedge N-1}\left(c\left(X_{t}, \pi^{* 1}\left(X_{t}\right), \hat{\pi}^{* 2}\left(X_{t}\right)\right)-\hat{\rho}\right)} \hat{\psi}\left(X_{\left(\check{\tau}\left(\hat{\mathcal{B}}_{2}\right) \wedge N\right)}\right)\right], \forall i \in \hat{\mathcal{B}}_{2}^{c} . \tag{4.28}
\end{equation*}
$$

Using the estimates of Lemma 3.1 and the dominated convergence theorem, taking $N \rightarrow \infty$ in (4.28), we obtain

$$
\begin{equation*}
\hat{\psi}(i)=E_{i}^{\pi^{* 1}, \hat{\pi}^{* 2}}\left[e^{\sum_{t=0}^{\check{t}\left(\hat{\mathcal{H}}_{2}\right)-1}\left(c\left(X_{t}, \pi^{* 1}\left(X_{t}\right), \hat{\pi}^{* 2}\left(X_{t}\right)\right)-\hat{\rho}\right)} \hat{\psi}\left(X_{\check{\tau}\left(\hat{\mathcal{B}}_{2}\right)}\right)\right], \forall i \in \hat{\mathcal{B}}_{2}^{c} . \tag{4.29}
\end{equation*}
$$

Since $\hat{\rho} \geq \rho^{*}$, from (4.8) we have

$$
\begin{equation*}
\psi^{*}(i) \geq E_{i}^{\pi^{* 1}, \hat{\pi}^{* 2}}\left[e^{\sum_{t=0}^{\check{r}\left(\hat{\mathcal{B}}_{2}\right)-1}\left(c\left(X_{t}, \pi^{* 1}\left(X_{t}\right), \hat{\pi}^{* 2}\left(X_{t}\right)\right)-\hat{\rho}\right)} \psi^{*}\left(X_{\check{\tau}\left(\hat{\mathcal{B}}_{2}\right)}\right)\right] \forall i \in \hat{\mathcal{B}}_{2}^{c} . \tag{4.30}
\end{equation*}
$$

Hence, from (4.29) and (4.30), it follows that

$$
\begin{equation*}
\psi^{*}(i)-\hat{k}_{1} \hat{\psi}(i) \geq E_{i}^{\pi^{* 1}, \hat{\pi}^{* 2}}\left[e^{\sum_{t=0}^{\check{\tau}\left(\mathcal{B}_{2}\right)-1}\left(c\left(X_{t}, \pi^{* 1}\left(X_{t}\right), \hat{\pi}^{* 2}\left(X_{t}\right)\right)-\hat{\rho}\right)}\left(\psi^{*}-\hat{k}_{1} \hat{\psi}\right)\left(X_{\check{\tau}\left(\hat{\mathcal{B}}_{2}\right)}\right)\right] \forall i \in \hat{\mathcal{B}}_{2}^{c} . \tag{4.31}
\end{equation*}
$$

Let $\hat{k}_{1}=\min _{\hat{\mathcal{B}}_{2}} \frac{\psi^{*}}{\hat{\psi}}$, thus we have $\left(\psi^{*}-\hat{k}_{1} \hat{\psi}\right) \geq 0$ in $\hat{\mathcal{B}}_{2}$ and for some $\hat{i}_{0} \in \hat{\mathcal{B}}_{2}$, $\left(\psi^{*}-\hat{k}_{1} \hat{\psi}\right)\left(\hat{i}_{0}\right)=0$. Therefore, from (4.31), we obtain that $\left(\psi^{*}-\hat{k}_{1} \hat{\psi}\right) \geq 0$ in $S$. Now since $\hat{\rho} \geq \rho^{*}$, from (4.20) and (4.27), we deduce that

$$
e^{\hat{\rho}}\left(\psi^{*}-\hat{k}_{1} \hat{\psi}\right)\left(\hat{i}_{0}\right) \geq\left[e^{c\left(\hat{i}_{0}, \pi^{* 1}\left(i_{0}\right), \hat{\pi}^{* 2}\left(i_{0}\right)\right)} \sum_{j \in S}\left(\psi^{*}-\hat{k}_{1} \hat{\psi}\right)(j) P\left(j \mid \hat{i}_{0}, \pi^{* 1}\left(\hat{i}_{0}\right), \hat{\pi}^{* 2}\left(\hat{i}_{0}\right)\right)\right]
$$

This gives us

$$
\begin{equation*}
0=\sum_{j \in S}\left(\psi^{*}-\hat{k}_{1} \hat{\psi}\right)(j) P\left(j \mid \hat{i}_{0}, \pi^{* 1}\left(\hat{i}_{0}\right), \hat{\pi}^{* 2}\left(\hat{i}_{0}\right)\right) \tag{4.32}
\end{equation*}
$$

Thus, in view of irreducibility property of the Markov chain under stationary Markov strategies, it follows that $\psi^{*}=\hat{k}_{1} \hat{\psi}$ in $S$. Hence from (4.20) and (4.26), it is easy to see that $\hat{\rho}=\rho^{*}$ for all $i \in S$. Therefore, we obtain

$$
\begin{align*}
\rho^{*}=\inf _{i \in S} \sup _{\pi^{2} \in \Pi_{a d}^{2}} \inf _{\pi^{1} \in \Pi_{a d}^{1}} g^{\pi^{1}, \pi^{2}}(i, c) & =\inf _{i \in S} \inf _{\pi^{1} \in \Pi_{a d}^{1}} \sup _{\pi^{2} \in \Pi_{a d}^{2}} g^{\pi^{1}, \pi^{2}}(i, c) \\
& =\inf _{i \in S} \sup _{\pi^{2} \in \Pi_{a d}^{2}} \partial^{\pi^{* 1}, \pi^{2}}(i, c) \\
& \geq \inf _{i \in S}{J \pi^{* 1}, \pi^{* 2}}^{*}(i, c) . \tag{4.33}
\end{align*}
$$

Now arguing as in [13, Lemma 2.6], it follows that for $\pi^{* 2} \in \Pi_{S M}^{2}$, there exists $\left(\psi^{\prime}, \rho^{\prime}\right) \in$ $L_{\mathcal{W}}^{\infty} \times \mathbb{R}_{+}, \psi^{\prime}>0$ satisfying

$$
\begin{equation*}
e^{\rho^{\prime}} \psi^{\prime}(i)=\inf _{\mu \in \mathcal{P}(U(i))}\left[e^{c\left(i, \mu, \pi^{* 2}(i)\right)} \sum_{j \in S} \psi^{\prime}(j) P\left(j \mid i, \mu, \pi^{* 2}(i)\right)\right], \tag{4.34}
\end{equation*}
$$

with

$$
\begin{equation*}
\rho^{\prime}=\inf _{i \in S} \inf _{\pi^{1} \in \Pi_{a d}^{1}} g^{\pi^{1}, \pi^{* 2}}(i, c) \tag{4.35}
\end{equation*}
$$

Thus, we have

$$
\begin{equation*}
\rho^{\prime}=\inf _{i \in S} \inf _{\pi^{1} \in \Pi_{a d}^{1}} \mathcal{J}^{\pi^{1}, \pi^{* 2}}(i, c) \leq \inf _{i \in S} \mathcal{J}^{\pi^{* 1}, \pi^{* 2}}(i, c) \leq \rho^{*} \tag{4.36}
\end{equation*}
$$

For any minimizing selector $\tilde{\pi}^{* 1}$ of (4.34), we obtain

$$
\begin{equation*}
e^{\rho^{\prime}} \psi^{\prime}(i)=\left[e^{c\left(i, \tilde{\pi}^{* 1}(i), \pi^{* 2}(i)\right)} \sum_{j \in S} \psi^{\prime}(j) P\left(j \mid i, \tilde{\pi}^{* 1}(i), \pi^{* 2}(i)\right)\right] . \tag{4.37}
\end{equation*}
$$

Also, arguing as in Lemma 4.1, for some finite set $\hat{\mathcal{B}}_{3} \supset \hat{\mathcal{B}}$, we deduce that

$$
\begin{equation*}
\psi^{\prime}(i)=E_{i}^{\tilde{\pi}^{* 1}, \pi^{* 2}}\left[e^{\left.\sum_{t=0}^{\check{\tau}\left(\hat{\mathcal{B}}_{3}\right)-1}\left(c\left(X_{t}, \tilde{\pi}^{* 1}\left(X_{t}\right)\right), \pi^{* 2}\left(X_{t}\right)\right)-\rho^{\prime}\right) d t} \psi^{\prime}\left(X_{\check{\tau}\left(\hat{\mathcal{B}}_{3}\right)}\right)\right], i \in \hat{\mathcal{B}}_{3}^{c} . \tag{4.38}
\end{equation*}
$$

From (4.20), we have

$$
\begin{equation*}
e^{\rho^{*}} \psi^{*}(i) \leq\left[e^{c\left(i, \tilde{\pi}^{* 1}(i), \pi^{* 2}(i)\right)} \sum_{j \in S} \psi^{*}(j) P\left(j \mid i, \tilde{\pi}^{* 1}(i), \pi^{* 2}(i)\right)\right], i \in S \tag{4.39}
\end{equation*}
$$

Also, from (4.8), it follows that

$$
\begin{equation*}
\psi^{*}(i) \leq E_{i}^{\tilde{\pi}^{* 11}, \pi^{* 2}}\left[e^{\sum_{t=0}^{\check{r}\left(\mathcal{B}_{3}\right)-1}\left(c\left(X_{t}, \tilde{\pi}^{* 1}\left(X_{t}\right), \pi^{* 2}\left(X_{t}\right)\right)-\rho^{*}\right) d t} \psi^{*}\left(X_{\left.\check{\tau}\left(\mathcal{B}_{3}\right)\right)}\right] \forall i \in \hat{\mathcal{B}}_{3}^{c} .\right. \tag{4.40}
\end{equation*}
$$

Therefore, by analogous arguments as above, using irreducibility property of the Markov chain, we get $\psi^{\prime}=\hat{k}_{2} \psi^{*}$, for some positive constant $\hat{k}_{2}$. Thus, from (4.20) and (4.34), it follows that

$$
\begin{equation*}
\rho^{*}=\rho^{\prime} \tag{4.41}
\end{equation*}
$$

Hence, by (4.33), (4.36) and (4.41), we obtain (4.21). This completes the proof of the theorem.

Now we prove the uniqueness of the eigenpair of the optimality equation (4.6) in the space $\mathbb{R}_{+} \times L_{W}^{\infty}$.

Lemma 4.2. Suppose that Assumptions 2.1 and 2.2 hold. Then the eigenpair $\left(\rho^{*}, \psi^{*}\right) \in$ $\mathbb{R}_{+} \times L_{W}^{\infty}$ is a unique solution of (4.6) (up to a scalar multiplication).

Proof. Let $(\tilde{\rho}, \tilde{\psi}) \in \mathbb{R}_{+} \times L_{W}^{\infty}, \tilde{\psi}>0$ be another solution of (4.6). Then using Fan's minimax theorem and (4.6), we get

$$
\begin{align*}
e^{\tilde{\rho}} \tilde{\psi}(i) & =\sup _{v \in \mathcal{P}(V(i))} \inf _{\mu \in \mathcal{P}(U(i))}\left[e^{c(i, \mu, v)} \sum_{j \in S} \tilde{\psi}(j) P(j \mid i, \mu, v)\right] \\
& =\inf _{\mu \in \mathcal{P}(U(i)))} \sup _{v \in \mathcal{P}(V(i))}\left[e^{c(i, \mu, v)} \sum_{j \in S} \tilde{\psi}(j) P(j \mid i, \mu, v)\right], i \in S . \tag{4.42}
\end{align*}
$$

There exists an outer minimizing selector $\tilde{\pi}^{* 1} \in \Pi_{S M}^{1}$ such that (4.42) can be written

$$
\begin{equation*}
e^{\tilde{\rho}} \tilde{\psi}(i)=\sup _{v \in \mathcal{P}(V(i))}\left[e^{c\left(i, \tilde{\pi}^{* 1}(i), v\right)} \sum_{j \in S} \tilde{\psi}(j) P\left(j \mid i, \tilde{\pi}^{* 1}(i), v\right)\right], i \in S \tag{4.43}
\end{equation*}
$$

We claim that $\rho^{*}=\tilde{\rho}$. If possible let us assume that $\tilde{\rho}<\rho^{*}$. Now from (4.8), for some finite set $\hat{\mathcal{B}}_{4} \supset \hat{\mathcal{B}}$, we get

$$
\begin{equation*}
\psi(i) \leq E_{i}^{\tilde{\pi}^{* 1}, \pi^{* 2}}\left[e^{\sum_{t=0}^{\check{\tau}\left(\hat{\mathcal{S}}_{4}\right)-1}\left(c\left(X_{t}, \tilde{\pi}^{* 1}\left(X_{t}\right), \pi^{* 2}\left(X_{t}\right)\right)-\rho^{*}\right)} \psi\left(X_{\check{\tau}\left(\hat{\mathcal{B}}_{4}\right)}\right)\right] \forall i \in \hat{\mathcal{B}}_{4}^{c}, \tag{4.44}
\end{equation*}
$$

where $\pi^{* 2}$ is an outer maximizing selector of (4.6) as in (4.20). Since $\rho^{*}>\tilde{\rho}$, in view of (4.18), by Dynkin's formula and Fatou's lemma from (4.43), we deduce that

$$
\begin{equation*}
\tilde{\psi}(i) \geq E_{i}^{\tilde{\pi}^{* 1}, \pi^{* 2}}\left[e^{\sum_{t=0}^{\check{\tau}\left(\hat{\mathcal{B}}_{4}\right)-1}\left(c\left(X_{t}, \tilde{\pi}^{* 1}\left(X_{t}\right), \pi^{* 2}\left(X_{t}\right)\right)-\rho^{*}\right)} \tilde{\psi}\left(X_{\check{\tau}\left(\hat{\mathcal{B}}_{4}\right)}\right)\right] \forall i \in \hat{\mathcal{B}}_{4}^{c} . \tag{4.45}
\end{equation*}
$$

Therefore, by analogous arguments as above (see (4.29)-(4.32) or (4.37)-(4.41)), using irreducibility property of the Markov chain and (4.6), (4.43), (4.44) and (4.45) and by taking $\hat{k}_{3}=\min _{\hat{\mathcal{B}}_{4}} \frac{\psi}{\psi^{*}}$, we get $\tilde{\psi}=\hat{k}_{3} \psi^{*}$. Thus, from (4.6) and (4.42), it follows that $\rho^{*}=\tilde{\rho}$. Hence we arrive at a contradiction and it contradicts to the fact that $\tilde{\rho}<\rho^{*}$. Thus, we obtain $\rho^{*} \leq \tilde{\rho}$.

Next, if possible let $\rho^{*}<\tilde{\rho}$. Then by analogous arguments, (by taking outer maximizer of (4.42) and outer minimizer of (4.6)), we will arrive at a contradiction to the fact that $\rho^{*}<\tilde{\rho}$. Hence, we deduce that

$$
\begin{equation*}
\rho^{*}=\tilde{\rho} \text { and } \psi^{*}=\hat{k}_{4} \tilde{\psi}, \tag{4.46}
\end{equation*}
$$

for some positive constant $\hat{k}_{4}$.
In particular, this implies that the eigenpair of (4.6) is unique up to a scalar multiplication.

Remark 4.1. In deriving (4.34), we used [13, Lemma 2.6]. It should be noted that the results of the paper [13] can be derived by using the assumptions of this paper (see [13, Remark 2.3]).

Next we prove the converse of the above theorem. That is, any saddle point equilibrium of our game problem will be a mini-max selector of the associated optimality equation.

Theorem 4.3. Suppose Assumptions 2.1 and 2.2 hold. Suppose there exists a saddle point equilibrium $\left(\hat{\pi}^{* 1}, \hat{\pi}^{* 2}\right) \in \Pi_{S M}^{1} \times \Pi_{S M}^{2}$, i.e., for all $i \in S$,

$$
\begin{align*}
& \mathcal{J}^{\hat{\pi}^{* 1}, \hat{\pi}^{* 2}}(i, c) \leq \mathcal{J}^{\pi^{1}, \hat{\pi}^{* 2}}(i, c), \text { for all } \pi^{1} \in \Pi_{a d}^{1}, \\
& \mathcal{J}^{\hat{\pi}^{* 1}, \hat{\pi}^{* 2}}(i, c) \geq \mathcal{J}^{\hat{\pi}^{* 1}, \pi^{2}}(i, c), \text { for all } \pi^{2} \in \Pi_{a d}^{2} . \tag{4.47}
\end{align*}
$$

Then $\left(\hat{\pi}^{* 1}, \hat{\pi}^{* 2}\right)$ is a mini-max selector of (4.6).

Proof. By Theorem 4.2 and (4.47), we have

$$
\begin{aligned}
\rho^{*}=\inf _{\pi^{1} \in \Pi_{a d}^{1}} \sup _{\pi^{2} \in \Pi_{a d}^{2}} \mathcal{g}^{\pi^{1}, \pi^{2}}(i, c) & \leq \sup _{\pi^{2} \in \Pi_{a d}^{2}} g^{\hat{\pi}^{* 1}, \pi^{2}}(i, c) \leq \mathcal{y}^{\hat{\pi}^{* 1}, \hat{\pi}^{* 2}}(i, c) \\
& \leq \inf _{\pi^{1} \in \Pi_{a d}^{1}} g^{\pi^{1}, \hat{\pi}^{* 2}}(i, c) \leq \sup _{\pi^{2} \in \Pi_{a d}^{2}} \inf _{\pi^{1} \in \Pi_{a d}^{1}} g^{\pi^{1}, \pi^{2}}(i, c)=\rho^{*}
\end{aligned}
$$

This implies that $\rho^{*}=\mathcal{J}^{\hat{\pi}^{* 1}, \hat{\pi}^{* 2}}(i, c)=\sup _{\pi^{2} \in \Pi_{a d}^{2}} \mathcal{f}^{\tilde{}^{* 1}, \pi^{2}}(i, c)=\inf _{\pi^{1} \in \Pi_{a d}^{1}} \mathcal{J}^{1}, \hat{\pi}^{* 2}(i, c)$. Now arguing as in Lemma 4.1 and Theorem 4.2, it follows that for $\hat{\pi}^{* 2} \in \Pi_{S M}^{2}$ there exists $\left(\rho^{\hat{\pi}^{* 2}}, \psi_{\hat{\pi}^{* 2}}^{*}\right) \in \mathbb{R}_{+} \times L_{\mathcal{W}}^{\infty}$ with $\psi_{\hat{\pi}^{* 2}}^{*}>0$ such that

$$
\begin{equation*}
e^{\rho^{\hat{\pi}^{* 2}}} \psi_{\hat{\pi}^{* 2}}^{*}(i)=\inf _{\mu \in \mathcal{P}(U(i))}\left[e^{c\left(i, \mu, \hat{\pi}^{* 2}(i)\right)} \sum_{j \in S} \psi_{\hat{\pi}^{* 2}}^{*}(j) P\left(j \mid i, \mu, \hat{\pi}^{* 2}(i)\right)\right], \tag{4.48}
\end{equation*}
$$

and $\rho^{\hat{\pi}^{* 2}}=\inf _{\pi^{1} \in \Pi_{a d}^{1}} \partial^{\pi^{1}, \hat{\pi}^{* 2}}(i, c)=\rho^{*}$. Thus for $\pi^{* 1}$ as in (4.20), we have

$$
\begin{equation*}
e^{\hat{\rho}^{* * 2}} \psi_{\hat{\pi}^{* 2}}^{*}(i) \leq\left[e^{c\left(i, \pi^{* 1}(i), \hat{\pi}^{* 2}(i)\right)} \sum_{j \in S} \psi_{\hat{\pi}^{* 2}}^{*}(j) P\left(j \mid i, \pi^{* 1}(i), \hat{\pi}^{* 2}(i)\right)\right] . \tag{4.49}
\end{equation*}
$$

Arguing as in Lemma 4.1, for some finite set $\hat{\mathcal{B}}_{5} \supset \hat{\mathcal{B}}$ it follows that

$$
\begin{equation*}
\psi_{\hat{\pi}^{* 2}}^{*}(i) \leq E_{i}^{\pi^{* 1}, \hat{\pi}^{* 2}}\left[e^{\sum_{t=0}^{\check{\tau}\left(\mathcal{B}_{5}\right)-1}\left(c\left(X_{t}, \pi^{* 1}\left(X_{t}\right), \hat{\pi}^{* 2}\left(X_{t}\right)\right)-\rho^{*}\right)} \psi_{\hat{\pi}^{* 2}}^{*}\left(X_{\check{\tau}\left(\hat{\mathcal{B}}_{4}\right)}\right)\right] \forall i \in \hat{\mathcal{B}}_{5}^{c} . \tag{4.50}
\end{equation*}
$$

Also, from (4.20), we deduce that

$$
\begin{equation*}
e^{\rho^{*}} \psi^{*}(i) \geq\left[e^{c\left(i, \pi^{* 1}(i), \hat{\pi}^{* 2}(i)\right)} \sum_{j \in S} \psi^{*}(j) P\left(j \mid i, \pi^{* 1}(i), \hat{\pi}^{* 2}(i)\right)\right] . \tag{4.51}
\end{equation*}
$$

By Dynkin's formula and Fatou's lemma (as in Lemma 4.1), we obtain

$$
\begin{equation*}
\psi^{*}(i) \geq E_{i}^{\pi^{* 1}, \hat{\pi}^{* 2}}\left[e^{\sum_{t=0}^{\check{\tau}\left(\mathcal{B}_{5}\right)-1}\left(c\left(X_{t}, \pi^{* 1}\left(X_{t}\right), \hat{\pi}^{* 2}\left(X_{t}\right)\right)-\rho^{*}\right)} \psi^{*}\left(X_{\check{\tau}\left(\hat{\mathcal{B}}_{5}\right)}\right)\right] \forall i \in \hat{\mathcal{B}}_{5}^{c} . \tag{4.52}
\end{equation*}
$$

Now, in view of (4.50) and (4.52) and applying the same technique as before (as in the proof of Theorem 4.2), it follows that $\psi^{*}=\hat{k}_{5} \psi_{\hat{\pi}^{* 2}}^{*}$, for some constant $\hat{k}_{5}>0$. Hence from (4.20) and (4.48), it is easy to see that $\hat{\pi}^{* 2}$ is an outer maximizing selector of (4.6). Similarly, one can show that $\hat{\pi}^{* 1}$ is an outer minimizing selector of (4.6). This completes the proof.

Now we are ready to prove Theorem 4.1.

Proof of Theorem 4.1. Existence of an eigenpair $\left(\rho^{*}, \psi^{*}\right)$ of Eq. (4.1) follows from Lemma 4.1. Uniqueness of the eigenpair of Eq. (4.1) is proved in Lemma 4.2. Also, from Theorem 4.2, we have Theorem 4.1(i) and Theorem 4.1(ii). Theorem 4.1(iii) follows from Theorem 4.3. This completes the proof.

## 5. Example

We present here an illustrative example in which all our assumptions hold, and the cost function is nonnegative and unbounded.

Example 5.1. Consider a controlled birth-and-death system in which the state variable stands for the total population size at time $t \geq 0$. Thus, the state space can be represented by $S:=$ $\{0,1,2, \ldots\}$. Suppose that there are two players, player 1 and player 2 , and they can control birth and death, respectively. Depending on the number of population in the system, player 1 can modify the number of births by choosing some action $u$, from the set $U(i)=\left[\delta, L_{1}\right]$. But this action results in a cost given by $\tilde{c}_{1}(i, u) \geq 0$ (or a reward $\tilde{c}_{1}(i, u) \leq 0$ ), if $i$ is the state of the system. On the other hand, player 2 can modify the number of deaths by choosing some action $v$ from the set $V(i)=\left[\delta, L_{1}\right]$. The action of player 2 incurs a cost given by $\tilde{c}_{2}(i, v) \geq 0$ (or a reward $\tilde{c}_{2}(i, v) \leq 0$ ). Also, in addition, assume that player 1 'owns' the system and he/she gets a cost $r(i):=\hat{p} \cdot i$ for each unit of time during which the system remains in the state $i \in S$, where $\hat{p}>0$ is a fixed cost per population.

We next formulate this model as a discrete-time Markov game. The corresponding transition stochastic kernel $P(j \mid i, u, v)$ and reward $c(i, u, v)$ for player 1 are given as follows: for $(0, u, v) \in \mathcal{K}(\mathcal{K}$ as in the game model (2.1)).

$$
\begin{equation*}
\sum_{j \in S} P(j \mid 0, u, v)=1, \text { and } P(j \mid 0, u, v)=e^{-\frac{j^{2}}{3}-3} \forall j \geq 1 \tag{5.1}
\end{equation*}
$$

Similarly, for $(1, u, v) \in \mathcal{K}$,

$$
P(j \mid 1, u, v)= \begin{cases}1-\frac{3 e^{-2} v}{2\left(L_{1}+L_{2}\right)}, \quad \text { if } j=0 \\ \frac{e^{-2} v}{2\left(L_{1}+L_{2}\right)}, & \text { if } j=1 \\ \frac{e^{-2} v}{2\left(L_{1}+L_{2}\right)} & \text { if } j=2 \\ \frac{e^{-2} v}{2\left(L_{1}+L_{2}\right)}, & \text { if } j=3 \\ 0, & \text { otherwise. }\end{cases}
$$

Also, for $(i, u, v) \in \mathcal{K}$ with $i \geq 2$,

$$
P(j \mid i, u, v)=\left\{\begin{array}{l}
\frac{u e^{-i}}{2\left(L_{1}+L_{2}\right)}, \text { if } j=i-1 \\
\frac{u e^{-i}+v e^{-2 i}}{2\left(L_{1}+L_{2}\right)} \text { if } j=i \\
\frac{v e^{-2 i}}{2\left(L_{1}+L_{2}\right)}, \text { if } j=i+1 \\
1-\frac{2\left(u e^{-i}+v e^{-2 i}\right)}{2\left(L_{1}+L_{2}\right)}, \text { if } j=0 \\
0, \quad \text { otherwise. }
\end{array}\right.
$$

$$
\begin{equation*}
c(i, u, v):=\hat{p} \cdot i+\tilde{c}_{1}(i, u)-\tilde{c}_{2}(i, v) \text { for }(i, u, v) \in \mathcal{K} . \tag{5.2}
\end{equation*}
$$

We make the following assumptions to ensure the existence of a pair of optimal strategies.
(I) The functions $\tilde{c}_{1}(i, u)$, and $\tilde{c}_{2}(i, v)$ are continuous with their respective variables for each fixed $i \in S$.
(II) Suppose that $\hat{p} \cdot i+\tilde{c}_{1}(i, u)-\tilde{c}_{2}(i, v) \geq 0$ for $(i, u, v) \in \mathcal{K}$ and $\hat{p}<\frac{1}{6}$. Also, assume that $f$, is a norm-like function, where $f(i):=\min _{(u, v) \in U(i) \times V(i)}\left[\tilde{c}_{2}(i, v)-\tilde{c}_{1}(i, u)\right]$ for all $i \in S$.
(III) We also consider an increasing sequence of finite subsets $\tilde{\mathcal{D}}_{n} \subset S$ such that $\cup_{n=1}^{\infty} \tilde{\mathcal{D}}_{n}=S$ and $0 \in \tilde{\mathcal{D}}_{n}$ for all $n \in \mathbb{N}$. We assume that for any pair $i, j \in \tilde{\mathcal{D}}_{n}$, the probability of hitting $j$ from $i$ before exiting $\tilde{\mathcal{D}}_{n}$ is bounded from below by some constant $\delta_{i j, n}>0$ for any stationary Markov strategy.

Proposition 5.1. Under conditions (I)-(III), the above controlled system satisfies the Assumptions 2.1 and 2.2. Hence by Theorem 4.1, there exists a saddle-point equilibrium for this controlled model.

Proof. Consider the Lyapunov function $\mathcal{W}(i):=e^{\frac{i^{2}+1}{6}}$ for $i \in S$. Then $\mathcal{W}(i) \geq 1$ for all $i \in S$. Now for each $i \geq 2$, and $(u, v) \in U(i) \times V(i)$, we have

$$
\begin{align*}
& \sum_{j \in S} P(j \mid i, u, v) \mathcal{W}(j) \\
&= P(i-1 \mid i, u, v) \mathcal{W}(i-1)+P(i \mid i, u, v) \mathcal{W}(i) \\
& \quad+P(i+1 \mid i, u, v) \mathcal{W}(i+1)+P(0 \mid i, u, v) \mathcal{W}(0) \\
&= \frac{1}{2\left(L_{1}+L_{2}\right)}\left[u e^{-i} e^{\frac{(i-1)^{2}}{6}+1}+e^{\frac{i^{2}}{6}+1}\left(u e^{-i}+v e^{-2 i}\right)+v e^{-2 i} e^{\frac{(i+1)^{2}}{6}+1}\right] \\
& \quad+e\left(1-\frac{2\left(u e^{-i}+v e^{-2 i}\right)}{2\left(L_{1}+L_{2}\right)}\right) \\
&= e^{\frac{i^{2}}{6}+1}\left[\frac{u e^{-i}}{2\left(L_{1}+L_{2}\right)} e^{-\frac{i}{3}+\frac{1}{6}}+\left(\frac{u e^{-i}+v e^{-2 i}}{2\left(L_{1}+L_{2}\right)}\right)\right. \\
&\left.\quad+\frac{v e^{-2 i}}{2\left(L_{1}+L_{2}\right)} e^{\frac{i}{3}+\frac{1}{6}}+e^{-\frac{i^{2}}{6}}\left(1-\frac{2\left(u e^{-i}+v e^{-2 i}\right)}{2\left(L_{1}+L_{2}\right)}\right)\right] \\
& \leq e^{i^{2}+1} e^{-\frac{i}{3}+\frac{1}{6}}\left[\frac{u e^{-i}}{2\left(L_{1}+L_{2}\right)}+\frac{u+v}{2\left(L_{1}+L_{2}\right)}+\frac{v}{2\left(L_{1}+L_{2}\right)}+\left(1-\frac{2\left(u e^{-i}+v e^{-2 i}\right)}{2\left(L_{1}+L_{2}\right)}\right)\right] \\
& \leq 4 e^{i^{\frac{2}{6}+1} e^{-\frac{i}{3}+\frac{1}{6}}} \\
& \leq e^{\left(\frac{i^{2}}{6}+1\right)-\frac{1}{3}(i+3)+4} \\
&= \mathcal{W}(i) e^{-\frac{1}{6}(i+3)-\frac{1}{6}(i+3)+4} \\
& \leq e^{-\frac{1}{6}(i+3)+4 I_{\mathcal{M}}(i)} \mathcal{W}(i) \leq e^{-\frac{1}{6}(i+3)} \mathcal{W}(i)+\max _{j \in \mathcal{M}} \mathcal{W}(j) e^{4} I_{\mathcal{M}}(i) \leq e^{-\tilde{\ell}(i)} \mathcal{W}(i)+\tilde{C} I_{\mathcal{M}}(i), \tag{5.3}
\end{align*}
$$

where $\tilde{\ell}(i)=\frac{1}{6}(i+3), \mathcal{M}:=\left\{i: 4-\frac{1}{6}(i+3)>0\right\}$, and $\tilde{C}=\max \left\{\max _{j \in \mathcal{M}} \mathcal{W}(j) e^{4}, e^{-2} \sum_{i \geq 1}\right.$ $\left.\left(e^{-\frac{i^{2}}{6}}-e^{-\frac{i^{2}}{3}}\right)+e\right\}$. It is clear that $0,1 \in \mathcal{M}$. Also, we have

$$
\begin{equation*}
\sum_{j \in S} P(j \mid 0, u, v) \mathcal{W}(j)=e P(0 \mid 0, u, v)+\sum_{j \geq 1} e^{-2} e^{-\frac{j^{2}}{6}} \leq \tilde{C} I_{\mathcal{M}}(i) \tag{5.4}
\end{equation*}
$$

By similar arguments as in (5.3), we have

$$
\begin{align*}
& \sum_{j \in S} P(j \mid 1, u, v) \mathcal{W}(j) \\
& =P(0 \mid 1, u, v) \mathcal{W}(0)+P(1 \mid 1, u, v) \mathcal{W}(1)+P(2 \mid 1, u, v) \mathcal{W}(2)+P(3 \mid 1, u, v) \mathcal{W}(3) \\
& =e\left(1-\frac{3 e^{-2} v}{2\left(L_{1}+L_{2}\right)}\right)+e^{\frac{1}{6}+1}\left(\frac{e^{-2} v}{2\left(L_{1}+L_{2}\right)}\right) \\
& \quad+e^{\frac{4}{6}+1}\left(\frac{e^{-2} v}{2\left(L_{1}+L_{2}\right)}\right)+e^{\frac{9}{6}+1}\left(\frac{e^{-2} v}{2\left(L_{1}+L_{2}\right)}\right) \\
& \leq e^{-\frac{1}{6}(1+3)+4 I_{\mathcal{M}}(1)} \mathcal{W}(1) \leq e^{-\frac{2}{3}} \mathcal{W}(1)+\max _{j \in \mathcal{M}} \mathcal{W}(j) e^{4} I_{\mathcal{M}(1)} \leq e^{-\tilde{\ell}(1)} \mathcal{W}(1)+\tilde{C} I_{\mathcal{M}(1)}(1) . \tag{5.5}
\end{align*}
$$

Now

$$
\begin{equation*}
\tilde{\ell}(i)-\max _{(u, v) \in U(i) \times V(i)} c(i, u, v)=\frac{1}{2}+\left(\frac{1}{6}-\hat{p}\right) i+\min _{(u, v) \in U(\cdot) \times V(\cdot)}\left[\tilde{c}_{2}(i, v)-\tilde{c}_{1}(i, u)\right] . \tag{5.6}
\end{equation*}
$$

We see from condition (II) and (5.6) that $\tilde{\ell}(i)-\sup _{(u, v) \in U(i) \times V(i)} c(i, u, v)$ is norm-like function. So, by condition (II), Eqs. (5.3), (5.4), (5.5), and (5.6), Assumption 2.1 is satisfied. Now, by the transition probability defined above and condition (III), Assumption 2.2(ii) is verified. Next, Assumption 2.2 (iii) is verified by (5.3), (5.4) and (5.5). Also, by the above construction of probability kernel, (5.2), and condition (I), $P(\cdot \mid i, u, v)$ and $c(i, u, v)$ are continuous in $(u, v) \in U(i) \times V(i)$ for all $i, j \in S$. Hence by Theorem 4.1, it follows that there exists a saddle-point equilibrium for this controlled model.

## 6. Eigenvalue problem for compact state space case

In this section we extend our results to compact state space $S$ without assuming any Lyapunov type stability assumptions since compact state space eliminates the need for any stability consideration. To this end, let us first introduce some notations. Let $\mathcal{C}^{+}(S):=\{f \in$ $\mathcal{C}(S): f(x) \geq 0 \forall x \in S\}$ denotes the closed cone of $\mathcal{C}(S)$, where $\mathcal{C}(S)$ denotes the Banach space of continuous maps $f: \mathcal{C}(S) \rightarrow \mathbb{R}$ with the supremum norm, denoted by $\|\cdot\|$. Thus $\mathcal{C}^{+}(S)$ defines a partial order on $\mathcal{C}(S)$, denoted $\succeq$, given by this: for any $f, g \in \mathcal{C}(S)$, we define $f \succeq g$ if $f-g \in \mathcal{C}^{+}(S)$, i.e., the partial ordering in $\mathcal{C}(S)$ with respect to the cone $\mathcal{C}^{+}(S)$. We write $f \succ g$ (equivalently, $g \prec f$ ) if $f \succeq g, f \neq g$, and we write $f \gg g$ (equivalently, $g \ll f)$ if $f-g$ is a strictly positive function in $\mathcal{C}(S)$ i.e., if $f-g \in \operatorname{interior}\left(\mathrm{C}^{+}(S)\right)$.

For our analysis we need to impose the following set of assumptions on the system.

## Assumption 6.1.

(i) The admissible action spaces $U(x)(\subset U)$ and $V(x)(\subset V)$ are compact for each $x \in S$.
(ii) The functions $P(D \mid x, u, v)$ and $c: \mathcal{K} \rightarrow \mathbb{R}$ are continuous in $(x, u, v) \in S \times U(x) \times$ $V(x), D \subseteq S$.
(iii) The maps $(x, u, v) \rightarrow \int_{S} f(y) P(d y \mid x, u, v), f \in \mathcal{C}(S)$ with $\|f\| \leq 1$ are equicontinuous.
(iv) We assume that the transition kernel $P(d y \mid x, u, v)$ of the Markov chain $\left\{X_{t}\right\}_{t \geq 0}$ has the full support for all $(x, u, v) \in \mathcal{K}$ i.e., $\operatorname{support}(P(d y \mid x, u, v))=S \forall(x, u, v) \in \mathcal{K}$.

Now, using the nonlinear version of the Krein-Rutman theorem, see [1, Theorem 2.2], [2, $47,48]$ we establish the existence of an eigenpair to the associated Shapley equation.

Theorem 6.1. Suppose Assumption 6.1 holds. Then there exists a unique eigenpair $\left(\rho^{*}, \psi^{*}\right) \in$ $\mathbb{R}_{+} \times \mathfrak{C}^{+}(S), \psi^{*} \in$ interior $\left(\mathrm{C}^{+}(S)\right)$ (unique up to a multiplicative constant) for the following nonlinear eigenequation

$$
\begin{align*}
e^{\rho^{*}} \psi^{*}(x) & =\inf _{\mu \in \mathcal{P}(U(x))} \sup _{v \in \mathcal{P}(V(x))}\left[e^{c(x, \mu, v)} \int_{S} \psi^{*}(y) P(d y \mid x, \mu, \nu)\right] \\
& =\sup _{\nu \in \mathcal{P}(V(x))} \inf _{\mu \in \mathcal{P}(U(x))}\left[e^{c(x, \mu, \nu)} \int_{S} \psi^{*}(y) P(d y \mid x, \mu, \nu)\right] \tag{6.1}
\end{align*}
$$

Furthermore, for any mini-max selector $\left(\pi^{* 1}, \pi^{* 2}\right) \in \Pi_{S M}^{1} \times \Pi_{S M}^{2}$ of (6.1) we have the following:

$$
\begin{align*}
\rho^{*} & =\inf _{x \in S} \sup _{\pi^{2} \in \Pi_{a d}^{2}} \inf _{\pi^{1} \in \Pi_{a d}^{1}} \mathfrak{g}^{1}, \pi^{2}(x, c)=\inf _{x \in S} \inf _{\pi^{1} \in \Pi_{a d}^{1}} \sup _{\pi^{2} \in \Pi_{a d}^{2}} \mathfrak{J}^{1}, \pi^{2}(x, c) \\
& =\inf _{x \in S} \inf _{\pi^{1} \in \Pi_{a d}^{1}} \partial^{\pi^{1}, \pi^{* 2}}(x, c)=\inf _{x \in S} \sup _{\pi^{2} \in \Pi_{a d}^{2}} \partial^{\pi^{* 1}, \pi^{2}}(x, c)=\mathfrak{g}^{* 1}, \pi^{* 2}(x, c), \tag{6.2}
\end{align*}
$$

and consequently $\left(\pi^{* 1}, \pi^{* 2}\right) \in \Pi_{S M}^{1} \times \Pi_{S M}^{2}$ is a saddle-point equilibrium.
Proof. Let us consider a mapping $\hat{T}: \mathcal{C}(S) \rightarrow \mathcal{C}(S)$ defined by

$$
\begin{equation*}
\hat{T} g(x)=\sup _{v \in \mathcal{P}(V(x))} \inf _{\mu \in \mathcal{P}(U(x))}\left[e^{c(x, \mu, \nu)} \int_{S} g(y) P(d y \mid x, \mu, \nu)\right] \tag{6.3}
\end{equation*}
$$

where $g \in \mathcal{C}(S)$ and $x \in S$.
Note that in view of Remark 2.1, the sets $\mathcal{P}(U(x))$ and $\mathcal{P}(V(x))$ are compact as well as convex. Also, the extreme points of $\mathcal{P}(U(x))$ and $\mathcal{P}(V(x))$ corresponds to the Dirac measures at points in $U(x)$ and $V(x)$ respectively. Hence,

$$
\begin{align*}
\hat{T} g(x) & =\sup _{v \in \mathcal{P}(V(x))} \inf _{\mu \in \mathcal{P}(U(x))}\left[e^{c(x, \mu, v)} \int_{S} g(y) P(d y \mid x, \mu, v)\right] \\
& =\sup _{v \in V(x)} \inf _{u \in U(x)}\left[e^{c(x, u, v)} \int_{S} g(y) P(d y \mid x, u, v)\right] \tag{6.4}
\end{align*}
$$

for details, see [1, p. 965].
Next, we show that the map $\hat{T}$ is well defined on $\mathcal{C}(S)$. So, it is sufficient to take the family $\{g \in \mathcal{C}(S):\|g\| \leq R\}$, for some $R>0$. Let $x, z \in S$ be arbitrary but fixed points. Then in view of (6.4), we have

$$
\begin{align*}
& |\hat{T} g(x)-\hat{T} g(z)| \\
& \leq e^{\|c\|} \sup _{v \in V} \sup _{u \in U} \sup _{g:\|g\| \leq R}\left|\int_{S} g(y) P(d y \mid x, u, v)-\int_{S} g(y) P(d y \mid z, u, v)\right| \\
& \quad+R \sup _{v \in V} \sup _{u \in U}\left|e^{c(x, u, v)}-e^{c(z, u, v)}\right| \tag{6.5}
\end{align*}
$$

In view of Assumption 6.1, it follows that the right hand side tends to zero when $x \rightarrow z$. Next, from the definition of the map $\hat{T}$, for any $g_{1}, g_{2} \in \mathcal{C}(S)$, it follows that $\left\|\hat{T}\left(g_{1}\right)-\hat{T}\left(g_{2}\right)\right\| \leq$ $e^{\|c\|}\left\|g_{1}-g_{2}\right\|$. Hence the map $\hat{T}$ is Lipschitz continuous map from $\mathcal{C}(S) \rightarrow \mathcal{C}(S)$.

Now, using Assumption 6.1, we prove the following properties of $\hat{T}$.
Let $g_{1} \succ g_{2}$ i.e., $g_{1} \succeq g_{2}, g_{1} \neq g_{2}$. Let $\pi^{* 1} \in \Pi_{S M}^{1}$ be such that

$$
\begin{align*}
& \sup _{v \in \mathcal{P}(V(x))} \inf _{\mu \in \mathcal{P}(U(x))}\left[e^{c(x, \mu, \nu)} \int_{S} g_{1}(y) P(d y \mid x, \mu, \nu)\right] \\
& =  \tag{6.6}\\
& \sup _{v \in \mathcal{P}(V(x))}\left[e^{c\left(x, \pi^{* 1}(x), v\right)} \int_{S} g_{1}(y) P\left(d y \mid x, \pi^{* 1}(x), \nu\right)\right] \forall x \in S .
\end{align*}
$$

Also, let $\pi^{* 2} \in \Pi_{S M}^{2}$ be such that

$$
\begin{align*}
& \sup _{v \in \mathcal{P}(V(x))}\left[e^{c\left(x, \pi^{* 1}(x), v\right)} \int_{S} g_{2}(y) P\left(d y \mid x, \pi^{* 1}(x), \nu\right)\right] \\
& =\left[e^{c\left(x, \pi^{* 1}(x), \pi^{* 2}(x)\right)} \int_{S} g_{2}(y) P\left(d y \mid x, \pi^{* 1}(x), \pi^{* 2}(x)\right)\right] \forall x \in S . \tag{6.7}
\end{align*}
$$

Then

$$
\begin{aligned}
\hat{T}\left(g_{1}\right)(x) & -\hat{T}\left(g_{2}\right)(x) \\
& \geq\left[e^{c\left(x, \pi^{* 1}(x), \pi^{* 2}(x)\right)} \int_{S}\left(g_{1}(y)-g_{2}(y)\right) P\left(d y \mid x, \pi^{* 1}(x), \pi^{* 2}(x)\right)\right] \\
& \geq e^{\alpha_{1}} \int_{S}\left(g_{1}(y)-g_{2}(y)\right) P\left(d y \mid x, \pi^{* 1}(x), \pi^{* 2}(x)\right)>0,
\end{aligned}
$$

since $g_{1} \succ g_{2}$ and $\operatorname{support}\left(P\left(d y \mid x, \pi^{* 1}(x), \pi^{* 2}(x)\right)\right)=S, \forall x \in S$, where $\alpha_{1}>0$ is the greatest lower bound of $c$ on $S$. So, $\hat{T}$ is strictly increasing.

By the definition of the map $\hat{T}$, it is easy to see that $\hat{T}(\lambda g)=\lambda \hat{T}(g)$ for all $\lambda>0$. For $M>e^{-\alpha_{1}}$ and $g \in \mathcal{C}^{+}(S)$ defined by $g(\cdot) \equiv 1, M \hat{T} g>g$. Using Assumption 6.1, in view of (6.5), by analogous arguments as in [1, p. 967-968], it is easy to say that the map $\hat{T}: \mathcal{C}(S) \rightarrow \mathcal{C}(S)$ is a compact operator. Hence by Theorem 3.1, there exists a nontrivial $\psi^{*} \in \mathrm{C}^{+}(S)$ and a constant $e^{\rho^{*}}>0$ such that $\hat{T} \psi^{*}=e^{\rho^{*}} \psi^{*}$ i.e,

$$
e^{\rho^{*}} \psi^{*}(x)=\sup _{v \in \mathcal{P}(V(x))} \inf _{\mu \in \mathcal{P}(U(x))}\left[e^{c(x, \mu, v)} \int_{S} \psi^{*}(y) P(d y \mid x, \mu, v)\right] \forall x \in S
$$

Thus, by Fan's minimax theorem [22], we have

$$
e^{\rho^{*}} \psi^{*}(x)=\inf _{\mu \in \mathcal{P}(U(x))} \sup _{v \in \mathcal{P}(V(x))}\left[e^{c(x, \mu, v)} \int_{S} \psi^{*}(y) P(d y \mid x, \mu, v)\right] \forall x \in S
$$

This implies that the pair $\left(\rho^{*}, \psi^{*}\right)$ satisfies (6.1).
Now, we claim that $\psi^{*}>0$. If not, then on contrary there exists a point $\tilde{x} \in S$ for which $\psi^{*}(\tilde{x})=0$. Again by continuity-compactness assumptions, there exists a mini-max selector $\left(\pi^{* 1}, \pi^{* 2}\right)$ such that (6.1) can be rewritten as

$$
e^{\rho^{*}} \psi^{*}(x)=\left[e^{c\left(x, \pi^{* 1}(x), \pi^{* 2}(x)\right)} \int_{S} \psi^{*}(y) P\left(d y \mid x, \pi^{* 1}(x), \pi^{* 2}(x)\right)\right] \quad \forall x \in S .
$$

So, we get

$$
\begin{equation*}
0=e^{\rho^{*}} \psi^{*}(\tilde{x})=\left[e^{c\left(\tilde{x}, \pi^{* 1}(\tilde{x}), \pi^{* 2}(\tilde{x})\right)} \int_{S} \psi^{*}(y) P\left(d y \mid \tilde{x}, \pi^{* 1}(\tilde{x}), \pi^{* 2}(\tilde{x})\right)\right] \tag{6.8}
\end{equation*}
$$

Since $\psi^{*}$ is nontrivial, there exists $\hat{x} \in S$ such that $\psi^{*}(\hat{x})>0$.
Now since $\operatorname{support}(P(d y \mid x, u, v))=S \forall(x, u, v) \in \mathcal{K}$, it follows that

$$
\left[\int_{S} \psi^{*}(y) P\left(d y \mid \tilde{x}, \pi^{* 1}(\tilde{x}), \pi^{* 2}(\tilde{x})\right)\right]>0
$$

Hence it contradicts (6.8). This establishes our claim.
In view of Assumption 6.1(iv), uniqueness of the eigenpair of Eqs. (6.1) can be proved easily by the analogous arguments as in [Lemma 4.2, Eqs. (4.42)-(4.46)]. To see this, suppose that $(\tilde{\rho}, \tilde{\psi}) \in \mathbb{R}_{+} \times \mathcal{C}^{+}(S), \tilde{\psi}>0$ is an another solution of (6.1) and if possible let $\tilde{\rho}<\rho^{*}$. Let $\hat{k}_{6}=\min _{S} \frac{\tilde{\psi}}{\psi^{*}}$, thus we have $\left(\tilde{\psi}-\hat{k}_{6} \psi^{*}\right) \geq 0$ in $S$ and for some $\hat{x}_{0} \in S$, $\tilde{\psi}\left(\hat{x}_{0}\right)-\hat{k}_{6} \psi^{*}\left(\hat{x}_{0}\right)=0$. Let $\tilde{\pi}^{* 1}$ and $\pi^{* 2}$ are outer minimizing and maximizing selectors of (6.1), corresponding to the eigenpair $(\tilde{\rho}, \tilde{\psi})$ and $(\rho, \psi)$, respectively. Thus we obtain

$$
\begin{equation*}
e^{\rho^{*}}\left(\tilde{\psi}-\hat{k}_{6} \psi^{*}\right)(x) \geq\left[e^{c\left(x, \tilde{\pi}^{* 1}(x), \pi^{* 2}(x)\right)} \int_{S}\left(\tilde{\psi}-\hat{k}_{6} \psi^{*}\right)(y) P\left(d y \mid x, \tilde{\pi}^{* 1}(x), \pi^{* 2}(x)\right)\right] . \tag{6.9}
\end{equation*}
$$

This implies that

$$
0=\int_{S}\left(\tilde{\psi}-\hat{k}_{6} \psi^{*}\right)(y) P\left(d y \mid \hat{x}_{0}, \tilde{\pi}^{* 1}\left(\hat{x}_{0}\right), \pi^{* 2}\left(\hat{x}_{0}\right)\right)
$$

Then we claim that $\tilde{\psi} \equiv \hat{k}_{6} \psi^{*}$. If not, then since support $(P(d y \mid x, u, v))=S \forall(x, u, v) \in \mathcal{K}$, it follows that

$$
\begin{equation*}
\int_{S}\left(\tilde{\psi}-\hat{k}_{6} \psi^{*}\right)(y) P\left(d y \mid \hat{x}_{0}, \tilde{\pi}^{* 1}\left(\hat{x}_{0}\right), \pi^{* 2}\left(\hat{x}_{0}\right)\right)>0 . \tag{6.10}
\end{equation*}
$$

So, we arrive at a contradiction and thus, we have $\tilde{\psi} \equiv \hat{k}_{6} \psi^{*}$. Hence we obtain $\rho^{*}=\tilde{\rho}$, which is a contradiction to the fact that $\tilde{\rho}<\rho^{*}$. This implies that $\rho^{*} \leq \tilde{\rho}$. Again, by similar arguments one can derive the analogous contradiction to the fact that $\rho^{*}<\tilde{\rho}$. Therefore, we deduce that $\rho^{*}=\tilde{\rho}$ and $\tilde{\psi} \equiv \hat{k}_{7} \psi^{*}$, for some $\hat{k}_{7}>0$. So, the eigenpair of Eq. (6.1) is unique (up to a scalar multiplication).

Let $\left(\pi^{* 1}, \pi^{* 2}\right) \in \Pi_{S M}^{1} \times \Pi_{S M}^{2}$ be a pair of outer mini-max selector of (6.1), satisfying

$$
\begin{align*}
e^{\rho^{*}} \psi^{*}(x) & =\sup _{v \in \mathcal{P}(V(x))}\left[e^{c\left(x, \pi^{* 1}(x), \nu\right)} \int_{S} \psi^{*}(y) P\left(d y \mid x, \pi^{* 1}(x), \nu\right)\right] \\
& =\inf _{\mu \in \mathcal{P}(U(x))}\left[e^{c\left(x, \mu, \pi^{* 2}(x)\right)} \int_{S} \psi^{*}(y) P\left(d y \mid x, \mu, \pi^{* 2}(x)\right)\right] \tag{6.11}
\end{align*}
$$

Therefore, by Dynkin's formula and (6.11) (as in [53, Lemma 3.1]), we obtain

$$
\begin{align*}
\psi^{*}(x) & \leq E_{x}^{\pi^{1}, \pi^{* 2}}\left[e^{\sum_{s=0}^{T-1}\left(c\left(X_{s}, \pi_{s}^{1}, \pi^{* 2}\left(X_{s}\right)\right)-\rho^{*}\right)} \psi^{*}\left(X_{T}\right)\right] \\
& \leq\left(\sup _{S} \psi^{*}\right) E_{x}^{\pi^{1}, \pi^{* 2}}\left[e^{\sum_{s=0}^{T-1}\left(c\left(X_{s}, \pi_{s}^{1}, \pi^{* 2}\left(X_{s}\right)\right)-\rho^{*}\right)}\right] . \tag{6.12}
\end{align*}
$$

Taking logarithm on both sides, dividing by $T$ and letting $T \rightarrow \infty$, we deduce that

$$
\begin{equation*}
\rho^{*} \leq \inf _{\pi^{1} \in \Pi_{a d}^{1}} \mathcal{J}^{\pi^{1}, \pi^{* 2}}(x, c) \leq \sup _{\pi^{2} \in \Pi_{a d}^{2}} \inf _{\pi^{1} \in \Pi_{a d}^{1}} \mathcal{J}^{1}, \pi^{2}(x, c) . \tag{6.13}
\end{equation*}
$$

Similarly, by Dynkin's formula and using (6.11), we get

$$
\begin{equation*}
\rho^{*} \geq \sup _{\pi^{2} \in \Pi_{a d}^{2}} \partial^{\pi^{* 1}, \pi^{2}}(x, c) \geq \inf _{\pi^{1} \in \Pi_{a d}^{1}} \sup _{\pi^{2} \in \Pi_{a d}^{2}} \mathcal{g}^{\pi^{1}, \pi^{2}}(x, c) . \tag{6.14}
\end{equation*}
$$

using (6.13) and (6.14), we have (6.2) and this gives us $\rho^{*}=\mathcal{J}^{\pi^{* 1}, \pi^{* 2}}(x, c)$. In particular, this implies that

$$
\mathcal{J}^{\pi^{* 1}, \pi^{2}}(x, c) \leq \mathcal{J}^{\pi^{* 1}, \pi^{* 2}}(x, c) \leq \mathcal{J}^{\pi^{1}, \pi^{* 2}}(x, c) .
$$

That is, $\left(\pi^{* 1}, \pi^{* 2}\right) \in \Pi_{S M}^{1} \times \Pi_{S M}^{2}$ is a saddle point equilibrium.
Next theorem shows that the converse statement of the above theorem is also true.

Theorem 6.2. Suppose Assumption 6.1 holds. Suppose there exists a saddle-point equilibrium $\left(\hat{\pi}^{* 1}, \hat{\pi}^{* 2}\right) \in \Pi_{S M}^{1} \times \Pi_{S M}^{2}$. Then $\left(\hat{\pi}^{* 1}, \hat{\pi}^{* 2}\right)$ is a mini-max selector of (6.1).

Proof. By analogous arguments as in Theorem 4.3, in view of Theorem 6.1 and definition of saddle-point, we have $\rho^{*}=\mathcal{J}^{\hat{\pi}^{* 1}, \hat{\pi}^{* 2}}(x, c)=\sup _{\pi^{2} \in \Pi_{a d}^{2}} \mathcal{J}^{\hat{\pi}^{* 1}, \pi^{2}}(x, c)=\inf _{\pi^{1} \in \Pi_{a d}^{1}} \partial^{\pi^{1}, \hat{\pi}^{* 2}}(x, c)$. Now arguing as in Theorem 6.1, it is easy to see that for $\hat{\pi}^{* 2} \in \Pi_{S M}^{2}$ there exists $\left(\rho^{\hat{\pi}^{* 2}}, \psi_{\hat{\pi}^{* 2}}^{*}\right) \in$ $\mathbb{R}_{+} \times \mathcal{C}^{+}(S)$ with $\psi_{\hat{\pi}^{* 2}}^{*}>0$ such that

$$
\begin{equation*}
e^{\rho^{\hat{\pi}^{* 2}}} \psi_{\hat{\pi}^{* 2}}^{*}(x)=\inf _{\mu \in \mathcal{P}(U(x))}\left[e^{c\left(x, \mu, \hat{\pi}^{* 2}(x)\right)} \int_{S} \psi_{\hat{\pi}^{* 2}}^{*}(y) P\left(d y \mid x, \mu, \hat{\pi}^{* 2}(x)\right)\right] \tag{6.15}
\end{equation*}
$$

and $\rho^{\hat{\pi}^{* 2}}=\inf _{\pi^{1} \in \Pi_{a d}^{1}} g^{\pi^{1}, \hat{\pi}^{* 2}}(x, c)=\rho^{*}$. Thus for $\pi^{* 1}$ as in (6.11), we have

$$
\begin{equation*}
e^{\rho^{\hat{\pi}^{* 2}}} \psi_{\hat{\pi}^{* 2}}^{*}(x) \leq\left[e^{c\left(x, \pi^{* 1}(x), \hat{\pi}^{* 2}(x)\right)} \int_{S} \psi_{\hat{\pi}^{* 2}}^{*}(y) P\left(d y \mid x, \pi^{* 1}(x), \hat{\pi}^{* 2}(x)\right)\right] \tag{6.16}
\end{equation*}
$$

Also, from (6.11), we deduce that

$$
\begin{equation*}
e^{\rho^{*}} \psi^{*}(x) \geq\left[e^{c\left(x, \pi^{* 1}(x), \hat{\pi}^{* 2}(x)\right)} \int_{S} \psi^{*}(y) P\left(d y \mid x, \pi^{* 1}(x), \hat{\pi}^{* 2}(x)\right)\right] . \tag{6.17}
\end{equation*}
$$

Let $\hat{k}_{8}=\min _{S} \frac{\psi^{*}}{\psi_{\hat{\pi}^{* 2}}^{*}}$, thus we have $\left(\psi^{*}-\hat{k}_{8} \psi_{\hat{\pi}^{* 2}}^{*}\right) \geq 0$ in $S$ and for some $\hat{x}_{0} \in S$, $\psi^{*}\left(\hat{x}_{0}\right)-\hat{k}_{8} \psi_{\hat{\pi}^{* 2}}^{*}\left(\hat{x}_{0}\right)=0$. Now, from (6.16) and (6.17), we deduce that

$$
0=\int_{S}\left(\psi^{*}-\hat{k}_{8} \psi_{\hat{\pi}^{* 2}}^{*}\right)(y) P\left(d y \mid \hat{x}_{0}, \pi^{* 1}\left(\hat{x}_{0}\right), \hat{\pi}^{* 2}\left(\hat{x}_{0}\right)\right)
$$

Then we claim that $\psi^{*} \equiv \hat{k}_{8} \psi_{\hat{\pi}^{*} 2}^{*}$. If not, then since $\operatorname{support}(P(d y \mid x, u, v))=S \forall(x, u, v) \in \mathcal{K}$, it follows that

$$
\begin{equation*}
\int_{S}\left(\psi^{*}-\hat{k}_{8} \psi_{\hat{\pi}^{* 2}}^{*}\right)(y) P\left(d y \mid \hat{x}_{0}, \pi^{* 1}\left(\hat{x}_{0}\right), \hat{\pi}^{* 2}\left(\hat{x}_{0}\right)\right)>0 . \tag{6.18}
\end{equation*}
$$

So, we arrive at a contradiction and so, $\psi^{*} \equiv \hat{k}_{8} \psi_{\hat{\pi}^{* 2}}^{*}$. Hence from (6.1) and (6.15), it is easy to see that $\hat{\pi}^{* 2}$ is an outer maximizing selector of (6.1). Similarly, one can show that $\hat{\pi}^{* 1}$ is an outer minimizing selector of (6.1). This completes the proof.

## 7. Conclusion

We have studied a risk-sensitive zero-sum stochastic game with ergodic cost criterion on countable/compact state space where the admissible action spaces $(U(x)$ and $V(x))$ are compact
metric spaces. Under certain assumptions we have established the existence of a saddle-point equilibrium and have completely characterized the same. Instead of employing the traditional vanishing discount asymptotics, we have pursued a direct approach involving the principal eigenpair of the corresponding Shapley equation.

It will be interesting to study the problem for locally compact state space. The major issues in extending our results from compact state space to locally compact state space is to prove the existence of a principal eigenpair to the associated Shapley equation. Following the arguments of this article (also from [1]), it is easy to see that the principal eigenpair exists for Dirichlet eigenvalue problems (in bounded domains). But the main difficulty here is to take limit from bounded domain to the unbounded domain. In countable state space, we use diagonalization argument to get the limit. This argument does not work in the general state space case. As it is pointed out in [1], that it will be a very challenging work to extend our results from compact state space to a locally compact state space. One possible way to overcome these difficulties is by imposing stronger equicontinuity condition on the transition kernels (as in Assumption 6.1(iii)), which will enable us to use Arzela-Ascoli theorem to pass to the limit.

## Declaration of competing interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

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