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From Cauchy's determinant formula to bosonic and fermionic immanant identities

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ABSTRACT

Cauchy's determinant formula (1841) involving $\det((1-u_i v_j)^{-1})$ is a fundamental result in symmetric function theory. It has been extended in several directions, including a determinantal extension by Frobenius (1882) involving a sum of two geometric series in $u_i v_j$. This theme also resurfaced in a matrix analysis setting in a paper by Horn (1969) – where the computations are attributed to Loewner – and in recent works by Belton et al. (2016) and Khare and Tao (2021). These formulas were recently unified and extended in Khare (2022) to arbitrary power series, with commuting/bosonic variables u_i, v_j.

In this note we formulate analogous permanent identities, and in fact, explain how all of these results are a special case of a more general identity, for any character – in fact, any complex class function – of any finite group that acts on the bosonic variables u_i and on the v_j via signed permutations. (We explain why larger linear groups do not work, via a – perhaps novel – "symmetric function" characterization of signed permutation matrices that holds over any integral domain.) We then provide fermionic analogues of these formulas, as well as of the closely related Cauchy product identities.

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1. Introduction

1.1. Post-1960 results: Entrywise positivity preservers and schur polynomials

The goal of this note is to extend some classical and modern symmetric function determinantal identities to other characters of the symmetric group (and its subgroups), and then to formulate and

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show fermionic counterparts of these. The origins of this work lie in classical identities by Cauchy and Frobenius, but also in a computation – see Theorem 1.1 – that originally appears in a letter by Charles Loewner to Josephine Mitchell on October 24, 1967 (as observed by the first-named author in the Stanford Library archives). Subsequently, this computation, and the broader result on "entrywise functions", appeared in print in the thesis of Loewner's Ph.D. student, Roger Horn — see also the proof of [6, Theorem 1.2], which Horn attributes to Loewner.

In his letter, Loewner explained that he was interested in understanding functions acting *entrywise* on positive semidefinite matrices (i.e., real symmetric matrices with non-negative eigenvalues) of a fixed size, and preserving positivity. Previously, results by Schur, Schoenberg, and Rudin had classified the dimension-free preservers, i.e., the entrywise maps preserving positivity in *all* dimensions [19,21,22]. In contrast, in a fixed dimension d, such a classification remains open to date, even for $d = 3$; moreover, Loewner's 1967 result is still state-of-the-art, in that it is (essentially) the only known necessary condition for a general entrywise function preserving positivity in a fixed dimension. We refer the reader to e.g. [10] for more details.

The present work begins by isolating from Loewner's positivity/analysis result, the following algebraic calculation. Fix an integer $n \geqslant 2$; given a matrix $A = (a_{ij})$, here and below $f[A]$ denotes the matrix with (i,j)-entry $f(a_{ij})$.

Theorem 1.1 (*Loewner*)**.** *Suppose $f : \mathbb{R} \to \mathbb{R}$ is a smooth function, $n \geqslant 2$, and let $\mathbf{u} = (u_1, \dots, u_n)^T \in \mathbb{R}^n$. Define the determinant function*

$$\Delta : \mathbb{R} \to \mathbb{R}, \qquad t \mapsto \det(f(tu_iu_j))_{i,j=1}^n = \det f[t\mathbf{u}\mathbf{u}^T].$$

Then $\Delta(0) = \cdots = \Delta^{(\binom{n}{2}-1)}(0) = 0$, and the next derivative is

$$\Delta^{(\binom{n}{2})}(0) = \binom{\binom{n}{2}}{1, 2, \dots, n-1} \prod_{i<j} (u_j - u_i)^2 \cdot f(0)f'(0)\cdots f^{(n-1)}(0). \tag{1.1}$$

In particular, if $f(t)$ is a convergent power series $\sum_{n\geqslant 0} f_n t^n$, then within a suitable radius of convergence,

$$\Delta(t) = t^{\binom{n}{2}} \prod_{i<j} (u_j - u_i)^2 \cdot f_0 f_1 \cdots f_{n-1} + \text{ higher order terms.}$$

The first term on the right-hand side of Eq. (1.1) is a multinomial coefficient, and the reader will recognize the next product as the square of a Vandermonde determinant for the matrix with entries u_i^{n-j}, $1 \leqslant i, j \leqslant n$. What the reader may find harder to recognize is that Eq. (1.1) contains a "hidden" Schur polynomial (these are defined presently) in the variables u_i: the simplest of them all, $s_{(0,\dots,0)}(\mathbf{u}) = 1$. In particular, if one goes even one derivative beyond Loewner's stopping point, one immediately uncovers other, nontrivial Schur polynomials. This is stated precisely in Theorem 1.2.

The presence of the lurking (simplest) Schur polynomial in (1.1) was suspected owing to very recent sequels to Loewner's matrix positivity result. First with Belton–Guillot–Putinar [2] and then with Tao [11], the first-named author found (the first) examples of polynomial maps with at least one negative coefficient, which preserve positivity in a fixed dimension when applied entrywise. These papers uncovered novel connections between polynomials that entrywise preserve positivity and Schur polynomials, and in particular, obtained expansions for $\det f[t\mathbf{u}\mathbf{v}^T]$ in terms of Schur polynomials, for all polynomials $f(t)$. This suggested revisiting the general case due to Loewner (in slightly greater generality, as above: for $\det f[t\mathbf{u}\mathbf{v}^T]$).

1.2. Pre-1900 results: Cauchy and Frobenius

We now go back in history and remind the reader of the first such determinantal identities involving Schur polynomials. Recall the well-known Cauchy determinant identity [3], [17, Chapter I.4, Example 6]: if B is the $n \times n$ matrix with entries $(1 - u_iv_j)^{-1} = \sum_{M\geqslant 0} (u_iv_j)^M$ for variables u_i, v_j with $1 \leqslant i, j \leqslant n$, then

$$\det B = V(\mathbf{u})V(\mathbf{v}) \sum_{\mathbf{m}} s_{\mathbf{m}}(\mathbf{u}) s_{\mathbf{m}}(\mathbf{v}), \tag{1.2}$$

where $V(\mathbf{u})$ for a finite tuple $\mathbf{u} = (u_i)_{i\geqslant 1}$ denotes the "Vandermonde determinant" $\prod_{i<j}(u_j - u_i)$, and the sum runs over all partitions $\mathbf{m}$ with at most n parts. Here, a partition $\mathbf{m} = (m_1, \dots, m_n)$ simply means a weakly decreasing sequence of nonnegative integers $m_1 \geqslant \cdots \geqslant m_n \geqslant 0$; and we use Cauchy's definition [9] for the Schur polynomial $s_{\mathbf{m}}(\mathbf{v})$, namely,

$$s_{\mathbf{m}}(v_1, \dots, v_n) := \frac{\det(v_j^{m_i+n-i})_{i,j=1}^n}{\det(v_j^{n-i})_{i,j=1}^n}.$$

(This definition differs from that in the literature, e.g. in [17].) Here and below, we restrict to n arguments v_j, to go with the n exponents m_i.

See also [14, Section 5] and the references therein, as well as [7,8,12,13,15,17] for other determinantal identities involving symmetric functions.

As discussed in Section 1.1, in this paper we focus on the specific form of the determinant in (1.2), i.e. where one applies to all $u_i v_j$ some power series (Eq. (1.2) considers the case of $f(x) = 1/(1-x) = \sum_{M\geqslant 0} x^M$), and then computes the determinant. For instance, if $f(x)$ has fewer than n monomials then $f[\mathbf{u}\mathbf{v}^T]$ is a sum of fewer than n rank-one matrices, hence is singular. (For more general polynomials – as mentioned above – the formula was worked out in [11].) Another such formula was shown by Frobenius [4], in fact in greater generality.[1] The formula appears in Rosengren–Schlosser [18, Corollary 4.7] as well, as a consequence of their Theorem 4.4; and it implies a more general determinantal identity than (1.2), with $(1-cx)/(1-x)$ replacing $1/(1-x)$ and the sum again running over all partitions with at most n parts:

$$\begin{aligned} &\det\left(\frac{1-cu_iv_j}{1-u_iv_j}\right)_{i,j=1}^n \qquad (1.3)\\ &= V(\mathbf{u})V(\mathbf{v})(1-c)^{n-1}\left(\sum_{\mathbf{m}\,:\,m_n=0} s_{\mathbf{m}}(\mathbf{u})s_{\mathbf{m}}(\mathbf{v}) + (1-c)\sum_{\mathbf{m}\,:\,m_n>0} s_{\mathbf{m}}(\mathbf{u})s_{\mathbf{m}}(\mathbf{v})\right). \end{aligned}$$

1.3. The present work

Given the many precursors listed above, it is natural to seek a more general identity, i.e. the expansion of $\det f[\mathbf{u}\mathbf{v}^T]$, where $f[\mathbf{u}\mathbf{v}^T]$ is the entrywise application of an arbitrary (formal) power series f to the rank-one matrix $\mathbf{u}\mathbf{v}^T = (u_iv_j)_{i,j=1}^n$. This question was recently answered by the first-named author – including additional special cases – again in the context of matrix positivity preservers.

Theorem 1.2 (*Khare, [10]*)**.** *Fix a unital commutative ring R and let t be an indeterminate. Let $f(t) := \sum_{M\geqslant 0} f_M t^M \in R[[t]]$ be an arbitrary formal power series. Given vectors $\mathbf{u}, \mathbf{v} \in R^n$ for some $n \geqslant 1$, we have:*

$$\det f[t\mathbf{u}\mathbf{v}^T] = V(\mathbf{u})V(\mathbf{v}) \sum_{M\geqslant 0} t^{M+\binom{n}{2}} \sum_{\mathbf{m}=(m_1,\dots,m_n)\vdash M} s_{\mathbf{m}}(\mathbf{u})s_{\mathbf{m}}(\mathbf{v}) \prod_{i=1}^n f_{m_i+n-i}, \qquad (1.4)$$

where $\mathbf{m} \vdash M$ means that $\mathbf{m}$ is a partition whose components sum to M.

The goal of this short note is to show that these identities hold more generally — not just for determinants, but also e.g. for permanents, where

$$\mathrm{perm}(A_{n\times n}) := \sum_{\sigma\in S_n} a_{1\sigma(1)} \cdots a_{n\sigma(n)}.$$

Thus we show below:

[1] Here one uses theta functions and obtains elliptic Frobenius–Stickelberger–Cauchy determinant (type) identities; see also [1,5].

Theorem 1.3. *With notation as in Theorem 1.2 (and over any commutative unital ring R), we have:*

$$n!\,\mathrm{perm}\,f[t\mathbf{u}\mathbf{v}^T] = \sum_{\mathbf{m}\in\mathbb{Z}^n_{\geqslant 0}} t^{m_1+\cdots+m_n}\,\mathrm{perm}(\mathbf{u}^{\circ\mathbf{m}})\,\mathrm{perm}(\mathbf{v}^{\circ\mathbf{m}})\prod_{i=1}^n f_{m_i},$$

where $\mathbf{v}^{\circ\mathbf{m}} := (v_j^{m_i})$ (and similarly for $\mathbf{u}^{\circ\mathbf{m}}$), and $\mathbf{m} \geqslant \mathbf{0}$ is interpreted coordinatewise.

We show this result as well as Theorem 1.2 by a common proof. In fact we go beyond permanents: we provide such an identity for an arbitrary character of an arbitrary subgroup G of S_n — and even of the hyperoctahedral group: $G \leqslant S_2 \wr S_n$. Thus, our proof differs from the approach in [10], and proceeds via group representation theory. We then produce a fermionic analogue of the bosonic immanant "master identity", in which the variables u_i anti-commute, as do the v_j. For quick reference, these identities are summarized in the following table (see Table 1).

Table 1
The first three rows provide formulas for an arbitrary formal power series applied entrywise to the matrix $t\mathbf{uv} = (tu_iv_j)_{i,j=1}^n$. The fourth row computes the product of $(1-u_iv_j)^{-1}$ or of $(1+u_iv_j)$. Two of these formulas can be found in earlier literature, see [10,17].

	Even (bosonic) variables	Odd (fermionic) variables
Determinant (for S_n)	(2.7) (see [10])	(3.3)
Permanent (for S_n)	(2.8)	(3.4)
Arbitrary immanants for subgroups of $S_2 \wr S_n$	(2.6)	(3.2)
(Bi)Product identities	(3.5) (see e.g. [17])	(3.6)

2. Immanant identities for bosonic variables

2.1. Establishing the setting

In order to state and prove our main results for arbitrary immanants of complex characters – and more generally, for their linear combinations (i.e., class functions) – we first establish the setting in which our results hold.

2.1.1. Ground ring — contains character values

The first step is to explain the degree of freedom in choosing the ground ring. Fix an integer $n \geqslant 1$ and a unital commutative subring R. Recall that given a complex character χ of the permutation group S_n, the *immanant* of a square matrix $A_{n\times n}$ – as defined by Littlewood and Richardson [16] – is

$$\mathrm{Imm}_\chi(A) := \sum_{\sigma\in S_n} \chi(\sigma)\prod_{i=1}^n a_{i,\sigma(i)}.$$

For this to act on $f(tu_iv_j)$ with $f \in R[[t]]$, we need R to "contain" the character values of χ. This is made precise as follows:

Definition 2.1. Given a finite group G and a complex character – or class function – $\psi := \sum_{\chi\in\widehat{G}_{\mathbb{C}}} a_\chi \chi$ (where the sum runs over the irreducible complex characters of G), define the *ring of character values* $R_\psi \subset \mathbb{C}$ to be the unital subring generated by all character values that occur in ψ:

$$R_\psi := \mathbb{Z}[\{\chi(g) : g \in G, a_\chi \neq 0\}] \subset \mathbb{C}. \tag{2.1}$$

In this paper we will work over arbitrary commutative R_ψ-algebras R. For instance, if $\psi = \chi$ is the determinant or permanent, then $R_\psi = \mathbb{Z}$, and so the determinant and permanent identities will hold over every commutative $\mathbb{Z}$-algebra – i.e. unital commutative ring – R, for all power series $f \in R[[t]]$. (Hence the theorems above are stated over all R.)

2.1.2. The Segre subalgebra — contains the immanant

The other point to establish is the algebra in which our immanant identities are to hold. For this, we begin by reminding the reader that we will work with arbitrary subgroups of the group of signed permutations $S_2 \wr S_n$. (To understand this choice of group $S_2 \wr S_n$ and not a larger one, see Section 2.3.) Now suppose $u_1, \dots, u_n, v_1, \dots, v_n$ are commuting variables. Recall that $S_2 \wr S_n$ acts on the tensor algebra $R\langle u_1, \dots, u_n\rangle$ via signed permutations on the span of the u_i, and similarly on $R\langle v_1, \dots, v_n\rangle$. Explicitly, write every element $g \in S_2 \wr S_n$ as: $g = D_g \sigma_g$, where $D_g = D_g^{-1}$ is a diagonal matrix with (i, i) diagonal entry $(D_g)_i \in \{\pm 1\}$, and $\sigma_g \in S_n$ is a permutation matrix. Now:

$$g \cdot \sum_{i=1}^n r_i u_i := \sum_{i=1}^n r_i \cdot (D_g)_{\sigma_g(i)} \cdot u_{\sigma_g(i)} = \sum_{j=1}^n r_{\sigma_g^{-1}(j)} \cdot (D_g)_j \cdot u_j, \tag{2.2}$$

and this is extended multiplicatively to $R\langle \mathbf{u}\rangle$. Moreover, this action preserves the two-sided ideals generated by

$$\{u_i \otimes u_j - u_j \otimes u_i : 1 \leqslant i, j \leqslant n\}, \qquad \{u_i \otimes u_j + u_j \otimes u_i : 1 \leqslant i < j \leqslant n\} \sqcup \{u_i \otimes u_i : 1 \leqslant i \leqslant n\}.$$

Hence denoting the free R-module $U := \sum_{i=1}^n R u_i$, the action of $S_2 \wr S_n$ on $R\langle \mathbf{u}\rangle$ descends to the quotient symmetric algebra $\mathrm{Sym}_R^\bullet(U)$ and quotient alternating algebra $\wedge_R^\bullet(U)$, via:

$$\begin{aligned} g(\mathbf{u}^{\mathbf{m}}) &= \prod_{i=1}^n ((D_g)_{\sigma_g(i)} \cdot u_{\sigma_g(i)})^{m_i}, \\ g(u_{i_1} \wedge \dots \wedge u_{i_d}) &= \prod_{j=1}^d (D_g)_{\sigma_g(i_j)} \cdot (u_{\sigma_g(i_1)} \wedge \dots \wedge u_{\sigma_g(i_d)}), \end{aligned} \tag{2.3}$$

for all non-negative integer tuples $\mathbf{m} = (m_1, \dots, m_n)$ and all integers $1 \leqslant i_1 < \dots < i_d \leqslant n$. Here and below, we define and use $\mathbf{u}^{\mathbf{m}} := \prod_{i=1}^n u_i^{m_i}$. Notice that this action preserves each graded component

$$\mathrm{Sym}_R^d(\mathbf{u}) := \mathrm{Sym}_R^d(U), \qquad \wedge_R^d(\mathbf{u}) := \wedge_R^d(U), \qquad d \geqslant 0. \tag{2.4}$$

The above holds verbatim for the v_j, i.e. for the free R module $V := \sum_{j=1}^n R v_j$. This action carries over to the tensor algebra $R\langle \mathbf{v}\rangle$ and its symmetric/alternating quotients.

Now fix a finite subgroup $G \leqslant S_2 \wr S_n$, an irreducible complex character χ of G, and a commutative R_χ-algebra R. Recall that one has the "minimal" pseudo-idempotent in the group algebra

$$E_\chi := \chi(1) \sum_{g \in G} \chi(g) g^{-1} \in R_\chi G$$

(see (2.1)), where "pseudo-idempotent" simply means that $E_\chi^2 = |G| E_\chi$ in R_χ (and hence in R_ψ for any class function ψ with $[\psi : \chi] \neq 0$). Below, we will act by E_χ on both the u_i and on the v_j, and so to keep track of which variables are acted upon, we denote by $E_\chi^{\mathbf{u}}$ the pseudo-idempotent acting on $R\langle \mathbf{u}\rangle$, and hence on each $\mathbf{u}^{\mathbf{m}}$ and each $\wedge_{j=1}^k u_{i_j}$ via (2.3). Similarly, $E_\chi^{\mathbf{v}}$ will denote the pseudo-idempotent acting on $R\langle \mathbf{v}\rangle$ and hence on its quotients.

Our goal is to extend Theorem 1.2 to all characters of all finite subgroups G of $S_2 \wr S_n$. To do so, we will make these pseudo-idempotents act on polynomials in both u_i and v_i, i.e. on the polynomial ring $R[\mathbf{u}, \mathbf{v}]$. More precisely, we work in its "Segre subring" (see (2.4))

$$\bigoplus_{d \geqslant 0} \mathrm{Sym}_R^d(\mathbf{u}) \otimes \mathrm{Sym}_R^d(\mathbf{v}) \hookrightarrow \mathrm{Sym}_R^{2d}(U \oplus V). \tag{2.5}$$

Note from above that $R\langle \mathbf{u}, \mathbf{v}\rangle = T_R^\bullet(U \oplus V)$ is indeed a $(S_2 \wr S_n) \times (S_2 \wr S_n)$-submodule, as is its quotient $\mathrm{Sym}_R^d(U \oplus V)$ for each $d \geqslant 0$ and hence the Segre subring. In particular, the pseudo-idempotents $E_\chi^{\mathbf{u}}$ act on it as explained in (2.12) and by fixing all factors of v_j:

$$E_\chi^{\mathbf{u}}(\mathbf{u}^{\mathbf{m}} \mathbf{v}^{\mathbf{m}'}) := E_\chi^{\mathbf{u}}(\mathbf{u}^{\mathbf{m}}) \cdot \mathbf{v}^{\mathbf{m}'} = \chi(1) \sum_{g \in G} \chi(g) g^{-1}(\mathbf{u}^{\mathbf{m}}) \cdot \mathbf{v}^{\mathbf{m}'},$$

and similarly for the $E_\chi^{\mathbf{v}}$.

2.2. The main theorem and its proof, for even variables

Having defined the Segre subring and the action of the two $E^{\bullet}_{\chi}$ on it, we can state the promised generalization of Theorem 1.2 to all subgroups $G \leqslant S_2 \wr S_n$ and all characters – in fact, class functions – ψ of G. We begin with ψ a multiplicity-free character:

Theorem 2.2. *Fix an integer $n \geqslant 1$, a subgroup $G \leqslant S_2 \wr S_n$ acting on the variables u_i (and on the v_j) by signed permutations, and a multiplicity-free complex character χ of G. Then for any commutative R_χ-algebra R – see (2.1) – and any formal power series $f \in R[[t]]$ (with t an indeterminate), one has:*

$$|G|E^{\mathbf{u}}_{\chi} \cdot \prod_{i=1}^{n} f(tu_iv_i) = |G|E^{\mathbf{v}}_{\overline{\chi}} \cdot \prod_{i=1}^{n} f(tu_iv_i) = \sum_{\mathbf{m}\in\mathbb{Z}^n_{\geqslant 0}} t^{|\mathbf{m}|} f_{\mathbf{m}} \cdot E^{\mathbf{u}}_{\chi}(\mathbf{u}^{\mathbf{m}}) \cdot E^{\mathbf{v}}_{\overline{\chi}}(\mathbf{v}^{\mathbf{m}}), \tag{2.6}$$

where the indeterminate t keeps track of the $\mathbb{Z}_{\geqslant 0}$-grading, we use the multi-index notation

$$\mathbf{m} = (m_1, \dots, m_n), \quad |\mathbf{m}| = m_1 + \cdots + m_n, \quad f_{\mathbf{m}} := \prod_i f_{m_i}, \quad \mathbf{u}^{\mathbf{m}} := \prod_i u_i^{m_i}, \quad \mathbf{v}^{\mathbf{m}} := \prod_i v_i^{m_i},$$

and $\mathbf{m} \geqslant \mathbf{0}$ is interpreted coordinatewise.

Observe that special cases of Eq. (2.6) yield Cauchy's determinantal formula, its analogue for permanents and immanants (for the power series $f_0(t) = 1/(1-t)$), and their generalizations to arbitrary power series. E.g. for χ the sign and trivial representation respectively (and $G = S_n$ for $n \geqslant 2$), the G-immanants have "orthogonal" expansions, respectively:

$$n!\det f[t\mathbf{u}\mathbf{v}^T] = n!\,V(\mathbf{u})V(\mathbf{v}) \sum_{\mathbf{m}\in\mathbb{Z}^n_{\geqslant 0}} t^{|\mathbf{m}|+\binom{n}{2}} \prod_{i=1}^{n} f_{m_i+n-i} \cdot s_{\mathbf{m}}(\mathbf{u}) s_{\mathbf{m}}(\mathbf{v}), \tag{2.7}$$

$$\begin{aligned} n!\,\mathrm{perm} f[t\mathbf{u}\mathbf{v}^T] &= \sum_{\mathbf{m}\in\mathbb{Z}^n_{\geqslant 0}} t^{|\mathbf{m}|} f_{\mathbf{m}} \cdot \mathrm{perm}(\mathbf{u}^{\circ\mathbf{m}})\,\mathrm{perm}(\mathbf{v}^{\circ\mathbf{m}}) && (2.8)\\ &= n! \sum_{\mathbf{m}\in\mathbb{Z}^n_{\geqslant 0},\ \mathbf{m}\ \text{non-increasing}} t^{|\mathbf{m}|} |\mathrm{Stab}_{S_n}(\mathbf{m})| f_{\mathbf{m}} \cdot m_{\mathbf{m}}(\mathbf{u}) m_{\mathbf{m}}(\mathbf{v}), && (2.9)\end{aligned}$$

for an arbitrary formal power series $f(t)$. (Here $m_{\mathbf{m}}(\mathbf{u})$ denotes the monomial symmetric polynomial.) These equalities hold over an arbitrary unital commutative ring, and hence one can work over $R = \mathbb{Q}[\mathbf{X}]$ a suitable polynomial ring, cancel $n!$ from all of these, then observe the "normalized" identity over the subring $R = \mathbb{Z}[\mathbf{X}]$, and finally, specialize the variables $\mathbf{X}$ to show these equalities over any unital commutative ring R.

Before proving Theorem 2.2, we make two observations. The first extends the theorem to all complex (finite-dimensional) characters of G. Even more generally:

Corollary 2.3. *Setting as in Theorem 2.2. Let $\psi = \sum_{\chi\in\widehat{G}_{\mathbb{C}}} a_\chi \chi$ be any complex class function of G, and R_ψ the corresponding ring as in (2.1). Defining*

$$E^{\mathbf{u}}_{\psi} := \sum_{\chi\in\widehat{G}_{\mathbb{C}}} a_\chi E^{\mathbf{u}}_{\chi}$$

and similarly $E^{\mathbf{v}}_{\overline{\psi}}$ (where we set $\overline{\psi} := \sum_\chi a_\chi \overline{\chi}$ – using a_χ and not $\overline{a}_\chi$), we have

$$|G|E^{\mathbf{u}}_{\psi} \cdot \prod_{i=1}^{n} f(tu_iv_i) = |G|E^{\mathbf{v}}_{\overline{\psi}} \cdot \prod_{i=1}^{n} f(tu_iv_i) = \sum_{\chi\in\widehat{G}_{\mathbb{C}}} a_\chi \sum_{\mathbf{m}\in\mathbb{Z}^n_{\geqslant 0}} t^{|\mathbf{m}|} f_{\mathbf{m}} \cdot E^{\mathbf{u}}_{\chi}(\mathbf{u}^{\mathbf{m}}) \cdot E^{\mathbf{v}}_{\overline{\chi}}(\mathbf{v}^{\mathbf{m}}) \tag{2.10}$$

for an arbitrary R_ψ-algebra R and any $f \in R[[t]]$ (and bosonic variables u_i, v_j).

While this identity is more general than (2.6), it also is an immediate consequence of it, by linearity. In fact, the proof (below) of the first equality in (2.6) will also carry over verbatim to (2.10).

Our next remark explains why – for $G = S_n$ or $S_2 \wr S_n$ – the above identities (2.6), (2.10) in fact hold over all rings.

Remark 2.4. Returning to (2.6), two pleasing special cases are when $G = S_n$ and $G = S_2 \wr S_n$ (the type A and B Weyl groups, respectively). In this case, Springer showed [23] that all irreducible complex G-representations can in fact be constructed over $\mathbb{Q}$. In particular, all character values are integers (since they are algebraic integers and rational), and so $R_\chi = \mathbb{Z}$. Thus, for G the Weyl group of type A or B, one has the immanant identity (2.6) – and hence the class function identity (2.10) – over arbitrary unital commutative rings.

We now show the above theorem. The proof has two ingredients: the first explains a key property of signed permutation matrices, when acting on symmetric functions in two sets of variables.

Lemma 2.5. *Fix an integer $n \geqslant 1$ and a unital commutative ring R. Given a signed permutation matrix $g = D \cdot \sigma \in GL_n(R)$ (see the lines before (2.2)), denote its action on the u_i via (2.2) by $g^{\mathbf{u}}$, and similarly define $g^{\mathbf{v}}$. These extend to actions on the ring of polynomials $R[u_1, \dots, u_n, v_1, \dots, v_n]$ and on its Segre subring. Then for every symmetric polynomial F in n variables, and all signed permutations g, we have:*

$$g^{\mathbf{u}} \cdot F(u_1 v_1, \dots, u_n v_n) = (g^{-1})^{\mathbf{v}} \cdot F(u_1 v_1, \dots, u_n v_n), \qquad \forall F \in R[w_1, \dots, w_n]^{S_n},\ g \in S_2 \wr S_n.$$

We defer the proof as we will also show the converse result, in Theorem 2.6 below.

With Lemma 2.5 in hand, we can complete the proof of the theorem above.

Proof of Theorem 2.2. We begin with an arbitrary power series $f(t) = \sum_{m \geqslant 0} f_m t^m \in R[[t]]$, and assert the equation

$$\prod_{i=1}^n f(t u_i v_i) = \sum_{\mathbf{m} \in \mathbb{Z}_{\geqslant 0}^n} t^{|\mathbf{m}|} f_{\mathbf{m}} \mathbf{u}^{\mathbf{m}} \mathbf{v}^{\mathbf{m}}. \tag{2.11}$$

Notice that Eq. (2.11) is (a) obvious, and (b) precisely the sought-for identity (Eq. (2.6)) corresponding to the trivial group $G = \{1\}$.

We now return to the original setting of a general subgroup $G \leqslant S_2 \wr S_n$ acting on the u_i and on the v_j via signed permutations (2.2) — and a multiplicity-free (complex) character χ of G. Working in the Segre subring (2.5), apply the operators $E^{\mathbf{u}}_\chi$ and $E^{\mathbf{v}}_{\overline{\chi}}$ to the above equation (2.11).[2] We now claim that both operations yield equal expressions on the left-hand side by reindexing. Indeed, apply Lemma 2.5 with

$$F(u_1 v_1, \dots, u_n v_n) = \prod_{i=1}^n f(t u_i v_i) \in R[u_1, \dots, u_n, v_1, \dots, v_n][[t]].$$

This yields the following calculation — e.g. in each t-degree separately:

$$\begin{aligned} E^{\mathbf{v}}_{\overline{\chi}} \cdot \prod_{i=1}^n f(t u_i v_i) &= \overline{\chi}(1) \sum_{h = g^{-1} \in G} \overline{\chi}(h^{-1}) (g^{-1})^{\mathbf{v}} \cdot F(u_1 v_1, \dots, u_n v_n) \\ &= \chi(1) \sum_{g \in G} \overline{\chi(g)} g^{\mathbf{u}} \cdot F(u_1 v_1, \dots, u_n v_n) = E^{\mathbf{u}}_\chi \cdot \prod_{i=1}^n f(t u_i v_i), \end{aligned}$$

since $\overline{\chi(g)} = \chi(g^{-1})$ for all $g \in G$.

[2] If $G = S_n$, then this precisely yields the corresponding immanant of the matrix $f[t \mathbf{u} \mathbf{v}^T]$.

This implies that both operations yield the same result *on the right-hand side of* (2.11) *as well*. In particular, $E^{\mathbf{u}}_{\chi} \cdot E^{\mathbf{v}}_{\overline{\chi'}} = 0$ when acting on (2.11), for irreducible complex characters $\chi \neq \chi'$, since

$$E^{\mathbf{u}}_{\chi} \cdot E^{\mathbf{v}}_{\overline{\chi'}} \cdot \sum_{\mathbf{m} \in \mathbb{Z}^n_{\geqslant 0}} t^{|\mathbf{m}|} f_{\mathbf{m}} \mathbf{u}^{\mathbf{m}} \mathbf{v}^{\mathbf{m}} = E^{\mathbf{u}}_{\chi} \cdot E^{\mathbf{v}}_{\overline{\chi'}} \cdot \prod_{i=1}^{n} f(tu_i v_i) = E^{\mathbf{u}}_{\chi} E^{\mathbf{u}}_{\chi'} \cdot \prod_{i=1}^{n} f(tu_i v_i),$$

and this vanishes by character orthogonality. This implies that $E^{\bullet}_{\chi}$ is also pseudo-idempotent for χ multiplicity-free ($E^2_{\chi} = |G|E_{\chi}$), and so applying either $|G|E^{\mathbf{u}}_{\chi}$ or $|G|E^{\mathbf{v}}_{\overline{\chi}}$ to the left-hand side of (2.11) is the same as applying $E^{\mathbf{u}}_{\chi} \cdot E^{\mathbf{v}}_{\overline{\chi}}$:

$$|G|E^{\mathbf{u}}_{\chi} \cdot \prod_{i=1}^{n} f(tu_i v_i) = (E^{\mathbf{u}}_{\chi})^2 \cdot \prod_{i=1}^{n} f(tu_i v_i) = E^{\mathbf{u}}_{\chi} \cdot E^{\mathbf{v}}_{\overline{\chi}} \prod_{i=1}^{n} f(tu_i v_i).$$

Therefore, the same observation applies to the right-hand side of (2.11) — which yields the result. □

2.3. Larger linear groups do not work

We now explain – as promised above – why Theorem 2.2 does not extend to other finite subgroups $G \leqslant GL_n$. The proof of Theorem 2.2 was in three steps:

(1) Lemma 2.5, which says that the actions of

$$g^{\mathbf{u}}, \ (g^{-1})^{\mathbf{v}} : R[w_1, \dots, w_n]^{S_n} \to \bigoplus_{d \geqslant 0} \mathrm{Sym}^d_R(\mathbf{u}) \otimes \mathrm{Sym}^d_R(\mathbf{v})$$

are the same, where $w_i = u_i v_i$ — **if** g is a signed permutation.

(2) This implies that the actions of $E^{\mathbf{u}}_{\chi}$ and $E^{\mathbf{v}}_{\overline{\chi}}$ are the same on the symmetric function $\prod_{i=1}^{n} f(tu_i v_i)$, for any multiplicity-free character χ of any $G \leqslant S_2 \wr S_n$.

(3) Now the pseudo-idempotence of $E^{\mathbf{u}}_{\chi}$ and $E^{\mathbf{v}}_{\overline{\chi}}$ implies the result.

Given Theorem 2.2, it is now natural to ask if this result can be extended from $G \leqslant S_2 \wr S_n$ to any finite matrix subgroup G of $GL_n(R)$. In greater detail: first note that the action of S_n (or $S_2 \wr S_n$) on the free R-module U extends to that of matrices $g = (m_{ij})_{i,j=1}^n \in GL_n$ via:

$$g \cdot \sum_{j=1}^{n} r_j u_j = \sum_{i=1}^{n} \left(\sum_{j=1}^{n} m_{ij} r_j \right) u_i. \tag{2.12}$$

In turn, this GL_n-action extends to all of $R\langle \mathbf{u} \rangle$ by multiplicativity, and then descends to a $GL_n(R)$-action on the quotient algebras $\mathrm{Sym}^{\bullet}_R(\mathbf{u})$, $\wedge^{\bullet}_R(\mathbf{u})$. These remarks apply equally to V and $U \oplus V$ in place of U, and then one can ask if Theorem 1.2 extends to characters of a finite subgroup $G \leqslant GL_n(R)$ that need not be contained in $S_2 \wr S_n$.

Here we show two negative results. The first is a converse to Step (1), and shows that over an integral domain, the conclusion of Lemma 2.5 holds only for signed permutations:

Theorem 2.6. *Fix a unital commutative ring R, an integer $n \geqslant 1$, and bosonic indeterminates u_i, v_i for $i = 1, \dots, n$. Given an element $g \in GL_n(R)$, each of the following assertions implies the next:*

(1) *g is a signed permutation: $g \in S_2 \wr S_n$.*

(2) $g^{\mathbf{u}} \cdot F(u_1 v_1, \dots, u_n v_n) = (g^{-1})^{\mathbf{v}} \cdot F(u_1 v_1, \dots, u_n v_n)$ *for all symmetric functions* $F \in R[w_1, \dots, w_n]^{S_n}$.

(3) $g^{\mathbf{u}} \cdot F(u_1 v_1, \dots, u_n v_n) = (g^{-1})^{\mathbf{v}} \cdot F(u_1 v_1, \dots, u_n v_n)$ *for the n elementary symmetric functions*

$$e_k(\mathbf{w}) = \sum_{1 \leqslant i_1 < i_2 < \cdots < i_k \leqslant n} w_{i_1} w_{i_2} \cdots w_{i_k}, \qquad 1 \leqslant k \leqslant n.$$

If moreover R is an integral domain, then all assertions are equivalent.

Remark 2.7. The group $S_2 \wr S_n$ of signed permutations affords several attractive properties over the reals $R = \mathbb{R}$, i.e. as a subgroup of $GL_n(\mathbb{R})$. In addition to its irreducible representations being constructible over $\mathbb{Q}$ (being the type B Weyl group; see Remark 2.4), signed permutation matrices enjoy characterizations in multiple fields. In linear algebra, they are precisely the orthogonal matrices with integer entries. In analysis, as a special case of the Banach–Lamperti theorem, they coincide with the linear isometries of the p-norms $(\mathbb{R}^n, \|\cdot\|_p)$ for each $p \in [1,\infty] \setminus \{2\}$. Now our Theorem 2.6 provides a "symmetric function" characterization in $GL_n(R)$ of the signed permutation matrices – over any integral domain R – that is novel to the best of our knowledge.

Proof of Theorem 2.6. We show a cyclic chain of implications, starting with (1) $\implies$ (2) (which was Lemma 2.5). Write $g = D\sigma$, where D is a diagonal matrix with (i,i) entry $\varepsilon_i \in \{\pm 1\}$. Now,

$$g^{\mathbf{u}} \cdot F(u_1v_1, \dots, u_nv_n) = D^{\mathbf{u}}\sigma^{\mathbf{u}} \cdot F(u_1v_1, \dots, u_nv_n) = F\left((\varepsilon_{\sigma(i)}u_{\sigma(i)}v_i)_{i=1}^n\right),$$

whereas

$$(g^{-1})^{\mathbf{v}} \cdot F(u_1v_1, \dots, u_nv_n) = (\sigma^{-1})^{\mathbf{v}}(D^{-1})^{\mathbf{v}} \cdot F\left((u_jv_j)_{j=1}^n\right) = F\left((u_j\varepsilon_j^{-1}v_{\sigma^{-1}(j)})_{j=1}^n\right).$$

Now permute the arguments here via: $j = \sigma(i)$, and use that $\varepsilon_j^{-1} = \pm 1 = \varepsilon_j$ together with the symmetry of F, to conclude the proof.

Clearly, (2) $\implies$ (3). Now we suppose R is an integral domain, say with quotient field $\mathbb{F}$, and show that (3) $\implies$ (1). We begin by recalling an observation on elementary symmetric functions that is required in this proof. Suppose an infinite field $\mathbb{K}$ contains pairwise distinct elements $w_1, w_2, \dots, w_n$ and pairwise distinct elements $w'_1, w'_2, \dots, w'_n$, whose elementary symmetric functions agree:

$$e_1(\mathbf{w}) = w_1 + \cdots + w_n = w'_1 + \cdots + w'_n = e_1(\mathbf{w}'), \quad e_2(\mathbf{w}) = e_2(\mathbf{w}'), \quad \dots, \quad e_n(\mathbf{w}) = e_n(\mathbf{w}').$$

Then the polynomials $(x - w_1)\cdots(x - w_n)$ and $(x - w'_1)\cdots(x - w'_n)$ coincide in $\mathbb{K}[x]$, hence so do their sets of roots in the field $\mathbb{K}$ – i.e., $\{w_i : 1 \leqslant i \leqslant n\} = \{w'_i : 1 \leqslant i \leqslant n\}$. We will apply this observation presently, with the (infinite) field being $\mathbb{K}' := \mathbb{F}(u_1, \dots, u_n, v_1, \dots, v_n)$.

Returning to the proof, let $g = (m_{ij})_{i,j=1}^n \in GL_n(R)$, and denote $\varepsilon := \det(g) \in R^\times$. Also write the adjugate matrix of g as $\operatorname{adj}(g) = (a_{ij})_{i,j=1}^n$, where a_{ij} equals $(-1)^{i+j}$ times the (j,i)-minor of g. In particular, $g^{-1} = (\varepsilon^{-1}a_{ij})_{i,j=1}^n$. Now compute, for F running over the elementary symmetric polynomials in n variables:

$$g^{\mathbf{u}} \cdot F(u_1v_1, \dots, u_nv_n) = F\left(\left(v_i \sum_{j=1}^n m_{ji}u_j\right)_{i=1}^n\right),$$
$$(g^{-1})^{\mathbf{v}} \cdot F(u_1v_1, \dots, u_nv_n) = F\left(\left(u_i\varepsilon^{-1} \sum_{j=1}^n a_{ji}v_j\right)_{i=1}^n\right).$$

We now apply the above observation applied to $\mathbb{K}'$; notice this is possible because as $g = (m_{ij})$ is invertible, no row or column is zero, and so the ith argument of $g^{\mathbf{u}} \cdot F$ is a nonzero multiple of v_i in $\mathbb{F}[u_1, \dots, u_n, v_1, \dots, v_n]$, but not of any other v_j. Similarly for the arguments of $(g^{-1})^{\mathbf{v}} \cdot F$. Hence by the above observation, there exists a permutation $\sigma \in S_n$ such that

$$v_{\sigma(i)} \sum_{j=1}^n m_{j\sigma(i)}u_j = u_i\varepsilon^{-1} \sum_{j=1}^n a_{ji}v_j, \qquad \forall 1 \leqslant i \leqslant n.$$

But this is possible in the rational function field $\mathbb{K}'$ only if the coefficients of every u_rv_s are equal on both sides. Thus, $m_{j\sigma(i)} = 0$ whenever $j \neq i$ (so g is necessarily a "generalized permutation matrix"). Moreover, equating the coefficients of $u_iv_{\sigma(i)}$ on both sides yields:

$$m_{i\sigma(i)} = \varepsilon^{-1}a_{\sigma(i)i} = \det(g)^{-1}a_{\sigma(i)i}. \tag{2.13}$$

Now expand the determinant of g along its $\sigma(i)$-column — where we saw above that only the ith entry is nonzero. Thus,

$$\varepsilon = \det(g) = \sum_{j=1}^{n} m_{j\sigma(i)} a_{\sigma(i)j} = m_{i\sigma(i)} a_{\sigma(i)i} \in R^{\times}.$$

Substituting this in (2.13) finally gives us:

$$m_{i\sigma(i)} = \det(g)^{-1} a_{\sigma(i)i} = m_{i\sigma(i)}^{-1} \quad \Longrightarrow \quad m_{i\sigma(i)} = \pm 1.$$

Thus, $g = (m_{ij})$ is a matrix with exactly one nonzero entry in each row and column, and each such entry is ± 1. Hence $g \in S_2 \wr S_n$, which shows (3) $\implies$ (1). $\square$

Theorem 2.6 shows that over an integral domain, Step (1) at the start of this discussion (i.e. of Section 2.3) fails to hold for any matrix g that is not a signed permutation. In order to check if the proof of Theorem 2.2 still goes through for larger matrix groups G, the next strategy to attempt would be to directly show Step (2) without showing Step (1). Namely, we would compute – and equate – the actions of $|G|E_\chi^{\mathbf{u}}, |G|E_{\overline{\chi}}^{\mathbf{v}}$ by summing over the entire group G, instead of using individual terms $g^{\mathbf{u}}, (g^{-1})^{\mathbf{v}}$ and reindexing via $h = g^{-1}$.

Unfortunately, this strategy also fails — already in "small" cases:

Example 2.8. As a simple (counter)example, we now show that (2.6) fails for $n = 1$ and

$$G = \{\exp(2\pi ki/3) : k = 0, 1, 2\} \cong \mathbb{Z}/3\mathbb{Z} \hookrightarrow S^1 \leqslant \mathbb{C}^{\times},$$

for a certain character. (Here, we work over $R = \mathbb{C}$.) More generally, let $d \geqslant 3$ and

$$G = \{\exp(2\pi ki/d) : k = 0, 1, \dots, d-1\} \cong \mathbb{Z}/d\mathbb{Z} \hookrightarrow S^1 \leqslant \mathbb{C}^{\times}.$$

Let the character $\chi(g) := g \in G$, and write $\zeta := \exp(2\pi i/d)$. Now $E_\chi^{u_1}$ acts on $f(tu_1v_1) = \sum_{m \geqslant 0} f_m t^m u_1^m v_1^m$ via:

$$\begin{aligned} E_\chi^{u_1} \cdot f(tu_1v_1) &= \sum_{g \in G} \sum_{m=0}^{\infty} f_m t^m v_1^m \cdot \chi(g^{-1})(gu_1)^m \\ &= \sum_{m=0}^{\infty} t^m \cdot f_m u_1^m v_1^m \cdot \sum_{a \in \mathbb{Z}/d\mathbb{Z}} \zeta^{a(m-1)}. \end{aligned}$$

Similarly,

$$E_{\overline{\chi}}^{v_1} \cdot f(tu_1v_1) = \sum_{m=0}^{\infty} t^m \cdot f_m u_1^m v_1^m \cdot \sum_{a \in \mathbb{Z}/d\mathbb{Z}} \zeta^{a(m+1)},$$

and for $d \geqslant 3$, this series differs from the preceding (formal) power series in $(2/d)$ths of the coefficients: for all $m \equiv \pm 1 \mod d$. $\square$

Theorem 2.6 and Example 2.8 explain our choice of working with $G \leqslant S_2 \wr S_n$ in formulating and proving the main results: larger groups do not work even for small n.

2.4. Further characterizations of the signed permutation matrices

For completeness, here we provide some explanation for why Lemma 2.5 and Theorem 2.6 characterize the group of signed permutations. Suppose $R = \mathbb{F}$ is an infinite field; we will examine the action of the group $GL_n(\mathbb{F})$ on the vector space $\mathcal{M}^{n \times n}$, where $\mathcal{M} \cong U \otimes V$ is the $\mathbb{F}$-span of the n^2 vectors $u_i v_j$.

The first point is that any $g \in GL_n(\mathbb{F})$ acts on all $u_i \in U$ via $g^{\mathbf{u}}$, which is simply matrix multiplication on $\mathbf{u}$, and hence on $\mathbf{u}\mathbf{v}^T$. The action of $g^{\mathbf{v}}$ on V is similar — but in this instance it is from the right, hence via its transpose; thus to make it a valid group action, we need to apply yet another anti-involution. It follows that the two actions $g \mapsto g^{\mathbf{u}}$ and $g \mapsto ((g^{-1})^T)^{\mathbf{v}}$ are indeed group actions, when thought of as acting on $\mathcal{M}^{n\times n}$ from the left and the right, respectively. Thus, Proposition 2.9 below serves as an explanation for why signed permutations are related to $g^{\mathbf{u}}$ acting on $\mathbf{u}$ and $((g^{-1})^T)^{\mathbf{v}}$ acting on $\mathbf{v}^T$, as in Theorem 2.6.

Second, via the observation in the proof of (3) $\implies$ (1) of Theorem 2.6, we are now interested classifying the $g \in GL_n(\mathbb{F})$ for which the diagonal entries in $g^{\mathbf{u}}(\mathbf{u}\mathbf{v}^T)$ and in $(\mathbf{u}\mathbf{v}^T)((g^{-1})^T)^{\mathbf{v}}$ coincide (up to permutation). Such g are characterized by Theorem 2.6 to be precisely the signed permutation matrices. As we now explain, the connection to symmetric function theory is essentially "solely" via the aforementioned observation. In other words, modulo this fact one obtains a "linear algebra" characterization of $S_2 \wr S_n \leqslant GL_n(\mathbb{F})$.

To explain, begin by fixing $g \in GL_n(\mathbb{F})$. Instead of asking that $g^{\mathbf{u}}$ and $((g^{-1})^T)^{\mathbf{v}}$ fix the set of diagonal entries of all matrices in $\mathcal{M}^{n\times n}$, we work alternately with rank one matrices $uv^T \in \mathbb{F}^{n\times n}$ instead of $\mathbf{u}\mathbf{v}^T \in \mathcal{M}^{n\times n}$. (Note that since the ground field is infinite, (symmetric) polynomials are the same as (symmetric) polynomial functions.) Since g is fixed, by relabelling v to $v_g := gv$ one can ask to classify those g such that the diagonal entries of

$$g \cdot (u(gv)^T) = g \cdot uv^T \cdot g^T \qquad \text{and of} \qquad (u(gv)^T) \cdot (g^{-1})^T = uv^T$$

agree as (multi)sets, for all vectors $u, v \in \mathbb{F}^n$. By linearity, this would imply the same fact with uv^T replaced by any matrix $A \in \mathbb{F}^{n\times n}$. Characterizing such g is the content of our next result, shown not just over $\mathbb{F}^{n\times n}$, but again in the generality of Theorem 2.6:

Proposition 2.9. *Fix a unital commutative ring R and an integer $n \geqslant 1$. Given an element $g \in GL_n(R)$, each of the following assertions implies the next:*

1. *g is a signed permutation: $g \in S_2 \wr S_n$.*
2. *For all $A \in R^{n\times n}$, gAg^T and A have the same multisets of diagonal entries.*
3. *For all $1 \leqslant i \leqslant n$, the matrices $gE_{ii}g^T$ and E_{ii} have the same multisets of diagonal entries, where E_{ii} is the matrix with (i, i) entry 1 and all other entries 0.*

If moreover R is an integral domain, then all assertions are equivalent.

Note that one can replace any "intermediate" subset

$$\{E_{11}, \dots, E_{nn}\} \subset S \subset R^{n\times n}$$

in the above characterization of $S_2 \wr S_n \leqslant GL_n(R)$ for R an integral domain. (Also, as the following proof shows, we require from the ground ring R not that it is an integral domain, but only that the square roots in R of 0 and 1 are $\{0\}$ and $\{\pm 1\}$, respectively.)

Proof. We show a cyclic chain of implications. If (1) holds, write $g = D\sigma$; then $g^{-1} = g^T = \sigma^T D$. Since D has diagonal entries ± 1, it is easy to check that DAD and A have the same multisets of diagonal entries. So do $\sigma A \sigma^T$ and A, and so (2) follows.

Clearly, (2) $\implies$ (3). We now assume (3), as well as that R is an integral domain, and show (1) — in two ways. The first is by direct computation: write the columns of $g \in GL_n(R)$ as: $g = [v_1|\cdots|v_n]$, with each v_j having entries $g_{ij} \in R$. Now if D is a diagonal matrix with (i, i) entry $d_i \in R$, then a direct computation reveals:

$$gDg^T = \sum_{j=1}^n d_j v_j v_j^T. \tag{2.14}$$

In particular, $gE_{jj}g^T = v_j v_j^T$, whose diagonal entries are $\{g_{ij}^2 : 1 \leqslant i \leqslant n\}$. By hypothesis, this multiset equals $\{1, 0, \dots, 0\}$, so there exists a map $\sigma : \{1, \dots, n\} \to \{1, \dots, n\}$ such that v_j equals the standard basis vector $e_{\sigma(j)}$ or else $-e_{\sigma(j)}$. Since g is invertible, σ has to be injective, hence $\sigma \in S_n$, and so $g \in S_2 \wr S_n$ as desired.

The second proof of (3) $\implies$ (1) is slightly different, and starts by considering the diagonal entries in (2.14): the (i,i) entry is:

$$(gDg^T)_{ii} = \sum_{j=1}^{n} d_j g_{ij}^2.$$

For each $D = E_{i'i'}$, this yields a system of n equations, which can be stated using the coefficient matrix (g_{ij}^2) applied to the standard coordinate basis vector $e_{i'} \in R^n$. Collecting these systems together for all i' yields

$$\begin{pmatrix} g_{11}^2 & g_{12}^2 & \cdots & g_{1n}^2 \\ g_{21}^2 & g_{22}^2 & \cdots & g_{2n}^2 \\ \vdots & \vdots & \ddots & \vdots \\ g_{n1}^2 & g_{n2}^2 & \cdots & g_{nn}^2 \end{pmatrix} \cdot [\, e_1 \,|\, e_2 \,|\, \dots \,|\, e_n \,] = [\, e_{\sigma(1)} \,|\, e_{\sigma(2)} \,|\, \dots \,|\, e_{\sigma(n)} \,]$$

for some map $\sigma : \{1, \dots, n\} \to \{1, \dots, n\}$. Thus for each j, $g_{ij}^2 = 0$ for $i \neq \sigma(j)$, and $g_{\sigma(j)j}^2 = 1$ in R. Hence $g_{\sigma(j)j} \in \{\pm 1\}$ for each j. Now if σ is not a bijection then $\sigma(j) = \sigma(j')$ for some $1 \leqslant j \neq j' \leqslant n$, in which case the j, j' columns of $g \in GL_n(R)$ are proportional. This is impossible, so σ is a bijection and each $g_{\sigma(j)j} = \pm 1$. Thus $g \in S_2 \wr S_n$, as desired. $\square$

3. Immanant identities for fermionic variables

Theorem 2.2 holds in the case of even/bosonic variables, i.e., where the u_i, v_j all commute among themselves. Our next result is an "odd"/fermionic analogue of Theorem 2.2, in which the u_i and v_j pairwise anti-commute: $u_i u_j = -u_j u_i$, and similarly for v_i, v_j and for u_i, v_j. (In particular, we also require $u_j^2 = v_j^2 = 0$ for $j \geqslant 1$.) We continue to work with a subgroup $G \leqslant S_2 \wr S_n$, and with a power series over some R_ψ-algebra when dealing with a complex class function ψ of G. We also fix an ordering of the fermionic variables u_i and the same one for the v_i, say increasing indices $(1, 2, \dots, n)$.

Since $u_j^2 = v_j^2 = 0 \ \forall j \geqslant 1$, without loss of generality $f(t) = f_0 + f_1 t$ is linear, and so the fermionic analogue of Eq. (2.11) is

$$\prod_{j=1}^{n} f(t u_j v_j) = \sum_{J \subset [n]} (-1)^{\binom{|J|}{2}} f_0^{n-|J|} (f_1 t)^{|J|} \mathbf{u}^J \mathbf{v}^J, \tag{3.1}$$

where $[n] := \{1, \dots, n\}$, the power of (-1) emerges from "taking the u_i past the v_j", and we use the notation

$$\mathbf{u}^J = \prod_{j \in J} u_j, \quad \mathbf{v}^J = \prod_{j \in J} v_j.$$

As in the case of even variables (and forgetting the role of t), Eq. (3.1) takes place inside the alternating algebra, or more precisely, inside the $G \times G$-submodule

$$\bigoplus_{d \geqslant 0} \wedge_R^d(\mathbf{u}) \otimes \wedge_R^d(\mathbf{v}) \subset \wedge_R^\bullet(M^1)$$

(see the discussion around (2.3)), where M^1 is the free R-module with basis $\{u_1, \dots, u_n, \ v_1, \dots, v_n\}$. (Here one begins with the aforementioned $(S_2 \wr S_n) \times (S_2 \wr S_n)$-module structure on $T_{\mathbb{C}}^\bullet(M^1)$.) In particular, the two pseudo-idempotents $E_\chi^{\mathbf{u}}, E_\chi^{\mathbf{v}}$ act on this alternating algebra. Now we claim that applying $E_\chi^{\mathbf{u}}$ or $E_{\overline{\chi}}^{\mathbf{v}}$ to the left-hand side of Eq. (3.1) yields the same expression. To see why, note that the proof of Lemma 2.5 goes through for fermionic variables as well, the key being that $u_i \cdot v_j$ is even for all i, j, hence polynomials in these are well-defined.

But this implies the same result on the right-hand sides of (3.1) — as in the bosonic case. Since $E_\chi^{\mathbf{u}}, E_{\overline{\chi}}^{\mathbf{v}}$ are pseudo-idempotents, this implies the sought-for "fermionic" immanant identity for irreducible characters — which we extend by linearity as in (2.10):

Theorem 3.1. *Fix an integer $n \geqslant 1$, a subgroup $G \leqslant S_2 \wr S_n$ (acting on u_i, v_j by signed permutations), and a complex class function $\psi = \sum_{\chi \in \widehat{G}_{\mathbb{C}}} a_\chi \chi$ of G. Let R_ψ be the corresponding ring as in (2.1), and R an R_ψ-algebra. Working with fermionic variables u_i, v_j, and for $f \in R[[t]]$ an arbitrary formal power series with t an indeterminate, one has:*

$$\begin{aligned} |G| E^{\mathbf{u}}_{\psi} \cdot \prod_{j=1}^n f(tu_jv_j) &= |G| E^{\mathbf{v}}_{\overline{\psi}} \cdot \prod_{j=1}^n f(tu_jv_j) \\ &= \sum_{\chi \in \widehat{G}_{\mathbb{C}}} a_\chi \sum_{J \subset [n]} (-1)^{\binom{|J|}{2}} f_0^{n-|J|} (f_1 t)^{|J|} E^{\mathbf{u}}_{\chi}(\mathbf{u}^J) \cdot E^{\mathbf{v}}_{\overline{\chi}}(\mathbf{v}^J). \end{aligned} \tag{3.2}$$

As in the bosonic case, a prominent special case is that of $G = S_n$, with χ the sign and trivial representations. By Remark 2.4, we may work over any commutative ring R. Since $x_{ij} := u_i v_j$ is still an even variable for all $1 \leqslant i, j \leqslant n$, the x_{ij} commute pairwise and so one can expand the determinant (and permanent) along any row or column. The expansions turn out to be "mirror images" that involve only the two largest or the two smallest powers of t:

Proposition 3.2. *Fix an integer $n \geqslant 2$, a unital commutative ring R, and $f_0, f_1 \in R$. Given odd variables u_i, v_j for $1 \leqslant i, j \leqslant n$ as above,*

$$\begin{aligned} &\det(f_0 + f_1 t u_i v_j)_{i,j=1}^n \\ &= t^n (-1)^{\binom{n}{2}} n! f_1^n \cdot u_1 \cdots u_n \cdot v_1 \cdots v_n \\ &\quad + t^{n-1} (-1)^{\binom{n-1}{2}} (n-1)! f_0 f_1^{n-1} \cdot \sum_{i=1}^n (-1)^{i-1} u_1 \cdots \widehat{u_i} \cdots u_n \cdot \sum_{j=1}^n (-1)^{j-1} v_1 \cdots \widehat{v_j} \cdots v_n. \end{aligned} \tag{3.3}$$

Similarly, the permanent of the above matrix equals

$$\operatorname{perm}(f_0 + f_1 t u_i v_j)_{i,j=1}^n = n! f_0^n + (n-1)! f_0^{n-1} f_1 t (u_1 + \cdots + u_n)(v_1 + \cdots + v_n). \tag{3.4}$$

Remark 3.3. Notice that this is not immediately connected to the even-variable case, since if one specializes the equation in (2.7) to $G = S_n$, χ the sign representation, and $f(t) = f_0 + f_1 t$, then already for $n \geqslant 3$ all products in (2.7) vanish, so we simply get zero there.

Proof of Proposition 3.2. While one can verify the claimed identities via explicit computations, we show them as corollaries of the "master immanant identity" (3.2), by computing the J-summand for every $J \subset [n]$. Since we work over any commutative R, we may assume without loss of generality that R has characteristic zero (see the discussion after (2.9)).

We begin with the determinant (3.3), i.e. with χ the sign character. Here, the term $E^{\mathbf{u}}_{\chi}(\mathbf{u}^J)$ equals a signed sum over permutations (the Laplace expansion), and one has three cases:

- When $J = [n]$, $E^{\mathbf{u}}_{\chi}(\mathbf{u}^J)$ yields $n!$ copies of $(f_1 t)^n$ times the *ordered* product $u_1 \cdots u_n$, and similarly for the v_j. Now dividing by $n!$ (since the left-hand side of (3.2) equals $n! \det(f(tu_iv_j))$) yields the first term on the right in (3.3).
- For each $J \subset [n]$ with $|J| = n-1$, a similar analysis shows that the $n!$ terms in $E^{\mathbf{u}}_{\chi}(\mathbf{u}^J)$ yield (with equal multiplicities) the ordered products $(-1)^{i-1} u_1 \cdots \widehat{u_i} \cdots u_n$, each with multiplicity $(n-1)!$. This also happens for $E^{\mathbf{v}}_{\chi}(\mathbf{v}^J)$; and there are $\binom{n}{n-1}$ choices of such subsets J. Thus the overall multiplicity for the term in (3.3) of t-degree $|J| = n-1$ is $n \cdot (n-1)!^2$. Now divide by $n!$ as above; this yields $(n-1)!$, and hence the second term on the right in (3.3).
- Finally, fix $J \subset \{1, \ldots, n\}$ with $|J| \leqslant n-2$. Also fix $i < i'$ in $[n] \setminus J$. Then $E^{\mathbf{u}}_{\chi}(\mathbf{u}^J)$ is a signed sum over $\sigma \in S_n$, which can be rewritten over the cosets of the reflection subgroup $\{1, (i, i')\}$. As multiplying by the transposition (i, i') reverses sign, this signed sum vanishes for every J of size at most $n-2$.

This shows the determinant formula (3.3). We now turn to the permanent (3.4); the proof here is similar and we write it out for completeness. Once again, there are three cases for $J \subset [n]$:

- When $J = \emptyset$, $E^{\mathbf{u}}_{\chi}(\mathbf{u}^J) = E^{\mathbf{u}}_{\chi}(1) = n!$, and similarly for $E^{\mathbf{v}}_{\chi}(\mathbf{v}^J)$. Now divide by $n!$ (since the left-hand side of (3.2) equals $n!$ times the permanent) to obtain the $n! f_0^n$ term in the claimed expansion.
- When $J = \{j\}$ is a singleton, a similar computation shows that $E^{\mathbf{u}}_{\chi}(u_j)$ yields $(n-1)!$ copies of each u_i. Add all of these terms, then do the same for the action of $E^{\mathbf{v}}_{\chi}$ on v_j (throughout, $J = \{j\}$ is fixed). Now summing over all j, and dividing by $n!$ as above, we get $(n-1)! f_0^{n-1} f_1 t(u_1 + \cdots + u_n)(v_1 + \cdots + v_n)$.
- Finally, if $|J| \geqslant 2$, we choose $j < j'$ in J, and write $E^{\mathbf{u}}_{\chi}(\mathbf{u}^J)$ as the (unsigned) sum over $\sigma \in S_n$, which can be rewritten over the cosets of the reflection subgroup $\{1, (j, j')\}$. As multiplying by the transposition (j, j') reverses sign, this $\square$

For completeness, we conclude this part with a fermionic counterpart of two related results for bosonic variables — Cauchy's *product* identities:

$$\prod_{i,j} \frac{1}{1 - u_i v_j} = \sum_{\mathbf{m}} s_{\mathbf{m}}(\mathbf{u}) s_{\mathbf{m}}(\mathbf{v}), \qquad \prod_{i,j} (1 + u_i v_j) = \sum_{\mathbf{m}} s_{\mathbf{m}}(\mathbf{u}) s_{\mathbf{m}'}(\mathbf{v}), \tag{3.5}$$

where $\mathbf{m}'$ is the dual partition to $\mathbf{m}$. In the fermionic case, since $u_i^2 = v_j^2 = 0$, the two left-hand expressions coincide:

Proposition 3.4. *Fix integers $n \geqslant 2$ and $k \geqslant 1$. Given odd variables u_i, v_j for $1 \leqslant i, j \leqslant n$,*

$$\prod_{i,j=1}^{n} \left(\frac{1}{1 - t u_i v_j} \right)^k = \prod_{i,j=1}^{n} (1 + t u_i v_j)^k = 1 + kt(u_1 + \cdots + u_n)(v_1 + \cdots + v_n). \tag{3.6}$$

Notice the similarity to (3.4). In fact, a similar identity holds for the more general product of factors $(f_0 \pm t f_1 u_i v_j)^{\pm 1}$, and we leave the details to the interested reader.

Proof. As mentioned above, the first equality holds because all variables are fermionic. We now prove the second equality; in doing so, note that all terms $t u_i v_j$ are even, and hence commute pairwise. Since moreover all $u_i^2 = v_j^2 = 0$, hence proving the second equality for $k = 1$ implies it for higher k by the binomial theorem. Thus, we assume henceforth that $k = 1$. Now viewing this product as a polynomial in t, the constant term is 1, and there are 2^{n^2} terms/monomials, each of which has t-degree at most n^2. In any monomial of t-degree $> n$, the pigeonhole principle yields a factor of a u_i^2 and a v_j^2, both of which vanish. Next, the linear terms in t clearly add up to $t(u_1 + \cdots + u_n)(v_1 + \cdots + v_n)$.

It remains to show that the coefficient of t^k vanishes, for all $2 \leqslant k \leqslant n$. For convenience, we multiply the even factors $(1 + u_i v_j)$ in lexicographic order $(1,1), (1,2), \ldots, (n,n)$. Then the coefficient of t^k consists of terms of the form

$$u_{i_1} v_{j_1} \cdots u_{i_k} v_{j_k} = (-1)^{\binom{k}{2}} u_{i_1} \cdots u_{i_k} \cdot v_{j_1} \cdots v_{j_k},$$

where $1 \leqslant i_1 < \cdots < i_k \leqslant n$ (since $u_i^2 = 0$ for all i) and $j_1, \ldots, j_k$ are pairwise distinct. It is now easy to see that this term is obtained in multiple ways, where one can pair the tuple $(j_1, j_2, \ldots, j_k)$ with $(j_2, j_1, \ldots, j_k)$ — and this procedure pairs off the terms into couples with opposite signs. Thus, the sum of all of these terms vanishes. Proceeding in this fashion, all quadratic and higher order terms in t vanish, proving the result. $\square$

Given the theory of symmetric functions, a natural follow-up to these fermionic and bosonic Cauchy product identities is the *nonsymmetric* analogue of the bosonic identity, see [20, Theorem 1.1 and Section 3]. We leave it to the interested reader to explore if there exists a fermionic counterpart to *loc. cit.*

3.1. The case of ε-commuting sets of odd/even variables

In the preceding set of formulas, the two sets of variables u_j, v_k were all odd/fermionic — whereas they were all even/bosonic in an earlier section. As a consequence, in both of these settings the variables $x_{ij} = u_i v_j$ commute in both of these settings, which makes the determinant well-defined regardless of how one expands it out.

In this concluding subsection (which is essentially an expanded remark), we derive analogous identities in a slightly more general setting. The point is to draw attention to a parameter that is implicit in the calculations in both of the above settings: the proportionality constant ε that one obtains when moving any u past any v. In the case of even/odd variables, we had specialized this parameter to equal the scalar $\varepsilon = \pm 1$, respectively. However, the computations in fact hold for arbitrary choice of ε in *either* setting, because the power of this scalar merely keeps track of how many u move past how many v. Thus, similar to the variable t that keeps track of the common homogeneity degree in the u's and the v's (separately), we now introduce another "bookkeeping" *indeterminate* ε, which ends up keeping track of the same information — but now via the number of exchanges of u's and v's. Notice, however, that the terms $x_{ij} = u_i v_j$ still pairwise commute, so that $\det f[\mathbf{u}\mathbf{v}^T]$ stays well-defined.

Thus, we now write down the "more general" formulas in the above two settings; the proofs are identical. In the case of **bosonic** u_i and v_j, if moreover

$$\varepsilon u_i v_j = v_j u_i, \quad \forall 1 \leqslant i, j \leqslant n,$$

then given a subgroup $G \leqslant S_2 \wr S_n$ and a multiplicity-free complex character χ of G, first fix an R_χ-algebra R. Now we work over the polynomial ring $R[\varepsilon]$, in the quotient of $T^\bullet_{R[\varepsilon]}(U \oplus V)$ by the two-sided ideal generated by

$$\{u_i \otimes u_j - u_j \otimes u_i,\ v_i \otimes v_j - v_j \otimes v_i, \quad v_j \otimes u_i - \varepsilon\, u_i \otimes v_j \ :\ 1 \leqslant i, j \leqslant n\};$$

notice this ideal is $(S_2 \wr S_n) \times (S_2 \wr S_n)$-stable. Then for any $f \in R[[t]]$ (or even $f \in R[\varepsilon][[t]]$),

$$|G| E^{\mathbf{u}}_\chi \cdot \prod_{i=1}^n f(tu_i v_i) = |G| E^{\mathbf{v}}_{\overline{\chi}} \cdot \prod_{i=1}^n f(tu_i v_i) = \sum_{\mathbf{m} \in \mathbb{Z}^n_{\geqslant 0}} \varepsilon^{\binom{|\mathbf{m}|}{2}} t^{|\mathbf{m}|} f_{\mathbf{m}} \cdot E^{\mathbf{u}}_\chi(\mathbf{u}^{\mathbf{m}}) \cdot E^{\mathbf{v}}_{\overline{\chi}}(\mathbf{v}^{\mathbf{m}}). \tag{3.7}$$

Specializing to $G = S_n$ and χ the sign character (and cancelling $n!$ as in the discussion following (2.8)),

$$\det f[t\mathbf{u}\mathbf{v}^T] = \sum_{\mathbf{m} \in \mathbb{Z}^n_{\geqslant 0},\ \mathbf{m} \text{decreasing}} \varepsilon^{\binom{|\mathbf{m}|}{2}} t^{|\mathbf{m}| + \binom{n}{2}} \prod_{i=1}^n f_{m_i + n - i} \cdot V(\mathbf{u}) s_{\mathbf{m}}(\mathbf{u}) \cdot V(\mathbf{v}) s_{\mathbf{m}}(\mathbf{v}). \tag{3.8}$$

Similarly, in the case of **fermionic** u_i and v_j, if moreover $\varepsilon u_i v_j = v_j u_i\ \forall i, j$, the analogous formula is:

$$|G| E^{\mathbf{u}}_\chi \cdot \prod_{j=1}^n f(tu_j v_j) = |G| E^{\mathbf{v}}_{\overline{\chi}} \cdot \prod_{j=1}^n f(tu_j v_j) = \sum_{J \subset [n]} \varepsilon^{\binom{|J|}{2}} f_0^{n-|J|} (f_1 t)^{|J|} \cdot E^{\mathbf{u}}_\chi(\mathbf{u}^J) \cdot E^{\mathbf{v}}_{\overline{\chi}}(\mathbf{v}^J) \tag{3.9}$$

for arbitrary $G \leqslant S_2 \wr S_n$, any multiplicity-free character χ of G, any R_χ-algebra R, and any $f \in R[[t]]$. Again specializing to $G = S_n$ and χ the sign character,

$$\begin{aligned} &\det(f_0 + f_1 t u_i v_j)_{i,j=1}^n \\ &= \varepsilon^{\binom{n}{2}} t^n n! f_1^n \cdot u_1 \cdots u_n \cdot v_1 \cdots v_n \\ &\quad + \varepsilon^{\binom{n-1}{2}} t^{n-1} (n-1)! f_0 f_1^{n-1} \cdot \sum_{i=1}^n (-1)^{i-1} u_1 \cdots \widehat{u_i} \cdots u_n \cdot \sum_{j=1}^n (-1)^{j-1} v_1 \cdots \widehat{v_j} \cdots v_n. \end{aligned} \tag{3.10}$$

Similar formulas hold for the permanents, in both the bosonic and fermionic settings. Moreover, the "master identities" (3.7) and (3.9) immediately extend to arbitrary complex class functions ψ of G, by linearity.

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