AN APPROXIMATION PROBLEM IN THE SPACE OF BOUNDED OPERATORS

ARPITA MAL

ABSTRACT. For Banach spaces X, Y, we consider a distance problem in the space of bounded linear operators $\mathcal{L}(X, Y)$. Motivated by a recent paper [19], we obtain sufficient conditions so that for a compact operator $T \in \mathcal{L}(X, Y)$ and a closed subspace $Z \subset Y$, the following equation holds, which relates global approximation with local approximation:

 $d(T, \mathcal{L}(X, Z)) = \sup\{d(Tx, Z) : x \in X, ||x|| = 1\}.$

In some cases, we show that the supremum is attained at an extreme point of the corresponding unit ball. Furthermore, we obtain some situations when the following equivalence holds:

 $T \perp_B \mathcal{L}(X, Z) \Leftrightarrow T^{**} x_0^{**} \perp_B Z^{\perp \perp} \Leftrightarrow T^{**} \perp_B \mathcal{L}(X^{**}, Z^{\perp \perp}),$

for some $x_0^{**} \in X^{**}$ satisfying $||T^{**}x_0^{**}|| = ||T^{**}||||x_0^{**}||$, where Z^{\perp} is the annihilator of Z. One such situation is when Z is an L^1 -predual space and an M-ideal in Y and T is a multi-smooth operator of finite order. Another such situation is when X is an abstract L_1 -space and T is a multi-smooth operator of finite order. Finally, as a consequence of the results, we obtain a sufficient condition for proximinality of a subspace Z in Y.

1. INTRODUCTION

In this paper, our aim is to study distance problem and Birkhoff-James orthogonality in the space of bounded linear operators. We approximate the distance of a bounded linear operator from a subspace of operators with the distance of its image from a subspace. To state the problem, we first introduce necessary notations and terminologies.

In this paper, X, Y denote real Banach spaces and Z denotes a closed subspace of Y. Let S_X and B_X denote the unit sphere and the unit ball of X, i.e., $S_X = \{x \in X : ||x|| = 1\}$ and $B_X = \{x \in X : ||x|| \leq 1\}$. Suppose E_X denotes the set of all extreme points of B_X and X^* denotes the dual space of X. We always assume that X is canonically embedded in its bidual X^{**} . The symbol $\mathcal{L}(X,Y)$ ($\mathcal{K}(X,Y)$) denotes the space of all bounded (compact) linear operators from X to Y. For $T \in \mathcal{L}(X,Y)$, $M_T = \{x \in S_X : ||Tx|| = ||T||\}$, the collection of all unit vectors of X, at which T attains its norm. For a non-zero element $x \in X$, suppose $J(x) = \{x^* \in S_{X^*} : x^*(x) = ||x||\}$. Recall that J(x) is a nonempty, weak*compact, convex subset of S_{X^*} . The symbol $E_{J(T)}$ denotes the set of all extreme points of J(T). A non-zero element $x \in X$ is said to be a multi-smooth point of finite order (or k-smooth) [8, 11] if J(x) contains finitely many (exactly

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k) linearly independent functionals. In other words, x is a multi-smooth point of finite order if Span(J(x)) is finite-dimensional. We say that x is smooth if J(x) is singleton and X is smooth if x is smooth for all non-zero $x \in X$. Similarly, a non-zero operator $T \in \mathcal{L}(X, Y)$ is a multi-smooth operator of finite order if Span(J(T)) is finite-dimensional, where $J(T) = \{f \in S_{\mathcal{L}(X,Y)^*} : f(T) = ||T||\}$. In particular, T is said to be a smooth operator if J(T) is singleton. For the study of smooth and multi-smooth (k-smooth) operators see [4, 5, 11, 12, 13, 16, 17, 24].

A classical problem in approximation theory is the distance problem. For $x \in X$ and a subspace $W \subset X$, $d(x, W) = \inf_{w \in W} ||x - w||$ is the distance of x from W. An element $w_0 \in W$ is said to be a best approximation to x out of W if $||x - w_0|| = d(x, W)$. We denote the collection of all best approximation(s) to x out of W by $\mathscr{L}_W(x)$, i.e., $\mathscr{L}_W(x) = \{w_0 \in W : ||x - w_0|| = d(x, W)\}$. A subspace $W \subset X$ is said to be a proximinal subspace of X if $\mathscr{L}_W(x) \neq \emptyset$ for all $x \in X$. Similarly, in operator space, for $T \in \mathcal{L}(X, Y)$ and a subspace $\mathscr{V} \subset \mathcal{L}(X, Y)$, $d(T, \mathscr{V}) = \inf_{S \in \mathscr{V}} ||T - S||$ and $\mathscr{L}_{\mathscr{V}}(T) = \{S_0 \in \mathscr{V} : ||T - S_0|| = d(T, \mathscr{V})\}$. For some recent study on best approximation and distance formula in operator spaces and C^* - algebra follow [3, 14, 21, 22]. Note that, the notions of best approximation and Birkhoff-James orthogonality are closely related. For $x, y \in X$, we say that x is Birkhoff-James orthogonal [2, 7] to y if $||x + \lambda y|| \ge ||x||$ for all scalars λ and we denote it by $x \perp_B y$. For a subspace $W \subset X$, we say that $x \perp_B W$ if $x \perp_B w$ for all $w \in W$. It is now straightforward to check that $w_0 \in \mathscr{L}_W(x)$ if and only if $x - w_0 \perp_B W$. Suppose that K is a compact Hausdorff space. Let C(K,Y) be the space of all continuous functions defined from K to Y equipped with the supremum norm. From a classical result [10, Th. 2.4] by Light and Cheney, we know that if $f \in C(K, Y)$, then

$$d(f, C(K, Z)) = \sup_{k \in K} d(f(k), Z) = d(f(k_0), Z)$$

for some $k_0 \in K$. The above distance formula provides a relation between global approximation and local approximation. Motivated by the above approximation result, recently in [19], Rao studied analogous problem in the space of bounded linear operators. More precisely, he raised the question that for $T \in \mathcal{K}(X, Y)$, when the following minimax formula holds:

(1)
$$d(T, \mathcal{L}(X, Z)) = \sup_{x \in S_X} d(Tx, Z).$$

It is also interesting to ask when the supremum in (1) is attained. Whenever the the supremum is attained at some $x_0 \in S_X$, we can further ask whether x_0 can be chosen from E_X . To answer these questions, we need to introduce a few notions. A Banach space X is said to be an L^1 -predual space [9] if X^* is isometrically isomorphic to $L^1(\mu)$ for a positive measure μ . A subspace Z of Y is said to be an M-ideal [6] in Y, if there is a linear projection $P: Y^* \to Y^*$ such that $||y^*|| = ||Py^*|| + ||y^* - Py^*||$ for all $y^* \in Y^*$ and $\ker(P) = Z^{\perp}$, where Z^{\perp} is the annihilator of Z, i.e., $Z^{\perp} = \{y^* \in Y^*: y^*(z) = 0 \forall z \in Z\}$. Note that, for any $x \in S_X$ and $S \in \mathcal{L}(X, Z)$, $||Tx - Sx|| \ge d(Tx, Z)$. Thus,

$$||T - S|| = \sup_{x \in S_X} ||Tx - Sx|| \ge \sup_{x \in S_X} d(Tx, Z).$$

Now, taking infimum over $S \in \mathcal{L}(X, Z)$, we get

(2)
$$d(T, \mathcal{L}(X, Z)) \ge \sup_{x \in S_X} d(Tx, Z).$$

In [19, Th. 1] Rao proved the existence of the minimax formula (1) with the attainment of the supremum at an extreme point of B_X assuming that X is a reflexive, separable Banach space, Z is an L^1 -predual space and also an M-ideal in Y. On the other hand, assuming that X is a separable Banach space, Z is an L^1 -predual space and an M-ideal in Y, Rao in [19, Th. 6] proved that under a local condition (a suitable smoothness condition) on T the minimax formula (1) holds. Both the proofs of these theorems are based on a lifting theorem from [6, Th. II.2.1].

Many researchers are devoted to the study of Birkhoff-James orthogonality in the space of operators (see [1, 15, 22, 23] and the references therein). Note that, if there exists a vector $x_0 \in M_T$ such that $Tx_0 \perp_B Z$, then for all $S \in \mathcal{L}(X, Z)$, $||T - S|| \geq ||Tx_0 - Sx_0|| \geq ||Tx_0|| = ||T||$. Therefore, in this case, $T \perp_B \mathcal{L}(X, Z)$. On the other hand, observe that if the supremum in (1) is attained at some $x_0 \in S_X$ and $T \perp_B \mathcal{L}(X, Z)$, then from $||T|| = d(T, \mathcal{L}(X, Z)) = d(Tx_0, Z) \leq ||Tx_0|| = ||T||$, it follows that $d(Tx_0, Z) = ||Tx_0||$, i.e., $Tx_0 \perp_B Z$. In [19], Rao used this approach to prove the implication $T \perp_B \mathcal{L}(X, Z) \Rightarrow Tx_0 \perp_B Z$ for some $x_0 \in M_T$, whenever the supremum in (1) is attained.

In this paper, we prove the existence of formula (1) whenever T is a multi-smooth operator of finite order and $T \perp_B \mathcal{L}(X, Z)$. In this case, we do not assume any restriction on the space X. We only assume that Z is an L^1 -predual space and an M-ideal in Y. Moreover, we prove the following equivalence

(3)
$$T \perp_B \mathcal{L}(X, Z) \Leftrightarrow T^{**} x_0^{**} \perp_B Z^{\perp \perp} \Leftrightarrow T^{**} \perp_B \mathcal{L}(X^{**}, Z^{\perp \perp})$$

for some $x_0^{**} \in M_{T^{**}}$. Furthermore, if X^* is assumed to be smooth, then x_0^{**} can be chosen from $M_{T^{**}} \cap E_{X^{**}}$ (see Theorem 2.2 and Remark 2.8). In addition, if Xis assumed to be reflexive, then we show that the supremum in (1) is attained at some $x_0 \in M_T$. On the other hand, we show that (1) holds for an arbitrary closed subspace Z of Y, if we assume that X^* is an L^1 -predual space (more generally, if X^{**} has L^1 -property according to Definition 2.5), T is a multi-smooth operator of finite order and $T \perp_B \mathcal{L}(X, Z)$. Moreover, we show that in this case, (3) holds for some $x_0^{**} \in M_{T^{**}} \cap E_{X^{**}}$ (see Theorem 2.6, Remark 2.8 and Remark 2.7). Finally, we provide another situation when formula (1) holds and the supremum is attained at some $x_0 \in M_T \cap E_X$. As a consequence of the results, we prove that if $\mathcal{L}(\ell_1^n, Z)$ is a proximinal subspace of $\mathcal{L}(\ell_1^n, Y)$, then Z is a proximinal subspace of Y, provided Z is an arbitrary closed subspace of Y and each non-zero element of Yis a multi-smooth point of finite order. We would like to mention that the approach used in this paper to prove the non-trivial part is completely different from [19]. In particular, we do not use the lifting theorem from [6, Th. II.2.1].

2. Main results

We begin this section with an easy proposition.

Proposition 2.1. Let X, Y be Banach spaces and Z be a closed subspace of Y. Let $T \in \mathcal{L}(X,Y)$. Suppose there exists $S \in \mathscr{L}_{\mathcal{L}(X,Z)}(T)$ such that T - S is smooth and $M_{T-S} \neq \emptyset$. Then there exist $x_0 \in E_X$, $y_0^* \in E_{Y^*} \cap Z^{\perp}$ such that the following hold.

$$d(T, \mathcal{L}(X, Z)) = \sup_{x \in S_X} d(Tx, Z) = d(Tx_0, Z)$$

= $\sup\{y^*(Tx) : x \in E_X, y^* \in E_{Y^*} \cap Z^{\perp}\} = y_0^*(Tx_0).$

Proof. From $S \in \mathscr{L}_{\mathcal{L}(X,Z)}(T)$, it follows that $T_0 \perp_B \mathcal{L}(X,Z)$, where $T_0 = T - S$. Since T_0 is smooth and $M_{T_0} \neq \emptyset$, from [24, Th. 3.3], we get $M_{T_0} = \{\pm x_0\}$ for some $x_0 \in S_X$. Observe that $x_0 \in E_X$. For otherwise, there exist $x_1, x_2 \in B_X$ such that $x_1 \neq x_2$ and $x_0 = \frac{1}{2}x_1 + \frac{1}{2}x_2$. Now, from

$$||T_0|| = ||T_0x_0|| \le \frac{1}{2}||T_0x_1|| + \frac{1}{2}||T_0x_2|| \le \frac{1}{2}||T_0|| + \frac{1}{2}||T_0|| = ||T_0||.$$

we get that $||T_0x_1|| = ||T_0x_2|| = ||T_0||$, i.e., $x_1, x_2 \in M_{T_0}$. Thus, $x_1 = -x_2$ and so $x_0 = 0$, a contradiction. Choose $x^* \in J(x_0)$. Let $z \in Z$. Consider $A \in \mathcal{L}(X, Z)$ defined as $Ax = x^*(x)z$ for all $x \in X$. Then $T_0 \perp_B A$. By [24, Th. 3.3], we get $T_0x_0 \perp_B Ax_0 \Rightarrow T_0x_0 \perp_B z$. Since $z \in Z$ is chosen arbitrarily, we have $T_0x_0 \perp_B Z$. Thus,

$$d(Tx_0, Z) = d(T_0x_0, Z) = ||T_0x_0|| = ||T_0|| = d(T_0, \mathcal{L}(X, Z)) = d(T, \mathcal{L}(X, Z)).$$

On the other hand,

(4)
$$\sup_{x \in S_X} d(Tx, Z) \ge d(Tx_0, Z) = d(T, \mathcal{L}(X, Z)) \ge \sup_{x \in S_X} d(Tx, Z),$$

where the last inequality follows from (2). This completes the proof of the first part.

Now, we prove the second part. Since T_0 is smooth, again from [24, Th. 3.3], we get, T_0x_0 is smooth, i.e., $J(T_0x_0) = \{y_0^*\}$ for some $y_0^* \in S_{Y^*}$. Since $J(T_0x_0)$ is convex, y_0^* is an extreme point of $J(T_0x_0)$. Observe that $J(T_0x_0)$ is an extremal subset of B_{Y^*} . Therefore, $y_0^* \in E_{Y^*}$. Now, from [7, Th. 2.1] and $T_0x_0 \perp_B Z$, it follows that $y_0^* \in Z^{\perp}$. Thus,

(5)
$$y_0^*(Tx_0) = y_0^*(Tx_0 - Sx_0) = y_0^*(T_0x_0) = ||T_0x_0|| = ||T_0|| = d(T, \mathcal{L}(X, Z)).$$

On the other hand, observe that for each $x \in E_X$, $y^* \in E_{Y^*} \cap Z^{\perp}$ and $A \in \mathcal{L}(X, Z)$, we have

$$y^*(Tx) = y^*(Tx - Ax) \le ||Tx - Ax|| \le ||T - A||.$$

Thus,

$$\sup\{y^*(Tx) : x \in E_X, y^* \in E_{Y^*} \cap Z^{\perp}\} \le \inf_{A \in \mathcal{L}(X,Z)} \|T - A\| = d(T, \mathcal{L}(X,Z)).$$

The above inequality together with (5) completes the proof of the second part. \Box

To prove the next theorem, we use the extremal structure of the unit ball of $\mathcal{K}(X,Y)^*$. From [20, Th. 1.3], we note that

(6)
$$E_{\mathcal{K}(X,Y)^*} = \{x^{**} \otimes y^* : x^{**} \in E_{X^{**}}, y^* \in E_{Y^*}\},\$$

where $x^{**} \otimes y^*(S) = x^{**}(S^*y^*)$ for $S \in \mathcal{L}(X, Y)$. Now, we are ready to prove our desired theorem.

Theorem 2.2. Let X, Y be Banach spaces. Suppose Z is a subspace of Y such that Z is an L^1 -predual space and an M-ideal in Y. Let $T \in \mathcal{K}(X,Y)$ be a multismooth operator of finite order. Suppose that $T \perp_B \mathcal{L}(X,Z)$. Then the following hold.

 $\begin{array}{l} \text{(i)} \ d(T, \mathcal{L}(X, Z)) = \sup_{x \in S_X} d(Tx, Z). \\ \text{(ii)} \ T^{**} x_0^{**} \perp_B Z^{\perp \perp} \ \text{for some} \ x_0^{**} \in M_{T^{**}}. \end{array}$

5

(iii) $T^{**} \perp_B \mathcal{L}(X^{**}, Z^{\perp\perp}).$

(iv) There exists $x_0^{**} \in M_{T^{**}}$ such that

$$d(T^{**}, \mathcal{L}(X^{**}, Z^{\perp \perp})) = \sup_{x^{**} \in S_{X^{**}}} d(T^{**}x^{**}, Z^{\perp \perp})$$

= $d(T^{**}x_0^{**}, Z^{\perp \perp}) = d(T, \mathcal{L}(X, Z)).$

Additionally, if we assume that X^* is smooth, then in (ii) and (iv), we may choose x_0^{**} from $M_{T^{**}} \cap E_{X^{**}}$.

Proof. (i) From $T \perp_B \mathcal{L}(X, Z)$ and [7, Th. 2.1], it follows that there exists $f \in J(T)$ such that f(A) = 0 for all $A \in \mathcal{L}(X, Z)$. Since T is a multi-smooth operator of finite order, $Span(E_{J(T)})$ is finite-dimensional. Now, J(T) being a non-empty, weak*compact, convex set, by the Krein-Milman theorem, we get

$$J(T) = \overline{conv}^{w^*}(E_{J(T)}) \subseteq \overline{Span}^{w^*}(E_{J(T)}) = \overline{Span}(E_{J(T)}) = Span(E_{J(T)}).$$

Using [25, Lem. 1.1, pp. 166], we get extreme points f_1, f_2, \ldots, f_h of the unit ball of $Span(E_{J(T)})$ and scalars $\lambda_1, \lambda_2, \ldots, \lambda_h > 0$ such that $\sum_{i=1}^h \lambda_i = 1$ and $f = \sum_{i=1}^h \lambda_i f_i$. Now, it is easy to check that $f_i \in E_{J(T)}$ for all $1 \le i \le h$. Since J(T) is an extremal subset of $B_{\mathcal{K}(X,Y)^*}$, each f_i is an extreme point of $B_{\mathcal{K}(X,Y)^*}$. Therefore, there exist $x_i^{**} \in E_{X^{**}}, y_i^* \in E_{Y^*}$ such that $f_i = x_i^{**} \otimes y_i^*$ for each $1 \le i \le h$. Now, $f_i \in J(T)$ implies that

$$||T|| = f_i(T) = x_i^{**} \otimes y_i^*(T) = x_i^{**}(T^*y_i^*) \le ||T^*y_i^*|| \le ||T^*|| = ||T||$$

which yields that $y_i^* \in M_{T^*}$ and $x_i^{**}(T^*y_i^*) = ||T^*y_i^*|| = ||T||$. Since Z is an M-ideal in Y, by [6, Rem. 1.13, pp. 11] we have $Y^* = Z^* \oplus_1 Z^{\perp}$ and by [6, Lem. 1.5, pp. 3] $E_{Y^*} = E_{Z^*} \cup E_{Z^{\perp}}$. Thus, $y_i^* \in E_{Z^*} \cup E_{Z^{\perp}}$ for all $1 \leq i \leq h$. We claim that for some $i, y_i^* \in E_{Z^{\perp}}$. If possible, suppose that $y_i^* \in E_{Z^*}$ for all i. Since Z is an L^1 -predual space, either $\{y_i^* : 1 \leq i \leq h\}$ is linearly independent or $y_i^* = \pm y_j^*$ for some $i \neq j$. In the next two paragraphs, we show that after suitable modification, we can write $f = \sum_{i=1}^h \lambda_i x_i^{**} \otimes y_i^*$, where $y_i^* \in E_{Y^*}$ and $\{y_i^* : 1 \leq i \leq h\}$ is linearly independent.

Now, suppose that X^* is smooth. Observe that if $y_i^* = y_j^*$ for some $i \neq j$, then $x_i^{**}(T^*y_i^*) = x_j^{**}(T^*y_i^*) = ||T^*y_i^*||$, i.e., $x_i^{**}, x_j^{**} \in J(T^*y_i^*)$. The smoothness of $T^*y_i^*$ yields that $x_i^{**} = x_j^{**}$. In this case, $\lambda_i x_i^{**} \otimes y_i^* + \lambda_j x_j^{**} \otimes y_j^* = (\lambda_i + \lambda_j) x_i^{**} \otimes y_i^*$. Therefore, in case X^* is smooth, if necessary changing the scalars suitably, we may write $f = \sum_{i=1}^h \lambda_i x_i^{**} \otimes y_i^*$, where $x_i^{**} \in E_{X^{**}}, y_i^* \in E_{Y^*}$ and $\{y_i^* : 1 \leq i \leq h\}$ is linearly independent.

Now, suppose that X^* is not smooth. Observe that, if $y_1^* = y_2^*$ holds, then considering $x^{**} = \lambda_1 x_1^{**} + \lambda_2 x_2^{**}$, we get $x^{**}(T^*y_1^*) = (\lambda_1 + \lambda_2) ||T^*y_1^*||$ and $||x^{**}|| = \lambda_1 + \lambda_2$. In that case, $f = (\lambda_1 + \lambda_2) \frac{x^{**}}{||x^{***}||} \otimes y_1^* + \lambda_3 x_3^{**} \otimes y_3^* + \ldots + \lambda_h x_h^{**} \otimes y_h^*$ and x^{**} may not belong to $E_{X^{**}}$. Therefore, in case X^* is not smooth, if necessary after suitable change, we may write $f = \sum_{i=1}^h \lambda_i x_i^{**} \otimes y_i^*$, where $x_i^{**} \in S_{X^{**}}, y_i^* \in E_{Y^*}$ and $\{y_i^* : 1 \le i \le h\}$ is linearly independent.

Now, choose $x^* \in X^*$ such that $x_1^{**}(x^*) \neq 0$ and $z_0 \in \bigcap_{i=2}^h \ker(y_i^*) \setminus \ker(y_1^*)$. Define $A \in \mathcal{L}(X, Z)$ by $A(x) = x^*(x)z_0$ for all $x \in X$. Therefore, $A^* : Z^* \to X^*$ is defined as $A^*z^* = z^*(z_0)x^*$ for all $z^* \in Z^*$. Now, from f(A) = 0, it follows that

$$\sum_{i=1}^{h} \lambda_i x_i^{**} \otimes y_i^*(A) = 0 \quad \Rightarrow \quad \sum_{i=1}^{h} \lambda_i x_i^{**}(A^* y_i^*) = 0$$
$$\Rightarrow \quad \sum_{i=1}^{h} \lambda_i x_i^{**}(y_i^*(z_0)x^*) = 0$$
$$\Rightarrow \quad \sum_{i=1}^{h} \lambda_i x_i^{**}(x^*)y_i^*(z_0) = 0$$
$$\Rightarrow \quad \lambda_1 x_1^{**}(x^*)y_1^*(z_0) = 0,$$

which is a contradiction. This proves our claim. Thus, we get $i \in \{1, 2, ..., h\}$ such that

(7) $y_i^* \in E_{Z^\perp} \cap M_{T^*} \text{ and } x_i^{**}(T^*y_i^*) = ||T||.$

Now, for each $x \in X$ and for each $z \in Z$,

$$||Tx - z|| \ge |y_i^*(Tx - z)| = |y_i^*(Tx)| = |T^*y_i^*(x)|.$$

Thus, taking infimum over $z \in Z$, we get for each $x \in X$,

$$||Tx|| \ge d(Tx, Z) \ge |T^*y_i^*(x)|.$$

In this inequality, taking supremum over $x \in S_X$, we have

$$||T|| \ge \sup_{x \in S_X} d(Tx, Z) \ge \sup_{x \in S_X} |T^* y_i^*(x)| = ||T^* y_i^*|| = ||T||.$$

Therefore, $||T|| = \sup_{x \in S_X} d(Tx, Z)$. On the other hand, from $T \perp_B \mathcal{L}(X, Z)$, it clearly follows that $d(T, \mathcal{L}(X, Z)) = ||T||$. This proves (i).

(ii) Let
$$u \in Z^{\perp \perp}$$
. Then using (7), we get $u(y_i^*) = 0$. Thus, for all $u \in Z^{\perp \perp}$,
 $\|T^{**}x_i^{**} - u\| \ge |(T^{**}x_i^{**} - u)y_i^*| = |T^{**}x_i^{**}(y_i^*)| = |x_i^{**}(T^*y_i^*)|$
 $\Rightarrow \quad d(T^{**}x_i^{**}, Z^{\perp \perp}) \ge |x_i^{**}(T^*y_i^*)| = \|T\| \text{ (taking infimum over } u \in Z^{\perp \perp})$
 $\Rightarrow \quad \|T^{**}\| \ge \|T^{**}x_i^{**}\| \ge d(T^{**}x_i^{**}, Z^{\perp \perp}) \ge \|T\| = \|T^{**}\|$
 $\Rightarrow \quad \|T^{**}\| = \|T^{**}x_i^{**}\| = d(T^{**}x_i^{**}, Z^{\perp \perp})$
 $\Rightarrow \quad T^{**}x_i^{**} \perp B Z^{\perp \perp} \text{ and } x_i^{**} \in M_{T^{**}}.$

(iii) Let $S \in \mathcal{L}(X^{**}, Z^{\perp \perp})$. Then using (ii), we get

$$||T^{**} - S|| \ge ||T^{**}x_0^{**} - Sx_0^{**}|| = ||T^{**}x_0^{**}|| = ||T^{**}||.$$

Thus, $T^{**} \perp_B \mathcal{L}(X^{**}, Z^{\perp \perp}).$

(iv) From (ii) it follows that $d(T^{**}x_0^{**}, Z^{\perp\perp}) = ||T^{**}x_0^{**}|| = ||T^{**}||$ and from (iii) it follows that $d(T^{**}, \mathcal{L}(X^{**}, Z^{\perp\perp})) = ||T^{**}||$. Therefore,

$$d(T, \mathcal{L}(X, Z)) = ||T|| = ||T^{**}|| = d(T^{**}x_0^{**}, Z^{\perp \perp})$$

= $d(T^{**}, \mathcal{L}(X^{**}, Z^{\perp \perp}))$
 $\geq \sup_{x^{**} \in S_{X^{**}}} d(T^{**}x^{**}, Z^{\perp \perp}) \text{ (similarly as (2))}$
 $\geq d(T^{**}x_0^{**}, Z^{\perp \perp}).$

This completes the proof.

As a simple consequence of Theorem 2.2, we get the next corollary.

Corollary 2.3. Let X, Y be Banach spaces. Suppose Z is a subspace of Y such that Z is an L^1 -predual space and an M-ideal in Y. Let $T \in \mathcal{K}(X, Y)$. Suppose there exists $S \in \mathscr{L}_{\mathcal{L}(X,Z)}(T)$ such that T - S is a multi-smooth operator of finite order. Then the following hold.

(i) $d(T, \mathcal{L}(X, Z)) = \sup_{x \in S_X} d(Tx, Z).$

(ii) There exists $x_0^{**} \in S_{X^{**}}$ such that

$$d(T^{**}, \mathcal{L}(X^{**}, Z^{\perp \perp})) = \sup_{x^{**} \in S_{X^{**}}} d(T^{**}x^{**}, Z^{\perp \perp})$$

= $d(T^{**}x_0^{**}, Z^{\perp \perp}) = d(T, \mathcal{L}(X, Z))$

Additionally, if we assume that X^* is smooth, then in (ii), we may choose x_0^{**} from $E_{X^{**}}$.

Proof. From $S \in \mathscr{L}_{\mathcal{L}(X,Z)}(T)$ it follows that $T - S \perp_B \mathcal{L}(X,Z)$. Now, using Theorem 2.2 for T - S, we get $d(T - S, \mathcal{L}(X,Z)) = \sup_{x \in S_X} d(Tx - Sx, Z)$. Since $S \in \mathcal{L}(X,Z)$, we have $d(T - S, \mathcal{L}(X,Z)) = d(T, \mathcal{L}(X,Z))$ and d(Tx - Sx,Z) = d(Tx,Z). Therefore, $d(T, \mathcal{L}(X,Z)) = \sup_{x \in S_X} d(Tx,Z)$, and thus (i) holds. Similarly, from (iv) of Theorem 2.2 we conclude that (ii) holds for some $x_0^* \in M_{(T-S)^{**}} \subseteq S_{X^{**}}$. If X^* is smooth, then from (iv) of Theorem 2.2 we conclude that (ii) holds for some $x_0^* \in M_{(T-S)^{**}} \cap E_{X^{**}} \subseteq E_{X^{**}}$.

The following corollary shows that in Theorem 2.2, if we additionally assume that X is reflexive, then the supremum in (1) is attained.

Corollary 2.4. Suppose that X, Y, Z, T are as in Theorem 2.2. Moreover, assume that X is reflexive. Then for some $x_0 \in M_T$, the following hold. (i) $d(T, \mathcal{L}(X, Z)) = \sup_{x \in S_X} d(Tx, Z) = d(Tx_0, Z)$. (ii) $Tx_0 \perp_B Z^{\perp \perp}$. (iii) $T \perp_B \mathcal{L}(X, Z^{\perp \perp})$.

Proof. From (i) and (iv) of Theorem 2.2 and the fact that $T^{**} = T$ on X, we get $x_0 \in M_T$ such that

$$d(T, \mathcal{L}(X, Z)) = \sup_{x \in S_X} d(Tx, Z) = d(Tx_0, Z^{\perp \perp}).$$

Now, the proof follows from the observation that Z is canonically embedded in $Z^{\perp\perp}$ and therefore,

$$d(Tx_0, Z^{\perp \perp}) \le d(Tx_0, Z) \le \sup_{x \in S_X} d(Tx, Z).$$

Note that an important part of Theorem 2.2 depends on a special property of the extreme points of Z^* , namely if $\{z_1^*, z_2^*, \ldots, z_n^*\} \subseteq E_{Z^*}$ such that $z_i^* \neq \pm z_j^*$ for $1 \leq i \neq j \leq n$, then the set $\{z_1^*, z_2^*, \ldots, z_n^*\}$ is linearly independent. Motivated by this property of the extreme points of an $L^1(\mu)$ space, we define this property of a Banach space as L^1 -property.

Definition 2.5. Let X be a Banach space. We say that X has L^1 -property if for any given $\{x_1, x_2, \ldots, x_n\} \subseteq E_X$, with $x_i \neq \pm x_j$ for $1 \leq i \neq j \leq n$, the set $\{x_1, x_2, \ldots, x_n\}$ is linearly independent.

Now, we present another situation, where the minimax formula (1) is satisfied. Most of the arguments of the following theorem are same as in Theorem 2.2. For the sake of convenience, we give a sketch of the proof here.

Theorem 2.6. Suppose X is a Banach space such that X^{**} satisfies L^1 -property. Let Y be an arbitrary Banach space and Z be a closed subspace of Y. Let $T \in \mathcal{K}(X,Y)$ be a multi-smooth operator of finite order. Suppose that $T \perp_B \mathcal{L}(X,Z)$. Then the conditions (i)-(iv) of Theorem 2.2 hold. Moreover, in this case we may choose x_0^{**} of (ii) and (iv) from $E_{X^{**}} \cap M_{T^{**}}$.

Proof. As in Theorem 2.2, we get $\lambda_i > 0, x_i^{**} \in E_{X^{**}}, y_i^* \in E_{Y^*} \cap M_{T^*}$ for $1 \le i \le h$ such that $\sum_{i=1}^h \lambda_i = 1, x_i^{**}(T^*y_i^*) = ||T||$ and $\sum_{i=1}^h \lambda_i x_i^{**}(A^*y_i^*) = 0$ for all $A \in \mathcal{L}(X, Z)$. Since X^{**} satisfies L^1 -property, without loss of generality, we may assume that $\{x_1^{**}, \ldots, x_h^{**}\}$ is linearly independent. Choose $x^* \in \cap_{i=2}^h \ker(x_i^{**}) \setminus \ker(x_1^{**})$. Let $z \in Z$ be arbitrary. Define $A \in \mathcal{L}(X, Z)$ by $A(x) = x^*(x)z$ for all $x \in X$. Then from $\sum_{i=1}^h \lambda_i x_i^{**}(A^*y_i^*) = 0$, we get $\lambda_1 x_1^{**}(x^*)y_1^*(z) = 0$. Thus, $y_1^*(z) = 0$. Therefore, $y_1^* \in Z^{\perp}$. The rest of the proof follows proceeding similarly as in Theorem 2.2.

Remark 2.7. Recall from [9] that a Banach space W is an L^1 -predual space if and only if W^{**} is isometrically isomorphic to C(K) for some extremally disconnected compact Hausdorff space K. Thus, W^{**} is again an L^1 -predual space. Hence, in the Theorem 2.6 we may consider $X = W^*$, where W is an L^1 -predual space. Suppose that, K is a Hyperstonian space and $N(K, \mathbb{R})^+$ is set of all positive normal regular Borel measures on K. Let $N(K, \mathbb{R}) = N(K, \mathbb{R})^+ - N(K, \mathbb{R})^+$. Then from Theorem [9, Th. 10, pp 95], it follows that $N(K, \mathbb{R})^*$ is isometrically isomorphic to C(K), which is an L^1 -predual space. Therefore, Theorem 2.6 in particular holds for $X = N(K, \mathbb{R})$. In other words, if X is an abstract L_1 -space, then X^* is an abstract M-space and $X^* = C(K)$ for some compact Hausdorff space K(see [9, pp. 97]). Since C(K) is an L_1 -predual space, Theorem 2.6 holds for an abstract L_1 - space X. For the definition of abstract L_1 -space, abstract M-space and related results see [9].

Remark 2.8. Note that from Theorem 2.2 (Theorem 2.6), we get the following implications

$$T \perp_B \mathcal{L}(X, Z) \Rightarrow T^{**} x_0^{**} \perp_B Z^{\perp \perp} \Rightarrow T^{**} \perp_B \mathcal{L}(X^{**}, Z^{\perp \perp}),$$

for some $x_0^{**} \in M_{T^{**}}$. Recall from [18, Prop. 1.11.14, pp 102] that Z^{**} is isometrically isomorphic to $Z^{\perp\perp}$. Therefore, the space $\mathscr{A} = \{S^{**} : S \in \mathcal{L}(X, Z)\}$ is a subspace of $\mathcal{L}(X^{**}, Z^{\perp\perp})$. Thus, $T^{**} \perp_B \mathcal{L}(X^{**}, Z^{\perp\perp})$ yields that $T^{**} \perp_B \mathscr{A}$, i.e., $T^{**} \perp_B S^{**}$ for all $S \in \mathcal{L}(X, Z)$. Now, from the equality $||T - S|| = ||T^{**} - S^{**}||$, we get that $T \perp_B \mathcal{L}(X, Z)$. Therefore, if X, Y, Z and T satisfy the conditions of Theorem 2.2 (respectively, Theorem 2.6), then we get the following equivalence:

$$T \perp_B \mathcal{L}(X, Z) \Leftrightarrow T^{**} x_0^{**} \perp_B Z^{\perp \perp} \Leftrightarrow T^{**} \perp_B \mathcal{L}(X^{**}, Z^{\perp \perp}),$$

for some $x_0^{**} \in M_{T^{**}}$ (respectively, $x_0^{**} \in M_{T^{**}} \cap E_{X^{**}}$).

In [19, Prop. 2], Rao proved that if the supremum in (1) is attained, then $d(T^{**}, \mathcal{L}(X^{**}, Z^{\perp\perp})) = \sup_{x^{**} \in S_{X^{**}}} d(T^{**}x^{**}, Z^{\perp\perp}) = d(T, \mathcal{L}(X, Z))$. Note that, Theorem 2.2 and Theorem 2.6 provide us other situations, where such equality holds even if the supremum in (1) is not attained.

Next, we show that if X is reflexive and Z is an arbitrary closed subspace of Y, then some restriction on the norm attainment set of T yields (1).

Theorem 2.9. Let X be a reflexive Banach space and Y be an arbitrary Banach space. Suppose Z is a closed subspace of Y. Let $T \in \mathcal{K}(X,Y)$ be a multismooth operator of finite order and $M_T \cap E_X = \{\pm x_i \in S_X : 1 \le i \le n\}$, where $\{x_1, x_2, \ldots, x_n\}$ is linearly independent. Suppose that $T \perp_B \mathcal{L}(X,Z)$. Then there exists $x_0 \in M_T \cap E_X$ such that the following hold. (i) $d(T, \mathcal{L}(X,Z)) = \sup_{x \in S_X} d(Tx,Z) = d(Tx_0,Z)$.

(ii)
$$Tx_0 \perp_B Z^{\perp \perp}$$
.

(iii) $T \perp_B \mathcal{L}(X, Z^{\perp \perp}).$

Proof. Since T is a multi-smooth operator of finite order, following similar arguments as Theorem 2.2, we get scalars $\lambda_1, \lambda_2, \ldots, \lambda_h > 0$ and $f_1, f_2, \ldots, f_h \in J(T) \cap E_{\mathcal{K}(X,Y)^*}$ such that $\sum_{i=1}^h \lambda_i = 1$ and $\sum_{i=1}^h \lambda_i f_i(A) = 0$ for all $A \in \mathcal{L}(X,Z)$. Since X is reflexive, from (6) it follows that

$$E_{\mathcal{K}(X,Y)^*} = \{ y^* \otimes x : y^* \in E_{Y^*}, x \in E_X \},\$$

where $y^* \otimes x(S) = y^*(Sx)$ for each $S \in \mathcal{L}(X, Y)$. Therefore, for $1 \leq i \leq h$, $f_i = y_i^* \otimes x_i$, where $y_i^* \in E_{Y^*}, x_i \in E_X$. Now, $y_i^* \otimes x_i \in J(T)$ implies that $x_i \in M_T$ and $y_i^* \in J(Tx_i)$. Without loss of generality, we may assume that $\{x_i : 1 \leq i \leq h\}$ is linearly independent. Now, choose $x^* \in X^*$ such that $x^*(x_i) = 0$ for all $2 \leq i \leq h$ and $x^*(x_1) \neq 0$. Choose an arbitrary $z \in Z$. Consider $A \in \mathcal{L}(X, Z)$ defined as $Ax = x^*(x)z$ for all $x \in X$. Then

$$\sum_{i=1}^{h} \lambda_i f_i(A) = 0 \quad \Rightarrow \quad \sum_{i=1}^{h} \lambda_i y_i^* \otimes x_i(A) = 0$$
$$\Rightarrow \quad \sum_{i=1}^{h} \lambda_i y_i^*(Ax_i) = 0$$
$$\Rightarrow \quad \sum_{i=1}^{h} \lambda_i y_i^*(z) x^*(x_i) = 0$$
$$\Rightarrow \quad \lambda_1 y_1^*(z) x^*(x_1) = 0 \Rightarrow y_1^*(z) = 0$$

Since $z \in Z$ is chosen arbitrarily, we get $y_1^* \in Z^{\perp}$. Since for each $u \in Z^{\perp \perp}$, $u(y^*) = 0$ holds, we get

0.

$$||Tx_1 + u|| \ge |(Tx_1 + u)y_1^*| = |y_1^*(Tx_1)| = ||Tx_1||.$$

Thus, $Tx_1 \perp_B Z^{\perp\perp}$. This proves (ii). Moreover, $Tx_1 \perp_B Z^{\perp\perp}$ implies that $Tx_1 \perp_B Z$, since Z is canonically embedded in $Z^{\perp\perp}$. Now, (i) follows from the following inequality:

$$d(Tx_1, Z) = ||Tx_1|| = ||T|| = d(T, \mathcal{L}(X, Z)) \ge \sup_{x \in S_X} d(Tx, Z) \ge d(Tx_1, Z),$$

where the first inequality follows from (2). To prove (iii), let $S \in \mathcal{L}(X, Z^{\perp \perp})$. Observe that

$$||T - S|| \ge ||Tx_1 - Sx_1|| \ge ||Tx_1|| = ||T||,$$

where the second inequality follows from $Tx_1 \perp_B Z^{\perp \perp}$. Thus $T \perp_B \mathcal{L}(X, Z^{\perp \perp})$. This completes the proof of the theorem. \Box

We immediately get the next corollary due to Theorem 2.9.

Corollary 2.10. Suppose $X = \ell_1^n$, Y is a Banach space and Z is a closed subspace of Y. Assume that each non-zero element of Y is a multi-smooth point of finite order. Let $T \in \mathcal{L}(X,Y) \setminus \mathcal{L}(X,Z)$ be such that $\mathscr{L}_{\mathcal{L}(X,Z)}(T) \neq \emptyset$. Then there exists $x_0 \in E_X$ such that

$$d(T, \mathcal{L}(X, Z)) = \sup_{x \in S_X} d(Tx, Z) = d(Tx_0, Z).$$

In particular, if $T \perp_B \mathcal{L}(X, Z)$ then $Tx_0 \perp_B Z^{\perp \perp}$, for some $x_0 \in M_T \cap E_X$.

Proof. Suppose that $(0 \neq) S \in \mathcal{L}(X, Y)$. Since $X = \ell_1^n$, $M_S \cap E_X$ is of the form $\{\pm x_i : 1 \leq i \leq h\}$, where $\{x_i : 1 \leq i \leq h\}$ is linearly independent. Moreover, $Sx_i \in Y$ is a multi-smooth point of finite order for each *i*. Therefore, using [16, Cor. 2.3] we get S is a multi-smooth operator of finite order. Thus, each non-zero operator of $\mathcal{L}(X,Y)$ satisfies the hypothesis of Theorem 2.9. Now, suppose that $S_0 \in \mathscr{L}_{\mathcal{L}(X,Z)}(T)$. Then $T - S_0 \perp_B \mathcal{L}(X,Z)$. Note that, since $T \notin \mathcal{L}(X,Z), T - S_0 \neq 0$. Now, using Theorem 2.9, we get $x_0 \in E_X$ such that

$$d(T - S_0, \mathcal{L}(X, Z)) = \sup_{x \in S_X} d(Tx - S_0 x, Z) = d(Tx_0 - S_0 x_0, Z)$$

$$\Rightarrow \quad d(T, \mathcal{L}(X, Z)) = \sup_{x \in S_X} d(Tx, Z) = d(Tx_0, Z).$$

On the other hand, if $T \perp_B \mathcal{L}(X, Z)$, then from (ii) of Theorem 2.9, it follows that $Tx_0 \perp_B Z^{\perp \perp}$ for some $x_0 \in M_T \cap E_X$. This completes the proof. \Box

Note that, in particular, Corollary 2.10 holds for a smooth Banach space Y. Furthermore, from [12, Cor. 2.2], it follows that every nonzero operator of $\mathcal{K}(H_1, H_2)$ is a multi-smooth operator of finite-order, where H_1, H_2 are Hilbert spaces. For other examples of Banach spaces Y, where each non-zero element is a multi-smooth point of finite order see [8, 11]. As a consequence of the results obtained here, we get the following corollary providing sufficient condition for proximinality of a subspace.

Corollary 2.11. Suppose $X = \ell_1^n$, Y is a Banach space and Z is a closed subspace of Y. Assume that each non-zero element of Y is a multi-smooth point of finite order. Suppose that $\mathcal{L}(X, Z)$ is a proximinal subspace of $\mathcal{L}(X, Y)$. Then Z is a proximinal subspace of Y.

Proof. Let $y \in Y \setminus Z$. Choose $x^* \in E_{X^*}$. Let $T \in \mathcal{L}(X,Y) \setminus \mathcal{L}(X,Z)$ be defined as $Tx = x^*(x)y$ for all $x \in X$. From the proximinality of $\mathcal{L}(X,Z)$ in $\mathcal{L}(X,Y)$, it follows that $\mathscr{L}_{\mathcal{L}(X,Z)}(T) \neq \emptyset$. Let $S \in \mathscr{L}_{\mathcal{L}(X,Z)}(T)$. Then $T - S \perp_B \mathcal{L}(X,Z)$. Therefore, using Corollary 2.10, we get $Tx_0 - Sx_0 \perp_B Z$ for some $x_0 \in E_X$. Observe that $|x^*(x_0)| = 1$, since $X = \ell_1^n$. Thus, $x^*(x_0)y - Sx_0 \perp_B Z$, which yields that $\frac{1}{x^*(x_0)}Sx_0 \in \mathscr{L}_Z(y)$. This completes the proof.

We would like to end the paper with the remark that Corollary 2.11 is motivated from [19, Cor. 5]. However, in [19, Cor. 5], one of the assumptions is that Z is an M-ideal in Y, which itself is a sufficient condition for the proximinality of Z in Y (see [6, Prop. 1.1, pp 50]). Here we emphasize that in Corollary 2.11, Z is assumed to be an arbitrary closed subspace of Y.

10

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Kolkata 700032, India.

Email address: arpitamalju@gmail.com