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Algebraic relations between partition functions and the j-function

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Abstract

We obtain identities and relationships between the modular j-function, the generating functions for the classical partition function and the Andrews *spt*-function, and two functions related to unimodal sequences and a new partition statistic we call the "signed triangular weight" of a partition. These results follow from the closed formula we obtain for the Hecke action on a distinguished harmonic Maass form $\mathscr{M}(\tau)$ defined by Bringmann in her work on the Andrews *spt*-function. This formula involves a sequence of polynomials in $j(\tau)$, through which we ultimately arrive at expressions for the coefficients of the j-function purely in terms of these combinatorial quantities.

Keywords: Partitions, Harmonic Maass forms, Modular forms, Spt function

1 Introduction and statement of results

Partitions, first and foremost combinatorial objects, permeate seemingly disparate areas of mathematics. The partition function $p(n)$ gives the number of ways to write n as the sum of unordered positive integers. The generating function for $p(n)$ is a weakly holomorphic modular form of weight $-1/2$, namely

$$\mathscr{P}(q) := \sum_{n\geq 0} p(n)q^{24n-1} = q^{-1}\prod_{n\geq 1}\frac{1}{1-q^{24n}} = \frac{1}{\eta(24\tau)}, \tag{1.1}$$

where $\eta(\tau)$ is Dedekind's eta-function and we use the convention $q = e^{2\pi i\tau}$. This is one indication of partitions' deep ties to number theory. Outside combinatorics and number theory, perhaps the most prominent role for partitions is in representation theory, where the theory of Young tableaux for partitions encodes the irreducible representations of all symmetric groups [12, Theorem 2.1.11].

Other modular forms and functions that were first studied in number theory have likewise appeared in the representation theory of finite groups. In particular, the modular j-function, whose Fourier expansion is

$$j(\tau) = \sum_{n\geq -1} c(n)q^n = q^{-1} + 744 + 196884q + 21493760q^2 + \cdots, \tag{1.2}$$

is well-known in number theory because the j-invariants, i.e. the values of $j(\tau)$ for $\tau \in \mathcal{H}$, parametrize isomorphism classes of elliptic curves over $\mathbb{C}$ [16, Proposition 12.11].

McKay famously observed that the first few coefficients of $j(\tau)$ satisfy striking relations such as

$$\begin{aligned} c(1) &= 196884 = 1 + 196883, \\ c(2) &= 21493760 = 1 + 196883 + 21296876, \end{aligned} \tag{1.3}$$

where the right-hand sides are linear combinations of dimensions of irreducible representations of the monster group M. Such expressions inspired Thompson to conjecture [17] that there is a monstrous moonshine module, an infinite-dimensional graded M-module $V^\natural = \bigoplus_{n \gg -1} V_n$ such that for $n \geq -1$, we have

$$c(n) = \dim(V_n).$$

Thompson further conjectured that, since the graded dimension is the graded trace of the identity element of M, the traces of other elements g may likewise be related to naturally-occuring q-series. This was refined by Conway and Norton in [11], who conjectured that for every element $g \in M$, the McKay-Thompson series

$$T_g(\tau) := \sum_{n=-1}^{\infty} \mathrm{Tr}(g|V_n)q^n$$

is the Hauptmodul which generates the function field for a genus 0 modular curve for a particular congruence subgroup $\Gamma_g \subset \mathrm{SL}_2(\mathbb{R})$. Borcherds proved the Conway–Norton conjecture for the Monster Moonshine Module in [6], an impactful result which, in part, solidifies the j-function's connection to the representation theory of M.

Since the j-function and partitions appear in both number theory and representation theory, one can ask if there is a relation between $c(n)$ and $p(n)$. In this paper, we discover that the coefficients of the Fourier expansion of both the j-function and a certain sequence of polynomials in j have a combinatorial description in terms of partitions of integers and unimodal sequences. This suggests the possibility of deeper connections between the representation theory of the symmetric group and the monster Lie algebra.

This research is inspired by recent work of Andrews [2] in which he defined $spt(n)$ to count the number of smallest parts among all integer partitions of n. For example, we can determine that $spt(4) = 10$ by counting the following underlined parts across all five partitions of 4:

$$\underline{4} = 3 + \underline{1} = \underline{2} + \underline{2} = 2 + \underline{1} + \underline{1} = \underline{1} + \underline{1} + \underline{1} + \underline{1}.$$

Following the notation of [15], we define a renormalized generating function for $spt(n)$ as

$$\mathcal{S}(q) := \sum_{n \geq 1} spt(n) q^{24n-1}. \tag{1.4}$$

Paralleling Ramanujan's notable congruences

$$\begin{aligned} p(5n+4) &\equiv 0 \pmod{5}, \\ p(7n+5) &\equiv 0 \pmod{7}, \\ p(11n+6) &\equiv 0 \pmod{11}, \end{aligned}$$

Andrews [2] showed that the *spt* function satisfies the congruences

$$\begin{aligned} spt(5n+4) &\equiv 0 \pmod 5, \\ spt(7n+5) &\equiv 0 \pmod 7, \\ spt(13n+6) &\equiv 0 \pmod{13}. \end{aligned}$$

Of further interest are the *spt*-function's rich families of congruences modulo all primes $\ell \geq 5$. As Ono proved in [15], if $\ell \geq 5$ is prime, $n \geq 1$, and $\left(\frac{-n}{\ell}\right) = 1$, then

$$spt\left(\frac{\ell^2 n - 1}{24}\right) \equiv 0 \pmod{\ell}. \tag{1.5}$$

Subsequent work by Ahlgren et al. [1] extended these congruences to arbitrary powers of ℓ. If $m \geq 1$, then

$$spt\left(\frac{\ell^{2m} n + 1}{24}\right) \equiv 0 \pmod{\ell^m}. \tag{1.6}$$

These congruences follow from studying a distinguished harmonic Maass form $\mathscr{M}(\tau)$ defined by Bringmann in [7] (see (2.1)). For background on harmonic Maass forms, we refer the reader to [8,14]. The function $\mathscr{M}(\tau)$ is of particular interest because its holomorphic part $M^+(\tau)$ involves the generating functions for both $p(n)$ and $spt(n)$; namely we have

$$M^+(\tau) = \mathcal{S}(q) + \frac{1}{12} q \frac{d}{dq} \mathscr{P}(q). \tag{1.7}$$

For weight 3/2 harmonic Maass forms with Nebentypus $\chi_{12} := \left(\frac{12}{\cdot}\right)$, we follow the normalization given in [15] to define the Hecke operators $T(\ell^2)$ of index ℓ^2 on a power series $f(\tau) = \sum_{n \gg -\infty} a(n) q^n$ by

$$f(\tau) \mid T(\ell^2) := \sum_{n \gg -\infty} \left[a(\ell^2 n) + \left(\frac{3}{\ell}\right)\left(\frac{-n}{\ell}\right) a(n) + \ell a(n/\ell^2) \right] q^n. \tag{1.8}$$

The congruences in (1.5) and (1.6) follow from the fact that

$$M^+(\tau) \mid T(\ell^2) \equiv \left(\frac{3}{\ell}\right) M^+(\tau) \pmod{\ell}. \tag{1.9}$$

Ono asked whether there exist explicit identities which imply (1.9). We answer this question. Using the standard notation $(q;q)_\infty := \prod_{n \geq 1}(1-q^n)$, we define a sequence of monic integer polynomials $B_m(x)$ of degree $(m-1)$ by

$$\begin{aligned} \mathcal{B}(x,q) = \sum_{m \geq 1} B_m(x) q^m &:= (q;q)_\infty \cdot \frac{1}{j(\tau) - x} \\ &= q + (x - 745)q^2 + (x^2 - 1489x + 357395)q^3 + \cdots. \end{aligned} \tag{1.10}$$

In terms of the Eisenstein series $E_4(\tau)$ and $E_6(\tau)$, as well as Ramanujan's Delta function $\Delta(\tau)$, we offer the following solution to Ono's problem.

Theorem 1.1 *If $\ell \geq 5$ is a prime and $\delta_\ell := \frac{\ell^2-1}{24}$, then*

$$M^+(\tau) \mid_{3/2} T(\ell^2) = \left(\frac{3}{\ell}\right)(1+\ell)M^+(\tau) - \frac{\ell}{12}\mathscr{P}(q) \cdot B_{\delta_\ell}(j(24\tau)) \cdot \frac{E_4^2(24\tau)E_6(24\tau)}{\Delta(24\tau)}.$$

Remark We note that the identity in the theorem immediately reduces to (1.9) modulo ℓ. Moreover, this result gives an expression for the Hecke action in terms of only the original mock modular form and the coefficient of $q^{-\ell^2}$ produced by the Hecke operator. Therefore, the resulting mock modular form is determined by a single term.

For notational clarity, we note that

$$-q\frac{d}{dq}j(\tau) = \frac{E_4^2(\tau)E_6(\tau)}{\Delta(\tau)} = q^{-1} - \sum_{n\geq 1} nc(n)q^n = q^{-1} - 196884q - 42987520q^2 + \cdots.$$

Thus, $B_{\delta_\ell}(j(24\tau)) \cdot \frac{E_4^2(24\tau)E_6(24\tau)}{\Delta(24\tau)}$ is completely determined by the coefficients of j. For convenience, we write

$$-q\frac{d}{dq}j(24\tau) = q^{-24} - 196884q^{24} - 42987520q^{48} + \cdots.$$

Example Here we illustrate Theorem 1.1 for the primes 5, 7, and 11. In the notation of [15], we define

$$M_\ell(\tau) := M^+(\tau) \mid_{3/2} T(\ell^2) - \left(\frac{3}{\ell}\right)(1+\ell)M^+(\tau). \tag{1.11}$$

For $\ell = 5$, note that $\delta_5 = 1$ and $B_1(x) = 1$. Therefore, we find that

$$M_5(\tau) = \frac{5}{12}\mathscr{P}(q) \cdot q\frac{d}{dq}j(24\tau) = -\frac{5}{12}q^{-25} - \frac{5}{12}q^{-1} + \frac{492205}{6}q^{23} + \cdots.$$

For $\ell = 7$, $\delta_7 = 2$ and $B_2(x) = x - 745$. Therefore, we have

$$M_7(\tau) = \frac{7}{12}\mathscr{P}(q) \cdot (j(24\tau) - 745) \cdot q\frac{d}{dq}j(24\tau) = -\frac{7}{12}q^{-49} - \frac{7}{12}q^{-1} + \frac{149078125}{12}q^{23} + \cdots.$$

For $\ell = 11$, $\delta_{11} = 5$ and

$$B_5(x) = x^4 - 2977x^3 + 2732795x^2 - 812685832x + 4947668669.$$

Therefore, we have

$$M_{11}(\tau) = \frac{11}{12}\mathscr{P}(q) \cdot B_5(j(24\tau)) \cdot q\frac{d}{dq}j(24\tau) = -\frac{11}{12}q^{-121} + \frac{11}{12}q^{-1} + \cdots.$$

In view of (1.11), the case $\ell = 5$ gives an expression for $M^+(\tau) \mid T(25)$ in terms of the coefficients $c(n)$ of the j-function, thus deriving an unexpected relationship between these coefficients and the values of $p(n)$ and $spt(n)$. Namely, we offer the following partition-theoretic counterparts to (1.3):

$$\begin{aligned} c(1) &= 196884 = 2 + 49 + 15708 + 181125, \\ c(2) &= 21493760 = \frac{1}{2}\Big(1 - 49 + 182 - 15708 - 181125 + 2405844 + 40778375\Big). \end{aligned}$$

The two identities above are examples of a more general theorem. To make this precise, it is important to illustrate how the summands above correspond to $p(n)$ and $spt(n)$. We require the following notation. For $n \geq 1$, we define

$$\begin{aligned} h_1(24n-1) &:= \frac{12}{5} spt(25n-1) + 5(24n-1)p(25n-1) \\ &\qquad + \mu_n \cdot \Big(\frac{12}{5} spt(n) + 5(24n-1)p(n)\Big), \\ h_2(25(24n-1)) &:= 12 spt(n) + (24n-1)p(n), \end{aligned} \tag{1.12}$$

where $\mu_n := 6 - \left(\frac{1-24n}{5}\right)$. We define $h_1(m) = 0$ if $m \not\equiv 23 \bmod 24$ and $h_2(m) = 0$ if $m \not\equiv 23 \bmod 24$ or if $m \not\equiv 0 \bmod 25$. We will also need the following function. For $n = 1$, we set $s(n) = 2$, and for $n > 1$, let

$$s(n) := \begin{cases} (-1)^{k+1} & \text{if } 24n = (6k+1)^2 - 25 \text{ or } 24n = (6k+1)^2 - 1 \text{ for some } k \in \mathbb{Z}, \\ 0 & \text{otherwise.} \end{cases}$$

Remark It is an easy exercise to confirm $s(n)$ is well-defined.

Then we have the following result.

Theorem 1.2 *If $n \geq 1$, then*

$$c(n) = \frac{s(n)}{n} + \frac{1}{n} \sum_{k \in \mathbb{Z}} \Big[(-1)^k h_1(24n - (6k+1)^2) + (-1)^k h_2(24n - (6k+1)^2)\Big].$$

Remark The formula in Theorem 1.2 bears a strong resemblance to another well-known expression for the coefficients of j. Work of Kaneko [13] shows for $n \geq 1$ that

$$c(n) = \frac{1}{n} \sum_{r \in \mathbb{Z}} \left[\mathbf{t}(n - r^2) - \frac{(-1)^{n+r}}{4} \mathbf{t}(4n - r^2) + \frac{(-1)^r}{4} \mathbf{t}(16n - r^2)\right], \tag{1.13}$$

where $\mathbf{t}$ are traces of singular moduli, i.e. the sums of the j-invariants of elliptic curves with complex multiplication. In view of the similarity of these expressions, it is natural to wonder whether Theorem 1.2 suggests a deep connection between partitions and traces of singular moduli.

In [3], Andrews related $spt(n)$ to a number of other combinatorial and number-theoretic functions. One connection of particular interest is the relationship of spt to strongly unimodal sequences. We ask whether this relationship reveals deeper connections to the j-function and representation theory.

A sequence of integers $\{a_k\}_{k=1}^s$ is a strongly unimodal sequence of size n if $\sum_{k=1}^s a_k = n$ and for some r it satisfies $0 < a_1 < a_2 < \cdots < a_r > a_{r+1} > a_{r+2} > \cdots > a_s > 0$. The rank of $\{a_k\}_{k=1}^s$ is $s - 2r + 1$, the number of terms after the maximal term minus the number of terms preceding it. The function $U(t;q)$ counts specific types of strongly unimodal sequences [10]. For $t = -1$,

$$U(-1;q) = \sum_{n\geq 1} u^*(n)q^n = q + q^2 - q^3 - 2q^5 + 2q^6 + \cdots,$$

where $u^*(n)$ is the difference of the number of even-rank strongly unimodal sequences of size n and the number of odd-rank strongly unimodal sequences of size n. Andrews proved in [3] that

$$U(-1;q) = -\sum_{n\geq 1} spt(n)q^n + 2A(q), \tag{1.14}$$

where

$$\begin{aligned} A(q) = \sum_{n\geq 1} a(n)q^n &:= \frac{1}{(q;q)_\infty} \sum_{n=1}^{\infty} \frac{(-1)^{n-1} n q^{\frac{n^2+n}{2}}}{1-q^n} \\ &= q + q^2 - q^3 + q^4 - q^5 + 4q^6 + \cdots. \end{aligned}$$

It is natural to ask what $A(q)$ is counting. We find that $A(q)$ is the generating function for a partition statistic that we call the "signed triangular weight" of a partition, a result which is of independent interest. Given a partition $\lambda \vdash N$, where we write the size of the partition as $|\lambda| := N$, let n_λ be the maximal number such that λ contains parts of size $1, 2, \ldots, n_\lambda$. Letting m_k denote the number of times that the part k appears in λ, we define the signed triangular weight of λ to be $t_s(\lambda) := \sum_{k=1}^{n_\lambda} (-1)^{k-1} k m_k$. If λ does not contain a part of size 1, then let $t_s(\lambda) = 0$.

Example Consider $\lambda = \{1, 2, 2, 3, 4, 5, 5, 8\}$. Then $\lambda \vdash 30$, $n_\lambda = 5$, and

$$t_s(\lambda) = 1 \cdot 1 - 2 \cdot 2 + 3 \cdot 1 - 4 \cdot 1 + 5 \cdot 2 = 6.$$

We prove the following result relating $t_s(\lambda)$ for all partitions λ of all positive integers to the series $A(q)$.

Theorem 1.3 *The following q-series identity is true:*

$$A(q) = \sum_{\lambda} t_s(\lambda) q^{|\lambda|}.$$

From this, we may conclude that $a(n) = \sum_{|\lambda|=n} t_s(\lambda)$. Given this relationship, the *spt* congruence given in (1.6) immediately implies the following result.

Corollary 1.4 *If* $\ell \geq 5$ *is prime,* $\left(\frac{-n}{\ell}\right) = 1$*, and* $m \geq 1$*, then*

$$u^*\left(\frac{\ell^{2m}n - 1}{24}\right) \equiv 2a\left(\frac{\ell^{2m}n - 1}{24}\right) \pmod{\ell^m}.$$

Combining our explicit expression for the action of the Hecke operator $T(25)$ in Theorem 1.1 and our combinatorial expressions for $c(n)$, we arrive at new expressions for the coefficients of $j(\tau)$ in terms of $p(n)$ and the coefficients of $a(n)$ and $u^*(n)$. For ease of notation, we define the functions

$$\begin{aligned} g_1(24n-1) &:= -\frac{12}{5}u^*(25n-1) + \frac{24}{5}a(25n-1) + 5(24n-1)p(25n-1) \\ &\quad + \mu_n \cdot \left(-\frac{12}{5}u^*(25n-1) + \frac{24}{5}a(25n-1) + 5(24n-1)p(n)\right), \\ g_2(25(24n-1)) &:= -12u^*(n) + 24a(n) + (24n-1)p(n), \end{aligned} \tag{1.15}$$

where as in (1.12), $g_1(m) = 0$ if $m \not\equiv 23 \bmod 24$ and $g_2(m) = 0$ if $m \not\equiv 23 \bmod 24$ and $m \not\equiv 0 \bmod 25$.

Corollary 1.5 *If $n \geq 1$, then*

$$c(n) = \frac{s(n)}{n} + \frac{1}{n}\sum_{k\in\mathbb{Z}} \Big[(-1)^k g_1(24n-(6k+1)^2) + (-1)^k g_2(24n-(6k+1)^2)\Big].$$

Example Using our result, we find the following identities:

$$\begin{aligned} c(1) = 196884 &= s(24) + \frac{168a(1) - 84u^*(1) + 161p(1)}{5} \\ &\quad + \frac{24}{5}a(24) - \frac{12}{5}u^*(24) + 115p(24) \\ c(2) = 21493760 &= \frac{1}{2}\Big(s(48) - \frac{168a(1) - 84u^*(1) + 161p(1)}{5} \\ &\quad + \frac{14a(2) - 7u^*(2) + 329p(2)}{5} \\ &\quad - \frac{24}{5}a(24) + \frac{12}{5}u^*(24) - 115p(24) + \frac{24}{5}a(49) - \frac{12}{5}u^*(49) + 235p(49)\Big). \end{aligned}$$

Question 1 Are the combinatorial interpretations of the coefficients of the j-function in Theorem 1.2 and Corollary 1.5 glimpses of hidden structure of the monster module? In particular, do $spt(n)$, $u^*(n)$, and $a(n)$ play roles in representation theory?

Remark After this paper was submitted, T. Matsusaka (Private communication, 2019) informed the authors that he has obtained further similar results along these lines which frame the *spt* function in terms of a weakly holomorphic Jacobi form. This structure also provides a connection to the formulation by Kaneko [13] of the j-function's coefficients using traces of singular moduli.

This paper is organized as follows. In Sect. 2, we investigate the specific harmonic Maass form $\mathscr{M}(\tau)$ and derive an expression for the action of the Hecke operator on its holomorphic part. To do this, we study canonical families of polynomials in $j(\tau)$ and explore the relationship of modular forms to modular functions on $\mathrm{SL}_2(\mathbb{Z})$. In Sect. 3, we prove Theorem 1.3. In Sect. 4, we prove Theorem 1.2 and Corollary 1.5.

2 Harmonic maass forms

2.1 Preliminaries

To motivate our study and to ground the methods used here, we begin by introducing the harmonic Maass form of interest for this paper. Recall that a weakly holomorphic modular form for a congruence subgroup Γ of $\mathrm{SL}_2(\mathbb{Z})$ is a function that is holomorphic on $\mathcal{H}$, whose poles, if any, are supported on the cusps of $\Gamma\backslash\mathcal{H}$, and which satisfies the corresponding modularity properties for its weight. If f is a weakly holomorphic modular form of weight k for Γ and Nebentypus χ, we write $f \in M_k^!(\Gamma, \chi)$.

Likewise, a smooth function $f : \mathcal{H} \to \mathbb{C}$ is a harmonic Maass form of weight k for Γ and χ if it satisfies the standard modular transformation laws, is annihilated by the harmonic Laplacian Δ_k, and has at most growth-order 1 exponential growth at each cusp on $\Gamma\backslash\mathcal{H}$. We denote the vector space of harmonic Maass forms of weight k for Γ and χ as $H_k(\Gamma, \chi)$.

Recalling the definitions of $\mathscr{P}(q)$ and $\mathcal{S}(q)$ in (1.1) and (1.4), we define $\mathscr{M}(\tau)$ following [15] as

$$\mathscr{M}(\tau) := \mathcal{S}(q) + \frac{1}{12} q \frac{d}{dq}\mathscr{P}(q) - \frac{i}{4\pi\sqrt{2}} \cdot \int_{-\bar{\tau}}^{i\infty} \frac{\eta(24z)}{[-i(z+\tau)]^{3/2}}\, dz. \tag{2.1}$$

By Theorem 2.1 of [15], $\mathscr{M}(\tau) \in H_{3/2}(\Gamma_0(576), \chi_{12})$, where $\chi_{12} := \left(\frac{12}{\cdot}\right)$. By $M^+(q)$ we denote the holomorphic part of $\mathscr{M}(\tau)$. This may be expressed as

$$M^+(q) := \mathcal{S}(q) + \frac{1}{12} q \frac{d}{dq}\mathscr{P}(q) = -\frac{1}{12}q^{-1} + \frac{35}{12}q^{23} + \frac{65}{6}q^{47} + \cdots.$$

2.2 The Hecke action

To understand the action of the Hecke operator on M^+, we will need the following result that produces a weakly holomorphic modular form involving $M^+(\tau) \mid T(\ell^2)$. We produce this modular form via the following result.

Lemma 2.1 *If*

$$M_\ell(\tau) := M^+(\tau) \mid T(\ell^2) - \left(\frac{3}{\ell}\right)(1+\ell)M^+(\tau),$$

then $M_\ell(\tau) \in M_{3/2}^!(\Gamma_0(576), \chi_{12})$.

Proof Up to a constant, the nonholomorphic part of $\mathscr{M}(\tau)$ is the period integral for $\eta(24\tau)$. Write $\tau = x + iy$ for $x, y \in \mathbb{R}$. Under the action of the differential operator $\xi_k := 2iy^k \frac{\overline{\partial}}{\partial\bar{\tau}}$, we have $\xi_{3/2}(\mathscr{M}) = -\frac{1}{8\pi}\eta(24\tau)$. Note that $\eta(24\tau)$ is an eigenform for Hecke operators of weight $1/2$ with eigenvalue $\chi_{12}(\ell)(1+\ell^{-1}) = (\frac{3}{\ell})(1+\ell^{-1})$. If we define

$$\mathscr{M}_\ell(\tau) := \mathscr{M}(\tau) \mid T(\ell^2) - \left(\frac{3}{\ell}\right)(1+\ell)\mathscr{M}(\tau),$$

we observe that

$$\xi_{3/2}\left[\mathscr{M}(\tau) \mid T(\ell^2) - \left(\frac{3}{\ell}\right)(1+\ell)\mathscr{M}(\tau)\right] = 0. \tag{2.2}$$

Here we have used the commutativity relation

$$\xi_k(f \mid_k T(\ell^2)) = \ell^{2k-2}(\xi_k f) \mid_{2-k} T(\ell^2)$$

for half-integral weight harmonic Maass forms given in Proposition 7.1 of [9]. Since the Hecke algebra preserves modularity, $\mathscr{M}(\tau) \mid T(\ell^2) \in H_{3/2}(\Gamma_0(576), \chi_{12})$. By (2.2), $\mathscr{M}_\ell(\tau)$ is in the kernel of $\xi_{3/2}$ and is therefore holomorphic on the upper half plane. Since the action of the Hecke and ξ operators are linear and thus split over the holomorphic and nonholomorphic parts of $\mathscr{M}(\tau)$, the same result holds for $M^+(\tau)$. In particular, $M_\ell(\tau) \in M^!_{3/2}(\Gamma_0(576), \chi_{12})$. □

2.3 Canonical polynomials in $j(\tau)$

We show that the set of all $B_m(j(\tau))$ form a convenient $\mathbb{C}$-basis for the ring of weakly holomorphic modular functions on $\mathrm{SL}_2(\mathbb{Z})$ as a $\mathbb{C}$-vector space. Recall that the ring of weakly holomorphic modular functions on $\mathrm{SL}_2(\mathbb{Z})$ is precisely the ring of complex polynomials in $j(\tau)$, i.e. $M^!_0(\mathrm{SL}_2(\mathbb{Z})) = \mathbb{C}[j(\tau)]$ [4, Theorem 2.8]. As defined in (1.10), we have

$$\begin{aligned} B_1(x) &= 1, \\ B_2(x) &= x - 745, \\ B_3(x) &= x^2 - 1489x + 357395. \end{aligned}$$

From these first few examples, the set of $B_m(x)$ appears to form a $\mathbb{C}$-basis for the polynomial ring $\mathbb{C}[x]$ as a $\mathbb{C}$-vector space, and hence the set of $B_m(j(\tau))$ would form a $\mathbb{C}$-basis for $M^!_0(\mathrm{SL}_2(\mathbb{Z}))$. In the following lemma, we show that this is indeed the case. To do so, we define the function

$$\alpha(q) := \frac{(q;q)_\infty}{-q\frac{d}{dq}j(\tau)} = q + O(q^2). \tag{2.3}$$

Lemma 2.2 *If $f(\tau)$ is a weakly holomorphic modular function on $\mathrm{SL}_2(\mathbb{Z})$ and is of the form*

$$f(\tau) = \alpha(q)\left(\sum_{n\gg-\infty}^{-1} t(n)q^n\right) + O(q), \tag{2.4}$$

then

$$f(\tau) = \sum_{n\gg-\infty}^{-1} t(n)B_{-n}(j(\tau)).$$

Remark The above lemma gives a clean formulation for modular functions f of the form given in (2.4) when the principal part of f/α is known.

Proof of Lemma 2.2 For each $m \geq 0$, note that there exists a unique weakly holomorphic modular function $j_m(\tau)$ on $\mathrm{SL}_2(\mathbb{Z})$ such that $j_m(\tau) = q^{-m} + O(q)$. The Faber polynomials $J_n(x)$ are the coefficients of the generating function

$$\sum_{n=0}^{\infty} J_n(x)q^n := \frac{E_4^2(\tau)E_6(\tau)}{\Delta(\tau)} \cdot \frac{1}{j(\tau)-x} = 1 + (x-744)q + \cdots.$$

By Corollary 4 in [5], $J_n(j(\tau)) = j_n(\tau)$ for all $n \geq 0$. By comparing the generating functions for $J_n(x)$ and $B_n(x)$ and using the identity (1.10), we see that

$$\alpha(q)\sum_{n\geq 0} J_n(x)q^n = \alpha(q)\cdot\frac{E_4^2(\tau)E_6(\tau)}{\Delta(\tau)}\cdot\frac{1}{j(\tau)-x} = (q;q)_\infty\cdot\frac{1}{j(\tau)-x} = \sum_{n\geq 1} B_n(x)q^n.$$

Since $\alpha(q) = q + O(q^2)$, we compare coefficients and deduce that for each $n \geq 1$,

$$\alpha(q)J_n(j(\tau)) = B_n(j(\tau)) = \alpha(q)q^{-n} + O(q).$$

And hence we can conclude that

$$f(\tau) = \alpha(q)\left(\sum_{n\gg-\infty}^{-1} t(n)q^n\right) + O(q) = \sum_{n\gg-\infty}^{-1} t(n)B_{-n}(j(\tau)).$$

2.4 Proof of Theorem 1.1

Note that we may write

$$M_\ell(z) = -\frac{\ell}{12}q^{-\ell^2} + \left(\frac{3}{\ell}\right)\frac{\ell}{12}q^{-1} + \sum_{\substack{n\geq 23\\ n\equiv 23 \bmod 24}} a_\ell(n)q^n, \tag{2.5}$$

where we observe that, since $\ell^2 \equiv 1 \bmod 24$, the nonzero coefficients of M_ℓ are supported on integral exponents that are 23 mod 24. Following this, we define

$$F_\ell(24\tau) := \eta^{\ell^2}(24\tau)M_\ell(\tau). \tag{2.6}$$

By Lemma 2.1, it is immediate that $F_\ell(24\tau)$ is a weakly holomorphic modular form of weight $\frac{\ell^2+3}{2}$ over $\Gamma_0(576)$ with trivial Nebentypus. In fact, by Theorem 2.2 in [15], $F_\ell(\tau)$ is a weight $\frac{\ell^2+3}{2}$ holomorphic modular form on $\mathrm{SL}_2(\mathbb{Z})$. We recall that the proof makes use of the observation that $F_\ell \in \mathbb{Z}[[q^{24}]]$ by construction, and that the behavior of F_ℓ under the matrix $S = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$ can be determined using a result of Bringmann in [7] which gives that $\mathscr{M}(\tau)$ is an eigenform of the Fricke involution.

2.4.1 Getting to Weight 0

Now that we have a holomorphic modular form of weight $\frac{\ell^2+3}{2}$ on all of $\mathrm{SL}_2(\mathbb{Z})$, we will leverage this information, along with some properties of the Eisenstein series E_{14} and the j-function, to produce a closed formula for the Hecke action. We first note that $\frac{\ell^2+3}{2} \equiv 2 \bmod 12$, and that likewise so is $E_4^2(\tau)E_6(\tau)$. To make use of this seemingly innocuous fact, define $\delta_\ell := \frac{\ell^2-1}{24}$ and note that

$$G_\ell(\tau) := E_4^2(\tau)E_6(\tau)\Delta^{\delta_\ell-1}(\tau) = q^{\delta_\ell-1} + \ldots \in M_{\frac{\ell^2+3}{2}}(\mathrm{SL}_2(\mathbb{Z})). \tag{2.7}$$

Since we now have another modular form of the same weight on $\mathrm{SL}_2(\mathbb{Z})$, we would like to prove that their quotient, $F_\ell(\tau)/G_\ell(\tau)$, is a weakly holomorphic modular function on $\mathrm{SL}_2(\mathbb{Z})$, which, coupled with our preceding characterization of the Faber polynomials, will allow for a unique expression of the quotient as a polynomial in $j(\tau)$.

Lemma 2.3 *The function $F_\ell(\tau)/G_\ell(\tau)$ is a polynomial in $j(\tau)$.*

Proof By construction, G_ℓ has a zero of degree 2 at $e^{2\pi i/3}$, a simple zero at i, and no other zeros in the fundamental domain $\mathcal{F}$ of $\mathrm{SL}_2(\mathbb{Z})$.

Since the weight of F_ℓ is $k = (\ell^2+3)/2 \equiv 2 \bmod 12$, we apply the transformation law under $S = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$ to get that $F_\ell(-1/i) = i^k F_\ell(i) = -F_\ell(i)$, hence $F_\ell(i) = 0$. Similarly, applying the transformation law under $\gamma = \begin{pmatrix} 0 & -1 \\ 1 & 1 \end{pmatrix}$ yields $F_\ell(e^{2\pi i/3}) = 0$. Differentiating both sides of $F_\ell(\gamma\tau) = (\tau+1)^k F_\ell(\tau)$ and letting $\tau = e^{2\pi i/3}$ gives that $\frac{d}{d\tau}F_\ell(\tau)|_{\tau=e^{2\pi i/3}} = 0$. Hence F_ℓ vanishes at $e^{2\pi i/3}$ with order at least 2. Therefore the quotient F_ℓ/G_ℓ has no poles in $\mathcal{F}$, and we may deduce that F_ℓ/G_ℓ is a weakly holomorphic modular form of weight 0 on $\mathrm{SL}_2(\mathbb{Z})$. Since the modular functions on $\mathrm{SL}_2(\mathbb{Z})$ are precisely the polynomials $\mathbb{C}[j(\tau)]$, we may conclude that F_ℓ/G_ℓ is a polynomial in $j(\tau)$. □

It remains to construct this polynomial in $j(\tau)$. Using the modular functions $B_{\delta_\ell}(j(24\tau))$, we arrive at the following conclusion:

$$\begin{aligned}
\frac{F_\ell(\tau)}{G_\ell(\tau)} &= \frac{\eta(\tau)^{\ell^2} M_\ell(\tau/24)}{E_4^2(\tau)E_6(\tau)\Delta^{\delta_\ell-1}(\tau)} \\
&= \frac{(q;q)_\infty}{-q\frac{d}{dq}j(\tau)} q^{1/24} M_\ell(\tau/24) \\
&= \frac{(q;q)_\infty}{-q\frac{d}{dq}j(\tau)}\left[-\frac{\ell}{12}q^{-\delta_\ell} + \left(\frac{3}{\ell}\right)\frac{\ell}{12} + O(q)\right] \\
&= \alpha(q)\left[-\frac{\ell}{12}q^{-\delta_\ell} + \left(\frac{3}{\ell}\right)\frac{\ell}{12} + O(q)\right] \\
&= -\frac{\ell}{12}B_{\delta_\ell}(j(\tau)),
\end{aligned}$$

where the last equality follows from Lemma 2.2. Hence we may rearrange to get the expression

$$M_\ell(\tau/24) = \frac{F_\ell(\tau)}{\eta^{\ell^2}(\tau)} = -\frac{\ell}{12}\frac{E_4^2(\tau)E_6(\tau)}{\Delta(\tau)}\eta^{-1}(\tau)B_{\delta_\ell}(j(\tau)). \tag{2.8}$$

Sending $\tau \mapsto 24\tau$ and using the fact that $\mathscr{P}(q) = \eta^{-1}(24\tau)$,

$$M_\ell(\tau) = \mathscr{P}(q)\left(\frac{E_4^2(24\tau)E_6(24\tau)}{\Delta(24\tau)}\right)\left[-\frac{\ell}{12}B_{\delta_\ell}(j(24\tau))\right].$$

We can finally conclude that the action of the Hecke operator $T(\ell^2)$ is

$$M^+(\tau)\,|_{3/2}\,T(\ell^2) = \left(\frac{3}{\ell}\right)(1+\ell)M^+(\tau) - \frac{\ell}{12}\mathscr{P}(q)\cdot B_{\delta_\ell}(j(24\tau))\cdot\frac{E_4^2(24\tau)E_6(24\tau)}{\Delta(24\tau)},$$

concluding the proof of Theorem 1.1. □

3 The signed triangular weight

In light of the connection between the generating functions of the Andrews *spt*-function and a particular class of unimodal sequences given in (1.14) mediated by the series $A(q)$, we present the proof of Theorem 1.3.

3.1 Proof of Theorem 1.3

We begin by examining the summation

$$\sum_{n\geq 1} \frac{q^{\frac{n(n+1)}{2}}}{1-q^n}. \tag{3.1}$$

By considering each summand to be of the form $q^{\frac{n(n+1)}{2}}(1+q^n+q^{2n}+q^{3n}+\cdots)$, note that if we formally expand the above power series as $\sum_{m\geq 1}\alpha(m)q^m$, then $\alpha(m)$ counts the number of ways to choose integers (n, k) with $n \geq 1, k \geq 0$ such that $m = T_n + kn$, where $T_n = n(n+1)/2$ denotes the nth triangular number. Similarly, the coefficient $\beta(m)$ of q^m in the formal expansion of

$$\sum_{n\geq 1} \frac{(-1)^{n-1} n q^{\frac{n(n+1)}{2}}}{1-q^n} := \sum_{m\geq 1} \beta(m) q^m \tag{3.2}$$

denotes a sum over all such pairs (n, k), weighted by the parity and size of n.

Multiplying the above series by the generating function $1/(q;q)_\infty$ for partitions then gives a formal power series

$$\frac{1}{(q;q)_\infty} \sum_{n\geq 1} \frac{(-1)^{n-1} n q^{\frac{n(n+1)}{2}}}{1-q^n} := \sum_{m\geq 1} \gamma(m) q^m \tag{3.3}$$

where $\gamma(m)$ runs over all partitions $\lambda \vdash m$ such that λ contains a subpartition consisting of the parts $\{1, 2, \ldots, n\}$ and also possibly k more parts of size n, for $n \geq 1$ and $k \geq 0$, but weighting this count by the parity and size of n. □

4 Combinatorial interpretations of the coefficients of $j(\tau)$

As we have now developed a variety of both combinatorial and number-theoretic objects, all of which are tied together by a class of polynomials in $j(24\tau)$, it is natural to ask if we may formalize and explicate this connection. To do this, we make use of both the standard definition of the Hecke operator on q-series expansions as well as the result of Theorem 1.1 in order to pull the functions $spt(n)$ and $p(n)$ through to the j-function. We restrict our attention to the case where $\ell = 5$ since $\delta_\ell = 1$ and $B_1(j(24\tau)) = 1$. While at first glance it may seem as though we have removed j from our expressions by looking at this case, we recall that

$$-q\frac{d}{dq} j(24\tau) = \frac{E_4^2(24\tau)E_6(24\tau)}{\Delta(24\tau)} = q^{-24} - \sum_{n\geq 1} nc(n)q^{24n}, \tag{4.1}$$

where $c(n)$ is the nth coefficient of the j-function. Thus, we need only solve for the $c(n)'s$ in terms of the combinatorial information given by $M^+(\tau)$ to arrive at our final conclusions.

4.1 Proof of Theorem 1.2

Writing the q-series expansion for $M^+(\tau)$ out in terms of $spt(n)$ and $p(n)$, we arrive at

$$M^+(\tau) = -\frac{1}{12}q^{-1} + \sum_{n\geq 1}\left[spt(n) + \frac{(24n-1)}{12}p(n)\right]q^{24n-1}.$$

For $n \geq 1$, we then write for ease

$$m(24n-1) := spt(n) + \frac{24n-1}{12}p(n). \tag{4.2}$$

Now we can describe the action of the Hecke operator $T(25)$ as follows:

$$\begin{aligned} M^+(\tau) \mid T(25) = &-\frac{5}{12}q^{-25} + \frac{1}{12}q^{-1} + \sum_{n\geq 1} 5m(24n-1)q^{25(24n-1)} \\ &+ \sum_{n\geq 1}\left[m(25(24n-1)) - \left(\frac{-24n+1}{5}\right)m(24n-1)\right]q^{24n-1}. \end{aligned}$$

Since $M_5(\tau) = M^+(\tau) \mid T(25) + 6M^+(\tau)$, we have

$$\begin{aligned} M_5(\tau) = &-\frac{5}{12}q^{-25} - \frac{5}{12}q^{-1} + \sum_{n\geq 1} 5\left[m(24n-1)\right]q^{25(24n-1)} \\ &+ \sum_{n\geq 1}\left[m(25(24n-1)) + \left[6 - \left(\frac{-24n+1}{5}\right)\right]m(24n-1)\right]q^{24n-1}. \end{aligned}$$

Thus, when $\ell = 5$, the statement of Theorem 1.1 reduces to

$$M_5(\tau) = -\frac{5}{12}\eta^{-1}(24\tau)\left[q^{-24} - \sum_{n\geq 1} nc(n)q^{24n}\right]$$

and we are able to rearrange as follows:

$$\begin{aligned} -q^{-24} + \sum_{n\geq 1} nc(n)q^{24n} = \eta(24\tau)\Bigg[&-q^{-25} - q^{-1} \\ &+ \frac{12}{5}\sum_{n\geq 1}\left[m(25(24n-1)) + \left[6 - \left(\frac{1-24n}{5}\right)\right]m(24n-1)\right]q^{24n-1} \\ &+ \sum_{n\geq 1} 12m(24n-1)q^{25(24n-1)}\Bigg]. \end{aligned}$$

Recall the definitions of $h_1(m)$ and $h_2(m)$ in (1.12). Using these, we define

$$\begin{aligned} \delta_1(n) &:= \sum_{k\in\mathbb{Z}}(-1)^k h_1(n-(6k+1)^2), \\ \delta_2(n) &:= \sum_{k\in\mathbb{Z}}(-1)^k h_2(n-(6k+1)^2). \end{aligned}$$

Then we may write

$$-q^{-24}+\sum_{n\geq 1} nc(n)q^{24n} = \sum_{n\geq 1}\delta_1(24n)q^{24n}+\sum_{n\geq 1}\delta_2(24(25n-1))q^{24(25n-1)}$$
$$-\sum_{n=-\infty}^{\infty}(-1)^n q^{(6n+1)^2-25}-\sum_{n=-\infty}^{\infty}(-1)^n q^{(6n+1)^2-1}.$$

We note that for $n\geq 1$,

$$\sum_{n\geq 1} s(n)q^{24n} = -\sum_{n=-\infty}^{\infty}(-1)^n q^{(6n+1)^2-25}-\sum_{n=-\infty}^{\infty}(-1)^n q^{(6n+1)^2-1}.$$

Thus, Theorem 1.2 follows by solving for $c(n)$. □

Proof of Corollary 1.5. This result follows immediately from Theorem 1.2 and the relation $spt(n)=-u^*(n)+2a(n)$.

Remark While the results above use only the action of the specific Hecke operator $T(25)$, one should note that the entire sequence of operators $T(\ell^2)$ generate similar results for the polynomials $B_{\delta_\ell}(j(24\tau)$. We outline this process below. We define

$$q\frac{d}{dq}j(24\tau)\cdot B_{\delta_\ell}(j(24\tau)) := \sum_{n\gg -\infty} r_\ell(n)q^{24n}.$$

Then likewise if

$$m_\ell(24n-1) := m(\ell^2(24n-1))+\left(\frac{3}{\ell}\right)\left[\left(\frac{-(24n-1)}{\ell}\right)-(1+\ell)\right]m(24n-1)$$
$$+\ell m((24n-1)/\ell^2),$$

we may write

$$M_\ell(\tau) = -\frac{\ell}{12}q^{-\ell^2}+\left(\frac{3}{\ell}\right)\frac{\ell}{12}q^{-1}+\sum_{n\geq 1} m_\ell(24n-1)q^{24n-1}.$$

Rewriting the result of Theorem 1.1, we have

$$\sum_{n\gg -\infty} r_\ell(n)q^{24n} = \frac{12}{\ell}\eta(24\tau)M_\ell(\tau). \tag{4.3}$$

Thus, expanding the right-hand side using the pentagonal number theorem allows one to solve for $r_\ell(n)$. By Theorem 1.2 and Corollary 1.5, the coefficients of $q\frac{d}{dq}j(24\tau)$ are known in terms of combinatorial quantities, and so the coefficients of $B_{\delta_\ell}(j(\tau))$ themselves can written as a sequence of combinatorial expressions as well.

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