Stabilizing Scheduling Policies for Networked Control Systems

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***Abstract*—This paper deals with the problem of allocating communication resources for Networked Control Systems (NCSs). We consider an NCS consisting of a set of discrete-time linear time-invariant plants whose stabilizing feedback loops are closed through a shared communication channel. Due to a limited communication capacity of the channel, not all plants can exchange information with their controllers at any instant of time. We propose a method to find periodic scheduling policies under which global asymptotic stability of each plant in the NCS is preserved. The individual plants are represented as switched systems, and the NCS is expressed as a weighted directed graph. We construct stabilizing scheduling policies by employing cycles on the underlying weighted directed graph of the NCS that satisfy appropriate contractivity conditions. We also discuss algorithmic design of these cycles.**

***Index Terms*—Asymptotic stability, directed graphs, networked control systems, scheduling policy, switched systems.**

I. Introduction

NETWORKED Control Systems (NCSs) are omnipresent in modern-day Cyber-Physical Systems (CPS) and Internet of Things (IoT) applications. While these applications typically involve a large-scale setting, the network resources are often limited. Consequently, multiple plants may need to share a communication channel (or network) for exchanging information with their remotely located controllers. Examples of communication networks with limited bandwidth include wireless networks (an important component of smart home, smart transportation, smart city, remote surgery, platoons of autonomous vehicles, etc.) and underwater acoustic communication systems. The often encountered scenario wherein the number of plants sharing a communication channel is higher than the capacity of the channel is called *medium access constraint*.

In this paper, we consider an NCS consisting of multiple discrete-time linear plants whose feedback loops are closed through a shared communication channel. A block diagram of such an NCS is shown in Fig. 1.

We assume that the plants are unstable in open-loop and asymptotically stable in closed-loop. Due to a limited communication capacity of the channel, only a few plants can exchange information with their controllers at any instant of time. Consequently, the remaining plants operate in open-loop at every time instant. Our objective is to allocate the shared communication channel to the set of plants in a manner so that stability of all plants is preserved. This task of efficient allocation of communication resources is commonly referred to as a *scheduling problem*, and the corresponding allocation scheme is called a *scheduling policy*, see, for example, [6] and [23] for real-world applications where scheduling problems arise naturally.

Scheduling policies that preserve the qualitative behavior of an NCS under limited communication and/or computation resources are widely researched upon, and tools from both control theory and communication theory have been explored; see the recent works [1], [9], [25], [30], [32] and the references therein. These policies can be broadly classified into two categories: *static* (also called *periodic*, *fixed*, or *open-loop*) and *dynamic* (also called *non-periodic*, or *closed-loop*) scheduling. In case of the former, a finite-length allocation scheme of the shared communication channel is determined off-line and is applied eternally in a periodic manner, while, in case of the latter, the allocation is determined based on some information (e.g., states, outputs, access status of sensors and actuators, etc.) about the plant. In this paper, we will focus on periodic scheduling policies that preserve global asymptotic stability (GAS) of all plants in an NCS. We will call such scheduling policies *stabilizing scheduling policies*. Static scheduling policies are easier to implement, often near-optimal, and guarantee activation of each sensor and actuator; see [13], [24], and [28] for detailed discussions. They are preferred for safety-critical control systems [24, and §2.5.1]. It is also observed in [27] and [28] that periodic phenomenon appears in non-periodic schedules.

For NCSs with continuous-time linear plants, stabilizing periodic scheduling policies are characterized using common Lyapunov functions [10] and piecewise Lyapunov-like functions with average dwell-time switching [22]. A more general case of co-designing a static scheduling policy and control action is addressed using combinatorial optimization with periodic control theory in [29] and linear matrix inequalities (LMIs) optimization with average dwell-time technique in [5]. In the discrete-time setting, the authors of [33] characterize periodic switching sequences that ensure reachability and observability of the

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plants under limited communication, and design an observer-based feedback controller for these periodic sequences. The techniques were later extended to the case of constant transmission delays [11] and a linear quadratic Gaussian (LQG) control problem [12]. Periodic sensor scheduling schemes that accommodate limited communication and adversary attacks have been studied recently in [31].

The main contribution of this paper lies in combining switched systems and graph theory to propose a new class of stabilizing scheduling policies for NCSs. We represent the individual (open-loop unstable) plants of an NCS as switched systems, where the switching is between their open-loop (unstable mode) and closed-loop (stable mode) operations. Clearly, within our setting, no switched system can operate in stable mode for the entire time as that will destabilize some of the plants in the NCS. The search for a stabilizing scheduling policy then becomes the problem of finding switching logics that obey the limitations of the shared channel and preserve stability. It is assumed that the exchange of information between a plant and its controller is not affected by communication uncertainties. In the recent past, graph-theoretic techniques have played an important role in designing stabilizing switching logics for switched systems (see e.g., [16], [17] and the references therein). In this paper, we associate a weighted directed graph with the NCS that captures the communication limitation of the shared channel, and design stabilizing switching logics for each plant in the NCS. Multiple Lyapunov-like functions are employed for analyzing the stability of the switched systems. The stabilizing switching logics form a stabilizing scheduling policy. The switching logics are combined in terms of a class of cycles on the underlying weighted directed graph of the NCS that satisfies appropriate contractivity properties. We also discuss algorithmic construction of these cycles.

In brief, our contributions are as follows.

- Given an NCS with discrete-time linear plants that exchange information with their stabilizing controllers through a shared channel of limited communication capacity, we present an algorithm to design a scheduling policy that preserves the GAS of each plant in the NCS. Our scheduling policy is periodic in nature, and relies on the existence of what we call a T-contractive cycle on the underlying weighted directed graph of the NCS. Periodic scheduling policies have proven to be immensely useful in process control, where many loops need to share a common communication resource to avoid the necessity of frequent network reconfigurations. In fact, periodic scheduling is an inherent feature of IEEE 802.15.4 networks [28], which underlie commercial standards, such as WirelessHart, ISA100.11a, and ZigBee. The use of cycles on a weighted directed graph makes our techniques numerically tractable; see Remark 11 for a detailed discussion.
- We address the algorithmic design of T-contractive cycles. Given the connectivity of the underlying weighted directed graph of the NCS and description of the individual plants, we fix a cycle on this graph and present an algorithm that designs multiple Lyapunov-like functions such that the above cycle is T-contractive. We also identify sufficient conditions on the multiple Lyapunov-like functions and channel constraints under which the existence of a T-contractive cycle is guaranteed.

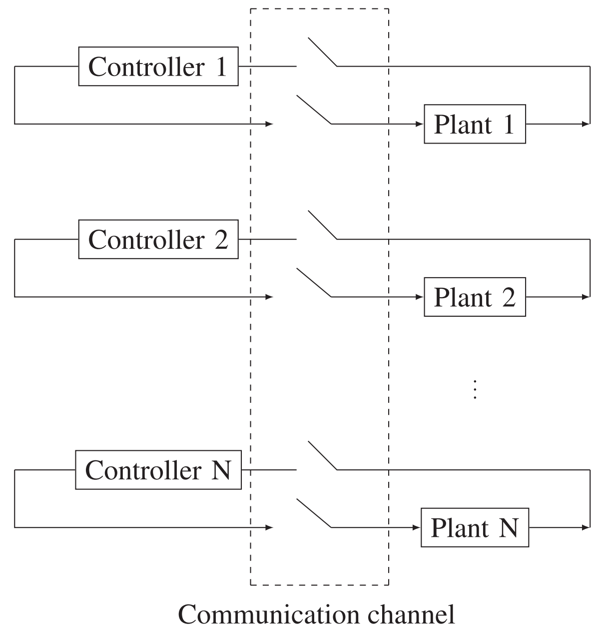

Fig. 1. Block diagram of NCS.

The remainder of this paper is organized as follows: in Section II, we formulate the problem under consideration, and describe the primary apparatus for our analysis. Our method for constructing stabilizing periodic scheduling policies appears in Section III. In Section IV, we discuss the algorithmic design of T-contractive cycles. A numerical example is presented in Section V to demonstrate our results. We conclude in Section VI with a brief discussion of future research directions. Proofs of our results appear in the Appendix.

Some notations used in this paper: $\mathbb{N} = \{1, 2, \ldots\}$ is the set of natural numbers, $\mathbb{N}_0 = \{0\} \cup \mathbb{N}$, and $\mathbb{R}$ is the set of real numbers. We let $]k_1 : k_2]$ denote the set $\{n \in \mathbb{N} \mid k_1 < n \leq k_2\}$. For a scalar m, let $|m|$ denote its absolute value, and for a set M, let $|M|$ denote its cardinality. Let $\|\cdot\|$ be the standard 2-norm and $^\top$ denote the transpose operation.

II. Preliminaries

We consider an NCS with N discrete-time linear plants. Each plant communicates with its remotely located controller through a shared communication channel (see Fig. 1). The plant dynamics are

$$x_i(t+1) = A_i x_i(t) + B_i u_i(t),\ x_i(0) = x_i^0,\ t \in \mathbb{N}_0 \quad (1)$$

where $x_i(t) \in \mathbb{R}^d$ and $u_i(t) \in \mathbb{R}^m$ are the vectors of states and inputs of the i-th plant at time t, respectively, $i = 1, 2, \ldots, N$. Each plant i employs a state-feedback controller given by $u_i(t) = K_i x_i(t)$, $i = 1, 2, \ldots, N$. The matrices $A_i \in \mathbb{R}^{d\times d}$, $B_i \in \mathbb{R}^{d\times m}$, and $K_i \in \mathbb{R}^{m\times d}$, $i = 1, 2, \ldots, N$ are known. The shared channel has limited communication capacity: at any time instant, only M plants $(0 < M < N)$ can access the channel. Consequently, N–M plants operate in open-loop at every time instant. We define $\mathcal{S} := \{s \in \{1, 2, \ldots, N\}^M \mid$ all elements of s are distinct$\}$ to be the set of vectors that consist of M distinct elements from $\{1, 2, \ldots, N\}$. We call the function $\gamma : \mathbb{N}_0 \to \mathcal{S}$ a scheduling policy. There exists a diverging sequence of times $0 =: \tau_0 < \tau_1 < \tau_2 < \cdots$ and a sequence of indices $s_0, s_1, s_2, \ldots$ with $s_j \in \mathcal{S}$, $j = 0, 1, 2, \ldots$ such that $\gamma(t) = s_j$ for $t \in [\tau_j : \tau_{j+1}[$, $j = 0, 1, 2, \ldots$. The role of γ is to specify, at every time t, M plants of the NCS which access the communication channel at that time. The remaining N–M plants operate in open-loop, in particular, with $u_i(t) = 0$.

Remark 1: One may also study a scheduling problem in the setting of the remaining N–M plants operating with $u_i(t) = \overline{u}$, where $\overline{u}$ is the last control input received before time t. However, in this paper, we consider open-loop evolution of a plant whenever it is not accessing the shared communication channel.

Assumption 1: The open-loop dynamics of each plant is unstable and each controller is stabilizing. More specifically, the matrices $A_i + B_i K_i$, $i = 1, 2, \ldots, N$ are Schur stable and the matrices A_i, $i = 1, 2, \ldots, N$ are unstable.[1]

Assumption 2: The shared communication channel is ideal in the sense that an exchange of information between plants and their controllers is not affected by communication uncertainties.

In view of Assumption 1, each plant in (1) operates in two modes: 1) stable mode when the plant has access to the shared communication channel and 2) unstable mode when the plant does not have access to the channel. Let us denote the stable and unstable modes of the i-th plant as i_s and i_u, respectively, $A_{i_s} = A_i + B_i K_i$ and $A_{i_u} = A_i, i = 1, 2, \ldots, N$. In this paper, we are interested in a scheduling policy that guarantees GAS of each plant in (1). In particular, we study the following problem:

Problem 1: Given the matrices $A_i, B_i, K_i, i = 1, 2, \ldots, N$, and a number $M(< N)$, find a scheduling policy that ensures GAS of each plant i in (1).

We will call a scheduling policy γ that is a solution to Problem 1, as a stabilizing scheduling policy. Recall that

Definition 1 ([15, Lemma 4.4]): The i-th plant in (1) is GAS for a given scheduling policy γ, if there exists a class $\mathcal{KL}$ function β_i such that the following inequality holds:

$$\|x_i(t)\| \leq \beta_i(\|x_i(0)\|, t) \text{ for all } x_i(0) \in \mathbb{R}^d \text{ and } t \geq 0.^2 \tag{2}$$

Toward solving Problem 1, we express individual plants in (1) as switched systems and associate a weighted directed graph with the NCS under consideration. Our solution to Problem 1 involves two steps:

- first, we present an algorithm that constructs a scheduling policy by employing what we call a T-contractive cycle on the underlying weighted directed graph of the NCS;
- second, we show that a scheduling policy obtained from our algorithm ensures GAS of each plant in (1).

[1] We call a matrix unstable, if it is not Schur stable.

[2] Recall classes of functions [15, Ch. 4]: $\mathcal{K} := \{\phi : [0, +\infty[\to [0, +\infty[\mid \phi$ is continuous, strictly increasing, $\phi(0) = 0\}$, $\mathcal{L} := \{\psi : [0, +\infty[\longrightarrow [0, +\infty[\mid \psi$ is continuous and $\psi(s) \searrow 0$ as $s \nearrow +\infty\}$, $\mathcal{KL} := \{\chi : [0, +\infty[^2 \longrightarrow [0, +\infty[\mid \chi(\cdot, s) \in \mathcal{K}$ for each s and $\chi(r, \cdot) \in \mathcal{L}$ for each $r\}$.

We also discuss the algorithmic design of T-contractive cycles. In the remainder of this section, we catalog our analysis tools.

A. Individual Plants and Switched Systems

The dynamics of the i-th plant in (1) can be expressed as a switched system [21, §1.1.2]

$$x_i(t+1) = A_{\sigma_i(t)} x_i(t),\ x_i(0) = x_i^0, \sigma_i(t) \in \{i_s, i_u\},\ t \in \mathbb{N}_0 \tag{3}$$

where the subsystems are $\{A_{i_s}, A_{i_u}\}$ and a switching logic $\sigma_i : \mathbb{N}_0 \to \{i_s, i_u\}$ satisfies

$$\sigma_i(t) = \begin{cases} i_s, & \text{if } i \text{ is an element of } \gamma(t) \\ i_u, & \text{otherwise.} \end{cases}$$

Clearly, a switching logic $\sigma_i, i = 1, 2, \ldots, N$ is a function of the scheduling policy γ. In order to ensure GAS of the individual plants, it therefore suffices to look for a γ that renders each σ_i stabilizing in the following sense: σ_i guarantees the GAS of the switched system (3) for each $i = 1, 2, \ldots, N$. We recall the following facts from recent literature:

Fact 1: [17, Fact 1] For each $i = 1, 2, \ldots, N$, there exist pairs (P_p, λ_p), $p \in \{i_s, i_u\}$, where $P_p \in \mathbb{R}^{d\times d}$ are symmetric and positive definite matrices, and $0 < \lambda_{i_s} < 1$, $\lambda_{i_u} \geq 1$, such that with

$$\mathbb{R}^d \ni \xi \longmapsto V_p(\xi) := \langle P_p \xi, \xi \rangle \in [0, +\infty[\tag{4}$$

we have

$$V_p(z_p(t+1)) \leq \lambda_p V_p(z_p(t)),\ \ t \in \mathbb{N}_0 \tag{5}$$

and $z_p(\cdot)$ solves the p-th recursion in (3), $p \in \{i_s, i_u\}$.

Fact 2: [17, Fact 2] For each $i = 1, 2, \ldots, N$, there exist $\mu_{pq} \geq 1$ such that

$$V_q(\xi) \leq \mu_{pq} V_p(\xi) \text{ for all } \xi \in \mathbb{R}^d \text{ and } p, q \in \{i_s, i_u\}. \tag{6}$$

The functions V_p, $p \in \{i_s, i_u\}$, $i = 1, 2, \ldots, N$ are called Lyapunov-like functions, and they are widely used in stability theory of switched and hybrid systems [4], [21]. We will use properties of these functions described in Facts 1 and 2 in our analysis toward deriving a stabilizing scheduling policy. The scalars λ_p, $p \in \{i_s, i_u\}$ give quantitative measures of (in)stability associated with (un)stable modes of operation of the i-th plant. Linear comparability of V_p's in (6) follows from the definition of V_p, $p \in \{i_s, i_u\}$ in (4). In [17, Prop. 1], a *tight* estimate of the scalars μ_{pq}, $p, q \in \{i_s, i_u\}$ was proposed to be $\lambda_{\max}(P_q P_p^{-1})$, where $\lambda_{\max}(M)$ denotes the maximum eigenvalue of a matrix $M \in \mathbb{R}^{d\times d}$.

B. NCS and Directed Graphs

Recall that a directed graph is a set of vertices connected by edges, where each edge has a direction associated with it. We connect a directed graph $\mathcal{G}(\mathcal{V}, \mathcal{E})$ with the NCS under consideration. $\mathcal{G}(\mathcal{V}, \mathcal{E})$ contains:

- a vertex set $\mathcal{V}$ consisting of $\binom{N}{M}$ vertices that are labeled distinctly. The label associated with a vertex v is given by $L(v) = \{\ell_v(1), \ell_v(2), \ldots, \ell_v(N)\}$, where $\ell_v(i) = i_s$ for any M elements and $\ell_v(i) = i_u$ for the remaining $N{-}M$ elements. Two labels $L(u)$ and $L(v)$ are equal if $\ell_u(i) = \ell_v(i)$ for all $i = 1, 2, \ldots, N$. By the term "distinct labeling", we mean that $L(u) = L(v)$ whenever $u = v \in \mathcal{V}$;

- an edge set $\mathcal{E}$ consisting of a directed edge (u, v) from every vertex $u \in \mathcal{V}$ to every vertex $v \in \mathcal{V}$, $v \neq u$.

The label $L(v)$ corresponding to a vertex $v \in \mathcal{V}$ gives a combination of M plants operating in stable mode and the remaining $N{-}M$ plants operating in unstable mode. Since $\mathcal{V}$ contains $\binom{N}{M}$ vertices and the label associated with each vertex is distinct, it follows that the set of vertex labels consists of all possible combinations of M plants accessing the communication channel and $N{-}M$ plants operating in open-loop. A directed edge (u, v) from a vertex u to a vertex v $(\neq u)$ corresponds to a transition from a set of M plants accessing the communication channel (as specified by $L(u)$) to another set of M plants accessing the communication channel (as specified by $L(v)$). In the sequel, we may abbreviate $\mathcal{G}(\mathcal{V}, \mathcal{E})$ as $\mathcal{G}$ if there is no risk of confusion.

We use functions $\overline{w} : \mathcal{V} \to \mathbb{R}^N$ and $\underline{w} : \mathcal{E} \to \mathbb{R}^N$ to associate weights to the vertices and edges of $\mathcal{G}$, respectively. They are defined as

$$\overline{w}(v) = \begin{pmatrix} \overline{w}_1(v) \\ \overline{w}_2(v) \\ \vdots \\ \overline{w}_N(v) \end{pmatrix}, \quad v \in \mathcal{V}, \quad \text{where}$$

$$\overline{w}_i(v) = \begin{cases} -|\ln \lambda_{i_s}|, & \text{if } \ell_v(i) = i_s, \\ |\ln \lambda_{i_u}|, & \text{if } \ell_v(i) = i_u, \end{cases} \quad i = 1, 2, \ldots, N \tag{7}$$

and

$$\underline{w}(u, v) = \begin{pmatrix} \underline{w}_1(u, v) \\ \underline{w}_2(u, v) \\ \vdots \\ \underline{w}_N(u, v) \end{pmatrix}, \quad (u, v) \in \mathcal{E}, \quad \text{where}$$

$$\underline{w}_i(u, v) = \begin{cases} \ln \mu_{i_s i_u}, & \text{if } \ell_u(i) = i_s \text{ and } \ell_v(i) = i_u \\ \ln \mu_{i_u i_s}, & \text{if } \ell_u(i) = i_u \text{ and } \ell_v(i) = i_s, \\ & \qquad i = 1, 2, \ldots, N. \\ 0, & \text{otherwise} \end{cases} \tag{8}$$

Here $\lambda_{i_s}, \lambda_{i_u}$, and $\mu_{i_s i_u}, \mu_{i_u i_s}$, $i = 1, 2, \ldots, N$ are as described in Facts 1 and 2, respectively.

Remark 2: We will aim for achieving the GAS of each switched system (3), $i = 1, 2, \ldots, N$. For this purpose, we will compensate the increase in $V_p, p \in \{i_s, i_u\}$ caused by activation of unstable mode i_u and switches between stable and unstable modes (i_s to i_u and i_u to i_s) by the decrease in $V_p, p \in \{i_s, i_u\}$ achieved by using the stable modes i_s, $i = 1, 2, \ldots, N$. As a natural choice, the vertex (subsystem) weights of $\mathcal{G}$ relate to the rate of increase/decrease of the Lyapunov-like functions V_p captured by the scalars λ_p, $p \in \{i_s, i_u\}$, and the edge (switch) weights of $\mathcal{G}$ relate to the "jump" between Lyapunov-like functions V_p and V_q, $p, q \in \{i_s, i_u\}$ captured by the scalars μ_{pq}, $p, q \in \{i_s, i_u\}$, $i = 1, 2, \ldots, N$.[3]

The weights associated with the vertices and edges of $\mathcal{G}$ are

$$\overline{w}(\overline{v}_1) = \begin{pmatrix} -|\ln \lambda_{1_s}| \\ -|\ln \lambda_{2_s}| \\ |\ln \lambda_{3_u}| \end{pmatrix}, \overline{w}(\overline{v}_2) = \begin{pmatrix} -|\ln \lambda_{1_s}| \\ |\ln \lambda_{2_u}| \\ -|\ln \lambda_{3_s}| \end{pmatrix},$$

$$\overline{w}(\overline{v}_3) = \begin{pmatrix} |\ln \lambda_{1_u}| \\ -|\ln \lambda_{2_s}| \\ -|\ln \lambda_{3_s}| \end{pmatrix}$$

and

$$\underline{w}(\overline{v}_1, \overline{v}_2) = \begin{pmatrix} 0 \\ \ln \mu_{2_s 2_u} \\ \ln \mu_{3_u 3_s} \end{pmatrix}, \quad \underline{w}(\overline{v}_1, \overline{v}_3) = \begin{pmatrix} \ln \mu_{1_s 1_u} \\ 0 \\ \ln \mu_{3_u 3_s} \end{pmatrix}$$

$$\underline{w}(\overline{v}_2, \overline{v}_1) = \begin{pmatrix} 0 \\ \ln \mu_{2_u 2_s} \\ \ln \mu_{3_s 3_u} \end{pmatrix}, \quad \underline{w}(\overline{v}_2, \overline{v}_3) = \begin{pmatrix} \ln \mu_{1_s 1_u} \\ \ln \mu_{2_u 2_s} \\ 0 \end{pmatrix}$$

$$\underline{w}(\overline{v}_3, \overline{v}_1) = \begin{pmatrix} \ln \mu_{1_u 1_s} \\ 0 \\ \ln \mu_{3_s 3_u} \end{pmatrix}, \quad \underline{w}(\overline{v}_3, \overline{v}_2) = \begin{pmatrix} \ln \mu_{1_u 1_s} \\ \ln \mu_{2_s 2_u} \\ 0 \end{pmatrix}.$$

Remark 3: With the construction of $\mathcal{G}$, it contains two directed edges (u, v) and (v, u) between every two vertices $u, v \in \mathcal{V}$. Employing an undirected graph instead of a directed one may appear to be a natural choice here. However, the use of directed edges allows us to distinguish easily between the transitions i_s to i_u and i_u to i_s, $i = 1, 2, \ldots, N$, and assign weights to the corresponding edges accordingly. Notice that since the vertex labels are distinct, for every two vertices $u, v \in \mathcal{V}$, there exists at least one i for which $\underline{w}_i(u, v)$ and $\underline{w}_i(v, u)$ are different, $i \in \{1, 2, \ldots, N\}$.

Recall that [3, p. 4] a *walk* on a directed graph $G(V, E)$ is an alternating sequence of vertices and edges $W = \tilde{v}_0, \tilde{e}_1, \tilde{v}_1, \tilde{e}_2, \tilde{v}_2, \ldots, \tilde{v}_{\ell-1}, \tilde{e}_\ell, \tilde{v}_\ell$, where $\tilde{v}_m \in V$, $\tilde{e}_m = (\tilde{v}_{m-1}, \tilde{v}_m) \in E$, $0 < m \leq \ell$. The *length* of a walk is its number of edges, counting repetitions, for example, the length of W is ℓ. The *initial vertex* of W is $\tilde{v}_0$ and the *final vertex* of W is $\tilde{v}_\ell$. If $\tilde{v}_\ell = \tilde{v}_0$, we say that the walk is *closed*. A closed walk is called a *cycle* if the vertices $\tilde{v}_k$, $0 < k < n$ are distinct from each other and $\tilde{v}_0$. We will use the following class of cycles on $\mathcal{G}$ to construct a stabilizing scheduling policy:

Definition 2: A cycle $W = v_0, (v_0, v_1), v_1, \ldots, v_{n-1}, (v_{n-1}, v_0), v_0$ on $\mathcal{G}(\mathcal{V}, \mathcal{E})$ is called *T-contractive* if there exist integers $T_{v_j} > 0$, $j = 0, 1, \ldots, n-1$, $2 \leq n \leq |\mathcal{V}|$ such that the following set of inequalities is satisfied:

$$\Xi_i(W) := \sum_{j=0}^{n-1} \overline{w}_i(v_j) T_{v_j} + \sum_{\substack{j=0 \\ v_n := v_0}}^{n-1} \underline{w}_i(v_j, v_{j+1}) < 0 \tag{9}$$

for all $i = 1, 2, \ldots, N$, where n is the length of W, $\overline{w}(v_j)$ is the weight associated with vertex v_j, $\overline{w}_i(v_j)$ is the i-th element of $\overline{w}(v_j)$, and $\underline{w}(v_j, v_{j+1})$ is the weight associated with edge

[3]The use of the absolute value and natural logarithm is explained in context; see Remark 15.

Algorithm 1: Construction of a Periodic Scheduling Policy.

Let $\mathcal{G}(\mathcal{V}, \mathcal{E})$ be a directed graph representation of the NCS described in Section II. Suppose that $\mathcal{G}(\mathcal{V}, \mathcal{E})$ admits a T-contractive cycle $W = v_0, (v_0, v_1), v_1, \ldots, v_{n-1}, (v_{n-1}, v_0), v_0$ (of length n) with T-factors $T_0, T_1, \ldots, T_{n-1}$.

Input: a T-contractive cycle $W = v_0, (v_0, v_1), v_1, \ldots, v_{n-1}, (v_{n-1}, v_0), v_0$ and corresponding T-factors $T_0, T_1, \ldots, T_{n-1}$

Output: a periodic scheduling policy γ

Step I: For each vertex v_j, $j = 0, 1, \ldots, n-1$, pick the elements i with label $\ell_{v_j}(i) = i_s$, $i = 1, 2, \ldots, N$, and construct M- dimensional vectors s_j, $j = 0, 1, \ldots, n-1$.

1: **for** $j = 0, 1, \ldots, n-1$ **do**
2: Set $p = 0$
3: **for** $i = 1, 2, \ldots, N$ **do**
4: **if** $\ell_{v_j}(i) = i_s$ **then**
5: Set $p = p + 1$ and $s_j(p) = i$
6: **end if**
7: **end for**
8: **end for**

Step II: Construct a scheduling policy using the vectors s_j, $j = 0, 1, \ldots, n-1$ obtained in Step I and the T-factors T_{v_j}, $j = 0, 1, \ldots, n-1$

9: Set $p = 0$ and $\tau_0 = 0$
10: **for** $q = pn, pn + 1, \ldots, (p+1)n - 1$ **do**
11: Set $\gamma(\tau_q) = s_{q-pn}$ and $\tau_{q+1} = \tau_q + T_{v_{q-pn}}$
12: Output τ_q and $\gamma(\tau_q)$
13: **end for**
14: Set $p = p + 1$ and go to 10.

(v_j, v_{j+1}), and $\underline{w}_i(v_j, v_{j+1})$ is the i-th element of $\underline{w}(v_j, v_{j+1})$, $i = 1, 2, \ldots, N$, $j = 0, 1, \ldots, n-1$. We call the scalar T_{v_j} as the T-factor of vertex v_j, $j = 0, 1, \ldots, n-1$.

We will employ the integers T_{v_j}, $j = 0, 1, \ldots, n-1$ to associate a time duration with every vertex v_j, $j = 0, 1, \ldots, n-1$ that appears in W. This time duration will determine how long a set of M plants can access the shared communication channel while preserving GAS of all plants in the NCS under consideration. In the present discrete-time setting, the association of integers with time durations is natural.

Remark 4: Definition 2 is an extension of [16, Def. 2] to a set of N-switched systems in the discrete-time setting. In [16], the notion of a contractive cycle with T-factors chosen from a given interval of real numbers was used to study input/output-to-state stability (IOSS) of continuous-time-switched nonlinear systems under dwell-time restrictions.

III. Stabilizing Periodic Scheduling Policies

The following algorithm is geared toward constructing a periodic scheduling policy. We will show that a scheduling policy obtained from this algorithm is stabilizing.

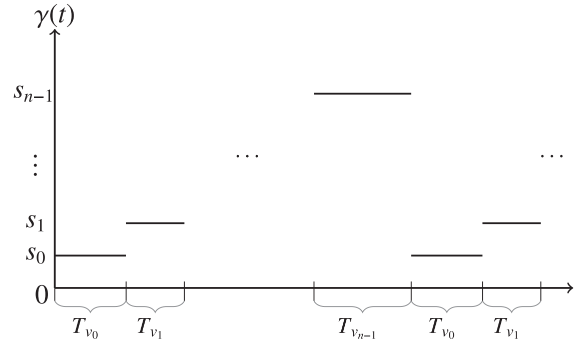

Fig. 2. Example scheduling policy: the activation of s_j corresponds to activation of the plants whose indices appear in s_j.

Given a set of matrices $A_i, B_i, K_i, i = 1, 2, \ldots, N$ and a number M, Algorithm 1 employs a T-contractive cycle $W = v_0, (v_0, v_1), v_1, \ldots, v_{n-1}, (v_{n-1}, v_0), v_0$ on $\mathcal{G}(\mathcal{V}, \mathcal{E})$ to construct a scheduling policy γ that specifies, at every time, $M(< N)$ plants that access the shared communication channel. The construction of γ involves two steps: in Step I, corresponding to each vertex v_j, $j = 0, 1, \ldots, n-1$, a vector s_j, $j = 0, 1, \ldots, n-1$ is created. The vector s_j contains the elements $i \in \{1, 2, \ldots, N\}$ for which $\ell_{v_j}(i) = i_s$, where $\ell_{v_j}(i)$ denotes the i-th element of the vertex label $L(v_j)$. Recall that by construction, each $L(v_j)$ contains $\ell_{v_j}(i) = i_s$ exactly for M i's. Consequently, the length of s_j is M, $j = 0, 1, \ldots, n-1$. In Step II, a scheduling policy γ is obtained from the vectors s_j, $j = 0, 1, \ldots, n-1$ and the T-factors T_{v_j}, $j = 0, 1, \ldots, n-1$. Sets of M plants corresponding to the elements in s_j access the shared communication channel for T_{v_j} duration of time, $j = 0, 1, \ldots, n-1$. In particular, the following mechanism is employed to construct values of γ on the intervals $[\tau_{pn} : \tau_{(p+1)n}[$, $p = 0, 1, \ldots$:

$$\left.\begin{aligned} \gamma(\tau_q) &= s_{q-pn} \\ \tau_{q+1} &= \tau_q + T_{v_q - pn} \end{aligned}\right\} q = pn, pn+1, \ldots, (p+1)n - 1.$$

Clearly, a scheduling policy γ constructed as above is periodic with period $\sum_{j=0}^{n-1} T_{v_j}$. A pictorial representation of a scheduling policy obtained from Algorithm 1 is given in Fig. 2.

The following theorem asserts that a scheduling policy obtained from Algorithm 1 is a solution to Problem 1.

Theorem 1: Consider an NCS described in Section II. Let the matrices $A_i, B_i, K_i, i = 1, 2, \ldots, N$ and a number $M(< N)$ be given. Then each plant in (1) is GAS under a scheduling policy γ obtained from Algorithm 1.

A proof of Theorem 1 is provided in the Appendix. For an NCS consisting of N discrete-time linear plants that are open-loop unstable and closed-loop stable, and a shared communication channel that allows access only to $M(< N)$ plants at every time instant, Algorithm 1 constructs a periodic scheduling policy that ensures the GAS of each plant in the NCS.

Remark 5: Our stabilizing scheduling policy is *static* and thereby easy to implement: A T-contractive cycle on the underlying weighted directed graph of the NCS is computed offline, and the scheduling policy is implemented by the following logics involving this cycle.

The existence of a stabilizing scheduling policy proposed in this section depends on the existence of a T-contractive cycle on the underlying directed graph $\mathcal{G}$ of the NCS. It is, therefore, of importance to study how to detect/design a T-contractive cycle on $\mathcal{G}$. We address this matter next.

IV. Algorithmic Design of T-Contractive Cycles

Given the weighted directed graph $\mathcal{G}$, the existence of a T-contractive cycle depends on two factors: connectivity of $\mathcal{G}$ (for existence of cycles) and the weights associated with the vertices and edges of $\mathcal{G}$ (for T-contractivity of cycles). Since $\mathcal{G}$ is a complete graph by construction, it necessarily admits cycles. Fix a cycle $W = v_0, (v_0, v_1), v_1, \ldots, v_{n-1}, (v_{n-1}, v_0), v_0$ on $\mathcal{G}$. The T-contractivity of W is guaranteed by the existence of integers $T_{v_j} > 0, j = 0, 1, \ldots, n-1$ such that condition (9) is satisfied. The existence of such T_{v_j}'s depends upon the vertex and edge weights $\overline{w}(v)$, $v \in \mathcal{V}$ and $\underline{w}(u, v)$, $(u, v) \in \mathcal{E}$ associated with $\mathcal{G}$. These weights are functions of the matrices P_p and the scalars λ_p, $p \in \{i_s, i_u\}$, $i = 1, 2, \ldots, N$.

Remark 6: Notice that the Lyapunov-like functions V_p and, consequently, the scalars λ_p, $p \in \{i_s, i_u\}$ and μ_{pq}, $p, q \in \{i_s, i_u\}$, $i = 1, 2, \ldots, N$ used in (9) are not unique. For each $i \in \{1, 2, \ldots, N\}$, we have that A_{i_s} is Schur stable and A_{i_u} is unstable. It is known that a Schur stable matrix $A \in \mathbb{R}^{d \times d}$ satisfies the following [2, Prop. 11.10.5]: for every symmetric and positive-definite matrix $Q \in \mathbb{R}^{d \times d}$, there exists a symmetric and positive-definite matrix $P \in \mathbb{R}^{d \times d}$ such that the discrete-time Lyapunov equation

$$A^{\top} P A - P + Q = 0 \tag{10}$$

holds. For a preselected symmetric and positive definite matrix Q_{i_s}, let P_{i_s} be the solution to (10) with $A = A_{i_s}$, $P = P_{i_s}$ and $Q = Q_{i_s}$; we put $V_{i_s}(\xi) := \xi^{\top} P_{i_s} \xi$ as the corresponding Lyapunov-like function. Direct calculations, along with an application of the standard inequality [2, Lemma 8.4.3], lead to the estimate $\lambda_{i_s} = 1 - \frac{\lambda_{\min}(Q_{i_s})}{\lambda_{\max}(P_{i_s})}$, which satisfies $0 < \lambda_{i_s} < 1$. Similarly, for the unstable matrix A_{i_u}, there exists $0 < \eta < 1$ such that ηA_{i_u} is Schur stable. Fix a symmetric and positive-definite matrix Q_{i_u}. Let P_{i_u} be the solution to (10) with $A = \eta A_{i_u}$, $P = P_{i_u}$ and $Q = Q_{i_u}$; we put $V_{i_u}(\xi) := \xi^{\top} P_{i_u} \xi$ as the corresponding Lyapunov-like function. A straightforward calculation gives an estimate $\lambda_{i_u} = \frac{1}{\eta^2} > 1$. Clearly, the choice of the matrices Q_p, $p \in \{i_s, i_u\}$ determines the choice of the matrices P_p, $p \in \{i_s, i_u\}$ and the scalars λ_p, $p \in \{i_s, i_u\}$, $i = 1, 2, \ldots, N$. In addition, the matrices $P_p, p \in \{i_s, i_u\}$ determine the scalars μ_{pq}, $p, q \in \{i_s, i_u\}$ as described in Section II. ∎

Recall that $\mathcal{G}$ has $\binom{N}{M}$ vertices. Consequently, depending on the values of N and M, one may need to design a T-contractive cycle on a "large" directed graph to implement the scheduling policy proposed in Section III. It is clear that checking for existence of T_{v_j}, $j = 0, 1, \ldots, n-1$ corresponding to all possible values of $\lambda_p, p \in \{i_s, i_u\}, \mu_{pq}, p, q \in \{i_s, i_u\}$, $i = 1, 2, \ldots, N$ for every cycle W on $\mathcal{G}$, is not numerically tractable. To overcome this issue, we will next address the design of a T-contractive cycle on $\mathcal{G}$ in two steps:

1) first, we identify conditions on the scalars $\lambda_p, p \in \{i_s, i_u\}$ and μ_{pq}, $p, q \in \{i_s, i_u\}$, $i = 1, 2, \ldots, N$ under which a cycle on $\mathcal{G}$ satisfying certain properties is T-contractive;
2) second, given the matrices A_i, B_i, K_i, $i = 1, 2, \ldots, N$, we present an algorithm to design the scalars λ_p, $p \in \{i_s, i_u\}$ and μ_{pq}, $p, q \in \{i_s, i_u\}$, $i = 1, 2, \ldots, N$ such that the above conditions are met.

Definition 3: A cycle $W = v_0, (v_0, v_1), v_1, \ldots, v_{n-1}$, (v_{n-1}, v_0), v_0 on $\mathcal{G}(\mathcal{V}, \mathcal{E})$ is called *candidate contractive*, if for each $i = 1, 2, \ldots, N$, there exists at least one v_j, $j \in \{0, 1, \ldots, n-1\}$ such that $\ell_{v_j}(i) = i_s$.

In view of Definition 2, for T-contractivity of $W = v_0, (v_0, v_1), v_1, \ldots, (v_{n-1}, v_0), v_0$, we require that the condition $\Xi_i(W) < 0$ holds for all $i = 1, 2, \ldots, N$. Since for each $i = 1, 2, \ldots, N$, the scalars $\ln \lambda_{i_u}$, $\ln \mu_{i_s i_u}$, $\ln \mu_{i_u i_s} \geq 0$, the existence of at least one v_j, $j \in \{0, 1, \ldots, n-1\}$ in W such that $\ell_{v_j}(i) = i_s$ is necessary. A candidate contractive cycle satisfies this property. Fix an $i \in \{1, 2, \ldots, N\}$. We let $\overline{N}_{pq}$ denote the total number of times $\ell_{v_j}(i) = p$ and $\ell_{v_{j+1}}(i) = q$ appear in W, $p, q \in \{i_s, i_u\}$, $j = 0, 1, \ldots, n-1$, $v_n := v_0$.

Observation 1: Let $W = v_0, (v_0, v_1), v_1, \ldots, v_{n-1}, (v_{n-1}, v_0), v_0$ be a candidate contractive cycle on $\mathcal{G}$. Suppose that there exist integers $T_{v_j} > 0$, $j = 0, 1, \ldots, n-1$ such that the following set of inequalities holds:

$$- |\ln \lambda_{i_s}| \left(\sum_{\substack{j=0,1,\ldots,n-1 | \\ \ell_{v_j}(i) = i_s}} T_{v_j} \right) + |\ln \lambda_{i_u}| \left(\sum_{\substack{j=0,1,\ldots,n-1 | \\ \ell_{v_j}(i) = i_u}} T_{v_j} \right)$$
$$+ (\ln \mu_{i_s i_u}) \overline{N}_{i_s i_u} + (\ln \mu_{i_u i_s}) \overline{N}_{i_u i_s} < 0, \quad i = 1, 2, \ldots, N \tag{11}$$

where the scalars λ_p, $p \in \{i_s, i_u\}$ and μ_{pq}, $p, q \in \{i_s, i_u\}$, $i = 1, 2, \ldots, N$ are as described in Facts 1 and 2, respectively. Then W is T-contractive with T-factors T_{v_j} associated with the vertices v_j, $j = 0, 1, \ldots, n-1$. ∎

In view of the definitions of vertex and edge weights $\overline{w}(v)$, $v \in \mathcal{V}$ and $\underline{w}(u, v)$, $(u, v) \in \mathcal{E}$ of $\mathcal{G}$, the above observation follows immediately from (9). A stabilizing scheduling policy γ constructed by employing cycle W is periodic with period $\sum_{j=0}^{n-1} T_{v_j}$. Notice that we do not consider the terms $\overline{N}_{pq}, p, q \in \{i_s, i_u\}$, $p = q$ for the candidate contractive cycle W, which is no loss of generality. Indeed, from [17, Prop. 1], we have that $\ln \mu_{i_s i_s} = \ln \mu_{i_u i_u} = 0$, $i = 1, 2, \ldots, N$.

Given the matrices A_i, B_i, K_i, $i = 1, 2, \ldots, N$, and a candidate contractive cycle W, our next algorithm finds pairs (P_p, λ_p), $p \in \{i_s, i_u\}$, $i = 1, 2, \ldots, N$ such that condition (11) holds.

Remark 7: The pairs (P_p, λ_p), $p \in \{i_s, i_u\}$ are solutions to the following set of bilinear matrix inequalities (BMIs):

$$A_{i_s}^{\top} P_{i_s} A_{i_s} - \lambda_{i_s} P_{i_s} \preceq 0, \;\; P_{i_s} \succ 0, \;\; 0 < \lambda_{i_s} < 1$$
$$A_{i_u}^{\top} P_{i_u} A_{i_u} - \lambda_{i_u} P_{i_u} \preceq 0, \;\; P_{i_u} \succ 0, \;\; \lambda_{i_u} \geq 1. \tag{12}$$

In general, solving BMIs is a numerically difficult task. We will use a grid-based approach, where the BMIs are transformed into LMIs—solution tools which are widely available.

Algorithm 2: Design of a T-contractive Cycle.

Consider the NCS described in Section II, and its underlying weighted directed graph $\mathcal{G}(\mathcal{V}, \mathcal{E})$.

Input: matrices A_i, B_i, K_i, $i = 1, 2, \ldots, N$, a candidate contractive cycle $W = v_0, (v_0, v_1), v_1, \ldots, v_{n-1}, (v_{n-1}, v_0), v_0$

Output: T-factors for W

Step I: Compute the matrices A_{i_s} and A_{i_u}, $i = 1, 2, \ldots, N$

for $i = 1, 2, \ldots, N$ **do**
  Set $A_{i_s} = A_i + B_i K_i$ and $A_{i_u} = A_i$
end for

Step II: Compute the integers $\overline{N}_{pq}$, $p, q \in \{i_s, i_u\}$, $i = 1, 2, \ldots, N$

for $i = 1, 2, \ldots, N$ **do**
  Compute $\overline{N}_{pq}$, $p, q \in \{i_s, i_u\}$ from W
end for

Step III: Fix a set of values for $\lambda_{i_s} \in]0, 1[$, $i = 1, 2, \ldots, N$

Fix a step-size $h_s > 0$ (small enough) and compute $k_s > 0$ such that k_s is the maximum integer satisfying $k_s h_s < 1$
for $i = 1, 2, \ldots, N$ **do**
  Set $\Lambda_i^S = \{h_s, 2h_s, \ldots, k_s h_s\}$
end for

Step IV: Fix a set of values for $\lambda_{i_u} \in [1, +\infty[$, $i = 1, 2, \ldots, N$

Fix a step-size $h_u > 0$ (small enough) and compute $k_u > 0$ such that k_u is the maximum integer satisfying $k_u h_u < 1$
for $i = 1, 2, \ldots, N$ **do**
  Set $\Lambda_i^U = \emptyset$
  for $\eta_i = h_u, 2h_u, \ldots, k_u h_u$ **do**
    if $\eta_i A_i$ is Schur stable **then**
      Add element $\frac{1}{\eta_i^2}$ to the set Λ_i^U
    end if
  end for
end for

Step V: Check for pairs (P_p, λ_p), $p \in \{i_s, i_u\}$, $i = 1, 2, \ldots, N$ under which W is T-contractive

for all pairs $(\lambda_{i_s}, \lambda_{i_u})$, $\lambda_{i_s} \in \Lambda_i^S$, $i_u \in \Lambda_i^U$, $i = 1, 2, \ldots, N$ **do**
  Solve the following feasibility problem in P_p, $p \in \{i_s, i_u\}$:

$$\text{minimize} \quad 1$$
$$\text{subject to} \quad \begin{cases} A_{i_s}^\top P_{i_s} A_{i_s} - \lambda_{i_s} P_{i_s} \preceq 0, \\ A_{i_u}^\top P_{i_u} A_{i_u} - \lambda_{i_u} P_{i_u} \preceq 0, \\ P_{i_s}, P_{i_u} \succ 0, \\ \kappa I \preceq P_{i_s}, P_{i_u} \preceq I, \kappa > 0 \end{cases} \tag{13}$$

  if there is a solution to (13) **then**
    Compute $\mu_{i_s i_u} = \lambda_{\max}(P_{i_u} P_{i_s}^{-1})$ and $\mu_{i_u i_s} = \lambda_{\max}(P_{i_s} P_{i_u}^{-1})$

Algorithm 2: *Continued.*

    Solve the following feasibility problem in T_{v_j}, $j = 0, 1, \ldots, n-1$:

$$\text{minimize} \quad 1$$
$$\text{subject to} \quad \begin{cases} T_{v_j} > 0, \ \ j = 0, 1, \ldots, n-1, \\ \text{condition (11)}. \end{cases} \tag{14}$$

    if there is a solution to (14) **then**
      Output T_{v_j}, $j = 0, 1, \ldots, n-1$ and exit Algorithm 2
    end if
  end if
end for

In Algorithm 2, we employ a grid-based approach to design the pairs (P_{i_s}, λ_{i_s}) and (P_{i_u}, λ_{i_u}) such that with the definition (4), inequality (5) holds.[4] Scalars λ_{i_s} and λ_{i_u} vary over the sets Λ_i^S and Λ_i^U, respectively. The elements of Λ_i^S belong to the interval]0,1[, while the set Λ_i^U is determined as follows: a scalar η_i varies over]0,1[with step-size h_u, and the estimates $\frac{1}{\eta_i^2}$ satisfying $\eta_i A_i$ is Schur stable are stored in Λ_i^U. For a fixed pair $(\lambda_{i_s}, \lambda_{i_u})$ with $\lambda_{i_s} \in \Lambda_i^S$ and $\lambda_{i_u} \in \Lambda_i^S$, the following set of LMIs is solved:

$$\begin{aligned} A_{i_s}^\top P_{i_s} A_{i_s} - \lambda_{i_s} P_{i_s} &\preceq 0 \\ A_{i_u}^\top P_{i_u} A_{i_u} - \lambda_{i_u} P_{i_u} &\preceq 0. \end{aligned} \tag{15}$$

If a solution to (15) is found, then the scalars $\mu_{i_s i_u}$ and $\mu_{i_u i_s}$ are computed using the estimates given in [17, Prop. 1]. The feasibility problem (14) is then solved with the above estimates of $\lambda_{i_s}, \lambda_{i_u}, \mu_{i_s i_u}, \mu_{i_u i_s}$. If there is a solution to (14), then the values of T_{v_j}, $j = 0, 1, \ldots, n-1$ are stored and Algorithm 2 terminates. Otherwise, the pair $(\lambda_{i_s}, \lambda_{i_u})$ is updated and the above process is repeated.

Remark 8: The condition $\kappa I \preceq P_{i_s}, P_{i_u} \preceq I$ in the feasibility problem (13) is not inherent to the inequalities (12). It is included for numerical reasons, in particular, $\kappa I \preceq P_{i_s}, P_{i_u}$ limits the condition numbers of P_{i_s} and P_{i_u} to κ^{-1}, and the condition $P_{i_s}, P_{i_u} \preceq I$ guarantees that the set of feasible P_{i_s}, P_{i_u} is bounded.

Remark 9: Notice that even if the step-sizes h_s and h_u are chosen to be very small, only a finite number of possibilities for (P_p, λ_p), $p \in \{i_s, i_u\}$, $i = 1, 2, \ldots, N$ are explored in Algorithm 2. Consequently, if no solution to the feasibility problem (14) is found, then it is not immediate whether there are indeed no pairs (P_{i_s}, λ_{i_s}) and (P_{i_u}, λ_{i_u}), $i = 1, 2, \ldots, N$ for the given matrices A_i, B_i, K_i such that there are integers T_{v_j}, $j = 0, 1, \ldots, n-1$ satisfying condition (11). Algorithm 2, therefore, offers only a partial solution to the problem of designing suitable matrices P_p and the scalars λ_p, $p \in \{i_s, i_u\}$, $i = 1, 2, \ldots, N$ in the sense that the algorithm does not conclude about their nonexistence. It is, therefore, of interest to identify

[4] Alternatively, one could also use the *path-following method* proposed in [8].

sufficient conditions under which the feasibility problem (14) admits a solution. We discuss this matter next.

The existence of a solution to the feasibility problem (14) depends on the choice of a candidate contractive cycle W and the scalars λ_p, $p \in \{i_s, i_u\}$, $i = 1, 2, \dots, N$.[5] The first component above is governed by the given numbers M and N. Recall that for a vertex $v \in \mathcal{V}$, $\ell_v(m)$ denotes the m-th element of its label $L(v)$. Let v^s denote the set of elements $j_1, j_2, \dots, j_M \in \{1, 2, \dots, N\}$ satisfying $\ell_v(j_p) = j_{p_s}$, $p = 1, 2, \dots, N$. Below, we propose a set of sufficient conditions on the scalars λ_p, $p \in \{i_s, i_u\}$, $i = 1, 2, \dots, N$ and the number M under which the feasibility problem (14) admits a solution.

Proposition 1: Let $M = 1$. Consider a candidate contractive cycle $W = v_0, (v_0, v_1), v_1, \dots, v_{N-1}, (v_{N-1}, v_0), v_0$ on $\mathcal{G}(\mathcal{V}, \mathcal{E})$ that satisfies $v_k^s \cap v_\ell^s = \emptyset$ for all $k, \ell = 0, 1, \dots, N-1$, $k \neq \ell$. Suppose that the scalars λ_p, $p \in \{i_s, i_u\}$, $i = 1, 2, \dots, N$ satisfy

$$|\ln \lambda_{i_s}| - (N-1)\,|\ln \lambda_{i_u}| > 0, \ \ i = 1, 2, \dots, N. \tag{16}$$

Then there exists $\tilde{T} \in \mathbb{N}$ such that the cycle W is T-contractive with $T_{v_j} = \tilde{T} > 0$, $j = 0, 1, \dots, N-1$.

Proposition 2: Let $M \geq N/2$. Consider a candidate contractive cycle $W = v_0, (v_0, v_1), v_1, (v_1, v_0), v_0$ on $\mathcal{G}(\mathcal{V}, \mathcal{E})$ that satisfies $v_1^s \supset \{1, 2, \dots, N\} \setminus v_0^s$. Suppose that the scalars λ_p, $p \in \{i_s, i_u\}$, $i = 1, 2, \dots, N$ satisfy

$$|\ln \lambda_{i_s}| - |\ln \lambda_{i_u}| > 0, \ \ i = 1, 2, \dots, N. \tag{17}$$

Then there exists $\tilde{T} \in \mathbb{N}$ such that the cycle W is T-contractive with $T_{v_0} = T_{v_1} = \tilde{T}$.

Proposition 1 deals with the case when exactly one plant is allowed to access the shared communication channel at any time instant, while Proposition 2 deals with the case where at least half of the total number of plants have access to the shared communication channel. In case of the former, a T-contractive cycle contains exactly one vertex v_j with $\ell_{v_j}(i) = i_s$ for each i, $j = 0, 1, \dots, N-1$, while in case of the latter, $\ell_{v_j}(i) = i_s$ for each i is accommodated in two vertices, $j = 0, 1$, $i = 1, 2, \dots, N$. Condition (17) is a relaxation of condition (16). We present concise proofs of Propositions 1 and 2 in the Appendix.

Example 1: Consider $N = 3$ with

$$(A_1, B_1, K_1) = \left(\begin{pmatrix} 0.2 & 0.7 \\ 1.6 & 0.1 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} -0.2752 & -0.6705 \end{pmatrix}\right)$$

$$(A_2, B_2, K_2) = \left(\begin{pmatrix} 1 & 0.1 \\ 0.1 & 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \begin{pmatrix} -0.9137 & -0.9505 \end{pmatrix}\right)$$

$$(A_3, B_3, K_3) = \left(\begin{pmatrix} 1.2 & 0.2 \\ 0.1 & 0.9 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} -1.0757 & -0.4839 \end{pmatrix}\right).$$

[5]Notice that while the scalars μ_{pq}, $p, q \in \{i_s, i_u\}$, $i = 1, 2, \dots, N$ affect the choice of T-factors that solve the feasibility problem (14), they do not affect the existence of a solution to (14). Indeed, given the scalars λ_p, $p \in \{i_s, i_u\}$ and μ_{pq}, $p, q \in \{i_s, i_u\}$, $i = 1, 2, \dots, N$, and a candidate contractive cycle $W = v_0, (v_0, v_1), v_1, \dots, v_{n-1}, (v_{n-1}, v_0), v_0$ on $\mathcal{G}$, if there exists $T_{v_j} = \tilde{T}$, $j = 0, 1, \dots, n-1$ such that condition (11) holds, then it follows that condition (11) holds for any $T' \geq \tilde{T}$.

Corresponding to $V_p(\xi) = \xi^\top P_p \xi$, $p \in \{i_s, i_u\}$, $i = 1, 2, 3$, we obtain the following estimates of the scalars λ_p, $p \in \{i_s, i_u\}$ and μ_{pq}, $p, q \in \{i_s, i_u\}$, $i = 1, 2, 3$:

$$\begin{aligned} \lambda_{1_s} &= 0.2787, & \lambda_{1_u} &= 1.5625, & \mu_{1_s 1_u} &= 4.1786, \\ \mu_{1_u 1_s} &= 1.5338, & \lambda_{2_s} &= 0.0859, & \lambda_{2_u} &= 1.2346, \\ \mu_{2_s 2_u} &= 23.5578, & \mu_{2_u 2_s} &= 1.9130, & \lambda_{3_s} &= 0.2147, \\ \lambda_{3_u} &= 2.0408, & \mu_{3_s 3_u} &= 3.6524, & \mu_{3_u 3_s} &= 2.5238. \end{aligned}$$

Let $M = 1$. We have that condition (16) holds. Indeed, $|\ln \lambda_{1_s}| - 2|\ln \lambda_{1_u}| = 0.3850 > 0$, $|\ln \lambda_{2_s}| - 2|\ln \lambda_{2_u}| = 2.0331 > 0$, $|\ln \lambda_{3_s}| - 2\,|\ln \lambda_{3_u}| = 0.1118 > 0$. The cycle $W_1 = v_0, (v_0, v_1), v_1, (v_1, v_2), v_2, (v_2, v_0), v_0$, where $\ell_{v_0}(1) = 1_s$, $\ell_{v_1}(2) = 2_s$ and $\ell_{v_2}(3) = 3_s$, is T-contractive with $T_{v_0} = T_{v_1} = T_{v_2} = \tilde{T} = 20$. We have $\Xi_1(W_1) = -6.0596$, $\Xi_2(W_1) = -36.85$, $\Xi_3(W_1) = -0.0154$.

Now, let $M = 2 (> N/2)$. Since the scalars λ_p, $p \in \{i_s, i_u\}$, $i = 1, 2, 3$ satisfy (16), it is immediate that (17) holds. The cycle $W_2 = v_0, (v_0, v_1), v_1, (v_1, v_0), v_0$, where $\ell_{v_0}(1) = 1_s$, $\ell_{v_0}(2) = 2_s$ and $\ell_{v_1}(2) = 2_s$, $\ell_{v_1}(3) = 3_s$, is T-contractive with $T_{v_0} = T_{v_1} = \tilde{T} = 5$. Indeed, $\Xi_1(W_2) = -2.2990$, $\Xi_2(W_2) = -24.5457$, $\Xi_3(W_2) = -1.9047$.

Remark 10: Both in Propositions 1 and 2, we consider the simplest setting where the T-factors associated with all vertices that appear in $W = v_0, (v_0, v_1), v_1, \dots, v_{n-1}, (v_{n-1}, v_0), v_0$, are the same. However, this choice of T-factors can also be extended to non-equal T_{v_j}, $j = 0, 1, \dots, n-1$. For instance, in Example 1, the candidate contractive cycle $W_2 = v_0, (v_0, v_1), v_1, (v_1, v_0), v_0$ is also T-contractive with $T_{v_0} = 5$ and $T_{v_1} = 4$. It follows that $\Xi_1(W_2) = -2.7452$, $\Xi_2(W_2) = -22.0911$, $\Xi_3(W_2) = -0.3662$.

Remark 11: Switched systems have appeared before in NCSs literature, see, for example, [5], [14], [22], [33], and average dwell-time switching logic has proven to be a useful tool. In the presence of unstable systems, stabilizing average dwell-time switching involves two conditions on $]0 : t]$ for every $t \in \mathbb{N}$ [26]: i) an upper bound on the number of switches and ii) a lower bound on the ratio of durations of activation of stable to unstable subsystems. In contrast, our design of a stabilizing scheduling policy involves the design of a T-contractive cycle on the underlying weighted directed graph of the NCS. To design these cycles, we solve the feasibility problems (13) and (14). Condition (9) does not involve nor imply restrictions on the behavior of a scheduling policy on every interval $]0 : t]$, $t \in \mathbb{N}$.

Remark 12: In the recent past, multiple Lyapunov-like functions and graph-theoretic tools are widely used to construct stabilizing switching logics for switched systems, see, for example, [16]–[18]. A weighted directed graph is associated with a family of systems and the admissible transitions between them, and a switching logic is expressed as an infinite walk on this weighted directed graph. Infinite walks, whose corresponding switching logics preserve stability, are constructed by employing negative weight cycles; see [16, §3], [17, §3], [18, §3] for details. In this paper, instead of studying GAS of *a* switched system, we analyze "simultaneous" GAS of N-switched systems, each containing one asymptotically stable and one unstable subsystem. For that purpose, a stabilizing scheduling policy is designed by incorporating multiple switching logics, each of which is stabilizing. Not surprisingly, the design of T-contractive cycles transcends beyond identifying negative weight cycles on a weighted directed graph: it involves the selection of T-factors that preserve GAS of all N plants, where every T-factor adds to the negativity of $\Xi_i(W)$ for M plants and to the positivity of $\Xi_i(W)$ for the remaining N–M plants. In addition, so far in the literature, negative weight cycles for the stability of *a* switched system are designed under the assumption that the Lyapunov-like functions V_p, $p \in \{i_s, i_u\}$ and the corresponding scalars $\lambda_p, p \in \{i_s, i_u\}, i \in \{1, 2, \ldots, N\}$ are "given"; see, for example, [16, Remark 9], [17, Remark 9], [18, §2.2] for discussions. In contrast, in this paper, we deal with the harder problem of identifying T-contractive cycles on $\mathcal{G}$, and design multiple Lyapunov-like functions V_p and the corresponding scalars $\lambda_p, p \in \{i_s, i_u\}, i = 1, 2, \ldots, N$ such that these cycles exist.

Remark 13: Optimal scheduling policies for remote state estimation in sensor networks have been studied recently in [7], [19], and [20]. In the context of our results, one can utilize properties of T-contractive cycles on $\mathcal{G}$ to achieve the optimal stability margin for a scheduling policy. Notice that the choice of T-factors for a T-contractive cycle on the underlying weighted directed graph of the NCS under consideration is not unique. In addition, the choice of a T-contractive cycle itself is not unique. It is clear that employing "any" T-contractive cycle W on $\mathcal{G}$ is sufficient to construct a stabilizing periodic scheduling policy as far as GAS of each plant i in (1) is concerned. Fix $i \in \{1, 2, \ldots, N\}$. Any T-contractive cycle yields $\Xi_i(W) = -\varepsilon_i$ for some $\varepsilon_i > 0$. We observe that as ε_i increases, the rate of convergence of $\|x_i(t)\|$ improves; see also the experimental results in Section V.

Remark 14: The nonuniqueness of T-factors and T-contractive cycles described in Remark 13 can be exploited to extend our results to the setting of a static scheduling policy with a nonperiodic structure. Indeed, suppose that W_1 and W_2 are two distinct (different in terms of either T-factors or vertices) T-contractive cycles on $\mathcal{G}$. Then a scheduling policy of a nonperiodic structure can be generated by concatenating W_1 and W_2, for example, $W_1W_2W_1W_2W_2W_1W_2W_2W_2\ldots$. Such a scheduling policy is static because the allocation sequences of the shared communication channel are computed offline, but the sequences are applied in a nonperiodic manner.

V. Numerical Experiment

A. The NCS

Consider an NCS with $N = 5$ discrete-time linear plants and a shared communication channel of limited capacity. The matrices $A_i \in \mathbb{R}^{2\times 2}$, $B_i \in \mathbb{R}^{2\times 1}$, and $K_i \in \mathbb{R}^{1\times 2}$, $i = 1, 2, 3, 4, 5$ are chosen as follows; numerical values are given in Table I.

- Elements of A_i are selected from the interval $[-2, 2]$ uniformly at random.
- Elements of B_i are selected by picking values from the $\{0, 1\}$.

TABLE I
Description of Individual Plants in the NCS

i	A_i	B_i	K_i	$\lvert\lambda(A_i)\rvert$	$\lvert\lambda(A_i + B_iK_i)\rvert$
1	$\begin{pmatrix} 1.0310 & 0.9725 \\ -0.4311 & 0.6219 \end{pmatrix}$	$\begin{pmatrix} 1 \\ 0 \end{pmatrix}$	$\begin{pmatrix} -0.9869 & -0.7541 \end{pmatrix}$	1.0298, 1.0298	0.3487, 0.3487
2	$\begin{pmatrix} 0.8375 & 1.0187 \\ -0.8959 & 0.7188 \end{pmatrix}$	$\begin{pmatrix} 0 \\ 1 \end{pmatrix}$	$\begin{pmatrix} 0.4978 & -1.0887 \end{pmatrix}$	1.2307, 1.2307	0.3095, 0.3095
3	$\begin{pmatrix} 1.2571 & -1.0259 \\ 1.7171 & -0.6001 \end{pmatrix}$	$\begin{pmatrix} 1 \\ 0 \end{pmatrix}$	$\begin{pmatrix} -0.7247 & 0.8152 \end{pmatrix}$	1.0036, 1.0036	0.2056, 0.2056
4	$\begin{pmatrix} 0.7569 & 0.9926 \\ -0.1978 & -1.6647 \end{pmatrix}$	$\begin{pmatrix} 1 \\ 1 \end{pmatrix}$	$\begin{pmatrix} -0.0933 & 0.8329 \end{pmatrix}$	0.6729, 1.5807	0.0826, 0.2508
5	$\begin{pmatrix} 0.5294 & -1.6098 \\ -0.8860 & 0.1875 \end{pmatrix}$	$\begin{pmatrix} 0 \\ 1 \end{pmatrix}$	$\begin{pmatrix} 0.9852 & -0.6016 \end{pmatrix}$	1.5649, 0.8480	0.3085, 0.1932

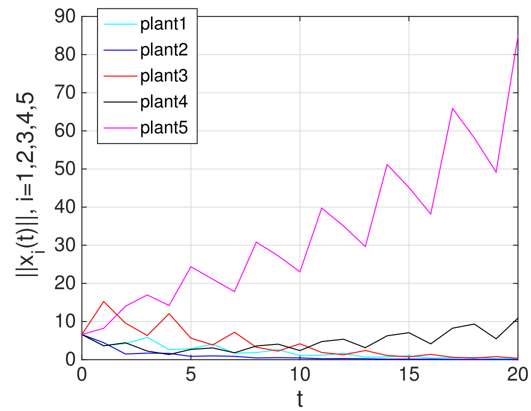

Fig. 3. Not all plants are GAS under round-robin scheduling.

TABLE II
Description of Scalars Admitting a Solution to the Feasibility Problem (14)

i	λ_{i_s}	λ_{i_u}	$\mu_{i_s i_u}$	$\mu_{i_u i_s}$
1	0.1360	1.2346	2.8452	1.3232
2	0.0720	1.2346	1.5681	1.3509
3	0.0715	1.2346	1.9025	1.3046
4	0.1757	2.7778	3.0854	1.1665
5	0.2430	2.7778	3.4664	1.1576

- It is ensured that the pair (A_i, B_i) is controllable; K_i is the discrete-time linear quadratic regulator for (A_i, B_i) with $Q_i = Q = 5I_{2\times 2}$ and $R_i = R = 1$.

Suppose that $M = 2$ plants are allowed to access the communication channel at every instant of time.

B. Non-triviality

We note that designing a stabilizing scheduling policy in the above setting is not a trivial problem. Indeed, consider a round-robin scheduling policy plants 1 and 2 followed by plants 2 and 3, and then followed by plants 4 and 5 accessing the channel, each combination being active for 1 unit of time. In Fig. 3, we demonstrate that plants 4 and 5 are unstable under this scheduling policy, and consequently, a careful design of γ is essential.

C. The Underlying Weighted Directed Graph

We construct the underlying directed graph $\mathcal{G}(\mathcal{V}, \mathcal{E})$ of the NCS under consideration. For the given setting, we have

TABLE III

DESCRIPTION OF DIFFERENT T-CONTRACTIVE CYCLES ON $\mathcal{G}$

j	v_0	v_1	v_2	T_{v_0}	T_{v_1}	T_{v_2}	$\Xi_1(W_j)$	$\Xi_2(W_j)$	$\Xi_3(W_j)$	$\Xi_4(W_j)$	$\Xi_5(W_j)$
1	$\overline{v}_5$	$\overline{v}_3$	$\overline{v}_9$	2	7	8	−10.5325	−1.3503	−23.9963	−0.67556	−0.73315
2	$\overline{v}_2$	$\overline{v}_6$	$\overline{v}_7$	3	8	9	−1.0769	−43.3456	−3.4224	−0.37122	−0.10453
3	$\overline{v}_8$	$\overline{v}_9$	$\overline{v}_1$	8	9	3	−1.0769	−3.5599	−43.3057	−0.37122	−0.10453

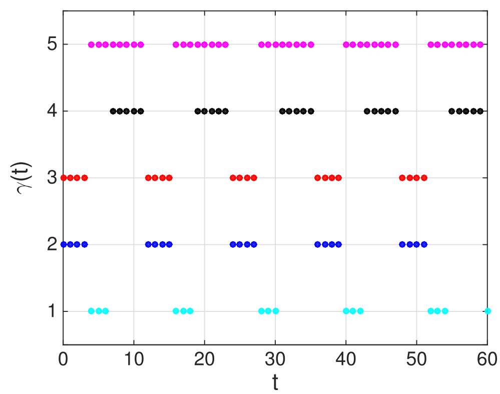

Fig. 4. Scheduling policy γ obtained from Algorithm 1.

$\binom{N}{M} = 10$. $\mathcal{G}$ consists of

- $\mathcal{V} = \{\overline{v}_1, \overline{v}_2, \dots, \overline{v}_{10}\}$ with
  $L(\overline{v}_1)=\{1_s, 2_s, 3_u, 4_u, 5_u\}$, $L(\overline{v}_2)=\{1_s, 2_u, 3_s, 4_u, 5_u\}$,
  $L(\overline{v}_3)=\{1_s, 2_u, 3_u, 4_s, 5_u\}$, $L(\overline{v}_4)=\{1_s, 2_u, 3_u, 4_u, 5_s\}$,
  $L(\overline{v}_5)=\{1_u, 2_s, 3_s, 4_u, 5_u\}$, $L(\overline{v}_6)=\{1_u, 2_s, 3_u, 4_s, 5_u\}$,
  $L(\overline{v}_7)=\{1_u, 2_s, 3_u, 4_u, 5_s\}$, $L(\overline{v}_8)=\{1_u, 2_u, 3_s, 4_s, 5_u\}$,
  $L(\overline{v}_9)=\{1_u, 2_u, 3_s, 4_u, 5_s\}$, $L(\overline{v}_{10})=\{1_u, 2_u, 3_u, 4_s, 5_s\}$,
  and
- $\mathcal{E} = \{(\overline{v}_p, \overline{v}_q),\ \ p, q = 1, 2, \dots, 10,\ \ p \neq q\}$.

D. A T-contractive Cycle

Fix a candidate contractive cycle $W = v_0, (v_0, v_1), v_1, (v_1, v_2), v_2, (v_2, v_0), v_0$ on $\mathcal{G}$, where $v_0 = \overline{v}_5$, $v_1 = \overline{v}_4$, $v_2 = \overline{v}_{10}$. We apply Algorithm 2 with $h_s = 0.0001$ and $h_u = 0.1$, and obtain that W is T-contractive with T-factors: $T_{v_0} = 4$, $T_{v_1} = 3$, $T_{v_2} = 5$. Indeed, $\Xi_1(W) = -2.7629$, $\Xi_2(W) = -8.0877$, $\Xi_3(W) = -7.9572$, $\Xi_4(W) = -0.2626$, and $\Xi_5(W) = -5.8414$. The corresponding values of the scalars λ_p, $p \in \{i_s, i_u\}$ and μ_{pq}, $p, q \in \{i_s, i_u\}$, $i = 1, 2, \dots, N$ are given in Table II.

E. The Scheduling Policy

A scheduling policy γ is obtained from Algorithm 1. γ is constructed by employing W, and it is periodic with period $T_{v_0} + T_{v_1} + T_{v_2} = 12$ units of time. In Fig. 4, we illustrate γ until time $t = 60$.

F. GAS of NCS

We choose 100 different initial conditions from the interval $[-10, 10]^2$ uniformly at random, and plot $(\|x_i(t)\|)_{t \in \mathbb{N}_0}$ under the scheduling policy γ, $i = 1, 2, 3, 4, 5$. Fig. 5 contains plots for $\|x_i(t)\|$, $i = 1, 2, 3, 4, 5$ until time $t = 60$. It is observed that the individual plants of the NCS under consideration are GAS under our scheduling policy.

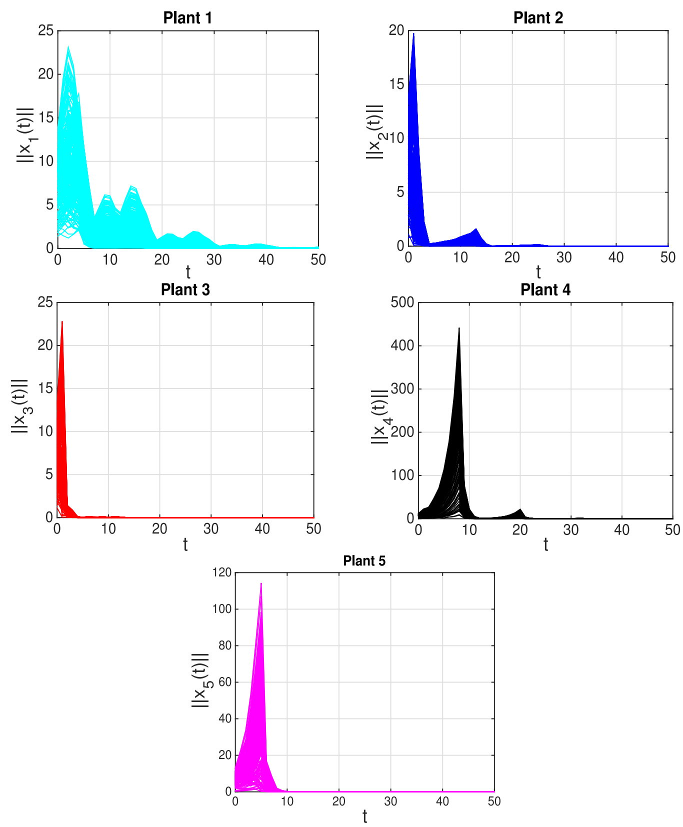

Fig. 5. Plot for $\|x_i(t)\|$ versus t for each plant $i = 1, 2, 3, 4, 5$.

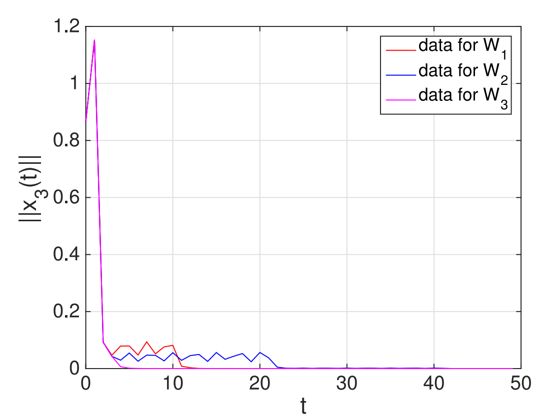

Fig. 6. Plot for $\text{average}_{x_3(0)} \|x_3(t)\|$ versus t corresponding to cycles W_j.

G. Comparison

We choose three distinct T-contractive cycles $W_j = v_0^{(j)}, (v_0^{(j)}, v_1^{(j)}), v_1^{(j)}, (v_1^{(j)}, v_2^{(j)}), v_2^{(j)}, (v_2^{(j)}, v_0^{(j)}), v_0^{(j)}$ on $\mathcal{G}$. The description of the cycles and the corresponding values of $\Xi_i(W_j)$, $j = 1, 2, 3$, $i = 1, 2, 3, 4, 5$ are given in Table III. We now illustrate that with smaller values of $\Xi_i(W_j)$, the rate of convergence of $\|x_i(t)\|$ to 0 becomes faster. For this purpose, we pick 10 different initial conditions $x_i(0)$ from the interval $[-1, 1]^2$ uniformly at random and simulate $(\|x_i(t)\|)_{t\in\mathbb{N}_0}$ for the cycles W_j. Fig. 6 contains plots for $\text{average}_{x_i(0)}(\|x_i(t)\|)_{t\in\mathbb{N}_0}$ for the plant $i = 3$ corresponding to the cycles W_j.

VI. Concluding Remark

In this paper, we presented a stabilizing scheduling policy for NCSs under medium-access constraints. A switched system representation is associated with the individual plants, and a weighted directed graph is associated with the NCS. Our scheduling policy is designed by employing a T-contractive cycle on the underlying weighted directed graph of the NCS. We also address an algorithmic construction of T-contractive cycles. Since our algorithm for designing T-contractive cycles does not conclude about their nonexistence, an important question is regarding the design of such cycles when our algorithm does not yield a solution for all choices of candidate contractive cycle on a weighted directed graph. Also, a natural extension of our work is to accommodate network-induced uncertainties, such as access delays and packet dropouts, in the feedback control loop. These aspects are currently under investigation and will be reported elsewhere.

Appendix

Proof of Theorem 1: Consider the NCS described in Section II and its underlying directed graph $\mathcal{G}(\mathcal{V}, \mathcal{E})$. Let $W = v_0, (v_0, v_1), v_1, \dots, v_{n-1}, (v_{n-1}, v_0), v_0$ be a T-contractive cycle on $\mathcal{G}$. Consider a scheduling policy γ obtained from Algorithm 1 constructed by employing W. We will show that each plant in (1) is GAS under γ.

Fix an arbitrary plant $i \in \{1, 2, \dots, N\}$. In view of the switched systems representation of plant i in (3), it suffices to show that the switching logic σ_i corresponding to γ ensures GAS of plant i.

Fix a time $t \in \mathbb{N}$. Recall that $0 =: \tau_0 < \tau_1 < \cdots$ are the points in time at which γ changes values. Let N_t^γ be the total number of times γ has changed its values on $]0 : t]$. In view of (5), we have

$$V_{\sigma_i(t)}(x_i(t)) \le \lambda_{\sigma_i(\tau_{N_t^\gamma})}^{t-\tau_{N_t^\gamma}} V_{\sigma_i(\tau_{N_t^\gamma})}(x_i(\tau_{N_t^\gamma})). \tag{18}$$

A straightforward iteration of (18) using (5) and (6) gives

$$V_{\sigma_i(t)}(x_i(t)) \le \left(\prod_{\substack{j=0 \\ \tau_{N_t^\gamma+1}:=t}}^{N_t^\gamma} \lambda_{\sigma_i(\tau_j)}^{\tau_{j+1}-\tau_j} \cdot \prod_{j=0}^{N_t^\gamma-1} \mu_{\sigma_i(\tau_j)\sigma_i(\tau_{j+1})} \right) V_{\sigma_i(0)}(x_i(0)). \tag{19}$$

The first term on the right-hand side of the above inequality is

$$\exp\left(\ln\left(\prod_{\substack{j=0 \\ \tau_{N_t^\gamma+1}:=t}}^{N_t^\gamma} \lambda_{\sigma_i(\tau_j)}^{\tau_{j+1}-\tau_j} \right) + \ln\left(\prod_{j=0}^{N_t^\gamma-1} \mu_{\sigma_i(\tau_j)\sigma_i(\tau_{j+1})} \right) \right).$$

Now

$$\ln\left(\prod_{\substack{j=0 \\ \tau_{N_t^\gamma+1}:=t}}^{N_t^\gamma} \lambda_{\sigma_i(\tau_j)}^{\tau_{j+1}-\tau_j} \right) = \sum_{\substack{j=0 \\ \tau_{N_t^\gamma+1}:=t}}^{N_t^\gamma} \left(\sum_{p\in\{i_s, i_u\}} 1_{\sigma_i(\tau_j)}(p)(\tau_{j+1}-\tau_j) \ln \lambda_p \right). \tag{20}$$

Let $D_s(s, t)$ and $D_u(s, t)$ denote the total durations number of time-steps of activation of the stable and unstable modes of i on $]s : t]$, respectively. Recall that $0 < \lambda_{i_s} < 1$ and $\lambda_{i_u} \ge 1$. Consequently, $\ln \lambda_{i_s} < 0$ and $\ln \lambda_{i_u} \ge 0$. Thus, the right-hand side of (20) is equal to

$$-|\ln \lambda_{i_s}|\, D_s(0, t) + |\ln \lambda_{i_u}|\, D_u(0, t). \tag{21}$$

Let $N_{pq}(s, t)$ denote the total number of transitions from subsystem (mode) p to subsystem (mode) q, $p, q \in \{i_s, i_u\}$ on $]s : t]$. We have

$$\ln\left(\prod_{j=0}^{N_t^\gamma-1} \mu_{\sigma_i(\tau_j)\sigma_i(\tau_{j+1})} \right) = \ln \mu_{i_s i_u} N_{i_s i_u}(0, t) + \ln \mu_{i_u i_s} N_{i_u i_s}(0, t) \tag{22}$$

since $\mu_{i_s i_s} = \mu_{i_u i_u} = 1$. Substituting (21) and (22) in (19), we obtain $V_{\sigma_i(t)}(x_i(t)) \le \psi_i(t) V_{\sigma_i(0)}(x_i(0))$, where

$$\mathbb{N} \ni t \mapsto \psi_i(t) := \exp\big(-|\ln \lambda_{i_s}|\, D_s(0, t) + |\ln \lambda_{i_u}|\, D_u(0, t) + \ln \mu_{i_s i_u} N_{i_s i_u}(0, t) + \ln \mu_{i_u i_s} N_{i_u i_s}(0, t) \big).$$

From the definition of V_p, $p \in \{i_s, i_u\}$ in (4) and properties of positive definite matrices [2, Lemma 8.4.3], it follows that $\|x_i(t)\| \le c\psi_i(t)\|x_i(0)\|$ for all $t \in \mathbb{N}_0$, where $c = \sqrt{(\max_{p\in\{i_s,i_u\}} \lambda_{\max}(P_p))/(\min_{p\in\{i_s,i_u\}} \lambda_{\min}(P_p))}$, where for a matrix $A \in \mathbb{R}^{d\times d}$, $\lambda_{\min}(A)$ denotes the minimum eigenvalue of A. By Definition 1, to establish GAS of (3), we need to show that $c\|x_i(0)\|\,\psi_i(t)$ can be bounded above by a class $\mathcal{KL}$ function. Toward this end, we already see that $c\|x_i(0)\|$ is a class $\mathcal{K}_\infty$ function. Therefore, it remains to show that $\psi_i(t)$ is bounded above by a function in class $\mathcal{L}$.

Recall that γ is constructed by employing a T-contractive cycle $W = v_0, (v_0, v_1), v_1, \dots, v_{n-1}, (v_{n-1}, v_0), v_0$ on $\mathcal{G}$, and T_{v_j}, $j = 0, 1, \dots, n-1$ are the T-factors associated with vertices v_j, $j = 0, 1, \dots, n-1$. Let $T_W := \sum_{j=0}^{n-1} T_{v_j}$, $t \ge mT_W$, $m \in \mathbb{N}_0$, and $\Xi_i(W) = -\varepsilon_i$, $\varepsilon_i > 0$, where $\Xi_i(W)$ is as defined in (9). With the construction of γ, we have

$$\begin{aligned}\psi_i(t) &= \exp(-|\ln\lambda_{i_s}|D_s(0,t) + |\ln\lambda_{i_u}|D_u(0,t)\\&\quad+\ln\mu_{i_s i_u}N_{i_s i_u}(0,t)+\ln\mu_{i_u i_s}N_{i_u i_s}(0,t))\\&= -|\ln\lambda_{i_s}|D_s(0,mT_W) - |\ln\lambda_{i_s}|D_s(mT_W,t)\\&\quad+|\ln\lambda_{i_u}|D_u(0,mT_W)+|\ln\lambda_{i_u}|D_u(mT_W,t)\\&\quad+\ln\mu_{i_s i_u}N_{i_s i_u}(0,mT_W)+\ln\mu_{i_s i_u}N_{i_s i_u}(mT_W,t)\\&\quad+\ln\mu_{i_u i_s}N_{i_u i_s}(0,mT_W)+\ln\mu_{i_u i_s}N_{i_u i_s}(mT_W,t). \quad (23)\end{aligned}$$

Notice that $-|\ln\lambda_{i_s}|D_s(0,mT_W)+|\ln\lambda_{i_u}|D_u(0,mT_W)+\ln\mu_{i_s i_u}N_{i_s i_u}(0,mT_W)+\ln\mu_{i_u i_s}N_{i_u i_s}(0,mT_W) = -|\ln\lambda_{i_s}|\, m\sum_{\substack{j:\ell_{v_j}(i)=i_s\\ j=0,1,\ldots,n-1}} T_{v_j} + |\ln\lambda_{i_u}|\, m\sum_{\substack{j:\ell_{v_j}(i)=i_u\\ j=0,1,\ldots,n-1}} T_{v_j} + \ln\mu_{i_s i_u} m\#(i_s\to i_u)_W + \ln\mu_{i_u i_s} m\#(i_u\to i_s)_W$, where $\#(p\to q)_W$ denotes the number of times a transition from a vertex v_j to a vertex v_{j+1} has occurred in W such that $\ell_{v_j}(i)=p$ and $\ell_{v_{j+1}}(i)=q$, $p,q\in\{i_s,i_u\}$, $p\neq q$. The right-hand side of the above equality can be rewritten as

$$m\left(-|\ln\lambda_{i_s}|\sum_{\substack{j:\ell_{v_j}(i)=i_s\\ j=0,1,\ldots,n-1}} T_{v_j} + |\ln\lambda_{i_u}|\sum_{\substack{j:\ell_{v_j}(i)=i_u\\ j=0,1,\ldots,n-1}} T_{v_j}\right.$$
$$\left.+\ln\mu_{i_s i_u}\#(i_s\to i_u)_W+\ln\mu_{i_u i_s}\#(i_u\to i_s)_W\right). \quad (24)$$

From the definition of weights associated with vertices and edges of $\mathcal{G}$, we have that the above expression is equal to $-m\varepsilon_i$. Also

$$\begin{aligned}&-|\ln\lambda_{i_s}|D_s(mT_W,t)+|\ln\lambda_{i_u}|D_u(mT_W,t)\\&+\ln\mu_{i_s i_u}N_{i_s i_u}(mT_W,t)+\ln\mu_{i_u i_s}N_{i_u i_s}(mT_W,t)\\&\le|\ln\lambda_{i_u}|(t-mT_W)+mn(\ln\mu_{i_s i_u}+\ln\mu_{i_u i_s}) := a \text{ (say)}. \quad (25)\end{aligned}$$

From (24) and (25), we obtain that the right-hand side of (23) is bounded above by $\exp\big(-m\varepsilon_i+a\big)$.

Let $\varphi_i:[0,\,t]\to\mathbb{R}$ be a function connecting $(0,\exp(a)+T_W)$, $(rT_W,\exp(-(r-1)\varepsilon_i+a))$, $(t,\exp(-m\varepsilon_i+a))$, $r=1,2,\ldots,m$, with straight line segments. By construction, φ_i is an upper envelope of $T\mapsto\psi_i(T)$ on $[0,t]$, is continuous, decreasing, and tends to 0 as $t\to+\infty$. Hence, $\varphi_i\in\mathcal{L}$.

Recall that $i\in\{1,2,\ldots,N\}$ was selected arbitrarily. It follows that our assertion holds for all plants i in (1).

Remark 15: The definition of the functions ψ_t, $i=1,2,\ldots,N$ clarifies the association of the natural logarithm with the scalars λ_p, $p\in\{i_s,i_u\}$ and μ_{pq}, $p,q\in\{i_s,i_u\}$ in the vertex and edge weights of $\mathcal{G}$, respectively, $i=1,2,\ldots,N$. The use of absolute values with $\ln\lambda_p$, $p\in\{i_s,i_u\}$, $i=1,2,\ldots,N$ allows for easy distinction between the positive and negative terms in ψ_i, $i=1,2,\ldots,N$.

Proof of Proposition 1: Let $M=1$. Fix a cycle $W=v_0,(v_0,v_1),v_1,\ldots,v_{N-1},(v_{N-1},v_0),v_0$ on $\mathcal{G}$ that satisfies $v_k^s\cap v_\ell^s=\emptyset$ for all $k,\ell=0,1,\ldots,N-1$, $l\neq\ell$. Clearly, W is a candidate contractive cycle on $\mathcal{G}$.

Without loss of generality, let us assume that $\ell_{v_{i-1}}(i)=i_s$, $i=1,2,\ldots,N$. Suppose that $T_{v_j}=\tilde{T}$, $j=0,1,\ldots,N-1$. By construction of W, the left-hand side of (11) is $-|\ln\lambda_{i_s}|T_{v_{i-1}}+|\ln\lambda_{i_u}|(\sum_{\substack{j=0\\ j\neq i-1}}^{N-1}T_{v_j})+\ln\mu_{i_s i_u}+\ln\mu_{i_u i_s}=(-|\ln\lambda_{i_s}|+(N-1)|\ln\lambda_{i_u}|)\tilde{T}+\ln\mu_{i_s i_u}+\ln\mu_{i_u i_s}$, $i=1,2,\ldots,N$. Since (16) holds, it is possible to choose an integer $\tilde{T}>0$ such that the above expression is strictly less than 0.

Proof of Proposition 2: Let $M\geq N/2$. Fix a cycle $W=v_0,(v_0,v_1),v_1,(v_1,v_0),v_0$ on $\mathcal{G}$ that satisfies $v_1^s\supset\{1,2,\ldots,N\}\setminus V_0^s$. Let $j_1,j_2,\ldots,j_M$ and $k_1,k_2,\ldots,k_{N-M}\in\{1,2,\ldots,N\}$ be the elements for which $\ell_{v_0}(j_p)=j_{p_s}$, $p=1,2,\ldots,M$ and $\ell_{v_1}(k_q)=k_{q_s}$, $q=1,2,\ldots,N-M$, respectively. We have $|\{j_1,j_2,\ldots,j_M\}|\geq|k_1,k_2,\ldots,k_{N-M}|$. It is immediate that W is a candidate contractive cycle.

Suppose that $T_{v_0}=T_{v_1}=\tilde{T}$. By construction of W, we have $\overline{N}_{i_s i_u}$, $\overline{N}_{i_u i_s}\in\{0,1\}$, $i=1,2,\ldots,N$. The left-hand side of (11) is bounded above by $(-|\ln\lambda_{i_s}|+|\ln\lambda_{i_u}|)\tilde{T}+\ln\mu_{i_s i_u}+\ln\mu_{i_u i_s}$. Since condition (17) holds, there exists $\tilde{T}>0$ such that the above expression is strictly less than 0. ■

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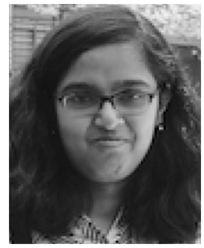

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