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A two level finite element method for Stokes constrained Dirichlet boundary control problem

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ABSTRACT

In this paper we present a finite element analysis for a Dirichlet boundary control problem governed by the Stokes equation. The Dirichlet control is considered in a convex closed subset of the energy space $\mathbf{H}^1(\Omega)$. Most of the previous works on the Stokes Dirichlet boundary control problem deals with either tangential control or the case where the flux of the control is zero. This choice of the control is very particular and their choice of the formulation leads to the control with limited regularity. To overcome this difficulty, we introduce the Stokes problem with outflow condition and the control acts on the Dirichlet boundary only hence our control is more general and it has both the tangential and normal components. We prove well-posedness and discuss on the regularity of the control problem. The first-order optimality condition for the control leads to a Signorini problem. We develop a two-level finite element discretization by using $\mathbf{P}_1$ elements (on the fine mesh) for the velocity and the control variable and P_0 elements (on the coarse mesh) for the pressure variable. The standard energy error analysis gives $\frac{1}{2}+\frac{\delta}{2}$ order of convergence when the control is in $\mathbf{H}^{\frac{3}{2}+\delta}(\Omega)$ space. Here we have improved it to $\frac{1}{2}+\delta$, which is optimal. Also, when the control lies in less regular space we derived optimal order of convergence up to the regularity. The theoretical results are corroborated by a variety of numerical tests.

1. Introduction

In this paper, we consider the following Dirichlet boundary control problem governed by Stokes equation

$$\min \ J(\mathbf{u},\mathbf{y}) = \frac{1}{2}\|\mathbf{u}-\mathbf{u}_d\|^2_{0,\Omega} + \frac{\rho}{2}\|\nabla \mathbf{y}\|^2_{0,\Omega} \tag{1.1}$$

subject to,

$$\begin{aligned} -\Delta \mathbf{u} + \nabla p &= \mathbf{f} \quad \text{in } \Omega,\\ \nabla\cdot\mathbf{u} &= 0 \quad \text{in } \Omega,\\ \mathbf{u} &= \mathbf{y} \quad \text{on } \Gamma_C,\\ \mathbf{u} &= \mathbf{0} \quad \text{on } \Gamma_D,\\ \frac{\partial \mathbf{u}}{\partial \mathbf{n}} - p\mathbf{n} &= \mathbf{0} \ \text{ on } \Gamma_N, \end{aligned} \tag{1.2}$$

with the control constraints

$$\mathbf{y}_a \le \gamma_0(\mathbf{y}) \le \mathbf{y}_b \text{ on } \Gamma_C.$$

In Section 2, we elaborate on the aforementioned problem. We propose a finite element approximation of the state and the control variable in order to discretize the above system. The first discussion on discretization of optimal control problems governed by partial differential equations was in the papers of Falk [14], Gevici [16]. Subsequently, many significant contributions have been made to this field. A control can act in the interior of a domain, in this case, we call distributed, or on the boundary of a domain, we call boundary (Neumann or Dirichlet) control problem. We refer to [11,22] for distributed control related problem, to [6,11] for the Neumann boundary control problem.

The Dirichlet boundary optimal control problems play an important role in the context of the computational fluid dynamics, see, e.g., [15, 21]. The *a priori* error analysis for such problems can be traced back to [7]. The literature on Dirichlet boundary control problem outlines various approaches. One typical method is to choose control from the $L^2(\partial\Omega)$-space:

$$\min_{y\in L^2(\Gamma)} J(u,y) := \frac{1}{2}\|u-u_d\|^2_{0,\Omega} + \frac{\rho}{2}\|y\|^2_{0,\partial\Omega}, \tag{1.3}$$

subject to Poisson problem

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$$\Delta u = f \ \text{in}\ \Omega, \quad u = y \ \text{on}\ \partial\Omega. \tag{1.4}$$

Due to the fact that the Dirichlet data is only in space $L^2(\partial\Omega)$, we need to understand the state equation (1.4) in an ultra-weak sense. This ultra weak formulation is easy to implement and usually results in optimal controls with low regularity. Especially, when the problem is posed on a convex polygonal domain, the control y exhibits layer behavior at the corners of the domain. This is because, it is determined by the normal derivative of the adjoint state, whereas in a non-convex polygonal domain the control may have singularity around a corner point, for more details one can see [7]. Another approach, as in [8], is the Robin boundary penalization which transforms the Dirichlet control problem into a Robin boundary control problem.

One other popular approach is to find controls from the energy space, i.e., $H^{1/2}(\partial\Omega)$:

$$\min_{y\in H^{1/2}(\partial\Omega)} J(u,y) := \frac{1}{2}\|u-u_d\|_{0,\Omega} + \frac{\rho}{2}|y|^2_{H^{1/2}(\partial\Omega)}, \tag{1.5}$$

we refer [24] for this approach. We can define the standard weak solution for the state equation (1.4) with this choice of control. This approach introduces the Steklov-Poincaré operator to establish a new cost functional. The Steklov-Poincaré operator transforms the Dirichlet data into a Neumann data by using harmonic extension of the Dirichlet data; but this type of abstract operator may cause some difficulties in numerical implementation. It is well known that for a given $y \in H^{1/2}(\partial\Omega)$ there exists a harmonic extension $u_y \in H^1(\Omega)$ such that $|y|_{H^{1/2}(\partial\Omega)}$ can be equivalently defined as

$$|y|_{H^{1/2}(\partial\Omega)} := \left\|\nabla u_y\right\|_{0,\Omega}.$$

This motivates to choose the control from $H^1(\Omega)$ space as in (1.1), i.e.,

$$\min_{y\in H^{1}(\Omega)} J(u,y) := \frac{1}{2}\|u-u_d\|^2_{0,\Omega} + \frac{\rho}{2}\|\nabla y\|^2_{0,\Omega}. \tag{1.6}$$

This approach for Dirichlet boundary control problem was first introduced in the paper [9]. Since the control is sought from the space $H^1(\Omega)$ they do not need the Steklov-Poincaré operator and hence this method is computationally very efficient. In the paper [9], one can find only unconstrained control, an improved analysis for constrained control can be found in [19].

In this article, we consider a Stokes equation with mixed boundary conditions and the control acts on the Dirichlet boundary only. In the literature of Stokes Dirichlet control problem, we can see that two types of control are chosen. The first one is tangential control i.e., the control acts only in the tangential direction of the boundary (see [17]). In [17] the authors propose hybridize discontinuous Galerkin (HDG) method to approximate the solution of a tangential Dirichlet boundary control problem with an L^2 penalty on the boundary control and here the controls are unconstrained. The second one is that the flux of controls is zero (i.e., $\int_{\partial\Omega}\mathbf{y}\cdot\mathbf{n} = 0$) [18]. The zero flux condition comes naturally on control since we have an impressibility condition and we have only Dirichlet boundary condition in the PDE. So, to hold this zero flux condition, the authors choose only tangential control as the first choice and as the other choice the authors take zero flux condition itself as a constraint in the space. Also, it is observed in many Navier-Stokes Dirichlet control problem that the authors use either tangential control or the zero flux condition on the control, for e.g., one can see [15,20]. This zero flux condition on the control reduces the regularity of the control discussed in [18]. To overcome this difficulty we introduce the Stokes equation with outflow condition and the control acts on the Dirichlet boundary only. Hence our control is more general and it has both the tangential and normal components. Also, we have introduced constraints in the control. Due to these constraints in the control, the optimal control satisfies a simplified Signorini problem:

$$-\rho\Delta\mathbf{y} = \mathbf{0} \quad \text{in} \quad \Omega, \tag{1.7a}$$

$$\mathbf{y} = \mathbf{0} \quad \text{on} \quad \Gamma_D \cup \Gamma_N, \tag{1.7b}$$

$$\mathbf{y}_a \le \boldsymbol{\gamma}_0(\mathbf{y}) \le \mathbf{y}_b \ \text{ a.e. on } \Gamma_C, \tag{1.7c}$$

further the following holds for almost every $x \in \Gamma_C$:

$$\text{if } \mathbf{y}_a < \mathbf{y}(x) < \mathbf{y}_b \quad \text{then} \quad \big(\boldsymbol{\mu}(\mathbf{y})\big)(x) = \mathbf{0}, \tag{1.7d}$$

$$\text{if } \mathbf{y}_a \le \mathbf{y}(x) < \mathbf{y}_b \quad \text{then} \quad \big(\boldsymbol{\mu}(\mathbf{y})\big)(x) \ge \mathbf{0}, \tag{1.7e}$$

$$\text{if } \mathbf{y}_a < \mathbf{y}(x) \le \mathbf{y}_b \quad \text{then} \quad \big(\boldsymbol{\mu}(\mathbf{y})\big)(x) \le \mathbf{0}, \tag{1.7f}$$

where the contact stress $\boldsymbol{\mu}(\mathbf{y}) = \rho\frac{\partial \mathbf{y}}{\partial \mathbf{n}} - \frac{\partial \boldsymbol{\phi}}{\partial \mathbf{n}} - r\mathbf{n}$, $(\boldsymbol{\phi}, r)$ is the adjoint variable and $\mathbf{y}_a$, $\mathbf{y}_b$ are vectors in $\mathbb{R}^2$ and Γ_C, Γ_D and Γ_N are subsets of $\partial\Omega$. As a result of this inequality in the contact boundary Γ_C, if we apply the standard error analysis for $\|\nabla(\mathbf{y}-\mathbf{y}_h)\|_{0,\Omega}$, we only achieve $\frac{1}{2}+\frac{\delta}{2}$ $(\delta > 0)$ order of convergence, when $\mathbf{y} \in \mathbf{H}^{\frac{3}{2}+\delta}(\Omega)$. However, this is not the optimal rate of convergence. Using the ideas in [12], we have derived in Theorem 4.3 that the control error $\|\nabla(\mathbf{y}-\mathbf{y}_h)\|_{0,\Omega}$ has $\frac{1}{2}+\delta$ order of convergence, which is optimal. Even if the control in the less regular space i.e., $\mathbf{y} \in \mathbf{H}^{\tau}(\Omega)$ with $1 < \tau \le 3/2$ we derive an optimal order of convergence up to the regularity. It is well known that it is challenging to find an inf-sup stable finite element pair for the Stokes problem. The degree of the polynomial and the regularity of the solution determine the order in which the solution converges. We were looking for a lower order polynomial (the best would be $\mathbb{P}_1/\mathbb{P}_0$) to approximate the Stokes equation because the regularity of our control problem is somewhat limited. We know that Taylor-Hood elements $\mathbb{P}_k/\mathbb{P}_{k-1}$ and $\mathbb{Q}_k/\mathbb{Q}_{k-1}$ with $k \ge 2$, $\mathbb{P}_1$-bubble/ $\mathbb{P}_1$ and etc. finite element are stable. Unfortunately the lowest order pair $\mathbb{P}_1/\mathbb{P}_0$ is not inf-sup stable because of the dimension of the discrete pressure space is far too rich for discrete divergence map to be onto. So our aim is to reduce the dimension of the pressure space so that the discrete divergence map becomes onto. So, we have used a two-level finite element method for the Stokes problem where the velocity and the control are approximated by a piecewise linear finite element space on a fine mesh and the pressure is approximated by a piecewise constant space on a coarse mesh. To this end, this way the point-wise control constraints are well respected. The theoretical results are corroborated by a variety of numerical tests

The rest of the article is structured as follows. In Section 2, we prove the existence and uniqueness of the solution of the optimal control problem and derive the continuous optimality system. In Section 3, we discuss the discrete optimal control problem. We derive *a priori* error estimates in Section 4. Section 5 is devoted to the numerical experiments.

2. Continuous control problem

In this section we briefly discuss the precise formulation of the optimization problem under consideration. Furthermore, we recall theoretical results on existence, uniqueness, and regularity of optimal solutions as well as optimality conditions. Before going to the analysis we need the following definitions:

2.1. Notations

Let Ω be a bounded convex polygonal domain in $\mathbb{R}^2$. Any function and space in bold notation can be understood in the vector form e.g., $\mathbf{x} := (x_1, x_2)$, $\mathbf{L}^2(\Omega) := [L^2(\Omega)]^2$ and $\mathbf{H}^1(\Omega) := [H^1(\Omega)]^2$. The norm and inner product on those spaces are defined component wise. The norm in the $\mathbf{L}^2(\Omega)$ space is denoted by $\|\cdot\|_{0,\Omega}$. Also, the norm on the Sobolev space $\mathbf{H}^k(\Omega)$ is denoted by $\|\cdot\|_{k,\Omega} (k > 0)$, see [10]. The trace of a vector valued function $\mathbf{x} \in \mathbf{H}^1(\Omega)$ is defined to be $\boldsymbol{\gamma}_0(\mathbf{x}) := (\gamma_0(x_1), \gamma_0(x_2))$, where $\gamma_0 : H^1(\Omega) \to L^2(\Gamma)$ is the trace operator whose range is $H^{1/2}(\Gamma)$. Let $\mathbf{x}$ and $\mathbf{y}$ be two functions, we say that $\mathbf{x} \le \mathbf{y}$ iff $x_1 \le y_1$ and $x_2 \le y_2$ almost everywhere in Ω.

Before going to the optimal control problem first, we will discuss the Stokes problem defined in (1.2). Here we will describe the problem more precisely. We have the following Stokes problem with mixed boundary conditions:

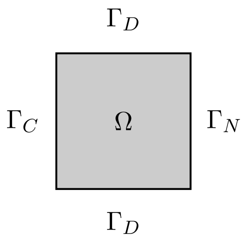

Fig. 2.1. The domain Ω.

$$-\Delta \mathbf{u} + \nabla p = \mathbf{f} \quad \text{in } \Omega, \tag{2.1a}$$

$$\nabla \cdot \mathbf{u} = 0 \quad \text{in } \Omega, \tag{2.1b}$$

$$\mathbf{u} = \mathbf{y} \quad \text{on } \Gamma_C, \tag{2.1c}$$

$$\mathbf{u} = \mathbf{0} \quad \text{on } \Gamma_D, \tag{2.1d}$$

$$\frac{\partial \mathbf{u}}{\partial \mathbf{n}} - p\mathbf{n} = \mathbf{0} \ \text{ on } \Gamma_N. \tag{2.1e}$$

Here Γ_D, Γ_C and Γ_N are three non-overlapping open subsets of the boundary $\partial\Omega$ with $\partial\Omega = \Gamma_C \cup \bar{\Gamma}_D \cup \Gamma_N$, Fig. 2.1 depicts an example of such a domain. We assume that, Γ_C be a straight line segment and one dimensional measure $|\Gamma_C| > 0$. The interior force $\mathbf{f} \in \mathbf{L}^2(\Omega)$.

Remark 2.1. The choice of the domain Ω in this problem is very specific because of the mixed boundary conditions in the Problem 2.1. It is known that the Neumann-Dirichlet transition points always impair the regularity of the solution. The singularity of those transition points depends on the data as well as the interior angle at that point. Here, we have taken the angle between Γ_D and Γ_N is always $\pi/2$ and Γ_C is a straight line segment so that, we can get a regular solution.

Define the test and trial space $\mathbf{V}$ by

$$\mathbf{V} := \mathbf{H}^1_{D\cup C}(\Omega) = \{\mathbf{x} \in \mathbf{H}^1(\Omega) : \gamma_0(\mathbf{x}) = \mathbf{0} \text{ on } \Gamma_D \cup \Gamma_C\}.$$

We choose controls from the following space:

$$\mathbf{Q} := \{\mathbf{x} \in \mathbf{H}^1(\Omega) : \gamma_0(\mathbf{x}) = \mathbf{0} \text{ on } \Gamma_D \cup \Gamma_N\}.$$

For given $\mathbf{y} \in \mathbf{Q}$, the weak formulation of (2.1) is as follows: find $(\mathbf{w}, p) \in \mathbf{V} \times L^2(\Omega)$ such that

$$\mathbf{u} = \mathbf{w} + \mathbf{y}, \tag{2.2a}$$

$$a(\mathbf{w}, \mathbf{z}) + b(\mathbf{z}, p) = (\mathbf{f}, \mathbf{z}) - a(\mathbf{y}, \mathbf{z}) \quad \text{for all } \mathbf{z} \in \mathbf{V}, \tag{2.2b}$$

$$b(\mathbf{w}, q) = -b(\mathbf{y}, q) \quad \text{for all } q \in L^2(\Omega), \tag{2.2c}$$

where $a(\mathbf{w}, \mathbf{z}) = \int_\Omega \nabla \mathbf{w} : \nabla \mathbf{z}\, dx$, $b(\mathbf{z}, p) = -\int_\Omega p \nabla \cdot \mathbf{z}\, dx$, and the matrix product $A : B := \sum_{i,j=1}^n a_{ij} b_{ij}$ when $A = (a_{ij})_{1\le i,j\le n}$ and $B = (b_{ij})_{1\le i,j\le n}$ with $(\cdot,\cdot)$ denotes the $\mathbf{L}^2(\Omega)$. The Babuska-Brezzi theorem [5,10] ensures the existence and uniqueness of the solution of (2.2). We define the solution map S by $S(\mathbf{f}, \mathbf{x}) := \mathbf{u}$, where $\mathbf{x} \in \mathbf{Q}$ be given and $\mathbf{u} = \mathbf{w} + \mathbf{x}$ solves (2.2).

Here we consider the energy cost functional $J : \mathbf{H}^1(\Omega) \times \mathbf{Q} \to \mathbb{R}$, which is defined in the equation (1.1). There the constant $\rho > 0$ is the regularizing parameter and $\mathbf{u}_d \in \mathbf{L}^2(\Omega)$ is a given target function. We seek control from the following constrained set:

$$\mathbf{Q}_{ad} := \{\mathbf{x} \in \mathbf{Q} : \mathbf{y}_a \le \gamma_0(\mathbf{x}) \le \mathbf{y}_b \text{ a.e. on } \Gamma_C\},$$

where $\mathbf{y}_a = (y_a^1, y_a^2)$, $\mathbf{y}_b = (y_b^1, y_b^2) \in \mathbb{R}^2$ satisfying $y_a^1 < y_a^2$ and $y_b^1 < y_b^2$. Furthermore, whenever the set Γ_D is nonempty, for compatibility we assume that $y_a^1, y_b^1 \le 0$ and $y_a^2, y_b^2 \ge 0$ in order that, the control set $\mathbf{Q}_{ad}$ is nonempty.

2.2. The model problem

Find $(\mathbf{u}, \mathbf{y}) \in \mathbf{H}^1_D(\Omega) \times \mathbf{Q}_{ad}$ such that

$$J(\mathbf{u}, \mathbf{y}) = \min_{(\mathbf{v},\mathbf{x}) \in \mathbf{H}^1_D(\Omega) \times \mathbf{Q}_{ad}} J(\mathbf{v}, \mathbf{x}), \tag{2.3}$$

subject to the condition that $\mathbf{v} = S(\mathbf{f}, \mathbf{x})$.

The reduced cost functional $j : \mathbf{Q} \to \mathbb{R}$ defined as

$$j(\mathbf{x}) := \frac{1}{2}\|S(\mathbf{f}, \mathbf{x}) - \mathbf{u}_d\|^2_{0,\Omega} + \frac{\rho}{2}\|\nabla \mathbf{x}\|^2_{0,\Omega}, \quad \mathbf{x} \in \mathbf{Q}_{ad}, \quad \rho > 0. \tag{2.4}$$

Differentiating the reduced cost functional j, we obtain

$$j'(\mathbf{y})(\mathbf{x}) = \lim_{t\to 0} \frac{j(\mathbf{y} + t\mathbf{x}) - j(\mathbf{y})}{t} = (S(\mathbf{f}, \mathbf{y}) - \mathbf{u}_d, S(\mathbf{0}, \mathbf{x})) + \rho\, a(\mathbf{y}, \mathbf{x}).$$

Theorem 2.2 *(Existence and uniqueness of the solution). There exists a unique solution of the control problem* (2.3)*.*

Proof. It is clear that the cost functional J is non negative. Set,

$$m = \inf_{(\mathbf{v},\mathbf{x}) \in \mathbf{H}^1_D(\Omega) \times \mathbf{Q}_{ad}} J(\mathbf{v}, \mathbf{x}).$$

Then there exists a minimizing sequence $(\mathbf{u}_n, \mathbf{y}_n)$ such that $J(\mathbf{u}_n, \mathbf{y}_n)$ converges to m and $\mathbf{u}_n = S(\mathbf{f}, \mathbf{y}_n)$. Since, the sequence $J(\mathbf{u}_n, \mathbf{y}_n)$ is convergent we can say that the sequences $\|\mathbf{u}_n - \mathbf{u}_d\|_{0,\Omega}$ and $\|\nabla \mathbf{y}_n\|_{0,\Omega}$ are also convergent and hence bounded. Now, $\mathbf{y}_n \in \mathbf{Q}_{ad}$ by using the Poincaré inequality we can conclude that the sequence $\mathbf{y}_n$ is bounded in $\mathbf{Q}$. Then there exists a subsequence of $\mathbf{y}_n$, still indexed by n to simplify the notation, and a function $\mathbf{y}$, such that $\mathbf{y}_n$ converges to $\mathbf{y}$ weakly in $\mathbf{Q}$. It is clear that the set $\mathbf{Q}_{ad}$ is closed and convex so it is weakly closed. Hence, $\mathbf{y} \in \mathbf{Q}_{ad}$. *A priori* estimate of the Stokes problem (2.2) from [4], we get

$$\|\nabla \mathbf{w}_n\|_{0,\Omega} + \|p_n\|_{0,\Omega} \le C(\|\mathbf{f}\|_{0,\Omega} + \|\nabla \mathbf{y}_n\|_{0,\Omega}), \tag{2.5}$$

where $\mathbf{u}_n = \mathbf{w}_n + \mathbf{y}_n$. Since, the sequence $\mathbf{y}_n$ is bounded in $\mathbf{Q}$ and using (2.5) we can conclude that the sequence $\mathbf{w}_n$ is bounded in $\mathbf{H}^1_0(\Omega)$. So, we can extract a subsequence of it and name it by $\mathbf{w}_n$ and it converges to $\mathbf{w}$. Thus we can extract a subsequence of p_n (call it p_n) corresponding to the subsequence of $\mathbf{w}_n$ such that, p_n weakly converges to p in $L^2(\Omega)$. Now we need to show that $\mathbf{w}$ is the corresponding candidate for the control $\mathbf{y}$. We have

$$\int_\Omega \nabla \mathbf{w}_n : \nabla \mathbf{v} + \int_\Omega p_n \nabla \cdot \mathbf{v} = \int_\Omega \mathbf{f} \cdot \mathbf{v} - \int_\Omega \nabla \mathbf{y}_n : \nabla \mathbf{v} \quad \text{for all } \mathbf{v} \in \mathbf{V}.$$

Using the above weak convergences, we can conclude that

$$\int_\Omega \nabla \mathbf{w} : \nabla \mathbf{v} + \int_\Omega p \nabla \cdot \mathbf{v} = \int_\Omega \mathbf{f} \cdot \mathbf{v} - \int_\Omega \nabla \mathbf{y} : \nabla \mathbf{v} \quad \text{for all } \mathbf{v} \in \mathbf{V}.$$

Hence, $\mathbf{u} = \mathbf{w} + \mathbf{y}$ is the corresponding state for the control $\mathbf{y}$. The weak lower semi continuity of the norm gives, $\|\nabla \mathbf{y}\|_{0,\Omega} \le \liminf_{n\to\infty} \|\nabla \mathbf{y}_n\|_{0,\Omega}$. Using the above weak convergences of $\mathbf{w}_n$ and $\mathbf{y}_n$ we can conclude that both sequences converge strongly in $\mathbf{L}^2(\Omega)$. Thus, $\mathbf{u}_n$ converges to $\mathbf{u}$ strongly in $\mathbf{L}^2(\Omega)$. Hence, we have

$$J(\mathbf{u}, \mathbf{y}) \le \lim_{n\to\infty} \frac{1}{2}\|\mathbf{u}_n - \mathbf{u}_d\|^2_{0,\Omega} + \frac{\rho}{2} \liminf_{n\to\infty} \|\nabla \mathbf{y}_n\|^2_{0,\Omega} = m.$$

This proves the existence of a control $\mathbf{y}$ such that $J(\mathbf{u}, \mathbf{y}) = m$. The uniqueness of the solution follows from the strict convexity of the cost functional. □

The following proposition establishes the first order optimality system:

Proposition 2.3 *(Continuous optimality system). The state, adjoint state, and control* $((\mathbf{u}, p), (\boldsymbol{\phi}, r), \mathbf{y}) \in (\mathbf{H}^1_D(\Omega) \times L^2(\Omega)) \times (\mathbf{V} \times L^2(\Omega)) \times \mathbf{Q}_{ad}$ *satisfy the optimality system*

$$\mathbf{u} = \mathbf{w} + \mathbf{y}, \quad \mathbf{w} \in \mathbf{V}, \tag{2.6a}$$

$$a(\mathbf{w},\mathbf{z}) + b(\mathbf{z},p) = (\mathbf{f},\mathbf{z}) - a(\mathbf{y},\mathbf{z}) \quad \forall\, \mathbf{z} \in \mathbf{V}, \tag{2.6b}$$

$$b(\mathbf{u},q) = 0 \quad \forall\, q \in L^2(\Omega), \tag{2.6c}$$

$$a(\mathbf{z},\boldsymbol{\phi}) - b(\mathbf{z},r) = (\mathbf{u}-\mathbf{u_d},\mathbf{z}) \quad \forall\, \mathbf{z} \in \mathbf{V}, \tag{2.6d}$$

$$b(\boldsymbol{\phi},q) = 0 \quad \forall\, q \in L^2(\Omega), \tag{2.6e}$$

$$\rho\, a(\mathbf{y},\mathbf{x}-\mathbf{y}) \geq a(\mathbf{x}-\mathbf{y},\boldsymbol{\phi}) - b(\mathbf{x}-\mathbf{y},r) - (\mathbf{u}-\mathbf{u_d},\mathbf{x}-\mathbf{y}) \quad \forall \mathbf{x} \in \mathbf{Q}_{ad}. \tag{2.6f}$$

Proof. The equations from (2.6a) to (2.6e) are optimal state and adjoint state equations. We only need to prove the last inequality (2.6f). The first order necessary optimality conditions yields

$$(\mathbf{u}-\mathbf{u_d}, S(\mathbf{0},\mathbf{x}-\mathbf{y})) + \rho\, a(\mathbf{y},\mathbf{x}-\mathbf{y}) \geq 0 \quad \forall \mathbf{x} \in \mathbf{Q}_{ad}.$$

The solution of the adjoint problem is defined by

$$a(\mathbf{z},\boldsymbol{\phi}) - b(\mathbf{z},r) = (\mathbf{u}-\mathbf{u_d},\mathbf{z}) \quad \text{for all } \mathbf{z} \in \mathbf{V},$$

$$b(\boldsymbol{\phi},q) = 0 \quad \text{for all } q \in L^2(\Omega)$$

Since $S(\mathbf{0},\mathbf{x}-\mathbf{y}) - (\mathbf{x}-\mathbf{y}) \in \mathbf{V}$ and $a(S(0,\mathbf{x}-\mathbf{y}),\boldsymbol{\phi}) = 0$, we obtain

$$\begin{aligned}(\mathbf{u}-\mathbf{u_d}, S(\mathbf{0},\mathbf{x}-\mathbf{y})) &= (\mathbf{u}-\mathbf{u_d}, S(\mathbf{0},\mathbf{x}-\mathbf{y})-(\mathbf{x}-\mathbf{y})) + (\mathbf{u}-\mathbf{u_d},\mathbf{x}-\mathbf{y})\\ &= (\mathbf{u}-\mathbf{u_d},\mathbf{x}-\mathbf{y}) - a(\mathbf{x}-\mathbf{y},\boldsymbol{\phi}) + b(\mathbf{x}-\mathbf{y},\mathbf{r}).\end{aligned}$$

This completes the proof. □

Remark 2.4 *(Control satisfies Signorini problem).* The optimal control $\mathbf{y}$ is the weak solutions of the simplified Signorini problem (1.7) defined in the introduction.

Remark 2.5 *(Regularity of Signorini problem).* The numerical analysis of any finite element method applied to the Signorini problem (1.7) requires the knowledge of the regularity of the solution $\mathbf{y}$. Since the work by Moussaoui and Khodja (see [23]), it is admitted that the Signorini condition may generate some singular behavior at the neighborhood of Γ_C. There are many factors that affect the regularity of the solution to the Signorini problem. Some of those factors are the regularity of the data, the mixed boundary conditions (e.g., Neumann-Dirichlet transitions), the corners in polygonal domains and the Signorini condition, which generates singularities at contact-noncontact transition points. In our model problem we assume Γ_C be a straight line segment. Let $\mathbf{p}$ be a contact-noncontact transition point in the interior of Γ_C, then the solution of Signorini problem (1.7) $\mathbf{y} \in \mathbf{H}^{\tau}(V_{\mathbf{p}})$ with $\tau < \frac{5}{2}$ and $V_{\mathbf{p}}$ be an open neighborhood of $\mathbf{p}$ (see [1, subsection 2.3], [2, section 2] and [23]). Let $\mathbf{p} \in \bar{\Gamma}_C \cap \bar{\Gamma}_D$ and $V_{\mathbf{p}}$ be a neighborhood of $\mathbf{p}$ in Ω such that $\mathbf{y}$ vanishes on $V_{\mathbf{p}} \cap \Gamma_C$ then the elliptic regularity theory on convex domain yields $\mathbf{y} \in \mathbf{H}^2(V_{\mathbf{p}})$ (see [1, subsection 2.3] and [25]). Now if $\mathbf{y}$ does not vanish on $V_{\mathbf{p}} \cap \Gamma_C$, then $\mathbf{p}$ be a contact-noncontact type transition point and hence $\mathbf{y} \in \mathbf{H}^{\tau}(V_{\mathbf{p}})$ with $\tau < 5/2$ (see [1, subsection 2.3] and [25]). The best we can expect is to obtain $\mathbf{y} \in \mathbf{H}^{\tau}(V_{\Gamma_C})$ with $\tau \leq 2$ and V_{Γ_C} is an open neighborhood of Γ_C (see [1,23]).

Remark 2.6 *(Regularity of the adjoint state variables).* The strong form of the adjoint state is the following:

$$\begin{aligned}-\Delta \boldsymbol{\phi} + \nabla r &= \mathbf{u}-\mathbf{u}_d \quad \text{in } \Omega,\\ \nabla \cdot \boldsymbol{\phi} &= 0 \quad \text{in } \Omega,\\ \boldsymbol{\phi} &= \mathbf{0} \quad \text{on } \Gamma_D \cup \Gamma_C,\\ \frac{\partial \boldsymbol{\phi}}{\partial \mathbf{n}} - r\mathbf{n} &= \mathbf{0} \text{ on } \Gamma_N.\end{aligned}$$

Since our domain under consideration has $\pi/2$ angle at Neumann-Dirichlet transition points and the given data are sufficiently regular so using Theorem A.1 of the paper [3] we can conclude that $\boldsymbol{\phi} \in \mathbf{H}^2(\Omega)$ and $r \in H^1(\Omega)$.

Remark 2.7 *(Regularity of the state variables).* We have seen from the Remark (2.5) that the control variable can have regularity up to $\mathbf{H}^2(\Omega)$. So, if we assume $\mathbf{y} \in \mathbf{H}^2(\Omega)$ and since we have $\pi/2$ angle at each transition (Dirichlet-Dirichlet and Neumann-Dirichlet) points and load $\mathbf{f} \in \mathbf{L}^2(\Omega)$ then, from the equations (2.6b)-(2.6c) we can conclude that $\mathbf{w} \in \mathbf{H}^{\frac{3}{2}+\delta}(\Omega)$ and $p \in H^{\frac{1}{2}+\delta}(\Omega)$ with $\delta > 0$ (see, [25]). Therefore, the velocity $\mathbf{u} \in \mathbf{H}^{\frac{3}{2}+\delta}(\Omega)$.

3. Discrete control problem

Let $\mathcal{T}_H$ be a shape-regular triangulation of the domain Ω into triangles K such that $\cup_{K \in \mathcal{T}_H} K = \bar{\Omega}$ see [5,10]. Also let $\mathcal{T}_h$ be a refinement of $\mathcal{T}_H$ by connecting all the midpoints of $\mathcal{T}_H$. The collection of interior edges of $\mathcal{T}_h$ is denoted by $\mathcal{E}_h^i$. The collection of Dirichlet, Neumann and Contact boundary edges of $\mathcal{T}_h$ are denoted by $\mathcal{E}_h^{b,D}$, $\mathcal{E}_h^{b,N}$ and $\mathcal{E}_h^{b,C}$ respectively. We define $\mathcal{E}_h = \mathcal{E}_h^i \cup \mathcal{E}_h^{b,D} \cup \mathcal{E}_h^{b,N} \cup \mathcal{E}_h^{b,C}$. A typical triangle is denoted by K and its diameter by h_T. Set $h = \max_{K \in \mathcal{T}_h} h_T$. The length of any edge $e \in \mathcal{E}_h$ will be denoted by h_e. The collection of all vertices of $\mathcal{T}_h$ is denoted by $\mathcal{V}_h$. The set of vertices on $\overline{\Gamma}_D$, Γ_N and Γ_C are denoted by $\mathcal{V}_h^D$, $\mathcal{V}_h^N$ and $\mathcal{V}_h^C$. Define the discrete space for velocity $\mathbf{V}_h \subset \mathbf{V}$ by

$$\mathbf{V}_h := \{\mathbf{v}_h \in \mathbf{V} : \mathbf{v}_h|_T \in \mathbf{P}_1(K)\ \forall K \in \mathcal{T}_h\},$$

and the discrete space for pressure is

$$M_H := \{p_H \in L^2(\Omega) : p_H|_T \in P_0(K)\ \forall K \in \mathcal{T}_H\},$$

and the discrete control space $\mathbf{Q}_h \subset \mathbf{Q}$ by

$$\mathbf{Q}_h := \{\mathbf{x}_h \in \mathbf{Q} : \mathbf{x}_h|_T \in \mathbf{P}_1(K),\ \forall K \in \mathcal{T}_h\},$$

where $P_0(K)$ is the space of constant polynomials on K and $\mathbf{P}_1(K)$ is the space of polynomials of degree ≤ 1 on the triangle K. The approximation of the control set is as follows

$$\mathbf{Q}_{ad}^h := \{\mathbf{x}_h \in \mathbf{Q}_h : \mathbf{y}_a \leq \mathbf{x}_h(z) \leq \mathbf{y}_b \text{ for all } z \in \mathcal{V}_h^C\}.$$

It is easy to check that $\mathbf{Q}_{ad}^h \subset \mathbf{Q}_{ad}$. Define the Lagrange interpolation $\mathcal{J}_h : \mathbf{C}(\bar{\Omega}) \to \mathbf{Q}_h$ by $\mathcal{J}_h(\mathbf{v}) = \sum_{z \in \mathcal{V}_h} \mathbf{v}(z)\boldsymbol{\phi}_z$, where $\boldsymbol{\phi}_z$ are basis functions of $\mathbf{Q}_h$ and $\mathbf{C}(\bar{\Omega})$ is the space of continuous functions on Ω. From now on it will be assumed that $C > 0$ be a constant independent of the mesh size h.

Proposition 3.1 *(Discrete optimality system). There exists a unique* $((\mathbf{w}_h, p_H), (\boldsymbol{\phi}_h, r_H), \mathbf{y}_h) \in (\mathbf{V}_h \times M_H) \times (\mathbf{V}_h \times M_H) \times \mathbf{Q}_{ad}^h$ *solves the following:*

$$\mathbf{u}_h = \mathbf{w}_h + \mathbf{y}_h, \quad \mathbf{w}_h \in \mathbf{V}_h, \tag{3.1a}$$

$$a(\mathbf{w}_h,\mathbf{z}_h) + b(\mathbf{z}_h,p_H) = (\mathbf{f},\mathbf{z}_h) - a(\mathbf{y}_h,\mathbf{z}_h) \quad \text{for all } \mathbf{z}_h \in \mathbf{V}_h, \tag{3.1b}$$

$$b(\mathbf{u}_h,q_H) = 0 \quad \text{for all } q_H \in M_H, \tag{3.1c}$$

$$a(\mathbf{z}_h,\boldsymbol{\phi}_h) - b(\mathbf{z}_h,r_H) = (\mathbf{u}_h-\mathbf{u}_d,\mathbf{z}_h) \quad \text{for all } \mathbf{z}_h \in \mathbf{V}_h, \tag{3.1d}$$

$$b(\boldsymbol{\phi}_h,q_H) = 0 \quad \text{for all } q_H \in M_H, \tag{3.1e}$$

$$\begin{aligned}\rho\, a(\mathbf{y}_h,\mathbf{x}_h-\mathbf{y}_h) \geq\ & a(\mathbf{x}_h-\mathbf{y}_h,\boldsymbol{\phi}_h) - b(\mathbf{x}_h-\mathbf{y}_h,r_H)\\ & - (\mathbf{u}_h-\mathbf{u}_d,\mathbf{x}_h-\mathbf{y}_h) \quad \text{for all } \mathbf{x}_h \in \mathbf{Q}_{ad}^h.\end{aligned} \tag{3.1f}$$

Proof. Before going to prove the existence of the optimal solution we need to check the inf-sup stability of the bilinear form b. The bilinear form b is inf-sup stable for the pair $(\mathbf{V}_h, M_H)$ because of the existence of the Fortin operator (see [13, Section 4.2.2]) $\boldsymbol{\pi}_h : \mathbf{V} \to \mathbf{V}_h$ defined by: For any vertex of the fine triangle which is a mid point of an edge E of the coarse triangle, we define

$$\int_E (\boldsymbol{\pi}_h \mathbf{v} - \mathbf{v}) = \mathbf{0}.$$

Also, we define $\boldsymbol{\pi}_h \mathbf{v}$ at the vertices $\{a_1, a_2, a_3\}$ of the coarse triangle by using Scott-Zhang interpolation. Let E_i be an edge containing the vertex a_i. The set $\{\phi_i^1, \phi_i^2\}$ denotes the restriction to E_i of the local shape functions associated with the nodes lying in E_i. Now consider the corresponding L^2– dual basis of $\{\phi_i^1, \phi_i^2\}$ is $\{\psi_i^1, \psi_i^2\}$ such that

$$\int_{E_i} \psi_i^k \phi_i^l = \delta_{kl} \quad 1 \le k, l \le 2.$$

Conventionally, set $\phi_i^1 = \phi_i$ and $\psi_i^1 = \psi_i$ for the node a_i. The nodal variables at vertices are defined by: $\boldsymbol{\pi}_h \mathbf{v}(a_i) = \int_{E_i} \psi_i \mathbf{v}$. Whenever a_i is at the boundary and in the intersection of many edges, it is important to pick the one edge such that $E_i \subseteq \partial\Omega$. The map $\boldsymbol{\pi}_h$ defined above is a well defined operator. The H^1 stability of $\boldsymbol{\pi}_h$ can be derived using standard scaling argument see, [13, Section 1.6.2]. The standard theory of optimal control problem [26,27] can be used to prove the existence and uniqueness of the solution. □

4. Error analysis

In this section is devoted to *a priori* error estimates. The convergence rate of the finite element approximation of the control problem depends on the regularity of the solution. It is clear from Remark 2.5, 2.6 and 2.7 that one can assume the solution $\mathbf{u} \in \mathbf{H}^{\frac{3}{2}+\delta}(\Omega)$, $p \in H^{\frac{1}{2}+\delta}(\Omega)$, $\boldsymbol{\phi} \in \mathbf{H}^{\frac{3}{2}+\delta}(\Omega)$, $r \in H^{\frac{1}{2}+\delta}(\Omega)$ and $\mathbf{y} \in \mathbf{H}^{\frac{3}{2}+\delta}(\Omega)$, where $0 < \delta \le 1/2$. To derive *a priori* error analysis, we introduce some projections as follows: Let $\mathbf{P}_h \mathbf{w} \in \mathbf{V}_h$, $\bar{\mathbf{P}}_h \boldsymbol{\phi} \in \mathbf{V}_h$, $R_H p \in M_H$ and $\bar{R}_H r \in M_H$ solve

$$a(\mathbf{P}_h \mathbf{w}, \mathbf{z}_h) + b(\mathbf{z}_h, R_H p) = \langle \mathbf{f}, \mathbf{z}_h \rangle - a(\mathbf{y}, \mathbf{z}_h) \quad \text{for all } \mathbf{z}_h \in \mathbf{V}_h, \tag{4.1a}$$

$$b(\mathbf{P}_h \mathbf{w}, q_H) = -b(\mathbf{y}, q_H) \quad \text{for all } q_H \in M_H, \tag{4.1b}$$

$$a_h(\mathbf{z}_h, \bar{\mathbf{P}}_h \boldsymbol{\phi}) - b_h(\mathbf{z}_h, \bar{R}_h r) = \langle \mathbf{u} - \mathbf{u}_d, \mathbf{z}_h \rangle_W \quad \text{for all } \mathbf{z}_h \in \mathbf{V}_h, \tag{4.1c}$$

$$b_h(\bar{\mathbf{P}}_h \boldsymbol{\phi}, q_H) = 0 \quad \text{for all } q_H \in M_H. \tag{4.1d}$$

The following theorem is the first step to get error estimates.

Theorem 4.1 *(Energy error estimate of control and L^2-estimate of velocity). Let $\mathbf{y}, \mathbf{u}$ be the continuous optimal control and state satisfy (2.6), $\mathbf{y}_h, \mathbf{u}_h$ be the discrete optimal control and state satisfy (3.1). Then there holds*

$$\begin{aligned} \rho \|\nabla(\mathbf{y} - \mathbf{y}_h)\|_{0,\Omega}^2 + \|\mathbf{u} - \mathbf{u}_h\|_{0,\Omega}^2 \le{} & C|a(\mathbf{y}, \mathbf{y} - \mathbf{x}_h) - a(\mathbf{x}_h - \mathbf{y}, \boldsymbol{\phi}) + b(\mathbf{y} - \mathbf{x}_h, r) \\ & - (\mathbf{u} - \mathbf{u}_d, \mathbf{x}_h - \mathbf{y})| \\ & + C\left(\|\nabla(\mathbf{y} - \mathbf{x}_h)\|_{0,\Omega}^2 + \|\nabla(\boldsymbol{\phi} - \bar{\mathbf{P}}_h \boldsymbol{\phi})\|_{0,\Omega}^2\right) \\ & + C\left(\|\mathbf{P}_h \mathbf{w} - \mathbf{w}\|_{0,\Omega}^2 + \|r - \bar{R}_H r\|_{0,\Omega}^2 + \|\mathbf{y} - \mathbf{x}_h\|_{0,\Omega}^2\right), \end{aligned} \tag{4.2}$$

for all $\mathbf{x}_h \in \mathbf{Q}_{ad}^h$.

Proof. A selection $\mathbf{x} = \mathbf{y}_h \in \mathbf{Q}_{ad}^h \subseteq \mathbf{Q}_{ad}$ in (2.6f), yields

$$\rho\, a(\mathbf{y}, \mathbf{y}_h - \mathbf{y}) \ge a(\mathbf{y}_h - \mathbf{y}, \boldsymbol{\phi}) - b(\mathbf{y}_h - \mathbf{y}, r) - (\mathbf{u} - \mathbf{u}_d, \mathbf{y}_h - \mathbf{y}). \tag{4.3}$$

Using (3.1f), we have

$$\begin{aligned} \rho\, a(\mathbf{y}_h, \mathbf{y} - \mathbf{y}_h) \ge{} & -\rho\, a(\mathbf{y}_h, \mathbf{x}_h - \mathbf{y}) + a(\mathbf{x}_h - \mathbf{y}_h, \boldsymbol{\phi}_h) - b(\mathbf{x}_h - \mathbf{y}_h, r_H) \\ & - (\mathbf{u}_h - \mathbf{u}_d, \mathbf{x}_h - \mathbf{y}_h) \quad \forall \mathbf{x}_h \in \mathbf{Q}_{ad}^h. \end{aligned} \tag{4.4}$$

Adding the equations (4.3) and (4.4), we find that for any $\mathbf{x}_h \in \mathbf{Q}_{ad}^h$

$$\begin{aligned} \rho\, a(\mathbf{y}_h - \mathbf{y}, \mathbf{y} - \mathbf{y}_h) \ge{} & -\rho\, a(\mathbf{y}_h, \mathbf{x}_h - \mathbf{y}) + a(\mathbf{x}_h - \mathbf{y}_h, \boldsymbol{\phi}_h) - b(\mathbf{x}_h - \mathbf{y}_h, r_H) \\ & - (\mathbf{u}_h - \mathbf{u}_d, \mathbf{x}_h - \mathbf{y}_h) + a(\mathbf{y}_h - \mathbf{y}, \boldsymbol{\phi}) - b(\mathbf{y}_h - \mathbf{y}, r) \\ & - (\mathbf{u} - \mathbf{u}_d, \mathbf{y}_h - \mathbf{y}) \\ \ge{} & \rho\, a(\mathbf{y} - \mathbf{y}_h, \mathbf{x}_h - \mathbf{y}) - \rho\, a(\mathbf{y}, \mathbf{x}_h - \mathbf{y}) + a(\mathbf{x}_h - \mathbf{y}, \boldsymbol{\phi}_h - \boldsymbol{\phi}) \\ & + a(\mathbf{x}_h - \mathbf{y}, \boldsymbol{\phi}) + a(\mathbf{y} - \mathbf{y}_h, \boldsymbol{\phi}_h - \boldsymbol{\phi}) - b(\mathbf{x}_h - \mathbf{y}_h, r_H - r) \\ & - b(\mathbf{x}_h - \mathbf{y}_h, r) - b(\mathbf{y} - \mathbf{y}_h, r_H - r) - (\mathbf{u}_h - \mathbf{u}_d, \mathbf{x}_h - \mathbf{y}) \\ & - (\mathbf{u}_h - \mathbf{u}, \mathbf{y} - \mathbf{y}_h) \\ \ge{} & \left(\rho\, a(\mathbf{y}, \mathbf{y} - \mathbf{x}_h) + a(\mathbf{x}_h - \mathbf{y}, \boldsymbol{\phi}) + b(\mathbf{y} - \mathbf{x}_h, r)\right. \\ & \left.- (\mathbf{u} - \mathbf{u}_d, \mathbf{x}_h - \mathbf{y})\right) + \rho a(\mathbf{y} - \mathbf{y}_h, \mathbf{x}_h - \mathbf{y}) \\ & + a(\mathbf{x}_h - \mathbf{y}, \boldsymbol{\phi}_h - \boldsymbol{\phi}) + a(\mathbf{y} - \mathbf{y}_h, \boldsymbol{\phi}_h - \bar{\mathbf{P}}_h \boldsymbol{\phi}) \\ & + a(\mathbf{y} - \mathbf{y}_h, \bar{\mathbf{P}}_h \boldsymbol{\phi} - \boldsymbol{\phi}) - b(\mathbf{x}_h - \mathbf{y}, r_H - r) - b(\mathbf{y} - \mathbf{y}_h, r_H - r) \\ & - (\mathbf{u}_h - \mathbf{u}, \mathbf{x}_h - \mathbf{y}) + \|\mathbf{u} - \mathbf{u}_h\|_{0,\Omega}^2 + (\mathbf{u}_h - \mathbf{u}, \mathbf{w}_h - \mathbf{w}) \\ \ge{} & \left(\rho\, a(\mathbf{y}, \mathbf{y} - \mathbf{x}_h) + a(\mathbf{x}_h - \mathbf{y}, \boldsymbol{\phi}) + b(\mathbf{y} - \mathbf{x}_h, r)\right. \\ & \left.- (\mathbf{u} - \mathbf{u}_d, \mathbf{x}_h - \mathbf{y})\right) + \rho a(\mathbf{y} - \mathbf{y}_h, \mathbf{x}_h - \mathbf{y}) \\ & + a(\mathbf{x}_h - \mathbf{y}, \boldsymbol{\phi}_h - \boldsymbol{\phi}) - b(\mathbf{y}_h - \mathbf{y}, r - \bar{R}_H r) \\ & + a(\mathbf{y} - \mathbf{y}_h, \bar{\mathbf{P}}_h \boldsymbol{\phi} - \boldsymbol{\phi}) - b(\mathbf{x}_h - \mathbf{y}, r_H - r) \\ & - (\mathbf{u}_h - \mathbf{u}, \mathbf{x}_h - \mathbf{y}) + \|\mathbf{u} - \mathbf{u}_h\|_{0,\Omega}^2 + (\mathbf{u}_h - \mathbf{u}, \mathbf{P}_h \mathbf{w} - \mathbf{w}). \end{aligned} \tag{4.5}$$

Using Cauchy-Schwarz inequality and Young's inequality in the equation (4.5), we obtain

$$\begin{aligned} \|\nabla(\mathbf{y} - \mathbf{y}_h)\|_{0,\Omega}^2 + \|\mathbf{u} - \mathbf{u}_h\|_{0,\Omega}^2 \le{} & C\left(\rho\, a(\mathbf{y}, \mathbf{x}_h - \mathbf{y}) + a(\mathbf{y} - \mathbf{x}_h, \boldsymbol{\phi}) + b(\mathbf{x}_h - \mathbf{y}, r)\right) \\ & - (\mathbf{u} - \mathbf{u}_d, \mathbf{y} - \mathbf{x}_h) + \|\nabla(\mathbf{y} - \mathbf{x}_h)\|_{0,\Omega}^2 \\ & + \|\mathbf{y} - \mathbf{x}_h\|_{0,\Omega}^2 + \|\nabla(\boldsymbol{\phi} - \boldsymbol{\phi}_h)\|_{0,\Omega}^2 \\ & + \|\nabla(\bar{\mathbf{P}}_h \boldsymbol{\phi} - \boldsymbol{\phi})\|^2 + \|r - \bar{R}_H r\|_{0,\Omega}^2 \\ & + \|r - r_H\|_{0,\Omega}^2 + \|\mathbf{P}_h \mathbf{w} - \mathbf{w}\|_{0,\Omega}^2. \end{aligned} \tag{4.6}$$

We need to estimate the terms $\|\nabla(\boldsymbol{\phi} - \boldsymbol{\phi}_h)\|_{0,\Omega}$ and $\|r - r_H\|_{0,\Omega}$. Introducing the projection we have

$$\|\nabla(\boldsymbol{\phi} - \boldsymbol{\phi}_h)\|_{0,\Omega} \le \|\nabla(\boldsymbol{\phi} - \bar{\mathbf{P}}_h \boldsymbol{\phi})\|_{0,\Omega} + \|\nabla(\bar{\mathbf{P}}_h \boldsymbol{\phi} - \boldsymbol{\phi}_h)\|_{0,\Omega}. \tag{4.7}$$

A subtraction of (2.6d) from (4.1c) yields

$$a(\mathbf{v}_h, \bar{\mathbf{P}}_h \boldsymbol{\phi} - \boldsymbol{\phi}_h) + b(\mathbf{v}_h, r_H - \bar{R}_H r) = (\mathbf{u} - \mathbf{u}_h, \mathbf{v}_h).$$

By taking $\mathbf{v}_h = \bar{\mathbf{P}}_h \boldsymbol{\phi} - \boldsymbol{\phi}_h$ in the above equation and using the fact that $b(\bar{\mathbf{P}}_h \boldsymbol{\phi} - \boldsymbol{\phi}_h, r_H - \bar{R}_H r) = 0$, we get $a(\bar{\mathbf{P}}_h \boldsymbol{\phi} - \boldsymbol{\phi}_h, \bar{\mathbf{P}}_h \boldsymbol{\phi} - \boldsymbol{\phi}_h) = (\mathbf{u} - \mathbf{u}_h, \bar{\mathbf{P}}_h \boldsymbol{\phi} - \boldsymbol{\phi}_h)$. Applying Cauchy-Schwarz inequality we find

$$\|\nabla(\bar{\mathbf{P}}_h \boldsymbol{\phi} - \boldsymbol{\phi}_h)\|_{0,\Omega} \le \|\mathbf{u} - \mathbf{u}_h\|_{0,\Omega}. \tag{4.8}$$

Hence,

$$\|\nabla(\boldsymbol{\phi} - \boldsymbol{\phi}_h)\|_{0,\Omega} \le \|\nabla(\boldsymbol{\phi} - \bar{\mathbf{P}}_h \boldsymbol{\phi})\|_{0,\Omega} + \|\mathbf{u} - \mathbf{u}_h\|_{0,\Omega}. \tag{4.9}$$

To estimate the term $\|r - r_H\|_{0,\Omega}$, we introduce the projection

$$\|r - r_H\|_{0,\Omega} \le \|r - \bar{R}_H r\|_{0,\Omega} + \|\bar{R}_H r - r_H\|_{0,\Omega}. \tag{4.10}$$

Using the inf-sup condition, we have

$$\beta \|\bar{R}_H r - r_H\|_{0,\Omega} \le \|\mathbf{u} - \mathbf{u}_h\|_{0,\Omega} + \|\nabla(\bar{\mathbf{P}}_h \boldsymbol{\phi} - \boldsymbol{\phi}_h)\|_{0,\Omega} \le 2\|\mathbf{u} - \mathbf{u}_h\|_{0,\Omega}. \tag{4.11}$$

In the above we have used the equation (4.8). Hence we have the following:

$$\|r - r_H\|_{0,\Omega} \le \|r - \bar{R}_H r\|_{0,\Omega} + 2\|\mathbf{u} - \mathbf{u}_h\|_{0,\Omega}. \tag{4.12}$$

Substituting (4.9) and (4.12) in (4.6) we get the desired result. □

Before going to derive the estimates for the terms on the right hand side of the equation (4.2) we need to introduce few notations. Let K be a triangle which shares an edge with Γ_C, define

$$\mathcal{N} = \{x \in K \cap \Gamma_C : \mathbf{y}_a < \mathbf{y}(x) < \mathbf{y}_b\},$$

and

$$\mathcal{C} = \{x \in K \cap \Gamma_C : \mathbf{y}(x) = \mathbf{y}_a\} \cup \{x \in K \cap \Gamma_C : \mathbf{y}(x) = \mathbf{y}_b\}.$$

The sets $\mathcal{C}$ and $\mathcal{N}$ are measurable as the function $\mathbf{y}|_{\Gamma_C}$ is continuous on Γ_C. Let $|\mathcal{C}|$ and $|\mathcal{N}|$ be their measures. Now we prove the following lemma, which will be useful in the error estimation of the control.

Lemma 4.2. *There holds*

$$|\rho\, a(\mathbf{y}, \mathbf{x}_h - \mathbf{y}) + a(\mathbf{y} - \mathbf{x}_h, \boldsymbol{\phi}) + b(\mathbf{x}_h - \mathbf{y}, r) - (\mathbf{u} - \mathbf{u}_d, \mathbf{y} - \mathbf{x}_h)| \le Ch^{1+2\delta}\big(\|\mathbf{y}\|^2_{\frac{3}{2}+\delta,\Omega} + \|\boldsymbol{\phi}\|^2_{\frac{3}{2}+\delta,\Omega} + \|r\|^2_{\frac{1}{2}+\delta,\Omega}\big).$$

Proof. A use of adjoint PDE (in Remark 2.6), equation (1.7), and integration by parts yields

$$\rho\, a(\mathbf{y}, \mathbf{x}_h - \mathbf{y}) + a(\mathbf{y} - \mathbf{x}_h, \boldsymbol{\phi}) + b(\mathbf{x}_h - \mathbf{y}, r) - (\mathbf{u} - \mathbf{u}_d, \mathbf{y} - \mathbf{x}_h) = \int_{\Gamma_C} \boldsymbol{\mu}(\mathbf{y})\,(\mathbf{x}_h - \mathbf{y})ds. \quad (4.13)$$

Choose $\mathbf{x}_h = \mathcal{J}_h\mathbf{y} \in \mathbf{Q}_h$, then the right hand side of (4.13) reads as and equals

$$\int_{\Gamma_C} \boldsymbol{\mu}(\mathbf{y})(\mathcal{J}_h\mathbf{y} - \mathbf{y})ds = \sum_{K\in\mathcal{T}_h} \int_{K\cap\Gamma_C} \boldsymbol{\mu}(\mathbf{y})(\mathcal{J}_h\mathbf{y} - \mathbf{y})ds.$$

Therefore it remains to estimate the following:

$$\int_{K\cap\Gamma_C} \boldsymbol{\mu}(\mathbf{y})(\mathcal{J}_h\mathbf{y} - \mathbf{y})ds \quad \text{for all } K \in \mathcal{T}_h. \quad (4.14)$$

Fix a triangle K, sharing an edge with Γ_C. Denote the length of the edge $e = T \cap \Gamma_C$ by h_e. Clearly, $|\mathcal{C}| + |\mathcal{N}| = h_e$. Now, if either $|\mathcal{C}|$ or $|\mathcal{N}|$ equals zero, then it is easy to see that the integral term in (4.14) vanishes. So we suppose that both $\mathcal{C}$ and $\mathcal{N}$ have positive measure in the following estimation of (4.14). Now we will derive some estimates for the term (4.14):

Estimate of (4.14) depending on $\mathcal{N}$: Applying Cauchy-Schwarz inequality, and estimation in (4.27) in Lemma 4.10 (see the subsection 4.1), and a standard interpolation estimate yields

$$\begin{aligned}\int_{K\cap\Gamma_C} \boldsymbol{\mu}(\mathbf{y})(\mathcal{J}_h\mathbf{y} - \mathbf{y})ds &\le \|\boldsymbol{\mu}(\mathbf{y})\|_{0,K\cap\Gamma_C}\, \|\mathcal{J}_h\mathbf{y} - \mathbf{y}\|_{0,K\cap\Gamma_C}\\ &\le C\frac{1}{|\mathcal{N}|^{\frac{1}{2}}} h_e^{\frac{1}{2}+\delta} |\boldsymbol{\mu}(\mathbf{y})|_{\delta,K\cap\Gamma_C}\, h^{1+\delta}|\mathbf{y}'|_{\delta,K\cap\Gamma_C}\\ &\le C\frac{1}{|\mathcal{N}|^{\frac{1}{2}}} h_e^{\frac{3}{2}+2\delta}\big(|\boldsymbol{\mu}(\mathbf{y})|^2_{\delta,K\cap\Gamma_C} + |\mathbf{y}'|^2_{\delta,K\cap\Gamma_C}\big). \quad (4.15)\end{aligned}$$

Estimate of (4.14) depending on $\mathcal{C}$: Using interpolation error estimation of $\mathcal{J}_h$, and estimations (4.27) and (4.30) in Lemma 4.10 (see the subsection 4.1), we obtain

$$\begin{aligned}\int_{K\cap\Gamma_C} \boldsymbol{\mu}(\mathbf{y})(\mathcal{J}_h\mathbf{y} - \mathbf{y})ds &\le \|\boldsymbol{\mu}(\mathbf{y})\|_{0,K\cap\Gamma_C}\, \|\mathcal{J}_h\mathbf{y} - \mathbf{y}\|_{0,K\cap\Gamma_C}\\ &\le C\,\|\boldsymbol{\mu}(\mathbf{y})\|_{0,K\cap\Gamma_C}\, h_e^{\frac{1}{2}}\, \|\mathbf{y}'\|_{L^1(K\cap\Gamma_C)}\\ &\le C\frac{1}{|\mathcal{C}|^{\frac{1}{2}}} h_e^{\frac{3}{2}+2\delta}\big(|\boldsymbol{\mu}(\mathbf{y})|^2_{\delta,K\cap\Gamma_C} + |\mathbf{y}'|^2_{\delta,K\cap\Gamma_C}\big). \quad (4.16)\end{aligned}$$

It is clear that either $|\mathcal{N}|$ or $|\mathcal{C}|$ is greater than or equal to $h_e/2$. Now by choosing the compatible estimation (4.15) or (4.16), we obtain

$$\int_{K\cap\Gamma_C} \boldsymbol{\mu}(\mathbf{y})(\mathcal{J}_h\mathbf{y} - \mathbf{y})ds \le Ch_e^{1+2\delta}\big(|\boldsymbol{\mu}(\mathbf{y})|^2_{\delta,K\cap\Gamma_C} + |\mathbf{y}'|^2_{\delta,K\cap\Gamma_C}\big).$$

By summing over all the triangles sharing an edge with Γ_C and applying trace results we obtain

$$\begin{aligned}\int_{\Gamma_C} \boldsymbol{\mu}(\mathbf{y})(\mathcal{J}_h\mathbf{y} - \mathbf{y})ds &\le Ch^{1+2\delta}\big(|\boldsymbol{\mu}(\mathbf{y})|^2_{\delta,\Gamma_C} + |\mathbf{y}'|^2_{\delta,\Gamma_C}\big)\\ &\le Ch^{1+2\delta}\left(\|\mathbf{y}\|^2_{\frac{3}{2}+\delta,\Omega} + \|\boldsymbol{\phi}\|^2_{\frac{3}{2}+\delta,\Omega} + \|r\|^2_{\frac{1}{2}+\delta,\Omega}\right).\end{aligned}$$

This completes the proof. □

Theorem 4.3 *(Energy error estimate of control and L^2-estimate of velocity). Let $\mathbf{y}, \mathbf{u}$ be the continuous optimal control and state (2.6), $\mathbf{y}_h, \mathbf{u}_h$ be the discrete optimal control and state (3.1). Then there holds*

$$\begin{aligned}\rho\,\|\nabla(\mathbf{y} - \mathbf{y}_h)\|_{0,\Omega} + \|\mathbf{u} - \mathbf{u}_h\|_{0,\Omega} \le C\big(&h^{\frac{1}{2}+\delta}\,\|\mathbf{y}\|_{\frac{3}{2}+\delta,\Omega}\\ &+ h^{\frac{1}{2}+\delta}\,\|\boldsymbol{\phi}\|_{\frac{3}{2}+\delta,\Omega} + h^{\frac{1}{2}+\delta}\,\|r\|_{\frac{1}{2}+\delta,\Omega}\\ &+ h^{\frac{3}{2}+\delta}\,\|\mathbf{y}\|_{\frac{3}{2}+\delta,\Omega} + h^{\frac{3}{2}+\delta}\,\|\mathbf{u}\|_{\frac{3}{2}+\delta,\Omega}\big).\end{aligned}$$

Proof. From Theorem 4.1 we have,

$$\begin{aligned}\rho\|\nabla(\mathbf{y} - \mathbf{y}_h)\|^2_{0,\Omega} + \|\mathbf{u} - \mathbf{u}_h\|^2_{0,\Omega} \le\, &C|a(\mathbf{y}, \mathbf{y} - \mathbf{x}_h) - a(\mathbf{x}_h - \mathbf{y}, \boldsymbol{\phi}) + b(\mathbf{y} - \mathbf{x}_h, r)\\ &- (\mathbf{u} - \mathbf{u}_d, \mathbf{x}_h - \mathbf{y})| + C\Big(\|\nabla(\mathbf{y} - \mathbf{x}_h)\|^2_{0,\Omega} + \|\nabla(\boldsymbol{\phi} - \bar{\mathbf{P}}_h\boldsymbol{\phi})\|^2_{0,\Omega}\Big)\\ &+ C\left(\|\mathbf{P}_h\mathbf{w} - \mathbf{w}\|^2_{0,\Omega} + \|r - \bar{R}_H r\|^2_{0,\Omega} + \|\mathbf{y} - \mathbf{x}_h\|^2_{0,\Omega}\right),\end{aligned} \quad (4.17)$$

for all $\mathbf{x}_h \in \mathbf{Q}^h_{ad}$. The first term in the right hand side of (4.17) has been estimated in Lemma 4.2. The estimate of $\|\nabla(\boldsymbol{\phi} - \bar{\mathbf{P}}_h\boldsymbol{\phi})\|_{0,\Omega}$, $\|\mathbf{P}_h\mathbf{w} - \mathbf{w}\|_{0,\Omega}$, and $\|r - \bar{R}_H r\|_{0,\Omega}$ follows from [13]. A selection, $\mathbf{x}_h = \mathcal{J}_h\mathbf{y}$ gives the required estimate of the second and the last term. Using all the estimates together we achieve the desired estimate. □

Theorem 4.4 *(Energy error estimate of velocity). Let $\mathbf{u}$ be the continuous optimal velocity satisfies (2.6a) and $\mathbf{u}_h$ be the discrete optimal velocity satisfies (3.1a). Then there holds*

$$\begin{aligned}\|\nabla(\mathbf{u} - \mathbf{u}_h)\|_{0,\Omega} \le C\big(&h^{\frac{1}{2}+\delta}\,\|\mathbf{y}\|_{\frac{3}{2}+\delta,\Omega} + h^{\frac{1}{2}+\delta}\,\|\boldsymbol{\phi}\|_{\frac{3}{2}+\delta,\Omega} + h^{\frac{1}{2}+\delta}\,\|r\|_{\frac{1}{2}+\delta,\Omega}\\ &+ h^{\frac{3}{2}+\delta}\,\|\mathbf{y}\|_{\frac{3}{2}+\delta,\Omega} + h^{\frac{3}{2}+\delta}\,\|\mathbf{u}\|_{\frac{3}{2}+\delta,\Omega}\big).\end{aligned}$$

Proof. Splitting the state $\mathbf{u} = \mathbf{w} + \mathbf{y}$ and the discrete state $\mathbf{u}_h = \mathbf{w}_h + \mathbf{y}_h$, we have

$$\|\nabla(\mathbf{u} - \mathbf{u}_h)\|_{0,\Omega} \le \|\nabla(\mathbf{w} - \mathbf{w}_h)\|_{0,\Omega} + \|\nabla(\mathbf{y} - \mathbf{y}_h)\|_{0,\Omega}.$$

Introducing the projection in the first term of the above equation we obtain

$$\|\nabla(\mathbf{w} - \mathbf{w}_h)\|_{0,\Omega} \le \|\nabla(\mathbf{w} - \mathbf{P}_h\mathbf{w})\|_{0,\Omega} + \|\nabla(\mathbf{P}_h\mathbf{w} - \mathbf{w}_h)\|_{0,\Omega}.$$

Subtraction of (3.1b) from (4.1a) yields

$$a(\mathbf{P}_h\mathbf{w} - \mathbf{w}_h, \mathbf{z}_h) + b(\mathbf{z}_h, R_H p - p_H) = a(\mathbf{y}_h - \mathbf{y}, \mathbf{z}_h) \quad \text{for all } \mathbf{z}_h \in \mathbf{V}_h. \quad (4.18)$$

By taking $\mathbf{z}_h = \mathbf{P}_h\mathbf{w} - \mathbf{w}_h$ in the above equation and using the fact that $b(\mathbf{P}_h\mathbf{w} - \mathbf{w}_h, R_H p - p_H) = 0$ we get $\|\nabla(\mathbf{P}_h\mathbf{w} - \mathbf{w}_h)\|^2_{0,\Omega} = a(\mathbf{y}_h - \mathbf{y}, \mathbf{P}_h\mathbf{w} - \mathbf{w}_h)$. Applying Cauchy-Schwarz inequality we find

$$\|\nabla(\mathbf{P}_h\mathbf{w} - \mathbf{w}_h)\|_{0,\Omega} \le \|\nabla(\mathbf{y}_h - \mathbf{y})\|_{0,\Omega}. \quad (4.19)$$

Hence,

$\|\nabla(\mathbf{u}-\mathbf{u}_h)\|_{0,\Omega} \le \|\nabla(\mathbf{w}-\mathbf{P}_h\mathbf{w})\|_{0,\Omega} + 2\|\nabla(\mathbf{y}-\mathbf{y}_h)\|_{0,\Omega}$.

Using the estimate of $\|\nabla(\mathbf{y}-\mathbf{y}_h)\|_{0,\Omega}$ from Theorem 4.1 and estimate of $\|\nabla(\mathbf{w}-\mathbf{P}_h\mathbf{w})\|_{0,\Omega}$ from [13] we have the required result. □

Theorem 4.5 *(Error estimate of pressure). Let p be the continuous optimal pressure satisfies (2.6b) and p_H be the discrete optimal pressure satisfies (3.1b). Then there holds*

$$\begin{aligned}\|p-p_H\|_{0,\Omega} \le & C\big(h^{\frac{1}{2}+\delta}\|\mathbf{y}\|_{\frac{3}{2}+\delta,\Omega} + h^{\frac{1}{2}+\delta}\|\boldsymbol{\phi}\|_{\frac{3}{2}+\delta,\Omega} + h^{\frac{1}{2}+\delta}\|r\|_{\frac{1}{2}+\delta,\Omega} \\ & + h^{\frac{3}{2}+\delta}\|\mathbf{y}\|_{\frac{3}{2}+\delta,\Omega} + h^{\frac{3}{2}+\delta}\|\mathbf{u}\|_{\frac{3}{2}+\delta,\Omega}\big).\end{aligned}$$

Proof. Introducing the projection we have, $\|p-p_H\|_{0,\Omega} \le \|p-R_Hp\|_{0,\Omega} + \|R_Hp-p_H\|_{0,\Omega}$. The estimate of $\|R_Hp-p_H\|_{0,\Omega}$ follows from the following:

$$\begin{aligned}\beta\|R_Hp-p_H\|_{0,\Omega} &\le \sup_{\mathbf{v}_h\in\mathbf{V}_h}\frac{b(\mathbf{v}_h, R_Hp-p_H)}{\|\mathbf{v}_h\|_1} \\ &\le \sup_{\mathbf{v}_h\in\mathbf{V}_h}\frac{a(\mathbf{y}_h-\mathbf{y},\mathbf{v}_h)-a(\mathbf{P}_h\mathbf{w}-\mathbf{w}_h,\mathbf{v}_h)}{\|\mathbf{v}_h\|_1} \\ &\le \|\nabla(\mathbf{y}-\mathbf{y}_h)\|_{0,\Omega} + \|\nabla(\mathbf{P}_h\mathbf{w}-\mathbf{w}_h)\|_{0,\Omega}. \end{aligned} \tag{4.20}$$

Using (4.19) in the above equation we have $\|p-p_H\|_{0,\Omega} \le \|p-R_Hp\|_{0,\Omega} + 2\|\nabla(\mathbf{y}-\mathbf{y}_h)\|_{0,\Omega}$. Using the estimates of $\|p-R_Hp\|_{0,\Omega}$ from [13] and $\|\nabla(\mathbf{y}-\mathbf{y}_h)\|_{0,\Omega}$ from Theorem 4.3 we get the desired result. □

Theorem 4.6 *(Error estimate of adjoint velocity). Let $\boldsymbol{\phi}$ be the continuous optimal adjoint velocity satisfies (2.6d) and $\boldsymbol{\phi}_h$ be the discrete optimal adjoint velocity satisfies (3.1d). Then there holds*

$$\begin{aligned}\|\nabla(\boldsymbol{\phi}-\boldsymbol{\phi}_h)\|_{0,\Omega} \le & C\big(h^{\frac{1}{2}+\delta}\|\mathbf{y}\|_{\frac{3}{2}+\delta,\Omega} + h^{\frac{1}{2}+\delta}\|\boldsymbol{\phi}\|_{\frac{3}{2}+\delta,\Omega} + h^{\frac{1}{2}+\delta}\|r\|_{\frac{1}{2}+\delta,\Omega} \\ & + h^{\frac{3}{2}+\delta}\|\mathbf{y}\|_{\frac{3}{2}+\delta,\Omega} + h^{\frac{3}{2}+\delta}\|\mathbf{u}\|_{\frac{3}{2}+\delta,\Omega}\big).\end{aligned}$$

Proof. Introducing the projection $\bar{\mathbf{P}}_h\boldsymbol{\phi}$, we obtain

$$\|\nabla(\boldsymbol{\phi}-\boldsymbol{\phi}_h)\|_{0,\Omega} \le \|\nabla(\boldsymbol{\phi}-\bar{\mathbf{P}}_h\boldsymbol{\phi})\|_{0,\Omega} + \|\nabla(\bar{\mathbf{P}}_h\boldsymbol{\phi}-\boldsymbol{\phi}_h)\|_{0,\Omega}. \tag{4.21}$$

A subtraction of (2.6d) from (4.1c) yields

$$a(\mathbf{v}_h, \bar{\mathbf{P}}_h\boldsymbol{\phi}-\boldsymbol{\phi}_h) + b(\mathbf{v}_h, r_H - \bar{R}_Hr) = (\mathbf{u}-\mathbf{u}_h, \mathbf{v}_h).$$

By taking $\mathbf{v}_h = \bar{\mathbf{P}}_h\boldsymbol{\phi}-\boldsymbol{\phi}_h$ in the above equation and using the fact that $b(\bar{\mathbf{P}}_h\boldsymbol{\phi}-\boldsymbol{\phi}_h, r_H-\bar{R}_Hr)=0$, we get $a(\bar{\mathbf{P}}_h\boldsymbol{\phi}-\boldsymbol{\phi}_h, \bar{\mathbf{P}}_h\boldsymbol{\phi}-\boldsymbol{\phi}_h) = (\mathbf{u}-\mathbf{u}_h, \bar{\mathbf{P}}_h\boldsymbol{\phi}-\boldsymbol{\phi}_h)$. Applying Cauchy-Schwarz inequality we find

$$\|\nabla(\bar{\mathbf{P}}_h\boldsymbol{\phi}-\boldsymbol{\phi}_h)\|_{0,\Omega} \le \|\mathbf{u}-\mathbf{u}_h\|_{0,\Omega}. \tag{4.22}$$

Hence,

$$\|\nabla(\boldsymbol{\phi}-\boldsymbol{\phi}_h)\|_{0,\Omega} \le \|\nabla(\boldsymbol{\phi}-\bar{\mathbf{P}}_h\boldsymbol{\phi})\|_{0,\Omega} + \|\mathbf{u}-\mathbf{u}_h\|_{0,\Omega}. \tag{4.23}$$

Using Theorem 4.3 and the estimate $\|\nabla(\boldsymbol{\phi}-\bar{\mathbf{P}}_h\boldsymbol{\phi})\|_{0,\Omega} \le h^{\frac{1}{2}+\delta}\|\boldsymbol{\phi}\|_{\frac{3}{2}+\delta,\Omega}$ from [13, Section 4.2], we achieve the desired result. □

Theorem 4.7 *(Error estimate of adjoint pressure). Let r be the continuous optimal adjoint pressure satisfies (2.6d) and r_H be the discrete optimal adjoint pressure satisfies (3.1d). Then there holds*

$$\begin{aligned}\|r-r_H\|_{0,\Omega} \le & C\big(h^{\frac{1}{2}+\delta}\|\mathbf{y}\|_{\frac{3}{2}+\delta,\Omega} + h^{\frac{1}{2}+\delta}\|\boldsymbol{\phi}\|_{\frac{3}{2}+\delta,\Omega} + h^{\frac{1}{2}+\delta}\|r\|_{\frac{1}{2}+\delta,\Omega} \\ & + h^{\frac{3}{2}+\delta}\|\mathbf{y}\|_{\frac{3}{2}+\delta,\Omega} + h^{\frac{3}{2}+\delta}\|\mathbf{u}\|_{\frac{3}{2}+\delta,\Omega}\big).\end{aligned}$$

Proof. Similar to the proof of Theorem 4.5. □

Remark 4.8. If the domain is not so smooth, e.g. the angle at transition (Dirichlet-Dirichlet and Neumann-Dirichlet) point is greater than $\pi/2$ or some bad polygonal structure, then there is a possibility that the solution could be less regular i.e., $\mathbf{u}\in\mathbf{H}^{\frac{3}{2}-\delta}(\Omega)$, $\boldsymbol{\phi}\in\mathbf{H}^{\frac{3}{2}-\delta}(\Omega)$, $r\in H^{\frac{1}{2}-\delta}(\Omega)$, and $\mathbf{y}\in\mathbf{H}^{\frac{3}{2}-\delta}(\Omega)$, where $0<\delta<1/2$. Then all the above *a priori* estimates hold true except Lemma 4.2. It is clear that if the solutions have the above regularity then (4.13) is not true because the right hand side of (4.13) does not make sense. So, to estimate the term

$$|\left(\rho\, a(\mathbf{y},\mathbf{x}_h-\mathbf{y}) + a(\mathbf{y}-\mathbf{x}_h,\boldsymbol{\phi}) + b(\mathbf{x}_h-\mathbf{y},r) - (\mathbf{u}-\mathbf{u}_d,\mathbf{y}-\mathbf{x}_h)\right)| \tag{4.24}$$

we use the following idea:

$$\begin{aligned}&\rho\, a(\mathbf{y},\mathbf{x}_h-\mathbf{y}) + a(\mathbf{y}-\mathbf{x}_h,\boldsymbol{\phi}) + b(\mathbf{x}_h-\mathbf{y},r) - (\mathbf{u}-\mathbf{u}_d,\mathbf{y}-\mathbf{x}_h) \\ &\qquad = \langle\rho\frac{\partial\mathbf{y}}{\partial\mathbf{n}} - \frac{\partial\boldsymbol{\phi}}{\partial\mathbf{n}} - r\mathbf{n}, \mathbf{x}_h-\mathbf{y}\rangle_{\delta,\Gamma_C} \\ &\qquad \le \|\boldsymbol{\mu}(\mathbf{y})\|_{H^{\delta}(\Gamma_C)'}\|\mathbf{x}_h-\mathbf{y}\|_{\delta,\Gamma_C}.\end{aligned} \tag{4.25}$$

Choosing $\mathbf{x}_h = \mathcal{J}_h\mathbf{y}$, we have $\|\mathbf{y}-\mathcal{J}_h\mathbf{y}\|_{\delta,\Gamma_C} \le h^{1-2\delta}\|\mathbf{y}\|_{3/2-\delta,\Omega}$. Using the trace estimate (discussed in Section 2) we have $\|\boldsymbol{\mu}(\mathbf{y})\|_{H^{\delta}(\Gamma_C)'} \le C\big(\|\mathbf{y}\|_{3/2-\delta,\Omega} + \|\boldsymbol{\phi}\|_{3/2-\delta,\Omega} + \|r\|_{1/2-\delta,\Omega}\big)$. Putting all these estimates in (4.25) we have

$$|\rho\, a(\mathbf{y},\mathbf{x}_h-\mathbf{y}) + a(\mathbf{y}-\mathbf{x}_h,\boldsymbol{\phi}) + b(\mathbf{x}_h-\mathbf{y},r) - (\mathbf{u}-\mathbf{u}_d,\mathbf{y}-\mathbf{x}_h)| \lesssim h^{1-2\delta}. \tag{4.26}$$

Thus, we have an optimal order (up to the regularity) of convergence of the term (4.13). Hence, Theorems 4.3-4.7 show the optimal order (up to the regularity) of convergence of control, state and adjoint state variables.

Remark 4.9. For the simplicity of the error analysis we choose the domain Ω very specific as shown in Fig. 2.1. But it is clear from the Remark 4.8 that our error analysis also works for solution with low regularity. This ensures that we can also work with another type of polygonal domain.

4.1. Some local L^2 and L^1 estimate for $\boldsymbol{\mu}(\mathbf{y})$ and $\mathbf{y}'$

Lemma 4.10. *Let h_e be the length of the edge $K\cap\Gamma_C$, and $|C|>0$ and $|\mathcal{N}|>0$. Then the following estimates hold:*

$$\|\boldsymbol{\mu}(\mathbf{y})\|_{0,K\cap\Gamma_C} \le \frac{1}{|\mathcal{N}|^{1/2}} h_e^{\frac{1}{2}+\delta}|\boldsymbol{\mu}(\mathbf{y})|_{\delta,K\cap\Gamma_C}, \tag{4.27}$$

$$\|\boldsymbol{\mu}(\mathbf{y})\|_{L^1(K\cap\Gamma_C)} \le \frac{|C|^{1/2}}{|\mathcal{N}|^{1/2}} h_e^{\frac{1}{2}+\delta}|\boldsymbol{\mu}(\mathbf{y})|_{\delta,K\cap\Gamma_C}, \tag{4.28}$$

$$\|\mathbf{y}'\|_{0,K\cap\Gamma_C} \le \frac{1}{|C|^{1/2}} h_e^{\frac{1}{2}+\delta}|\mathbf{y}'|_{\delta,K\cap\Gamma_C}, \tag{4.29}$$

$$\|\mathbf{y}'\|_{L^1(K\cap\Gamma_C)} \le \frac{|\mathcal{N}|^{1/2}}{|C|^{1/2}} h_e^{\frac{1}{2}+\delta}|\mathbf{y}'|_{\delta,K\cap\Gamma_C}, \tag{4.30}$$

where $\boldsymbol{\mu}(\mathbf{y}) = \rho\frac{\partial\mathbf{y}}{\partial\mathbf{n}} - \frac{\partial\boldsymbol{\phi}}{\partial\mathbf{n}} - r\mathbf{n}$ and $\mathbf{y}' := (y_1', y_2')$ be the tangential derivative of $\mathbf{y}$ on $K\cap\Gamma_C$.

Proof. Let us start with the L^2-estimate of $\boldsymbol{\mu}(\mathbf{y})$

$$\begin{aligned}\|\boldsymbol{\mu}(\mathbf{y})\|^2_{0,K\cap\Gamma_C} &= \int_{K\cap\Gamma_C}|\boldsymbol{\mu}(\mathbf{y})(\xi)|^2 d\xi \\ &= \int_C |\boldsymbol{\mu}(\mathbf{y})(\xi)|^2 d\xi \quad (\boldsymbol{\mu}(\mathbf{y})=0 \text{ on } \mathcal{N}) \\ &= \frac{1}{|\mathcal{N}|}\int_C\int_{\mathcal{N}}|\boldsymbol{\mu}(\mathbf{y})(\xi)-\boldsymbol{\mu}(\mathbf{y})(\eta)|^2 d\eta\, d\xi \\ &\le \frac{1}{|\mathcal{N}|}\sup_{C\times\mathcal{N}}|\xi-\eta|^{1+2\delta}\int_C\int_{\mathcal{N}}\frac{|\boldsymbol{\mu}(\mathbf{y})(\xi)-\boldsymbol{\mu}(\mathbf{y})(\eta)|^2}{|\xi-\eta|^{1+2\delta}}d\eta\, d\xi\end{aligned}$$

$$\leq \frac{1}{|\mathcal{N}|} h_e^{1+2\delta} |\boldsymbol{\mu}(\mathbf{y})|^2_{\delta,K\cap\Gamma_C},$$

which proves (4.27). Now we prove (4.28) as follows

$$\begin{aligned}\int_{K\cap\Gamma_C} |\boldsymbol{\mu}(\mathbf{y})| &= \int_C |\boldsymbol{\mu}(\mathbf{y})|,\\ &\leq |C|^{1/2}\,\|\boldsymbol{\mu}(\mathbf{y})\|_{0,C},\\ &\leq |C|^{1/2}\,\|\boldsymbol{\mu}(\mathbf{y})\|_{0,K\cap\Gamma_C},\\ &\leq \frac{|C|^{1/2}}{|\mathcal{N}|^{1/2}} h_e^{\frac{1}{2}+\delta} |\boldsymbol{\mu}(\mathbf{y})|_{\delta,K\cap\Gamma_C}.\end{aligned}$$

The L^2-estimate of $\mathbf{y}'$ is derived as follows

$$\begin{aligned}\|\mathbf{y}'\|^2_{0,K\cap\Gamma_C} &= \int_{K\cap\Gamma_C} |\mathbf{y}'(\xi)|^2 d\xi\\ &= \int_{\mathcal{N}} |\mathbf{y}'(\xi)|^2 d\xi \quad (\mathbf{y}'=0 \text{ on } C)\\ &= \frac{1}{|C|}\int_{\mathcal{N}}\int_C |\mathbf{y}'(\xi)-\mathbf{y}'(\eta)|^2 d\eta\, d\xi\\ &\leq \frac{1}{|C|}\sup_{C\times\mathcal{N}} |\xi-\eta|^{1+2\delta}\int_{\mathcal{N}}\int_C \frac{|\mathbf{y}'(\xi)-\mathbf{y}'(\eta)|^2}{|\xi-\eta|^{1+2\delta}} d\eta\, d\xi\\ &\leq \frac{1}{|C|} h_e^{1+2\delta} |\mathbf{y}'|^2_{\delta,K\cap\Gamma_C},\end{aligned}$$

which proves (4.29). One can easily derive the estimate (4.30) by using (4.29). □

5. Numerical experiments

In this section, we are going to validate the a priori error estimates for the error in the control, state, and adjoint state numerically. Here we consider two model examples with known exact solutions. For the numerical experiments we slightly modify the optimal control problem which is as follows:

$$\text{minimize } \tilde{J}(\mathbf{w},\mathbf{x}) = \frac{1}{2}\|\mathbf{w}-\mathbf{u}_d\|^2_{0,\Omega} + \frac{\rho}{2}\|\nabla(\mathbf{x}-\mathbf{y}_d)\|^2_{0,\Omega},$$

subject to the PDE,

$$\begin{aligned} -\Delta\mathbf{w}+\nabla p &= \mathbf{f} \quad \text{in } \Omega, &&(5.1a)\\ \nabla\cdot\mathbf{w} &= 0 \quad \text{in } \Omega,\\ \mathbf{w} &= \mathbf{x} \quad \text{on } \Gamma_C,\\ \mathbf{w} &= \mathbf{0} \quad \text{on } \Gamma_D,\end{aligned}$$

the control set is given by

$$\mathbf{Q}_{ad} := \{\mathbf{x}\in\mathbf{H}^1(\Omega) : \gamma_0(\mathbf{x}) = \mathbf{0} \text{ on } \Gamma_D, \mathbf{y}_a \leq \gamma_0(\mathbf{x}) \leq \mathbf{y}_b \text{ a.e. on } \Gamma_C\},$$

where the function $\mathbf{y}_d$ is given and the boundary $\partial\Omega = \Gamma_C \cup \bar{\Gamma}_D$. Consequently, the discrete optimality system takes the form

$$\begin{aligned} \mathbf{u}_h &= \mathbf{w}_h + \mathbf{y}_h, \quad \mathbf{w}_h \in \mathbf{V}_h,\\ a(\mathbf{w}_h,\mathbf{z}_h) + b(\mathbf{z}_h,p_H) &= (\mathbf{f},\mathbf{z}_h) - a(\mathbf{y}_h,\mathbf{z}_h) \quad \text{for all } \mathbf{z}_h\in\mathbf{V}_h,\\ b(\mathbf{u}_h,q_H) &= 0 \quad \text{for all } q_H\in M_H,\\ a(\mathbf{z}_h,\boldsymbol{\phi}_h) - b(\mathbf{z}_h,r_H) &= (\mathbf{u}_h-\mathbf{u}_d,\mathbf{z}_h) \quad \text{for all } \mathbf{z}_h\in\mathbf{V}_h,\\ b(\boldsymbol{\phi}_h,q_H) &= 0 \quad \text{for all } q_H\in M_H,\\ \rho\, a(\mathbf{y}_h,\mathbf{x}_h-\mathbf{y}_h) &\geq a(\mathbf{x}_h-\mathbf{y}_h,\boldsymbol{\phi}_h) - b(\mathbf{x}_h-\mathbf{y}_h,r_H)\\ &\quad - (\mathbf{u}_h-\mathbf{u}_d,\mathbf{x}_h-\mathbf{y}_h) \quad \text{for all } \mathbf{x}_h\in\mathbf{Q}^h_{ad},\end{aligned}$$

Table 5.1
Energy errors and convergence rates of the state variable for Example 5.1.

h	$\|\nabla(\mathbf{u}-\mathbf{u}_h)\|_{0,\Omega}$	order	H	$\|p-p_H\|_{0,\Omega}$	order
0.2500	1.2670	0	0.5000	0.3327	0
0.1250	0.7124	0.8307	0.2500	0.7025	-1.0782
0.0625	0.3502	1.0243	0.1250	0.3584	0.9709
0.0312	0.1770	0.9843	0.0625	0.1878	0.9323
0.0156	0.0888	0.9955	0.0312	0.0949	0.9852

Table 5.2
Energy errors and convergence rates of the adjoint state variable for Example 5.1.

h	$\|\nabla(\boldsymbol{\phi}-\boldsymbol{\phi}_h)\|_{0,\Omega}$	order	H	$\|r-r_H\|_{0,\Omega}$	order
0.2500	2.3420	0	0.5000	0.0153	0
0.1250	1.3102	0.8380	0.2500	0.7643	-5.6448
0.0625	0.6934	0.9181	0.1250	0.3786	1.0136
0.0312	0.3537	0.9711	0.0625	0.1975	0.9386
0.0156	0.1777	0.9928	0.0312	0.0995	0.9890

where, $\mathbf{V}_h := \{\mathbf{v}_h\in\mathbf{H}^1_0(\Omega) : \mathbf{v}_h|_T\in\mathbf{P}_1(K)\ \forall K\in\mathcal{T}_h\}$ and $\mathbf{Q}^h_{ad} = \mathbf{Q}_h\cap\mathbf{Q}_{ad}$. The set $\mathbf{Q}_h$ is defined by $\mathbf{Q}_h = \{\mathbf{x}_h\in\mathbf{H}^1(\Omega) : \gamma_0(\mathbf{x}) = \mathbf{0} \text{ on } \Gamma_D, \mathbf{x}_h|_T\in\mathbf{P}_1(K),\ \forall K\in\mathcal{T}_h\}$.

Example 5.1. Let the computational domain $\Omega = (0,1)^2$, and $\Gamma_D = (0,1)\times\{0\}$, $\Gamma_C = \partial\Omega\backslash\Gamma_D$. We choose the constants $\rho = 10^{-2}$, $\mathbf{y}_a = (-4,0)$, and $\mathbf{y}_b = (0,2.5)$. The state and adjoint state variables are given by

$$\mathbf{u} = \mathbf{y} = \begin{pmatrix} -\exp(x)(y\cos(y)+\sin(y))\\ \exp(x)y\sin(y)\end{pmatrix}, \quad p = \sin(2\pi x)\sin(2\pi y), \tag{5.3}$$

and

$$\boldsymbol{\phi} = \begin{pmatrix} (\sin(\pi x))^2\sin(\pi y)\cos(\pi y)\\ -(\sin(\pi y))^2\sin(\pi x)\cos(\pi x)\end{pmatrix}, \quad r = \sin(2\pi x)\sin(2\pi y). \tag{5.4}$$

We choose $\mathbf{u}$ and $\boldsymbol{\phi}$ such that $\nabla\cdot\mathbf{u} = \nabla\cdot\boldsymbol{\phi} = 0$ in Ω and $\boldsymbol{\phi} = \mathbf{0}$ on $\partial\Omega$. The data of the problem are chosen such that $\mathbf{f} = -\Delta\mathbf{u}+\nabla p$, $\mathbf{u}_d = \mathbf{u}+\Delta\boldsymbol{\phi}+\nabla r$ and $\mathbf{y}_d = \mathbf{y}$.

The discrete solution is computed on several uniform grids with mesh sizes $h = \frac{1}{2^i}, i = 2,\dots,6$ for the velocity variable and mesh sizes $H = 2h$ for the pressure variable. To solve the optimal control problem numerically, we have used the primal-dual active set algorithm. The continuous and discrete approximations for the state velocity variables using conforming $\mathbf{P}_1$ (in fine mesh) finite elements are shown in Fig. 5.1. The continuous and discrete approximations for the pressure variables using conforming P_0 (in coarse mesh) finite elements are shown in Fig. 5.3(A). The continuous and discrete approximations for the adjoint state velocity variables using conforming $\mathbf{P}_1$ (in fine mesh) finite elements are shown in Fig. 5.2. The continuous and discrete approximations for the pressure variables using conforming P_0 (in coarse mesh) finite elements are shown in Fig. 5.3(B). The continuous and discrete approximations for the control variables using conforming $\mathbf{P}_1$ (in fine mesh) finite elements are shown in Fig. 5.4.

Tables 5.1 and 5.2 show the computed errors and orders of convergence of the state and adjoint state variables respectively for the Example 5.1. The errors and orders of convergence of the control variable are shown in Table 5.3. The numerical convergence rates with respect to the energy norm for the state, adjoint state and control variables are linear as predicted theoretically.

Example 5.2. In this example, we report the results of numerical tests carried out for the L-shaped domain $\Omega = (-1,1)^2\setminus((0,1)\times(-1,0))$ and $\Gamma_C = \partial\Omega$. We choose the constants $\rho = 10^{-2}$, $\mathbf{y}_a = (-3,-3)$, $\mathbf{y}_b = (4,4)$ and the exact state

$$\mathbf{u} = r^\alpha\begin{pmatrix}(1+\alpha)\sin(\theta)\omega(\theta)+\cos(\theta)\omega'(\theta)\\ -(1+\alpha)\cos(\theta)\omega(\theta)+\sin(\theta)\omega'(\theta)\end{pmatrix},$$

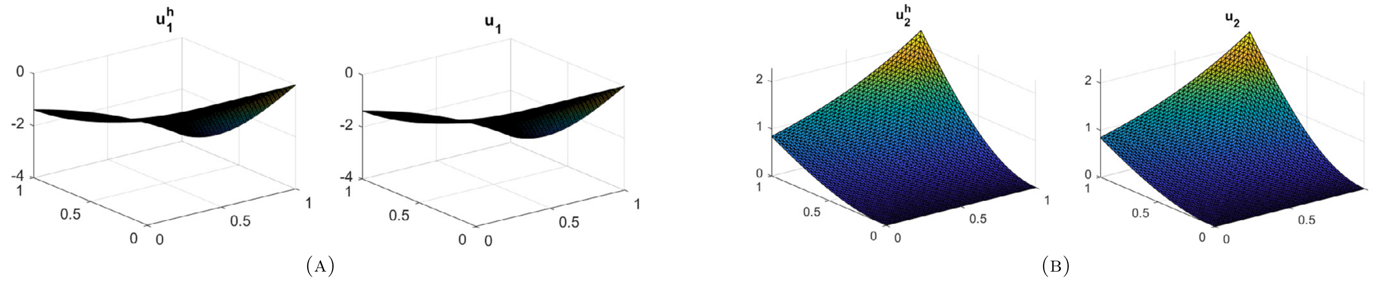

Fig. 5.1. (A) Discrete state ($\mathbf{u}_1^h$) and continuous state ($\mathbf{u}_1$) for Example 5.1 (B) Discrete state ($\mathbf{u}_2^h$) and continuous state ($\mathbf{u}_2$) for Example 5.1.

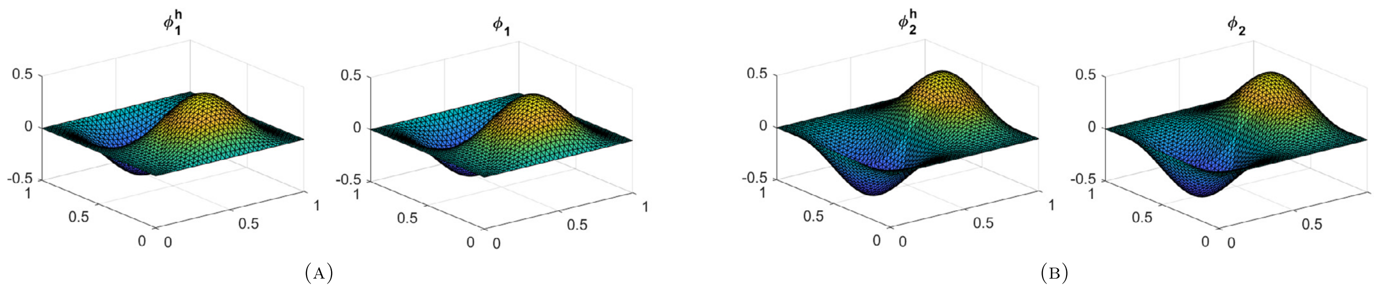

Fig. 5.2. (A) Discrete adj-state ($\boldsymbol{\phi}_1^h$) and continuous adj-state ($\boldsymbol{\phi}_1$) for Example 5.1 (B) Discrete adj-state ($\boldsymbol{\phi}_2^h$) and continuous adj-state ($\boldsymbol{\phi}_2$) for Example 5.1.

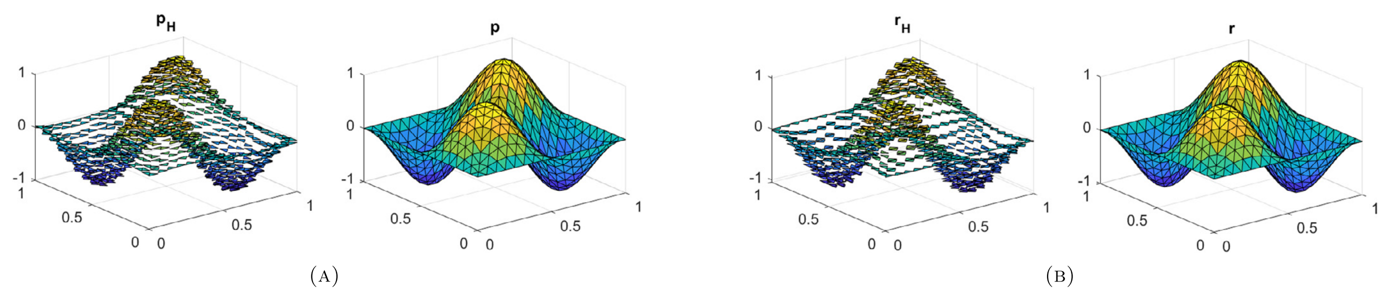

Fig. 5.3. (A) Discrete pressure (p_H) and continuous pressure (p) for Example 5.1 (B) Discrete adj-pressure (r_H) and continuous adj-pressure (r) for Example 5.1.

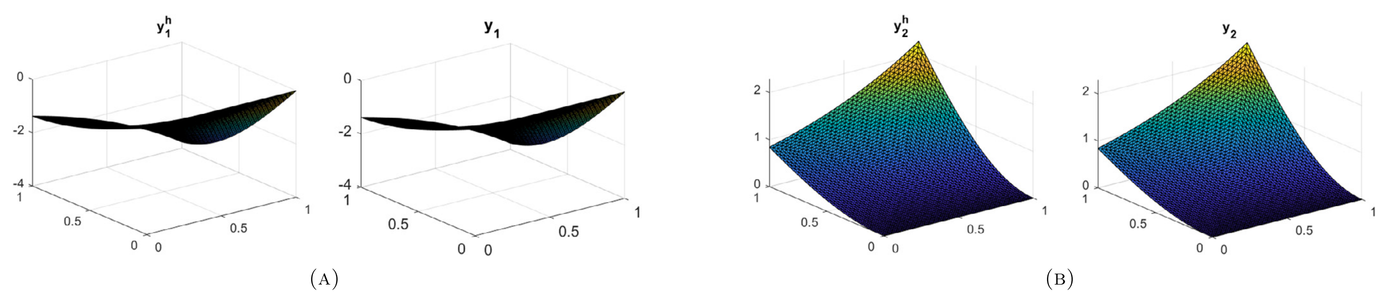

Fig. 5.4. (A) Discrete control ($\mathbf{y}_1^h$) and continuous control ($\mathbf{y}_1$) for Example 5.1 (B) Discrete control ($\mathbf{y}_2^h$) and continuous control ($\mathbf{y}_2$) for Example 5.1.

Table 5.3
Energy errors and convergence rates of the control variable for Example 5.1.

h	$\|\nabla(\mathbf{y}-\mathbf{y}_h)\|_{0,\Omega}$	order
0.2500	1.2307	0
0.1250	0.6167	0.9969
0.0625	0.3085	0.9993
0.0312	0.1543	0.9998
0.0156	0.0771	1.0000

$$p = -r^{\alpha-1}((1+\alpha)^2\omega'(\theta) + \omega'''(\theta))/(1-\alpha),$$

where

$$\omega(\theta) = 1/(1+\alpha)\sin(\alpha+1)\theta)\cos(\alpha w) - \cos((\alpha+1)\theta)$$

$$+ 1/(1+\alpha)\sin(\alpha-1)\theta)\cos(\alpha\omega) - \cos((\alpha-1)\theta)$$

and $\alpha = 856399/1572864$ and $w = 3\pi/2$. The adjoint variables $\boldsymbol{\phi}$, r are considered as same as in Example 5.1.

$$\mathbf{f} = -\Delta\mathbf{u} + \nabla p - \mathbf{y}, \text{ and } \mathbf{u}_d = \mathbf{u} + \Delta\boldsymbol{\phi} + \nabla r. \tag{5.5}$$

This problem is defined on the L-shaped domain, and the derivative of $\mathbf{u}$ and p have singularity at the origin. The velocity $\mathbf{u}$ and the control $\mathbf{y}$ are in $H^{1+s}(\Omega)$ and the pressure p is in the space $H^s(\Omega)$ with $0 < s < 1$. In the same way as in Example 5.1, we produce a sequence of meshes. Tables 5.4 and 5.5 show the computed errors and orders of convergence of the state and adjoint state variables respectively for Example 5.2. The errors and orders of convergence of the control variable are shown in Table 5.6. It can be observed that the convergence rate for the state

Table 5.4
Energy errors and convergence rates of the state variable for Example 5.2.

h	$\|\nabla(\mathbf{u}-\mathbf{u}_h)\|_{0,\Omega}$	order	H	$\|p-p_H\|_{0,\Omega}$	order
0.3536	1.5056	0	0.7071	0.8023	0
0.1768	1.1718	0.3616	0.3536	0.6433	0.3187
0.0884	0.8519	0.4600	0.1768	0.3239	0.9898
0.0442	0.6001	0.5054	0.0884	0.1773	0.8691
0.0221	0.4170	0.5252	0.0442	0.1073	0.7249

Table 5.5
Energy errors and convergence rates of the adjoint state variable for the Example 5.2.

h	$\|\nabla(\boldsymbol{\phi}-\boldsymbol{\phi}_h)\|_{0,\Omega}$	order	H	$\|r-r_H\|_{0,\Omega}$	order
0.3536	5.4415	0	0.7071	0.0265	0
0.1768	3.0894	0.8167	0.3536	0.7780	-4.8773
0.0884	1.8792	0.7172	0.1768	0.8775	-0.1735
0.0442	0.9968	0.9147	0.0884	0.4986	0.8154
0.0221	0.5065	0.9768	0.0442	0.2572	0.9553

Table 5.6
Energy errors and convergence rates of the control variable for the Example 5.2.

h	$\|\nabla(\mathbf{y}-\mathbf{y}_h)\|_{0,\Omega}$	order
0.3536	1.3446	0
0.1768	0.9556	0.4926
0.0884	0.6677	0.5172
0.0442	0.4624	0.5302
0.0221	0.3187	0.5370

and control have been deteriorated for the above choice of non-regular solutions.

6. Conclusions

In this article, we propose an energy space based approach to formulate the Dirichlet boundary optimal control problem governed by the Stokes equation. Most of the previous work in the Stokes Dirichlet boundary control problem authors took either tangential control or flux of control is zero. This choice of control is very restrictive and those conditions on the control reduce the regularity of the control. To overcome this difficulty we introduce the Stokes problem with mixed boundary conditions and the control acts on the Dirichlet boundary only hence our control is more general and it has both the tangential and normal components. We discuss well-posedness and regularity results for the control problem. The first order necessary optimality condition results in a simplified Signorini type problem for control variable. We develop a finite element discretization by using $\mathbf{P}_1$ elements (in the fine mesh) for the velocity and control variable and P_0 elements (in the coarse mesh) for the pressure variable. The standard error analysis gives $\frac{1}{2}+\frac{\delta}{2}$ order of convergence for the control. Here we have improved it to $\frac{1}{2}+\delta$, which is optimal up to regularity. The theoretical results are corroborated by a variety of numerical tests.

Data availability

No data was used for the research described in the article.

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