1 Introduction

Active matter is the focus of intense current interest due to its dramatic mechanical and statistical properties, such as giant number fluctuations, wave propagation without conventional inertia, sustained spontaneous oscillations, instability of simple liquid-crystalline order in bulk fluid, motile topological defects, motility-induced phase separation and its generalisations and viscosity reduction through internally generated stresses. The origin of these properties is the energy supply at the microscopic level, directly to the constituent particles, unlike in conventional nonequilibrium systems such as sheared fluids, which are powered through their periphery. The energy transduction responsible for the active character of a particle could be wholly internal, as in living organisms, or it could take place in the region of contact of the particle with its surroundings, as with self-phoretic colloids in a fluid. Quincke rollers or vibrated grains. Active particles are commonly elongated and can therefore form orientationally ordered states, of which the simplest are nematic and polar uniaxial liquid crystals. Diverse mechanisms underlie the ordering of active particles: in living active systems like bird flocks, fish schools and herds, the process is behavioural and based on mutual sensing. In active granular systems, which exemplify dry active matter, steric and collisional effects give rise to alignment. In colloidal rollers, a restricted version of the hydrodynamic interaction is responsible.

Considerable progress has been made towards understanding active polar systems since the discovery of a flocking phase transition in an agent-based model, including field-theoretic arguments towards the existence of long-range order in 2D flocks and predictions of highly anisotropic sound waves – through the interplay of the concentration and broken-symmetry fields – and anomalously large number fluctuations in the ordered phase. Recently, Geyer et al. observed sound waves in the suspension of active colloidal rollers. Bertin et al., in a Boltzmann-equation construction of the Toner–Tu equations, discovered that the dependence of the local ordering tendency on the local density inevitably led to a linear instability of the ordered phase, with wavevectors parallel to the ordering direction, just past the mean-field flocking transition. Indeed, a banded phase is widely observed to intervene between the isotropic and the uniform ordered phases in agent-based numerical simulations and in experiments on rolling-colloid flocks, and arises as well in a variety of theoretical models.
Here, we study a two-dimensional active polar monolayer consisting of tapered rods adrift in a sea of spherical beads. The energy input to the particles is provided by a vertically vibrated supporting surface, and the rods, by virtue of their shape, transduce this vibration into directed movement in the plane. The spherical beads mediate an aligning interaction between the rods, and otherwise behave like passive particles that move if pushed or dragged by the polar rods. In earlier work on this system, we discovered a nonequilibrium phase transition from the isotropic state to an ordered, coherently moving flock, which took place when the concentration of the spherical beads exceeded a critical value that decreased with increasing concentration of rods. Our experimental results were supported by a hydrodynamic theory and numerical simulations incorporating the detailed Newtonian mechanics of the particles and boundaries, including vibration, inelasticity and static friction. In this paper, we offer a detailed exploration of the phase diagram, mode structure and spatiotemporal correlations of this system, primarily in simulations but supported by key experimental findings. There are of course many parameters one could consider varying, such as the concentrations of rods and beads, the rotational diffusivity of the rods, and coefficients of friction between the particles and the substrate, several of which feed into the speed of the rods, and thus mainly change the effective clock speed of the dynamics. We restrict our studies to the dependence on rod area fraction \( \Phi_r \) and the bead area fraction \( \Phi_b \). Our numerical studies all employ periodic boundary conditions (PBCs) in the horizontal plane, thus eliminating the role of the lateral boundaries of the sample. In principle, our system is distinct from a single-component Vicsek model because it has two conserved species, and is also distinct from confined flocks in incompressible fluid because the bead fluid is compressible and its density is a relevant control parameter.

Here is a summary of our main results. (i) Large-scale simulation studies over a range of \( \Phi_r \) and \( \Phi_b \), under periodic boundary conditions, reveal that a banded state intervenes between the isotropic state and the homogeneous ordered phase, for \( \Phi_r \) above a threshold value \( \Phi_{r \text{c}} \), as \( \Phi_b \) is increased. For the system sizes we explored, even for a long-channel geometry with the aspect ratio as large as 16, we see a single coherently moving band rich in both rods and beads, and not a periodic array. We cannot of course say if an even larger simulation will reveal a long-period striped state as in ref. 14.

Detailed studies of band morphology can be found in the body of the paper. Below \( \Phi_{r \text{c}} \), the system appears, within our resolution, to undergo a phase transition from a disordered to an ordered state directly, as discussed in ref. 35. (ii) We present experimental evidence for a banded flock, to our knowledge for the first time in dry granular matter. (iii) In the homogeneous ordered state, our numerical studies reveal giant number fluctuations and a spectrum of propagating modes. The observed wavevector dependence of the damping rates of the modes differs from the predictions of the Toner–Tu theory, which does however capture other broad features such as the scaling of the number fluctuations, and direction-dependent wavestres. (iv) At higher values of total concentration of the rods and the beads, we observe phase segregation into bead-rich and aligned rod-rich regions, both moving coherently. The remainder of the paper is organised as follows: in Section 2, we discuss our numerical and experimental methods. In Section 3, we present our detailed results. We summarise and suggest future directions in the last Section 4.

2 Methods

2.1 Experiment

Our experimental cell has a shallow circular geometry, made of hardened aluminium alloy. The particles are confined to two dimensions using a glass lid, which is fixed on the external perimeter of the circle at a height of \( w = 1.2 \text{ mm} \) above the base. We use the “flower” geometry to prevent clustering of particles on the cell boundary. The cell is mounted on a permanent magnet shaker (LDS 406/8) and is shaken at a fixed frequency \( f = 200 \text{ Hz} \) and amplitude \( a_\text{p} \). The amplitude of the resulting sinusoidal acceleration \( a = a_\text{p}(2\pi f)^2/g \), measured by an accelerometer (PCB Piezoelectronics 352B02), is chosen to be 7.0 in the units of the gravitational acceleration of the earth. Our “self-propelled” polar particle, which we call a “rod” henceforth, is a brass rod, \( \ell = 4.5 \text{ mm} \) long and 1.1 mm in diameter at its thick end, as shown in Fig. 1(a). The tilt of the rod with respect to the horizontal transduces the energy of vertical vibration into in-plane propulsion, and its geometrical polarity, that is, fore-aft asymmetry, ensures that this propulsion is biased towards one end of the rod, specifically the narrow end. The other particles in our experiment are spherical beads of aluminium, 0.8 mm in diameter, which do not show any in-plane dynamics when vibrated vertically. We created an annular geometry by inserting a circular disk of 5 cm diameter in the middle of the experimental cell. We expected this geometry to stabilize flocking along the azimuthal direction and therefore to favour bands.

![Fig. 1](image-url)
2.2 Simulation

Our numerical simulations are based on a mechanically faithful reproduction of the microscopic dynamics of each particle. We assume that all the particles and walls are perfectly rigid. Therefore, all interactions are instantaneous events. All the collisions are inelastic with prescribed restitution and Coulomb friction, and the gravitational acceleration of the Earth is taken into account in our simulations. The vibrating base and lid are modelled as horizontal walls moving in the vertical direction with their z-coordinates changing with time \( t \) as \( a_0 \cos 2\pi ft + a_0 \) and \( a_0 \cos 2\pi ft + a_0 + w \), respectively. We do not use the event-driven method,\(^{50}\) often preferred for granular systems at low density, but use instead a time-driven algorithm.\(^{31}\) The latter is more appropriate for our dense system, where an event-driven approach would require a large number of computations to predict the time of the next particle–particle and particle–wall collision.\(^{46}\) Since the rod in experiments has a complicated shape (see Fig. 1a), several calculations are required to detect the collisions between the rods. Therefore, in order to simplify the collision rules and speed up the collision detection process, we construct the rod as an array of overlapping spheres (see Fig. 1b).

The ballistic motion of the particles is governed by Newtonian rigid-body dynamics. The Impulse-Based Rigid Body Collision Model\(^{52,53}\) is implemented to calculate post-collision velocities for all the collisions. We write an MPI-based parallel code to simulate our system: the simulation box is divided into many equal-sized sub-boxes and the dynamics of the particles in different sub-boxes are dealt with by different computer cores. At each step, data for the particles at the boundaries of each sub-box are communicated to the neighbour sub-boxes to execute the collisions between the particles across the sub-boxes. We use VMD software\(^{54}\) to construct all the simulation movies and snapshots. To achieve the best imitation of the single-particle dynamics of rods and the beads in experiments, we choose the following values of restitution and friction coefficients \( \mu \) and \( \epsilon \) (Table 1): we choose the size of the spheres such that the rod in simulations is a close mimic of the original rod in experiments: the tail of the rod is made of seven spheres of diameter 1.1 mm, and the head and the middle parts each consist of three spheres of diameter 0.72 mm and 0.88 mm, respectively (see Fig. 1b). The beads are represented by spheres of diameter 0.8 mm. The mass densities of the rods and beads are 8.7 gm cm\(^{-3}\) and 2.7 gm cm\(^{-3}\) corresponding to brass and aluminium, respectively. The values of \( w, f \) and \( a_0 \) are set to 1.2 mm, 200 Hz and 0.04 mm, respectively, as in experiments. In experiments, imperfections in the shape of the rods and the substrate roughness lead to the diffusive nature of the orientation of the rods. Numerical simulations lack such imperfections.

### Table 1: Values of restitution and friction coefficients

<table>
<thead>
<tr>
<th>Collision</th>
<th>( \mu )</th>
<th>( \epsilon )</th>
</tr>
</thead>
<tbody>
<tr>
<td>Particle–particle</td>
<td>0.05</td>
<td>0.3</td>
</tr>
<tr>
<td>Rod-base (or lid)</td>
<td>0.03</td>
<td>0.1</td>
</tr>
<tr>
<td>Rod-boundary</td>
<td>0.01</td>
<td>0.3</td>
</tr>
<tr>
<td>Bead-wall (base, lid or boundary)</td>
<td>0.01</td>
<td>0.3</td>
</tr>
</tbody>
</table>

Therefore, we supply the rods with noisy angular velocity \( \omega_z = \omega_{0z} \) in the z direction each time they collide with the base or the lid. Here, \( \omega_{0z} \) is the relative velocity at the contact point normal to the contact plane, \( \epsilon \) is a control parameter and \( \eta = \pm 1 \) with equal probability. Fig. 1c shows that the orientation autocorrelation function \( C(t) \) of a single rod decays exponentially with time \( t \) and the relaxation time \( \tau \) decreases with \( \epsilon \) as \( \sim \epsilon^{-2} \) (see Fig. 1d). The value of \( \epsilon \) is set to 0.015, as in ref. 35.

### 3 Results

We first present our findings based on the numerical simulations and then discuss the bands seen in our experiments. In the rest of the paper, all lengths are scaled by the rod length \( \ell \).

#### 3.1 Simulations

The simulations are performed with the periodic box of dimension \( L = 56\ell / \ell \) unless otherwise mentioned. The largest systems we studied had \( L = 112\ell / \ell \) with 7200 rods and 300 000 beads. We here discuss the observed phase diagram (see Section 3.1.1), examine the ordering transition at small \( \Phi_l \) (see Section 3.1.2), and then study the properties of bands (see Section 3.1.3), and then present the sound wave spectrum (see Section 3.1.4) and large density fluctuations (see Section 3.1.5) in the highly ordered phase.

##### 3.1.1 Phase diagram

Fig. 2a, a phase diagram in the plane of \( \Phi_i \) and \( \Phi_{b, s} \) gives an overview of the behaviour of the system in various regimes. We mainly observe four phases in our system: disordered, homogeneous ordered, banded, and phase-separated (see Fig. 2b–e). The disordered phase, which is found at small values of \( \Phi_i \) and \( \Phi_{b, s} \) is structureless at very low rod densities (see Fig. 2b) but displays a few randomly moving highly dense and locally ordered swarms at higher values of \( \Phi_i \) (see Fig. 11 of the Appendix), presumably incomplete MIPS.\(^{35,56}\) For \( \Phi_i > \Phi_{i, c} \approx 0.13 \), the band phase is present between the ordered and disordered state (see Fig. 2c). In Section 3.1.3, we present detailed observations on this phase. Ordered states are observed at higher values of \( \Phi_b \) (see Fig. 2d). The velocity field of the bead medium plays an important role in achieving the ordered state,\(^{15}\) as can be seen strikingly through a careful choice of initial conditions. When we perform the simulation of a system that is initially at rest, with the rods in an aligned state, the rods immediately start moving with a constant speed but the beads take some time to pick up their steady state in-plane velocity. Therefore, initially, the bead flow is not enough to keep the rods ordered, and the polar order parameter decreases because the rods start disordering. Since they are dragged and pushed by the rods, the beads acquire some speed after some time, and the polar order parameter again increases due to the ordering enhanced by the bead flow (see Fig. 12 of the Appendix). More interestingly, the rods can flock at ultra-low \( \Phi_i \) (as low as \( \approx 0.03 \)) if \( \Phi_b \) is high enough (see Fig. 13 of the Appendix). At very high densities, phase segregation into bead-rich and (ordered) rod-rich regions is observed (see Fig. 2e and Movie S1, ESI†). This regime occurred in our earlier experiments,\(^{36}\) where it was characterised as jamming.
because the motile rods were immobilised as they pushed up against the sample boundary. A detailed study of the system in this regime will be presented in a separate paper. At higher values of $\Phi_i$ ($\geq 0.19$), the rods are condensed into large dense ordered swarms for $\Phi_b \ll 1$, through a MIPS-like mechanism (see Fig. 2f).

Finite-size effects are significant in our study. The observed value of $\Phi_i^*$ and the boundaries of the phases are influenced by system size $L$. We find that $\Phi_b^*$ decreases as $L$ is increased but we find no banded state at $\Phi_i = 0.06$ even for system size as big as $L = 112r$ (or around 457 rod diameters). We do not pursue the question of whether $\Phi_b^*$ can reach zero for very large systems, as seen in agent based numerical simulations.14

### 3.1.2 Phase transition at low $\Phi_i$

In order to measure the ordering of the rods, we first define the order parameter of the rods as $P \equiv \langle |\mathbf{n}(t)|\rangle$, where $\mathbf{n}(t)$ is the orientation unit vector of the $i$th rod and $\langle \rangle$ denotes the average over all rods and many configurations in the steady state. Our simulations suggest a direct disorder-to-order phase transition without bands at low $\Phi_b (\ll \Phi_i^*)$, although we cannot strictly rule out bands on a much larger length scale. In Fig. 3a, we plot the order parameter $P$ as a function of $\Phi_b$ for $\Phi_i = 0.03, 0.06, 0.09$ and 0.11. $P$ visibly increases with $\Phi_b$. In Fig. 3b, $P$ is plotted as a function of $\Phi_b$ for $L = 28, 42, 84$ and 112, at $\Phi_i = 0.06$. The profile becomes sharper with increasing $L$ and the graphs for different $L$ don’t intersect with each other; this is consistent with a continuous phase transition, but our size range is too limited to conclude further.

We now calculate the polar order parameter correlation function defined as $G(r) \equiv \langle \mathbf{n}(t) \cdot \mathbf{n}(t) \rangle$. Here, the averaging is performed over all the pairs of the rods separated by distance $r$. Fig. 3c, which shows $G(r)$ vs. $r$ for different values of $\Phi_b$ at $\Phi_i = 0.06$ and $L = 112$, illustrates that $G(r)$ decays to a nonzero constant at high $\Phi_b$ corresponding to long range order, and vanishes for large $r$ at low $\Phi_b$ in disordered states.35 Thus, our numerical experiments are consistent with long-range order in two dimensions, as argued by Toner et al.30,41

### 3.1.3 Properties of bands.

At high rod densities ($\Phi_i \geq \Phi_i^*$), a phase is seen between the order and the disordered states in which a single highly ordered and highly dense stripe of the rods and the beads extended over the length of the simulation box – a band – is observed to be moving perpendicular to its own long axis, amidst a disordered background also consisting of a bead–rod mixture (see Fig. 2e, and Movie S2, ESI†). In this system, up to $L = 112$, the segregation results in a single band

![Fig. 2](image-url)
unlike the periodically arranged many bands demonstrated by Vicsek particles. The bands are generally aligned along the sides of the simulation box but also could be in an arbitrary direction at large values of $\Phi_r$ (see Fig. 14 of the Appendix). For convenience of analysis, we study the regime in which the bands are parallel to the length of the simulation box. Let $R_0^r(t)$ and $R_0^i(t)$ be the positions of the $i$th rod and the $i$th bead, respectively, and $V_i(t)$ be the velocity of the $i$th bead, at time $t$. Respectively, the coarse grained number densities for the rods and beads are defined as

$$\rho(r, t) = \frac{1}{P} \int_{\text{cell}} d^2 r \sum_i \delta(r - R_0^i(t)),$$

and

$$\sigma(r, t) = \frac{1}{P} \int_{\text{cell}} d^2 r \sum_i \delta(r - R_0^i(t))n_i(t),$$

where $\sum$ stands for a sum over all the particles and the integration is taken over a square cell of length $l$ centred at position $r$. Similarly, the polar order parameter field for the rods and the velocity field for the beads are given by

$$p(r, t) = \frac{1}{P\rho(r, t)} \int_{\text{cell}} d^2 r \sum_i \delta(r - R_0^i(t))n_i(t),$$

and

$$v(r, t) = \frac{1}{P\rho(r, t)} \int_{\text{cell}} d^2 r \sum_i \delta(r - R_0^i(t))V_i(t).$$

In order to quantify the density and ordering profile of the rods, we define $\phi_0(x) = \langle \sigma(r, t) \rangle$ and $\mathcal{P}(x) \equiv \langle \mathcal{P}_0(x), \mathcal{P}_r(x) \rangle = \langle p(r, t) \rangle$, where we assume that the band is moving along the $x$ axis and the angle bracket represents the average over the $y$ direction and time in a frame moving with the band. Similarly, the density profile and the velocity profile of the beads are measured by $\phi_b(x) = \langle \rho(r, t) \rangle$ and $\mathcal{V}(x) \equiv \langle \mathcal{V}_r(x), \mathcal{V}_i(x) \rangle = \langle v(r, t) \rangle / \max\|v(r, t)\|$. We divide the simulation box into cells of length one to calculate the value of these functions. A graphical representation of the bands can be found in Fig. 4a showing the typical density profile of the rods in a band moving in the $x$ direction: the scaled density profile of the rods $\phi_r(x) \equiv \phi_0(x)/\max[\phi_0(x)]$ decays faster at the front than at the back, i.e., the band is asymmetric with the front sharper than the back. The value of $\mathcal{P}_0(x)$ is close to 1 in the band region and fluctuates around 0 elsewhere, and $\mathcal{P}_r(x)$ remains close to zero everywhere in the simulation box, indicating that the rods in the band region are aligned along the direction of the motion of the band and randomly oriented elsewhere. An interesting feature of these bands is that even the medium particles, the beads, form bands co-centred with the band of the rods (see Fig. 4b), which is an aspect that cannot arise in the related colloidal rollers. In Fig. 4c, we plot $\langle \mathcal{V}_r(x), \mathcal{V}_i(x) \rangle$ and $\langle \mathcal{P}_r(x), \mathcal{P}_i(x) \rangle$ as a function of $x$, for $\phi_0 = 0.15$ and $\phi_r = 0.10$. Again, in the band region, the average velocity of beads is high along the direction of the band and vanishingly small elsewhere. Also, the profile of $\mathcal{V}(x)$ is quite similar to that of $\mathcal{P}(x)$. We further explore the effect of $\phi_0$ and the total area fraction $\phi_0 = \phi_r + \phi_b$ on the bands. The band becomes wider and denser with increasing $\phi_0$, at fixed $\phi_r$ (see Fig. 4d). The condensation of the particles in a single band has also been observed in the active ising systems but, in contrast, more than one band was observed in Vicsek systems with the number of bands increasing with system size and particle density. The velocity of the bands $v_{\text{band}}$ does not change significantly with $\phi_r$, but the average velocity of the rods lying in the band region $v_r$ increases, probably because of the suppression of the transverse fluctuations of the orientation of the rods due to the increasing density in the band region (see Fig. 4e). The band moves faster than the rods occupying the band because of the significant velocity gradient at its boundaries. For a given value of $\phi_0$, the band widens with increasing $\phi_b$, suggesting a tendency to dissolve the band into a homogeneous ordered state at high enough $\phi_0$ (see Fig. 4f). Correspondingly, $v_r$ decreases with $\phi_b$ due to an enhancement in the transverse fluctuations and $v_{\text{band}}$ rises due to a decrease in the density gradient across the band boundaries (see Fig. 4g). A shoulder-like trend is found in $G(r)$ vs. $r$ as a result of the rectangular shape of the bands, indicating the typical width of the band (see Fig. 15 of the Appendix). The nature of the phase transition at the boundaries of the segregated regime remains unclear.

Bands in our system do not appear to arise through the instability proposed in ref. 42. Movie S2 (ESI) suggests a
different mechanism. Initially, small swarms with internal alignment are observed, moving randomly in the isotropic background, with both rods and beads joining and leaving them at their boundary. When the swarms come alongside one another, they unite through lateral exchange of particles to form a band.

Our simulations with a long periodic channel of size $14\ell \times 224\ell$ find a single band at $F_r = F_b = 0.20$, which sometimes "calves" a second band that then dissolves (see Movie S3, ESI†). We cannot however rule out multiple bands on an even longer length scale. In order to explore the possibility of the bands in experiments, we execute simulations with the annular geometry and the outer boundary the same as in our experiments. Fig. 5 and Movie S4 (ESI†) show that the bands should be seen in experiments as well. We will present our experimental result on the bands in subsection 16.

3.1.4 Wavelike excitations. For a statistical characterisation of excitations in the ordered phase, we calculate the dynamic and static orientation structure factors. Let $n_i(r,t)$ be the component of the orientation of the $i$th rod normal to the direction of the flock. The field for the normal orientation fluctuations is then given by

$$\delta p_{\perp}(r,t) = \sum_i n_i^j(t)\delta(\mathbf{R}_i^j(t) - r) - \sum_i \delta(\mathbf{R}_i^j(t) - r), \quad (5)$$

with $n_i^j(t)$ the component of the orientation of the $i$th rod normal to the direction of the flock. The field for the normal orientation fluctuations is then given by
where the sum is taken over all the rods. The static orientation structure factor $S_0(q)$ is defined as

$$S_0(q) = \frac{1}{N} \langle \delta p_\parallel (q,t) \cdot \delta p_\perp (-q,-t) \rangle_t,$$

(6)

where $\langle \rangle_t$ stands for average over time $t$ and

$$\delta p_\parallel (q,t) = \sum_i n_i^\parallel (t) \exp(-i q \cdot \mathbf{R}_i(t))$$

(7)

is the Fourier transform of $\delta p_\parallel (t)$ in space. The expression for the dynamic orientation structure factor $S(q,\omega)$ reads

$$S(q,\omega) = \frac{1}{N} \langle \delta p_\parallel (q,\omega) \cdot \delta p_\perp (-q,-\omega) \rangle,$$

(8)

where the angle bracket stands for an average over time or configuration and $\delta p_\perp (q,\omega)$ is the Fourier transform of $\delta p_\perp (q,t)$ in time:

$$\delta p_\perp (q,\omega) = \int dt \exp(i \omega t) \sum_i n_i^\perp (t) \exp(-i q \cdot \mathbf{R}_i(t)).$$

(9)

In Fig. 6a, $S(q,\omega)$ vs. $\omega$ is reported for modes with wavevectors $q$ parallel and antiparallel to the directions $\theta = 0$, $\pi/3$ and $\pi/2$ with respect to the flocking direction, for $\Phi_0 = 0.11$, $\Phi_\infty = 0.60$ and $L = 112$. We also construct the heat maps for $S(q,\omega)$ as functions of $q$ and $\omega$ along these three directions (see Fig. 6b): the single mode for $\theta = 0$ and two modes for the other cases are clearly visible. Correspondingly, $S(q,\omega)$ as a function of $\omega$ shows two peaks in all the directions, which merge into one along the direction of the flock. The heat map for the static structure factor for orientation $S_0(q)$ is presented in the bottom-right panel of Fig. 6b, which reveals that the waves are highly anisotropic. Fig. 6c shows that $S_0(q)$ scales as $q^{-1}$ with $\alpha = 1.4$ and $\beta = 1.2$ but differ from the values $\alpha = 1.4$ and $\beta = 1.33$ calculated from the large-scale simulations of the Vicsek model.\textsuperscript{58} The speed of the propagating mode of wave vector $q$ is defined as

$$v_\perp(q) = \omega_\perp(q)/q,$$

(10)

where $\omega_\perp(q)$ is the position of the peak corresponding to the mode in the $S(q,\omega)$ vs. $\omega$ plot. We find that $\omega_\perp(q)$ is proportional to $q$ for small $q$ (see Fig. 7a, b and c for $\theta = 0$, $\pi/3$ and $\pi/2$, respectively), except where $v_\perp(q = 0)$ is very small, e.g., wavevectors at $4\pi/3$ to the flocking direction. The width of the peak $\Delta \omega_\perp(q)$ displays a dependence consistent with a power law $q^\delta$. The value of $\delta$ is $\beta_\perp = 1.6$ for $\theta = 0$, $0.6$ and $0.9$ for the two peaks at $\theta = \pi/3$ and $\beta_\perp = 0.6$ for both the peaks at $\theta = \pi/2$ (insets of Fig. 7a–c). This is in contrast to the Toner–Tu\textsuperscript{8} prediction $\beta_\perp = 2$ and $\beta_\perp = 1.2$ as well as $\beta_\parallel = 1.4$ and $\beta_\parallel = 1.33$ found in the large-scale simulation studies of the Vicsek model.\textsuperscript{58} Fig. 7d shows the angular dependence of $v_\perp(\theta)$ at $q = 45(2\pi/L)$. Two loops intersecting at $\theta = 0$ correspond to two wave modes. All these observations agree reasonably with the theoretical predictions\textsuperscript{8} except the wavenumber dependence of the sound peak widths. The broken-symmetry “sound” waves have already been detected in the numerical simulations of Vicsek-style models\textsuperscript{9} but observing them in our numerical model, which is mechanically realistic rather than agent-based, suggests that these modes should play a significant role in real granular-matter experiments. Given both the limited dynamic range of wave-numbers in our study and the observed disagreement between Toner–Tu theory and the presumably definitive numerical measurements\textsuperscript{58} on the Vicsek model, we will not dwell further on the issue of the scaling of sound-peak widths here.

A different wavelike aspect is seen in Movie S5 (ESI†), for $\Phi_0 = 0.19$ and $\Phi_\infty = 0.51$, in the form of undulating structures,\textsuperscript{59} with considerable lateral wandering, on a uniform ordered background. These do not appear to be phase-coherent, and the measurements of $S(q,\omega)$ do not show peaks corresponding to oscillations at an intrinsic, system-size-independent frequency.
However, their presence suggests an incipient excitability of the uniform flock towards oscillations, which we propose to explore further through our numerical studies and by examining the Toner–Tu equations in the regime of bending instabilities.60

3.1.5 Large density fluctuations. In the thermal equilibrium systems away from critical points, in the thermodynamic limit, the particle number fluctuations in the grand canonical ensemble grow as the square root of the average number of the particles \( \langle N \rangle \) as the system size is increased, i.e. root-mean-square deviation

\[
\Delta N = \sqrt{\langle N^2 \rangle - \langle N \rangle^2} = \langle N \rangle^{1/2},
\]

where \( N \) is the instantaneous number of the particles and \( \langle \cdot \rangle \) represents the ensemble average for a given system size. In contrast, active systems show anomalous number fluctuation properties: hydrodynamical theories suggest that \( \Delta N \) for active polar and apolar systems is proportional to \( \langle N \rangle^{1/2} \) in isotropic states but the broken-symmetry states demonstrate large number fluctuations growing as \( \langle N \rangle^{1/2} \)(see Fig. 8a for \( \Phi_1 = 0.06 \) and \( \Phi_0 = 0.24 \)). Fig. 8a presents \( \Delta N/\sqrt{\langle N \rangle} \) vs. \( \langle N \rangle \) for three different ordered states: the number fluctuations are larger than the thermal systems with \( a \sim 0.6 \) but weaker than the ones predicted by Toner–Tu theory. More detailed studies with finite-size scaling are required for a definitive exponent estimate. Here, we should make it clear that the large number fluctuations in order states are not due to a segregation as seen in Fig. 2e and h; a typical ordered state has homogenous rod density (see Fig. 8b).

In order to quantify the lifetime of the number fluctuations, we also calculate the density autocorrelation function defined as

\[
C_d(t) = \langle \sigma(r,t) - \sigma_0(0) \rangle \langle \sigma(r,0) - \sigma_0(0) \rangle,
\]

where \( \sigma_0(0) \) is the average density at position 0 and \( \langle \cdot \rangle \) represents the average over space. The coarse-grained number density for the rods \( \sigma(r,t) \) has been defined in Section 3.1.3. Fig. 9 suggests that \( C_d(t) \) shows two exponential decays. We do not understand this observation. A linearized treatment by Narayan et al.6 would give a logarithmic decay, while Toner–Tu theory would give a slower \( t^{-2/3} \) in two dimensions. Neither of these forms gives a reasonable fit to our data. The finite size effects could be the reason we do not see agreement with the existing theories.

3.2 Experiments

As we showed in earlier work,35 a flocking transition can be triggered at very low area fraction of motile polar rods by increasing the concentration of non-motile beads (see Fig. 16 of the Appendix). Here, motivated by our simulations, we perform experiments in annular geometry at varying \( \Phi_0 \) and \( \Phi_1 \) and find that for parameter values shortly after the onset of flocking, a band does emerge, with appearance typically as in Fig. 10b–f and Movie S6 (ESI†). The band is identified as the location of the bands in the \( \Phi_1 - \Phi_0 \) plane is depicted in the phase diagram in Fig. 2a. Although the observation of band formation in our experiments
is robust, limitations on system size make a more systematic study impractical.

4 Future directions and summary

The research presented in this article was motivated by multiple aims. One was to study collective motion with both self-propulsion and alignment arising for purely mechanical rather than behavioural reasons. The second was to carry out such a study in a system in which – unlike in the Quinke rollers – the strength of interactions could be tuned simply, in this case by changing the bead concentration. Third was to see if the band formation just after the ordering onset could be modulated or suppressed. Despite the limitations on system size, we believe we have succeeded in achieving the above aims. Two regimes remain to be explored in more detail. One is the limit of no beads, not discussed here, where a patchy MIPS precursor possibly leads to a dilute flock at low noise (see Fig. 17 of the Appendix and Movie S7, ESI†). The other is the bead–rod phase separation that occurs at a high area fraction of both species.

We close with a summary. In our detailed numerical simulations, we observed four distinct phases of our system: disordered, banded and ordered, homogeneous and ordered, and bead–rod phase separated. The existence of the broken symmetry state at very low rod concentration is a key result. The banded state is similar to the one seen in a system of polar disks or in rolling colloids: only single bands are observed, which are generally aligned with the boundary of the periodic box but also could be in an arbitrary direction at high rod concentration. Moreover, in these systems, the band absorbs the bead medium as well, for which there is no equivalent in the other systems like polar disks and colloidal rollers. The width of the bands increases with the rod concentration. We also find the first experimental realisation of bands in a dry granular flocking system. We also explore the anisotropic propagating waves and large number fluctuations in the highly

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**Fig. 9** Density autocorrelation function $C_s(t)$ scaled by its value at $t = 0$ as a function of $t$. Here, $\Phi_r = 0.11$, $\Phi_b = 0.60$ and $L = 112$ and the size of the cell used for coarse-graining is $2a$.

**Fig. 10** Images from the experiments performed with rods and beads in annular geometry: (a) an isotropic phase and (b–d) bands at different $\Phi_r$ and $\Phi_b$. It can be seen from (b–d) for $\Phi_r = 0.15$, and (e and f) for $\Phi_r = 0.20$, that the size of the band increases with $\Phi_r$ as observed in our simulations.
ordered state. Some interesting departures are observed with respect to Toner–Tu theory, especially as regards the scaling of damping rate with wavenumber.

Conflicts of interest

There are no conflicts to declare.

Appendix

Fig. 11 A disordered phase at low $\Phi_r$ and high $\Phi_b$ having locally ordered swarms. Here, $\Phi_r = 0.13$ and $\Phi_b = 0.07$.

Fig. 12 Time evolution of the polar order parameter $P(t)$ of the system, which is perfectly ordered with zero velocity at $t = 0$. Initially, the ordering of the rods starts lessening due to the absence of the bead flow. In some time, as soon as the beads pick up the velocity as they are pushed and dragged by the rods, the rods begin to realign due to the interaction with the bead flow. Correspondingly, the order parameter $P(t)$ initially decreases and then increases after some time.

Fig. 13 Ordered state at extremely low rod density $\Phi_r = 0.03$ with $\Phi_b = 0.72$. The inset shows the crystalline order in the bead medium.

Fig. 14 A band moving in a direction not parallel to the sides of the periodic box at $\Phi_r = 0.19$ and $\Phi_b = 0.11$. 
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Notes and references

56 In the present context MIPS is simply joint condensation of rods and beads, and should not be confused with the demixing of the two species, which we call segregation, observed at higher area fraction.
59 We thank an anonymous referee for drawing our attention to this feature.