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Rational Penta-Inner Functions and the Distinguished Boundary of the Pentablock

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Abstract
In this note, we give a description of rational maps from the open unit disc $\mathbb{D}$ to the pentablock that map the boundary of $\mathbb{D}$ to the distinguished boundary of the pentablock. We also obtain a new characterization of the distinguished boundary of the pentablock.

Keywords Rational inner functions · Symmetrized bidisc · Pentablock · Distinguished boundary

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1 Introduction

In 2015, Agler, Lykova and Young introduced a new bounded domain called pentablock in [6]. The pentablock is a subdomain of $\mathbb{C}^3$ denoted by $\mathcal{P}$ and defined as the image of the domain $\{A \in M_2(\mathbb{C}) : \|A\| < 1\}$ under the mapping

$$\pi : A = [a_{ij}] \mapsto (a_{21}, \operatorname{tr}(A), \det(A)).$$

We denote the closure of $\mathcal{P}$ by $\overline{\mathcal{P}}$. The set $\mathcal{P} \subset \mathbb{C}^3$ is non-convex, polynomially convex, and star-like about the origin, see [6]. The pentablock is an inhomogeneous domain, see [23]. The complex geometry and function theory of the pentablock were further developed in [6, 23, 26, 27].

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Attempts to solve particular cases of the $\mu-$synthesis problem have also led to the study of two other domains namely the *symmetrized bidisc*

$$\mathbb{G} := \{(\mathrm{tr}(A), \det(A)) : A = [a_{ij}]_{2\times 2}, \|A\| < 1\} \subset \mathbb{C}^2,$$

see [7] and the *tetrablock*

$$\mathbb{E} := \{(a_{11}, a_{22}, \det(A)) : A = [a_{ij}]_{2\times 2}, \|A\| < 1\} \subset \mathbb{C}^3,$$

see [1]. We denote the closure of $\mathbb{G}$ by Γ. The set $\mathbb{G}$ and $\mathbb{E}$ are polynomially convex and non-convex domains. The symmetrized bidisc and the tetrablock have attracted a considerable amount of interest in recent years. For a greater exposition on these domains, see [1, 4, 7, 9, 14–16, 19, 22, 24, 25].

Let $\Omega \subset \mathbb{C}^d$ be a bounded polynomially convex domain with closure $\overline{\Omega}$. Let $A(\Omega)$ be the algebra of continuous scalar functions on $\overline{\Omega}$ that are holomorphic in Ω. A *boundary* for Ω is a subset C of $\overline{\Omega}$ such that every function in $A(\Omega)$ attains its maximum modulus on C. The *distinguished boundary* of Ω, to be denoted by $b\Omega$ (some authors write $b\overline{\Omega}$), is the smallest closed boundary of Ω.

The distinguished boundaries of the symmetrized bidisc and the tetrablock were found in [7] and [1] to be

$$\begin{aligned} b\Gamma &= \{(s, p) \in \mathbb{C}^2 : |s| \leq 2, s = \bar{s}p, |p| = 1\} \\ &= \{(\mathrm{tr}(U), \det(U)) : U = [u_{ij}]_{2\times 2}, U \text{ is a unitary}\} \end{aligned}$$

and

$$b\mathbb{E} = \{(u_{11}, u_{22}, \det(U)) : U = [u_{ij}]_{2\times 2}, U \text{ is a unitary}\},$$

respectively. A key fact used in the above descriptions of distinguished boundaries is that the set of 2×2 unitary matrices is the distinguished boundary of the 2×2 matrix operator-norm unit ball. It was shown in reference [6] that the sets

$$K_0 = \left\{(a, s, p) \in \mathbb{C}^3 : (s, p) \in b\Gamma, |a| = \sqrt{1 - \frac{1}{4}|s|^2}\right\}$$

and

$$K_1 = \left\{(a, s, p) \in \mathbb{C}^3 : (s, p) \in b\Gamma, |a| \leq \sqrt{1 - \frac{1}{4}|s|^2}\right\}$$

both are boundaries of the pentablock. It was further shown in reference [6] that the set K_0 is the distinguished boundary of the pentablock while

$$K_1 = \{(u_{21}, \mathrm{tr}(U), \det(U)) : U = [u_{ij}]_{2\times 2}, U \text{ is a unitary}\}.$$

This suggests that, unlike in the cases of the symmetrized bidisc and tetrablock, the distinguished boundary of the pentablock is attuned to a certain special class of unitary matrices rather than whole class of unitary matrices. This note finds exactly that special class that describes K_0 via the map π. This, in turn, leads to a new description of the distinguished boundary of the pentablock.

Let $\mathbb{T}$ denote the unit circle in the complex plane $\mathbb{C}$. An analytic map $x = (x_1, \ldots, x_d) : \mathbb{D} \to \overline{\Omega}$ is called a *rational* $\Omega-$*inner* (some authors call it *rational* $\overline{\Omega}-$*inner*) function if each x_i is a rational function with poles outside $\overline{\mathbb{D}}$ and

$$(x_1(\lambda), \ldots, x_d(\lambda)) \in b\Omega$$

for all $\lambda \in \mathbb{T}$. In [17], W. Blaschke studied the rational $\mathbb{D}-$inner functions and proved that all rational $\mathbb{D}-$inner functions are of the form

$$B(z) := e^{i\theta} \prod_{j=1}^{n} \frac{z - a_j}{1 - \overline{a_j} z}$$

for some $a_1, a_2, ..., a_n \in \mathbb{D}$ and $\theta \in [0, 2\pi]$. Functions of this form are well-known to be the finite Blaschke product. For a survey of results, see [18]. If $\Omega = \mathbb{D}^d$, then it follows from $d = 1$ case that all rational $\mathbb{D}^d-$inner functions are of the form

$$(B_1(z), \ldots B_d(z))$$

for some finite Blaschke products $B_1, \ldots, B_d$. A description of rational $\Gamma-$inner functions is given by Agler–Lykova–Young, see [3]. Alsalhi–Lykova gave a description of rational $\mathbb{E}-$inner functions, see [13]. In Sect. 3, we give a description of rational $\mathcal{P}-$inner functions, see Theorem 3.9.

Sometime after this paper was finished and uploaded to arXiv, [12] appeared on arXiv. There is an overlap of one result of our paper with [12]. Theorem 3.9 also appears there. The proofs are different. Fejér-Riesz Theorem is used in [12] whereas our proof uses a study of the zeros and poles of certain functions.

2 A New Characterization of the Distinguished Boundary

In the following theorem, we shall give a characterization of points in $b\mathcal{P}$. The proof of the theorem will manifest a recipe to construct a 2×2 unitary matrix $U = [u_{ij}]$ for any $(a, s, p) \in b\mathcal{P}$ such that $(a, s, p) = (u_{21}, \operatorname{tr}(U), \det(U))$.

Theorem 2.1 *For $(a, s, p) \in \mathbb{C}^3$, the following are equivalent:*

(1) $(a, s, p) \in b\mathcal{P}$,

(2) There exists a unique unitary matrix $U = \begin{pmatrix} u_{11} & u_{12} \\ u_{21} & u_{22} \end{pmatrix}$ such that

$$u_{11} = u_{22} \quad \textit{and} \quad (a, s, p) = (u_{21}, \operatorname{tr}(U), \det(U)).$$

Proof First, we shall prove that (1) $\Rightarrow$ (2). Let $(a, s, p) \in b\mathcal{P}$. Since $b\mathcal{P} = K_0$, we have

$$|s| \leq 2, \quad s = \bar{s}p, \quad |p| = 1 \quad \text{and} \quad |a|^2 = 1 - \frac{|s|^2}{4}.$$

In order to find the desired matrix $U = [u_{ij}]_{2\times 2}$, we need to solve the following four equations in four variables.

$$u_{11} - u_{22} = 0, \quad u_{21} = a, \quad u_{11} + u_{22} = s \quad \text{and} \quad u_{11}u_{22} - u_{12}u_{21} = p.$$

If $a \neq 0$, then we get a unique solution

$$(u_{11}, u_{12}, u_{21}, u_{22}) = \left(\frac{s}{2}, \frac{s^2 - 4p}{4a}, a, \frac{s}{2}\right).$$

A simple computation will show that the matrix U is unitary. If $a = 0$, then the set of solutions is

$$\{(u_{11}, u_{12}, u_{21}, u_{22}) = \left(\frac{s}{2}, \lambda, 0, \frac{s}{2}\right) : \lambda \in \mathbb{C}, s^2 = 4p\}.$$

Since $|p| = 1$, we get $|s| = 2$ and hence the matrix $U = [u_{ij}]_{2\times 2}$ is unitary if and only if $\lambda = 0$.

Now we shall prove that (2) $\Rightarrow$ (1). Let $U = [u_{ij}]_{2\times 2}$ be a unitary matrix with

$$u_{11} = u_{22} \qquad \text{and} \qquad (a, s, p) = (u_{21}, \operatorname{tr}(U), \det(U)).$$

Since U is a unitary, we get that $(s, p) = (\operatorname{tr}(U), \det(U)) \in b\Gamma$, also

$$4|a|^2 + |s|^2 = 4|u_{21}|^2 + |\operatorname{tr}(U)|^2 = 4(|u_{21}|^2 + |u_{11}|^2) = 4.$$

This proves that $(a, s, p) \in b\mathcal{P}$. $\square$

3 Rational $\mathcal{P}-$Inner Functions

In this section, we give a description of rational $\mathcal{P}-$inner functions. First, recall that, a rational map $x = (x_1, x_2, x_3) : \mathbb{D} \to \overline{\mathcal{P}}$ is said to be *rational* $\mathcal{P}-$*inner* if

$$\big(x_1(\lambda), x_2(\lambda), x_3(\lambda)\big) \in b\mathcal{P}$$

for all $\lambda \in \mathbb{T}$. Note that if $(s, p) \in \Gamma$ and $\alpha \in \mathbb{D}$, then $1 - s\alpha + p\alpha^2 \neq 0$, see [8]. For each $\alpha \in \mathbb{D}$, we define a function $\Psi_\alpha : \mathbb{C} \times \Gamma \to \mathbb{C}$ by

$$\Psi_\alpha(a, s, p) = \frac{a(1 - |\alpha|^2)}{1 - s\alpha + p\alpha^2}.$$

The function Ψ_α is analytic in $\mathbb{C} \times \mathbb{G}$ and continuous on $\mathbb{C} \times \Gamma$. One of the main results of [6] contains several characterization of a point to be in $\overline{\mathcal{P}}$. We recall the one characterization which we shall use later.

Theorem 3.1 [6, Theorem 5.3] *For* $(a, s, p) \in \mathbb{C} \times \Gamma$, *the following are equivalent:*

(1) $(a, s, p) \in \overline{\mathcal{P}}$,
(2) $|\Psi_\alpha(a, s, p)| \le 1$ *for all* $\alpha \in \mathbb{D}$.

For any positive integer n and for any polynomial f of degree less than or equal to n, we define the polynomial $f^{\sim n}$ by the formula,

$$f^{\sim n}(\lambda) = \lambda^n \overline{f\left(\frac{1}{\overline{\lambda}}\right)}.$$

For a $\mathbb{C}-$valued rational function $x = f/g$, where f and g are relatively prime polynomials, we define $\deg(x)$ to be the maximum of $\deg(f)$, $\deg(g)$. Note that if x is a finite Blashcke product, then $\deg(x)$ is same as number of Blaschke factors in the product. The following theorem gives a description of rational $\Gamma-$inner functions.

Theorem 3.2 [3, Proposition 2.2] *Let* $h = (s, p)$ *be a rational* $\Gamma-$*inner function with* $\deg(p) = n$. *Then there exist polynomials D and N such that*

(1) $\deg(D), \deg(N) \le n$
(2) $N^{\sim n}(\lambda) = N(\lambda)$ *on* $\overline{\mathbb{D}}$,
(3) $D(\lambda) \neq 0$ *on* $\overline{\mathbb{D}}$,
(4) $|N(\lambda)| \le 2|D(\lambda)|$ *on* $\overline{\mathbb{D}}$,
(5) $s = \frac{N}{D}$ *on* $\overline{\mathbb{D}}$, *and*
(6) $p = \frac{D^{\sim n}}{D}$ *on* $\overline{\mathbb{D}}$.

Conversely, if N and D are polynomials satisfying (1), (2), (4) *above,* $D(\lambda) \neq 0$ *on* $\mathbb{D}$, *and s and p are defined by* (5) *and* (6) *respectively, then* $h = (s, p)$ *is a rational* $\Gamma-$*inner function with* $\deg(p) = n$.

Furthermore, a pair of polynomials N' and D' satisfies (1) – (6) *if and only if there exists a non-zero real number t such that $N = tN'$ and $D = tD'$.*

Note that if $x = (x_1, x_2, x_3)$ is a rational $\mathcal{P}-$inner function, then in particular,

(1) $(x_2(\lambda), x_3(\lambda)) \in \mathbb{G}$ for every $\lambda \in \mathbb{D}$; and
(2) $(x_2(\lambda), x_3(\lambda)) \in b\Gamma$ for every $\lambda \in \mathbb{T}$.

Consequently, it is necessary for $x = (x_1, x_2, x_3)$ to be rational $\mathcal{P}-$inner that (x_2, x_3) be $\Gamma-$inner. The latter class is completely understood in view of Theorem 3.2. Thus, our job reduces to understanding just the first coordinate of a rational $\mathcal{P}$-inner function. This is what we do in the following sequence of preliminary results.

Lemma 3.3 *If* (x_2, x_3) *is a rational* $\Gamma-$*inner function and* x_1 *is a rational function with poles outside* $\overline{\mathbb{D}}$ *such that*

$$|x_1(\lambda)|^2 = 1 - \frac{|x_2(\lambda)|^2}{4}$$

for all $\lambda \in \mathbb{T}$, *then* $x = (x_1, x_2, x_3)$ *is a rational* $\mathcal{P}-$*inner function.*

Proof First note that $x(\lambda) = \big(x_1(\lambda), x_2(\lambda), x_3(\lambda)\big) \in b\mathcal{P}$ for all $\lambda \in \mathbb{T}$. We need to show that $\big(x_1(\lambda), x_2(\lambda), x_3(\lambda)\big) \in \overline{\mathcal{P}}$ for all $\lambda \in \mathbb{D}$. Fix $\alpha \in \mathbb{D}$ and consider the map $\Psi_\alpha \circ x : \overline{\mathbb{D}} \to \mathbb{C}$. The map $\Psi_\alpha \circ x$ is analytic in $\mathbb{D}$ and continuous on $\overline{\mathbb{D}}$. Since $x(\lambda) \in b\mathcal{P} \subset \overline{\mathcal{P}}$ for $\lambda \in \mathbb{T}$, by Theorem 3.1, for all $\lambda \in \mathbb{T}$ we get

$$|\Psi_\alpha\big(x(\lambda)\big)| = |\Psi_\alpha\big(x_1(\lambda), x_2(\lambda), x_3(\lambda)\big)| \leq 1$$

for all $\alpha \in \mathbb{D}$. By the maximum modulus principle, for $\lambda \in \mathbb{D}$ we get

$$|\Psi_\alpha\big(x(\lambda)\big)| = |\Psi_\alpha\big(x_1(\lambda), x_2(\lambda), x_3(\lambda)\big)| \leq 1$$

for all $\alpha \in \mathbb{D}$. Again by Theorem 3.1, $x(\lambda) = \big(x_1(\lambda), x_2(\lambda), x_3(\lambda)\big) \in \overline{\mathcal{P}}$ for all $\lambda \in \overline{\mathbb{D}}$. Thus, $x = (x_1, x_2, x_3)$ is a rational map from $\mathbb{D}$ to $\overline{\mathcal{P}}$ which sends $\mathbb{T}$ into $b\mathcal{P}$. This proves that $x = (x_1, x_2, x_3)$ is a rational $\mathcal{P}-$inner function. □

Now, we shall give some examples of rational $\mathcal{P}-$inner functions.

Example 3.4 Let B be a finite Blaschke product. Then the function $x : \mathbb{D} :\to \overline{\mathcal{P}}$ defined by

$$x(\lambda) = \big(B(\lambda), 0, B(\lambda)\big)$$

is rational $\mathcal{P}-$inner.

Proof It is easy to see that $(0, B(\lambda))$ is a rational $\Gamma-$inner function. Now we show that, for $\lambda \in \mathbb{T}$, the point $x(\lambda)$ lies in $b\mathcal{P}$. Here

$$x_1(\lambda) = B(\lambda), \quad x_2(\lambda) = 0, \quad \text{and} \quad x_3(\lambda) = B(\lambda).$$

Since $|B(\lambda)| = 1$ on the circle, it follows that

$$|x_1(\lambda)|^2 = 1 = 1 - \frac{|x_2(\lambda)|^2}{4}.$$

Thus, by Lemma 3.3, x is a rational $\mathcal{P}-$inner function. □

The following lemma gives a class of rational $\mathcal{P}-$inner functions.

Lemma 3.5 *Let* $\beta \in \mathbb{T}$. *Then the map* $x : \mathbb{D} \to \overline{\mathcal{P}}$ *by the setting*

$$\lambda \mapsto \left(\frac{\beta - \overline{\beta}\lambda}{2}, \beta + \overline{\beta}\lambda, \lambda\right)$$

is rational $\mathcal{P}-$*inner.*

Proof By virtue of Lemma 3.3, we need to show that (x_2, x_3) is a $\Gamma-$inner function, and the following equality holds for $\lambda \in \mathbb{T}$,

$$4|x_1(\lambda)|^2 + |x_2(\lambda)|^2 = 4.$$

Here,

$$x_1(\lambda) = \frac{\beta - \overline{\beta}\lambda}{2}, \quad x_2(\lambda) = \beta + \overline{\beta}\lambda \text{ and } \quad x_3(\lambda) = \lambda.$$

Note that, for $\lambda \in \mathbb{T}$, $x_2(\lambda) = \overline{x_2(\lambda)}x_3(\lambda)$, $|x_3(\lambda)| = 1$, and $|x_2(\lambda)| \leq 2$. So the map (x_2, x_3) maps $\mathbb{T}$ into $b\Gamma$. Since $(x_2(\lambda), x_3(\lambda)) \in \Gamma$ for all $\lambda \in \mathbb{D}$, it follows that (x_2, x_3) is a rational $\Gamma-$inner function. Now, for $\lambda \in \mathbb{T}$,

$$\begin{aligned} |x_1(\lambda)|^2 &= x_1(\lambda)\overline{x_1(\lambda)} = 1/4(\beta - \overline{\beta}\lambda)(\overline{\beta} - \beta\overline{\lambda}) \\ &= \frac{1}{4}\left[|\beta|^2 - \overline{\beta}^2\lambda - \beta^2\overline{\lambda} + |\beta|^2|\lambda|^2\right] \\ &= \frac{1}{2} - \frac{1}{4}\left[\overline{\beta}^2\lambda + \beta^2\overline{\lambda}\right]. \end{aligned} \tag{3.1}$$

We also have

$$\begin{aligned} |x_2(\lambda)|^2 &= x_2(\lambda)\overline{x_2(\lambda)} = (\beta + \overline{\beta}\lambda)(\overline{\beta} + \beta\overline{\lambda}) \\ &= |\beta|^2 + \beta^2\overline{\lambda} + \overline{\beta}^2\lambda + |\beta|^2|\lambda|^2 \\ &= 2 + \beta^2\overline{\lambda} + \overline{\beta}^2\lambda \end{aligned} \tag{3.2}$$

Thus, from Eqs. (3.1) and (3.2), for all $\lambda \in \mathbb{T}$,

$$4|x_1(\lambda)|^2 + |x_2(\lambda)|^2 = 4.$$

$\square$

The next two lemmas give some more examples of rational $\mathcal{P}-$inner functions. These will also be used in the proof of the main theorem of this section.

Lemma 3.6 *If $x = (x_1, x_2, x_3)$ is a rational $\mathcal{P}-$inner function, then $x_B \overset{\text{def}}{=} (Bx_1, x_2, x_3)$ is also a rational $\mathcal{P}-$inner function for any finite Blaschke product B.*

Proof Since (x_1, x_2, x_3) is a rational $\mathcal{P}-$inner function, (x_2, x_3) is a $\Gamma-$inner function. For $\lambda \in \mathbb{T}$,

$$\begin{aligned} 4|Bx_1(\lambda)|^2 + |x_2(\lambda)|^2 &= 4|B(\lambda)|^2|x_1(\lambda)|^2 + |x_2(\lambda)|^2 \\ &= 4|x_1(\lambda)|^2 + |x_2(\lambda)|^2 \\ &= 4. \end{aligned}$$

Thus, by Lemma 3.3, $x_B = (Bx_1, x_2, x_3)$ is a rational $\mathcal{P}-$inner function. □

Lemma 3.7 *If B is a finite Blaschke product, x_1 is a rational function with poles outside $\overline{\mathbb{D}}$ and (Bx_1, x_2, x_3) is a rational $\mathcal{P}-$inner function, then (x_1, x_2, x_3) is also a rational $\mathcal{P}-$inner function.*

Proof Since (Bx_1, x_2, x_3) is a rational $\mathcal{P}-$inner function, (x_2, x_3) is a $\Gamma-$inner function. For $\lambda \in \mathbb{T}$,

$$\begin{aligned} 4|x_1(\lambda)|^2 + |x_2(\lambda)|^2 &= 4|B(\lambda)|^2|x_1(\lambda)|^2 + |x_2(\lambda)|^2 \\ &= 4|Bx_1(\lambda)|^2 + |x_2(\lambda)|^2 \\ &= 4. \end{aligned}$$

Thus, by Lemma 3.3, (x_1, x_2, x_3) is a rational $\mathcal{P}-$inner function. □

If $f(z) = \sum_{i=1}^n a_i z^i$ is a polynomial, then define

$$f^\vee(z) = \sum_{i=1}^n \overline{a_i} z^i.$$

If f_1, f_2 are two polynomials and $r = f_1/f_2$ is a rational function, then define $r^\vee = f_1^\vee/f_2^\vee$. The following proposition is an intermediate step to prove the main theorem of this section.

Proposition 3.8 *Let $x = (x_1, x_2, x_3)$ be a rational $\mathcal{P}-$inner function. Let $x_1 = B\frac{f_1}{g_1}$ where B is a Blaschke product and f_1, g_1 are relatively prime polynomials such that f_1/g_1 has no Blaschke factor. Then the following hold.*

(1) If $g_1(a) = 0$, then $x_1^\vee(1/a) \neq 0$; and
(2) if $x_2 = f_2/g_2$, where f_2 and g_2 are relatively prime polynomials, then $g_1 = tg_2$ for some non-zero constant t.

Proof Let $x = (x_1, x_2, x_3)$ be a rational $\mathcal{P}-$inner function. Let $g_1(a) = 0$. Suppose if possible $x_1^\vee(1/a) = 0$. This implies that $f_1^\vee(1/a) = 0$, which in turn implies that $f_1(1/\overline{a}) = 0$, this together with $g_1(a) = 0$, imply that f_1/g_1 has a Blaschke factor, which is a contradiction. Hence, $x_1^\vee(1/a) \neq 0$. This proves (1).

Since $x = (x_1, x_2, x_3)$ is a rational $\mathcal{P}-$inner function, (x_2, x_3) is a $\Gamma-$inner function. Therefore, x_2 and x_3 satisfy

$$x_2(\lambda) = \overline{x_2(\lambda)}x_3(\lambda) = x_2^\vee(\overline{\lambda})x_3(\lambda) = x_2^\vee(1/\lambda)x_3(\lambda)$$

for all $\lambda \in \mathbb{T}$. Since the first and last terms are rational functions,

$$x_2(\lambda) = x_2^\vee(1/\lambda)x_3(\lambda) \qquad \text{for all } \lambda \in \mathbb{C}.$$

Hence,

$$x_2(a) \neq 0 \Rightarrow x_2^\vee(1/a) \neq 0.$$

Since $x = (x_1, x_2, x_3)$ is a rational $\mathcal{P}-$inner function, x_1, x_2 satisfy

$$x_1(\lambda)\overline{x_1(\lambda)} = 1 - \frac{1}{4}x_2(\lambda)\overline{x_2(\lambda)}$$
$$\Rightarrow x_1(\lambda)x_1^{\vee}(\bar{\lambda}) = 1 - \frac{1}{4}x_2(\lambda)x_2^{\vee}(\bar{\lambda})$$

for all $\lambda \in \mathbb{T}$. This implies

$$x_1(\lambda)x_1^{\vee}(1/\lambda) = 1 - \frac{1}{4}x_2(\lambda)x_2^{\vee}(1/\lambda) \quad \text{for all } \lambda \in \mathbb{T}. \tag{3.3}$$

Since both the left hand side and the right hand side are rational functions in Eq. (3.3), it follows that

$$x_1(\lambda)x_1^{\vee}(1/\lambda) = 1 - \frac{1}{4}x_2(\lambda)x_2^{\vee}(1/\lambda) \qquad \text{for all } \lambda \in \mathbb{C}.$$

For $m \geq 1$, we have

$$(\lambda - a)^{m-1}x_1(\lambda)x_1^{\vee}(1/\lambda) = (\lambda - a)^{m-1}\left(1 - \frac{1}{4}x_2(\lambda)x_2^{\vee}(1/\lambda)\right) \tag{3.4}$$

for all $\lambda \in \mathbb{C}$.

Let a be a pole of x_1 of multiplicity $m \geq 1$. Clearly, $|a| > 1$. Hence $|1/a| < 1$, and so $x_1^{\vee}$ and $x_2^{\vee}$ are analytic at $1/a$. Also by part-1 of the proposition $x_1^{\vee}(1/a) \neq 0$. Therefore, on letting $\lambda \to a$ in (3.4), we get

$$(\lambda - a)^{m-1}x_2(\lambda) \to \infty.$$

Thus a is a pole of x_2 of multiplicity at least m.

Let a be a pole of x_2 of multiplicity $m \geq 1$. Again on letting $\lambda \to a$ in Eq. (3.4) we get that a is a pole of x_1 of multiplicity at least m. This proves that g_1 and g_2 have same zeros with same multiplicities. Hence $g_1 = tg_2$ for some non-zero constant t.

□

Now we are ready to prove the main result of this section.

Theorem 3.9 *If $x = (x_1, x_2, x_3)$ is a rational $\mathcal{P}-$inner function and the degree of x_3 is n, then there exist polynomials N_1, N_2, D and a finite Blaschke product B such that*

(1) $(x_2, x_3) = \left(\frac{N_2}{D}, \frac{D^{\sim n}}{D}\right)$ *is a $\Gamma-$inner function,*
(2) $x_1 = B\frac{N_1}{D}$ *on $\overline{\mathbb{D}}$,*
(3) $|N_1(\lambda)|^2 = |D(\lambda)|^2 - \frac{1}{4}|N_2(\lambda)|^2$ *on $\mathbb{T}$, and*
(4) $\deg(N_1) \leq n$.

Conversely, if N_1, N_2, and D are polynomials satisfying (1) *and* (3) *above, then* $(\frac{N_1}{D}, \frac{N_2}{D}, \frac{D^{\sim n}}{D})$ *is a rational $\mathcal{P}-$inner function and the degree of $\frac{D^{\sim n}}{D}$ is equal to n.*

Furthermore, a triple of polynomials N_1', N_2' and D' satisfy $(1)-(4)$ *if and only if there exists a non-zero real number t such that*

$$N_1 = tN_1', \quad N_2 = tN_2' \quad \text{and} \quad D = tD'.$$

Proof Let $x = (x_1, x_2, x_3)$ be a rational $\mathcal{P}-$inner function and the degree of x_3 be n. Then (x_2, x_3) is a rational $\Gamma-$inner function. By Theorem 3.2, there exist two polynomials N_2 and D of degree less than or equal to n such that

$$(x_2, x_3) = \left(\frac{N_2}{D}, \frac{D^{\sim n}}{D}\right).$$

This proves condition (1). Note that $D(\lambda) \neq 0$ for all $\lambda \in \overline{\mathbb{D}}$. Since x_1 is a rational function with poles outside $\overline{\mathbb{D}}$, we have

$$x_1 = B\frac{f}{g}$$

where B is a finite Blaschke product and f, g are relatively prime polynomials such that f/g does not contain any Blaschke factor. By Proposition 3.8, g can be taken to be D. Let us denote f by N_1. Thus,

$$x_1 = B\frac{N_1}{D}.$$

This proves condition (2).

Since $\big(x_1(\lambda), x_2(\lambda), x_3(\lambda)\big) \in b\mathcal{P}$ for all $\lambda \in \mathbb{T}$, we have

$$|x_1(\lambda)|^2 = 1 - \frac{1}{4}|x_2(\lambda)|^2.$$

By virtue of conditions (1) and (2), we have

$$\begin{aligned} \left|\frac{N_1(\lambda)}{D(\lambda)}\right|^2 &= 1 - \frac{1}{4}\left|\frac{N_2(\lambda)}{D(\lambda)}\right|^2 \\ \Rightarrow |N_1(\lambda)|^2 &= |D(\lambda)|^2 - \frac{1}{4}|N_2(\lambda)|^2 \end{aligned} \tag{3.5}$$

for all $\lambda \in \mathbb{T}$. This proves condition (3).

From Eq. (3.5), it follows that

$$N_1(\lambda)N_1^{\vee}(\bar{\lambda}) = D(\lambda)D^{\vee}(\bar{\lambda}) - \frac{1}{4}N_2(\lambda)N_2^{\vee}(\bar{\lambda}).$$

This is same as

$$N_1(\lambda)N_1^{\vee}(1/\lambda) = D(\lambda)D^{\vee}(1/\lambda) - \frac{1}{4}N_2(\lambda)N_2^{\vee}(1/\lambda) \tag{3.6}$$

for all $\lambda \in \mathbb{T}$. Since $N_1(0) \neq 0$, the coefficient of $\lambda^{\deg(N_1)}$ is non-zero in $N_1(\lambda)N_1^{\vee}(1/\lambda)$, which is the highest degree coefficient in this expression. Since the degree of the right hand side in Eq. (3.6) is at most n, we get $\deg(N_1) \leq n$. This proves condition (4).

Proof of the converse follows from Theorem 3.2 and Lemma 3.3.

Finally, suppose a triple of polynomials N_1', N_2' and D' satisfy (1)$-$(4). By Theorem 3.2, there exists a non-zero real number t such that $N_2 = tN_2'$ and $D = tD'$. Using (2) we get $N_1 = tN_1'$. The converse is straightforward. □

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Author Contributions Both the authors contributed equally.

Declarations

Conflict of interest The authors declare that they have no conflict of interest.

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