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Beurling's theorem on the Heisenberg group

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Abstract. We formulate and prove an analogue of Beurling's theorem for the Fourier transform on the Heisenberg group. As a consequence, we deduce Hardy and Cowling-Price theorems.

1. Introduction

This article deals with the formulation and proof of an analogue of Beurling's theorem for the Fourier transform on the Heisenberg group and the related uncertainty principles of Hardy and Cowling-Price. We also state and prove a version of Beurling's theorem for Hermite expansions.

1.1. Beurling's theorem for the Heisenberg group

There are several uncertainty principles for the Fourier transform on $\mathbb{R}^n$ such as theorems of Hardy, Cowling and Price and Beurling. Each of the first two of these theorems can be deduced as corollaries of the third one which we recall now. Beurling's theorem states that there is no nontrivial function which satisfies

$$\int_{\mathbb{R}^n}\int_{\mathbb{R}^n} |f(y)|\,|\hat{f}(\xi)|\,e^{|(y,\xi)|}\,dy\,d\xi < \infty. \tag{1.1}$$

A proof of this theorem in the one dimensional case was published by Hörmander in [8]. Later Bonami et al. [1] extended the result to higher dimensions. See the memoir [4] by Demange for very general results on uncertainty principles dealing with tempered distributions. Recently, Hedenmalm [7] came up with a very elegant

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and simple proof of Beurling's theorem, which we will recall later for the convenience of the reader, see Section 3.1.

Observe that the Beurling's condition (1.1) can be restated in the following form:

$$w_0(\hat{f},y) =: \int_{\mathbb{R}^n} |\hat{f}(\xi)| e^{|(y,\xi)|}\, d\xi, \quad \int_{\mathbb{R}^n} |f(y)| w_0(\hat{f},y)\, dy < \infty.$$

The function $w_0(\hat{f},y)$ is an increasing function of $|y|$ and dominates $|f(x+iy)|$:

$$|f(x+iy)| \le (2\pi)^{-n/2} \int_{\mathbb{R}^n} |\hat{f}(\xi)| e^{|(y,\xi)|}\, d\xi \le C\, w_0(\hat{f},y).$$

Here we have tacitly used the fact that under the Beurling's condition, f extends to $\mathbb{C}^n$ as a holomorphic function. Once the holomorphic extendability of f is given, Hedenmalm's proof goes through under the modified Beurling's condition

$$\sup_{x\in\mathbb{R}^n} |f(x+iy)| \le C\, w(\hat{f},y), \quad \int_{\mathbb{R}^n} |f(y)| w(\hat{f},y)\, dy < \infty, \tag{1.2}$$

for some $w(\hat{f},y)$ which is increasing as a function of $|y|$. In particular, we can take

$$w_1(\hat{f},y) = \sup_{|u|\le|y|} \int_{\mathbb{R}^n} |\hat{f}(\xi)|\, e^{-u\cdot\xi}\, d\xi. \tag{1.3}$$

Thus Beurling's theorem can be stated in the following form.

Theorem 1.1. *There is no nontrivial function f on $\mathbb{R}^n$ for which*

$$\int_{\mathbb{R}^n} |f(y)| \Big(\sup_{|u|\le|y|} \int_{\mathbb{R}^n} |\hat{f}(\xi)|\, e^{-u\cdot\xi} d\xi \Big)\, dy < \infty. \tag{1.4}$$

In this article our aim is to prove an analogue of Theorem 1.1 for the Fourier transform on the Heisenberg group. In order to state our result, let us recall some definitions and set the stage. Given a function f on the Heisenberg group $\mathbb{H}^n$, whose underlying manifold is just $\mathbb{R}^{2n+1}$, its Fourier transform $\hat{f}(\lambda)$ is the operator valued function on $\mathbb{R}^*$ defined by

$$\hat{f}(\lambda) = \int_{\mathbb{R}^{2n+1}} f(x,u,t)\, \pi_\lambda(x,u,t)\, dx\, du\, dt$$

where $\pi_\lambda(x,u,t) = e^{i\lambda t}\pi_\lambda(x,u)$ are the Schrödinger representations, all realised on the same Hilbert space $L^2(\mathbb{R}^n)$. The operators $\pi_\lambda(x,u)$ have natural extensions to complex values as unbounded operators $\pi_\lambda(z,w)$, where $z=x+iy$ and $w=u+iv$. We let $f^\lambda(x,u)$ stand for the inverse Fourier transform of f in the t-variable so that

$$\hat{f}(\lambda) = \int_{\mathbb{R}^{2n}} f^\lambda(x,u)\, \pi_\lambda(x,u)\, dx\, du = \pi_\lambda(f^\lambda).$$

We need to make a preliminary assumption on $\hat{f}(\lambda)$ which allows us to holomorphically extend $f^\lambda(x,u)$ to $\mathbb{C}^n\times\mathbb{C}^n$.

Let $Sp(n,\mathbb{R})$ and $O(2n,\mathbb{R})$ stand for the symplectic group and orthogonal group acting on $\mathbb{R}^{2n}$. Then the compact group $K=Sp(n,\mathbb{R})\cap O(2n,\mathbb{R})$ has an action on $\mathbb{H}^n$ given by $\sigma(x,u,t)=(\sigma(x,u),t)$ which is an automorphism of $\mathbb{H}^n$. This action has a natural extension to $\mathbb{C}^n\times\mathbb{C}^n$ and so it makes sense to talk about $\sigma(z,w)$ for $z,w\in\mathbb{C}^n$. On the Fourier transform of $\hat{f}(\lambda)$ we impose the condition

$$\int_K \|\pi_\lambda(\sigma(z,w))\hat{f}(\lambda)\|_{HS}^2\, d\sigma<\infty \tag{1.5}$$

where $d\sigma$ is the Haar measure on K. Using Gutzmer's formula for special Hermite expansions (see Theorem 2.1), we will show that (1.5) allows us to extend $f^\lambda(x,u)$ holomorphically. For a linear operator T on $L^2(\mathbb{R}^n)$ we let $\mathcal{D}(T)$ and $\mathcal{R}(T)$ stand for its domain and range. We also let $\|T\|_1=\operatorname{tr}|T|, |T|=(T^*T)^{1/2}$ stand for the trace norm of the operator T. Having set up the notations, here is our Beurling's theorem.

Theorem 1.2. *For $f\in L^2(\mathbb{H}^n)$, assume that $\mathcal{R}(\hat{f}(\lambda))\subset\mathcal{D}(\pi_\lambda(z,w))$ and $\pi_\lambda(z,w)\hat{f}(\lambda)$ is of trace class for all $(z,w)\in\mathbb{C}^{2n}$ such that the assumption (1.5) holds. If the condition*

$$\int_{\mathbb{R}^{2n}} |f^\lambda(y,v)| \quad \Big(\sup_{|(y',v')|\le|(y,v)|}\|\pi_\lambda(i(y',v'))\hat{f}(\lambda)\|_1\Big)\, dy\, dv<\infty \tag{1.6}$$

holds for almost every $\lambda\in\mathbb{R}^*$, then $f=0$.

Unlike the Euclidean case treated in Bonami et al. [1] and Hedenmalm [7] we have managed to extend only the super-critical case of the Beurling's theorem. In [1] functions of the form $P(x)e^{-\langle Ax,x\rangle}$ where P is a polynomial and A is a real symmetric are characterised in terms of the integrability of $f(x)\hat{f}(y)$. In [7] the author also proves a general version which also treats the critical case. It is unlikely that our methods will suffice to treat such critical cases unless combined with new ideas. We plan to take up these issues in a future work.

Remark 1.3. Thus the role of the function (1.3) for $\mathbb{R}^n$ is played by the function

$$w_\lambda(\hat{f},(y,v))=\sup_{|(y',v')\le|(y,v)|}\|\pi_\lambda(i(y',v'))\hat{f}(\lambda)\|_1 \tag{1.7}$$

for the Heisenberg group. An examination of the proof will show that in the above theorem we can replace $w_\lambda(\hat{f},(y,v))$ by any radial increasing $F_\lambda(y,v)$ satisfying the condition

$$|\operatorname{tr}\big(\pi_\lambda(x+iy,u+iv)^*\hat{f}(\lambda)\big)|\le C_\lambda e^{\frac{1}{2}\lambda(u\cdot y-v\cdot x)}\, F_\lambda(y,v) \tag{1.8}$$

(Note that $w_\lambda(\hat{f},(y,v))$ clearly satisfies this condition in view of the relation

$$\pi_\lambda(iy,iv)\pi_\lambda(x,u)=\pi_\lambda(x+iy,y+iv)e^{-\frac{1}{2}\lambda(u\cdot y-v\cdot x)}$$

and the fact that $\pi_\lambda(iy,iv)$ are self-adjoint.) Thus the conclusion of the Theorem 1.2 holds under the assumption that

$$\int_{\mathbb{R}^{2n}} |f^\lambda(y,v)| \quad F_\lambda(y,v)\,dy\,dv<\infty. \tag{1.9}$$

It is this flexibility with the choice of $F_\lambda(y,v)$ that allows us to deduce theorems of Hardy and Cowling-Price from Theorem 1.2.

As we mentioned in the beginning, the well known uncertainty principles of Hardy and Cowling-Price for the Fourier transform on $\mathbb{R}^n$ can be easily deduced from Beurling's theorem. For the convenience of the reader, let us recall these results. For $a>0$ let q_a be the heat kernel for the Laplacian Δ on $\mathbb{R}^n$ so that

$$q_a(x)=(4\pi a)^{-n/2}e^{-\frac{1}{4a}|x|^2}, \quad \widehat{q_a}(\xi)=e^{-a|\xi|^2}.$$

Hardy's uncertainty principle [6] reads as follows: Suppose $f\in L^1(\mathbb{R}^n)$ satisfies

$$|f(x)|\le c_1\, q_a(x), \quad |\hat{f}(\xi)|\le c_2\,\widehat{q_b}(\xi).$$

Then $f=0$ whenever $a<b$ and when $a=b$ it is a constant multiple of q_a. In the case of Cowling-Price theorem [3], the condition reads as

$$f(x)=g(x)q_a(x), \quad \hat{f}(\xi)=h(\xi)\widehat{q_b}(\xi), \quad g,h\in L^2(\mathbb{R}^n)$$

and the conclusion is that $f=0$ whenever $a<b$. Analogues of these two theorems are known for the Fourier transform on the Heisenberg group.

In the case of the Heisenberg group, the sublaplacian $\mathcal{L}$ is the natural analogue of Δ and hence the role of q_a is played by the heat kernel p_a associated to $\mathcal{L}$. It is well known that $\widehat{p_a}(\lambda)=e^{-aH(\lambda)}$ where $H(\lambda)=-\Delta+\lambda^2|x|^2$ is the scaled Hermite operator on $\mathbb{R}^n$ and Hardy's theorem for the Heisenberg group is formulated in terms of the Hermite semigroup $e^{-aH(\lambda)}$. The following analogue of Hardy's theorem proved in [17] can be deduced as an easy corollary from the general version of Beurling's theorem mentioned in Remark 1.3.

Corollary 1.4. (Hardy's theorem) *Suppose a function f on $\mathbb{H}^n$ satisfies the following two conditions*: (i) $|f(x,u,t)|\le p_a(x,u,t)$, (ii) $\hat{f}(\lambda)^*\hat{f}(\lambda)\le Ce^{-2bH(\lambda)}$. *Then $f=0$ whenever $a<b$.*

Ideally, we would like to replace the condition (i) by $|f^\lambda(x,u)|\le c_\lambda\, p_a^\lambda(x,u)$ to arrive at the same conclusion. We remark that this has been already proved by other means; see Theorem 2.9.4 in [17]. However, we are not able to verify the Beurling condition in this situation. The conclusion remains valid even if we replace (i) by $f(x,u,t)=g*_3 p_a(x,u,t)$ for some function g on $\mathbb{H}^n$ for which $g^\lambda\in L^\infty(\mathbb{R}^{2n})$ for all λ. Here we have used the notation $g*_3 h$ to stand for the convolution in the central variable. By taking the inverse Fourier transform in the central variable, we see that $|f^\lambda(x,u)|=|g^\lambda(x,u)|p_a^\lambda(x,u)\le\|g^\lambda\|_\infty\, p_a^\lambda(x,u)$ which is enough to prove the above theorem by other means. Moreover, by a theorem of Douglas [5], condition (ii) takes the form $\hat f(\lambda)=B_\lambda e^{-bH(\lambda)}$ for some bounded linear operator B_λ. By replacing g^λ by L^2 functions and B_λ by Hilbert-Schmidt operators, we get an analogue of Cowling-Price theorem on the Heisenberg group.

Corollary 1.5. (Cowling-Price theorem) *Suppose a function f on $\mathbb{H}^n$ satisfies the following two conditions: for every $\lambda\in\mathbb{R}^*$,* (i) *$f(x,u,t)=h(x,u,t)\,p_a(x,u,t)$,* (ii) *$\hat f(\lambda)=B_\lambda\, e^{-bH(\lambda)}$ where $h\in L^2(\mathbb{H}^n)$ and B_λ are Hilbert-Schmidt. Then $f=0$ whenever $a<b$.*

As in the case of Theorem 1.4, this theorem will also be deduced from Beurling's theorem. For the general L^p-L^q version of Cowling-Price theorem, with a different proof, we refer to the work Parui-Thangavelu [13].

Remark 1.6. As every Hilbert-Schmidt operator is the Weyl transform of an $L^2(\mathbb{R}^{2n})$ function we can state the conditions of the above theorem as: (i) $f=hp_a$, and (ii) $f=g*p_b$, where $g,h\in L^2(\mathbb{H}^n)$. We also remark that the case $a=b$ remains open.

The same ideas lead to an interesting version of Beurling's theorem for Hermite expansions. Given $f\in L^2(\mathbb{R}^n)$ we let P_kf stand for the orthogonal projection of f onto the eigenspace of the Hermite operator $H=-\Delta+|x|^2$ with eigenvalue $(2k+n)$. The Plancherel theorem for Hermite expansions reads as

$$\|f\|_2^2=\sum_{k=0}^\infty\|P_kf\|_2^2,\quad f(x)=\sum_{k=0}^\infty P_kf(x). \tag{1.10}$$

Let $L_k^{n-1}(t)$ stand for the Laguerre polynomials of type $(n-1)$ and define the Laguerre functions $\varphi_k^{n-1}(z)$ on $\mathbb{C}^n$ by

$$\varphi_k^{n-1}(z)=L_k^{n-1}(\frac{1}{2}z^2)e^{-\frac14 z^2},\quad z^2=\sum_{j=1}^n z_j^2.$$

Then we can prove the following theorem for Hermite expansions:

Theorem 1.7. (Beurling-Hermite) *There is no non trivial $f\in L^2(\mathbb{R}^n)$ for which*

$$\sum_{k=0}^{\infty}\int_{\mathbb{R}^n}|f(y)|\|P_kf\|_2\sqrt{\varphi_k^{n-1}(2iy)}\,dy<\infty. \tag{1.11}$$

Thus we see that the role of $e^{|(y,\xi)|}$ is played by $\sqrt{\varphi_k^{n-1}(2iy)}$ which is natural in view of Gutzmer's formula for the Hermite expansions, see [19]. As an immediate corollary of this theorem we can obtain Hardy's theorem for Hermite expansions.

Corollary 1.8. (Hardy-Hermite) *Suppose $f\in L^2(\mathbb{R}^n)$ satisfies the following two conditions*:

$$|f(x)|\le Ce^{-\frac{1}{2}a|x|^2},\quad \|P_kf\|_2\le Ce^{-\frac{1}{2}b(2k+n)}$$

for some $a,b>0$. Then $f=0$ whenever $a\tanh b>1$.

Remark 1.9. For a different proof of this result see Theorem 1.4.7 in [17] where even the equality case has been considered. However, we cannot deduce the equality case from the Beurling's theorem we have proved.

1.2. An uncertainty principle for weighted Bergman spaces

Though we have stated Hardy's theorem and Cowling-Price theorem as corollaries, they cannot be deduced easily from Theorem 1.2. Instead, we will deduce them as corollaries of a theorem concerning elements of twisted Bergman spaces; see Theorem 1.11 below. In order to motivate this result, let us return to the euclidean case and consider the Hardy conditions

$$|f(x)|\le c_1\,q_a(x),\quad |\hat{f}(\xi)|\le c_2\,\widehat{q_b}(\xi),$$

stated in terms of the heat kernel q_a. Note that both f and $\widehat{q_b}$ extend to $\mathbb{C}^n$ as entire functions and Fourier inversion gives the estimate

$$|f(x+iy)|\le(2\pi)^{-n/2}\int_{\mathbb{R}^n}|\hat{f}(\xi)|\,e^{-y\cdot\xi}\,d\xi\le C\int_{\mathbb{R}^n}e^{-b|\xi|^2}\,e^{-y\cdot\xi}\,d\xi=C'q_b(iy).$$

Thus when $a<b$ the condition (1.2) is satisfied with $w(\hat{f},y)=q_b(iy)$ and we obtain Hardy's theorem.

In the case of Cowling-Price theorem we are working under the assumptions

$$f(x)=g(x)q_a(x),\quad \hat{f}(\xi)=h(\xi)\widehat{q_b}(\xi),\quad g,h\in L^2(\mathbb{R}^n).$$

Consequently, $f=\varphi*q_b$ where $\widehat{\varphi}=h$, and hence f belongs to the image of $L^2(\mathbb{R}^n)$ under the Segal-Bargmann or heat kernel transform which takes φ into the holomorphic extension of $\varphi*q_b$. It is well know that this image is weighted Bergman space $\mathcal{B}_b(\mathbb{C}^n)$ consisting of all entire functions which are square integrable on $\mathbb{C}^n$ with respect to the weight function $q_{2b}(2y)$. The reproducing kernel for this Hilbert space is given by $K(z,w)=q_{2b}(z-\overline{w})$ and from general theory of such Hilbert spaces we get the estimate

$$|f(x+iy)|^2 \le c_1\, K(z,z) = c_1\, q_{2b}(2iy) = c_2\, (q_b(iy))^2.$$

Thus we can again take $w(\hat{f},y)=q_b(iy)$ in (1.2) and Cowling-Price theorem follows.

More generally, theorems of Hardy and Cowing-Price can be put in the general frame work of weighted Bergman spaces $\mathcal{B}(\mathbb{C}^n,w)$ which are invariant under the natural action of $\mathbb{R}^n$ by translations. This invariance forces w to be a function of y alone and the reproducing kernel takes the form $K(z,w)=k(z-\overline{w})$ for some function k. We are interested in the situation when $\mathcal{B}(\mathbb{C}^n,w)$ occurs as the image of $L^2(\mathbb{R}^n)$ under a transform $T_qf(z)=f*q(z)$ for some nice function $q\in L^1(\mathbb{R}^n)$. Under these circumstances we have

$$\int_{\mathbb{C}^n} |f*q(z)|^2\, w(y)\, dz = c_n \int_{\mathbb{R}^n} |f(x)|^2\, dx. \tag{1.12}$$

Moreover, the reproducing kernel is given by $K(z,w)=q*q(z-\overline{w})$ and we can prove the following theorem.

Theorem 1.10. *Let $q\in L^1(\mathbb{R}^n)$ be such that the image of $L^2(\mathbb{R}^n)$ under the transform $T_qf(z)=f*q(z)$ is a weighted Bergman space $\mathcal{B}(\mathbb{C}^n,w)$ where w depends only on y. Then there is no nontrivial $f\in L^2(\mathbb{R}^n)$ for which*

$$\int_{\mathbb{R}^n} |f*q(y)|\, \sqrt{q*q(2iy)}\, dy < \infty$$

under the extra assumption that $q*q(iy)$ is an increasing function of $|y|$.

As observed above, both Hardy and Cowling-Price theorems correspond to the Segal-Bargmann transform $T_bf(z)=f*q_b(z)$. This theorem can be easily proved by showing that $f*q$ satisfies the modified Beurling condition (1.2), see Remark 3.1.

The relation between the weight function w and the reproducing kernel k can be read out by considering the orbital integrals of F. Consider the action of the Euclidean motion group $M(n)$ on $\mathbb{R}^n$ given by $g.\,y=(x+\sigma y)$ for $g=(x,\sigma)\in M(n), \sigma\in SO(n)$. This action has a natural extension to $\mathbb{C}^n$ simply given by $g.(u+iv)= g.\,u+ig.\,v$. The orbital integrals of $|F|^2$ are then defined by

$$O_{|F|^2}(z) = \int_{M(n)} |F(g.\,z)|^2\, dg.$$

As it turns out, $O_{|F|^2}(x+iy)$ depends only on y and is radial in this variable and given explicitly in terms of $\hat{f}$ where f is just the restriction of F to $\mathbb{R}^n$.

$$O_{|F|^2}(z)=c_n \int_{\mathbb{R}^n} |\hat{f}(\xi)|^2 \frac{J_{n/2-1}(2i|y||\xi|)}{(2i|y||\xi|)^{n/2-1}}\, d\xi$$

where J_α stands for the Bessel function of order α. When the weight function $w(y)$ is radial, its Fourier transform is given by Hankel transform and hence we have

$$\int_{\mathbb{C}^n} |F(z)|^2\, w(y)\, dz = \int_{\mathbb{R}^n} O_{|F|^2}(y)\, w(y)\, dy = c_n \int_{\mathbb{R}^n} |\hat{f}(\xi)|^2\, \widehat{w}(2i\xi)\, d\xi.$$

By defining q by the relation $\hat{q}(\xi)=(w(2i\xi))^{-1/2}$ we can factorise f as $f=h*q$ and from the above relation we get

$$c_n \int_{\mathbb{R}^n} |\hat{h}(\xi)|^2\, d\xi = \int_{\mathbb{C}^n} |F(z)|^2\, w(y)\, dz.$$

Thus we see that $\mathcal{B}(\mathbb{C}^n, w)$ can be identified with the image of $L^2(\mathbb{R}^n)$ under the transform $h(x)\to h*q(z)$. We also note that the relation between q and w allows to calculate $q*q$ in terms of w as follows:

$$q*q(2iy) = c_n \int_{\mathbb{R}^n} e^{2y\cdot\xi}\, \widehat{w}(2i\xi)^{-1}\, d\xi.$$

From this, it is clear that $q*q(2iy)$ is a radial increasing function. The above arguments are formal but we can make them rigorous by imposing certain assumptions either on w or on q. As this is not our main concern here, we do not pursue this line of investigation further.

We can prove an analogue of Theorem 1.10 for the Heisenberg group, the formulation of which requires the setting of twisted Bergman spaces. Given a positive radial weight function $w_\lambda(y,v)$ on $\mathbb{R}^{2n}$ we consider the twisted Bergman space $\widetilde{\mathcal{B}}_\lambda(\mathbb{C}^{2n}, w_\lambda)$ consisting of entire functions $F(z,w)$ on $\mathbb{C}^{2n}$ such that

$$\|F\|^2 = \int_{\mathbb{C}^{2n}} |F(z,w)|^2 w_\lambda(y,v)\, e^{\lambda(u\cdot y - v\cdot x)}\, dz\, dw < \infty.$$

Such spaces are invariant under twisted translations and we restrict ourselves to the situation where they can be realised as the image of $L^2(\mathbb{R}^{2n})$ under the map $\widetilde{T}_{q_\lambda} f(z,w) = f*_\lambda q_\lambda(z,w)$ for a nice kernel function $q_\lambda(x,u)$ in such a way that

$$\int_{\mathbb{C}^{2n}} |f*_\lambda q_\lambda(z,w)|^2 w_\lambda(y,v)\, e^{\lambda(u\cdot y - v\cdot x)}\, dz\, dw = c_n \int_{\mathbb{R}^{2n}} |f(x,u)|^2\, dx\, du. \tag{1.13}$$

Under these assumptions, the reproducing kernel for the space $\widetilde{\mathcal{B}}_\lambda(\mathbb{C}^{2n}, w_\lambda)$ takes the form

$$K((z,w),(z',w')) = e^{-\lambda(w\cdot z'-z\cdot w')} q_\lambda *_\lambda q_\lambda(z-\overline{z'}, w-\overline{w'}).$$

With these notations, we formulate and prove the following analogue of Theorem 1.10.

Theorem 1.11. *Let $q\in L^1(\mathbb{H}^n)$ be such that for every λ the image of $L^2(\mathbb{R}^{2n})$ under the transform $\widetilde{T}_{q^\lambda} f(z,w) = f *_\lambda q^\lambda(z,w)$ is a twisted Bergman space $\widetilde{\mathcal{B}}_\lambda(\mathbb{C}^{2n}, w_\lambda)$ where $w_\lambda(y,v)$ is radial. Then there is no nontrivial $f\in L^2(\mathbb{H}^n)$ such that for all $\lambda\in\mathbb{R}^*$*

$$\int_{\mathbb{R}^{2n}} |f^\lambda *_\lambda q^\lambda(y,v)| \sqrt{q^\lambda *_\lambda q^\lambda(2iy,2iv)}\, dy\, dv < \infty$$

under the extra assumption that $q^\lambda *_\lambda q^\lambda(iy,iv)$ is an increasing function of $|(y,v)|$.

The twisted Bergman spaces were introduced by Krötz-Thangavelu-Xu [12] in the context of Segal-Bargmann transform $T_b f(z,w,\zeta) = f * p_b(z,w,\zeta)$ where p_b is the heat kernel associated to the sublaplacian $\mathcal{L}$ on $\mathbb{H}^n$. The associated twisted Bergman spaces correspond to the kernel p_b^λ and the weight function is given by $w_\lambda(y,v) = p_{2b}^\lambda(2y,2v)$. As in the Euclidean case the reproducing kernel is given by $p_{2b}^\lambda(2iy,2iv)$. As a consequence, the condition in Theorem 1.11 takes the form

$$\int_{\mathbb{R}^{2n}} |f^\lambda *_\lambda p_b^\lambda(y,v)| \sqrt{p_{2b}^\lambda(2iy,2iv)}\, dy\, dv < \infty.$$

Hardy and Cowling-Price theorems for the Heisenberg group, namely Corollaries (1.4) and (1.5), will be deduced from the above theorem by taking $q = p_b$; see Section 3.3.

Here is the plan of the paper. After recalling the preliminaries on the Heisenberg group and collecting all the relevant results needed in the next section, we will prove the theorems in Section 3. There is a vast literature on uncertainty principles including Hardy, Cowling-Price and Beurling theorems. Our bibliography is kept selective for the sake of brevity. We refer the reader to the survey articles Fernandez-Bertolin and Malinnikova [9], Bonami and Demange [2] and Folland and Sitaram [11] for further references.

2. Preliminaries on the Heisenberg group

2.1. Heisenberg group and the Fourier transform

We develop the required background for the Heisenberg group. General references for this section are the monographs [16], [17] and the book [10] of Folland.

Let $\mathbb{H}^n := \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}$ be the $(2n+1)$- dimensional Heisenberg group with the group law

$$(x,u,t)(y,v,s) = (x+y, u+v, t+s+\frac{1}{2}(u\cdot y - v\cdot x)).$$

This is a step two nilpotent Lie group where the Lebesgue measure $dx\,du\,dt$ on $\mathbb{R}^{2n+1}$ serves as the Haar measure. The representation theory of $\mathbb{H}^n$ is well-studied in the literature. In order to define Fourier transform, we use the Schrödinger representations as described below.

For each non zero real number λ we have an infinite dimensional representation π_λ realised on the Hilbert space $L^2(\mathbb{R}^n)$. These are explicitly given by

$$\pi_\lambda(x,u,t)\varphi(\xi) = e^{i\lambda t} e^{i\lambda(x\cdot\xi + \frac{1}{2}x\cdot u)}\varphi(\xi+u),$$

where $\varphi \in L^2(\mathbb{R}^n)$. These representations are known to be unitary and irreducible. Moreover, by a theorem of Stone and Von-Neumann, (see e.g., [10]) upto unitary equivalence these account for all the infinite dimensional irreducible unitary representations of $\mathbb{H}^n$ which act as $e^{i\lambda t} I$ on the center. Also there is another class of finite dimensional irreducible representations which do not contribute to the Plancherel measure. Hence we will not describe them here.

The Fourier transform of a function $f \in L^1(\mathbb{H}^n)$ is the operator valued function defined on the set of all nonzero reals, $\mathbb{R}^*$ given by

$$\hat{f}(\lambda) = \int_{\mathbb{H}^n} f(x,u,t)\pi_\lambda(x,u,t)\,dx\,du\,dt.$$

Note that $\hat{f}(\lambda)$ is a bounded linear operator on $L^2(\mathbb{R}^n)$. It is known that when $f \in L^1 \cap L^2(\mathbb{H}^n)$ its Fourier transform is actually a Hilbert-Schmidt operator and one has

$$\int_{\mathbb{H}^n} |f(x,u,t)|^2\,dx\,du\,dt = (2\pi)^{-n-1}\int_{-\infty}^{\infty} \|\hat{f}(\lambda)\|_{HS}^2 |\lambda|^n\,d\lambda \tag{2.1}$$

where $\|.\|_{HS}$ denotes the Hilbert-Schmidt norm. The above allows us to extend the Fourier transform as a unitary operator between $L^2(\mathbb{H}^n)$ and the Hilbert space of Hilbert-Schmidt operator valued functions on $\mathbb{R}$ which are square integrable with respect to the Plancherel measure $d\mu(\lambda) = (2\pi)^{-n-1}|\lambda|^n d\lambda$.

For suitable functions f on $\mathbb{H}^n$ we also have the following inversion formula: with the notation $\pi_\lambda(x,u) = \pi_\lambda(x,u,0)$

$$f(x,u,t) = \int_{-\infty}^{\infty} e^{-i\lambda t}\,\mathrm{tr}(\pi_\lambda(x,u)^*\hat{f}(\lambda))\,d\mu(\lambda). \tag{2.2}$$

From the definition $\pi_\lambda(x,u,t)=e^{i\lambda t}\pi_\lambda(x,u)$ and hence it follows that

$$\hat{f}(\lambda)=\int_{\mathbb{C}^n} f^\lambda(x,u)\pi_\lambda(x,u)\,dx\,du$$

where f^λ stands for the inverse Fourier transform of f in the central variable:

$$f^\lambda(x,u):=\int_{-\infty}^{\infty} e^{i\lambda t} f(x,u,t)\,dt.$$

This suggests that for functions g on $\mathbb{R}^{2n}$, we consider the following operator

$$\pi_\lambda(g):=\int_{\mathbb{R}^{2n}} g(x,u)\pi_\lambda(x,u)\,dx\,du$$

known as the Weyl transform of g. With these notations we note that $\hat{f}(\lambda)=\pi_\lambda(f^\lambda)$. We have the Plancherel formula for the Weyl transform (See [17, 2.2.9, page no-49]):

$$\|g\|_2^2=(2\pi)^{-n}|\lambda|^n\|\pi_\lambda(g)\|_{HS}^2,$$

for $g\in L^2(\mathbb{R}^{2n})$ and the inversion formula reads as

$$g(x,u)=(2\pi)^{-n}|\lambda|^n\,\mathrm{tr}(\pi_\lambda(x,u)^*\pi_\lambda(g)). \tag{2.3}$$

We will make use of these properties in the proof of Beurling's theorem on $\mathbb{H}^n$.

The convolution on the Heisenberg group gives rise to a family of convolutions on $\mathbb{R}^{2n}$ called twisted convolutions. Recall that

$$f*g(x,u,t)=\int_{\mathbb{H}^n} f\big(x-u,y-v,t-s-\frac{1}{2}(u\cdot y-v\cdot x)\big)g\big(y,v,s\big)\,du\,dv\,ds.$$

By calculating the inverse Fourier transform of $f*g(x,u,t)$ in the last variable, we get

$$\big(f*g\big)^\lambda(x,u)=\int_{\mathbb{R}^{2n}} f^\lambda(x-u,y-v)g^\lambda(y,v)e^{i\frac{\lambda}{2}(u\cdot y-v\cdot x)}\,dy\,dv=f^\lambda*_\lambda g^\lambda(x,u).$$

The integral on the right is called the λ-twisted convolution of f^λ with g^λ. We call the function $f^\lambda(x-u,y-v)e^{i\frac{\lambda}{2}(u\cdot y-v\cdot x)}$ the λ-twisted translate of f^λ by (u,v). The well known relation $\widehat{f*g}(\lambda)=\hat{f}(\lambda)\hat{g}(\lambda)$ leads to the formula $\pi_\lambda(f^\lambda*_\lambda g^\lambda)=\pi_\lambda(f^\lambda)\pi_\lambda(g^\lambda)$.

2.2. Hermite and Laguerre functions

We are tempted to define $\pi_\lambda(z,w)$ for complex values $(z,w)\in\mathbb{C}^{2n}$ by the action

$$\pi_\lambda(z,w)\varphi(\xi)=e^{i\lambda(z\cdot\xi+\frac{1}{2}z\cdot w)}\varphi(\xi+w).$$

This will make sense only if $\varphi(z)$ can be defined for complex z and the resulting function $e^{i\lambda(z\cdot\xi+\frac{1}{2}z\cdot w)}\varphi(\xi+w)$ also belongs to $L^2(\mathbb{R}^n)$. Fortunately, there is a dense subspace of $L^2(\mathbb{R}^n)$ on which we can define $\pi_\lambda(z,w)$. This subspace consists of finite linear combinations of the Hermite functions $\Phi_\alpha^\lambda(\xi)=|\lambda|^{n/4}\Phi_\alpha(|\lambda|^{1/2}\xi)$ where

$$\Phi_\alpha(\xi)=c_\alpha\, H_\alpha(\xi)e^{-\frac{1}{2}|\xi|^2}.$$

Here $H_\alpha(\xi)$ are the Hermite polynomials on $\mathbb{R}^n$ and c_α are normalising constants. These functions $\Phi_\alpha^\lambda, \alpha\in\mathbb{N}^n$ which are eigenfunctions of $H(\lambda)=-\Delta+\lambda^2|\xi|^2$ with eigenvalues $(2|\alpha|+n)|\lambda|$, form an orthonormal basis for $L^2(\mathbb{R}^n)$. As H_α are polynomials, we can extend $\Phi_\alpha(\xi)$ to $\mathbb{C}^n$ by setting $\Phi_\alpha(z)=c_\alpha H_\alpha(z)e^{-\frac{1}{2}z^2}$ where $z^2=\sum_{j=1}^n z_j^2$. With this we have

$$\pi_\lambda(z,w)\Phi_\alpha^\lambda(\xi)=e^{i\lambda(z\cdot\xi+\frac{1}{2}z\cdot w)}\Phi_\alpha^\lambda(\xi+w).$$

These functions $\pi_\lambda(z,w)\Phi_\alpha^\lambda$ turn out to be in $L^2(\mathbb{R}^n)$ and hence $\pi_\lambda(z,w)$ are densely defined unbounded operators. The formal adjoint of $\pi_\lambda(z,w)$ can be calculated and we have $\pi_\lambda(z,w)^*=\pi_\lambda(-\bar{z},-\overline{w})$. Thus for $(y,v)\in\mathbb{R}^{2n}$ the operators $\pi_\lambda(i(y,v))$ are self-adjoint.

The $L^2(\mathbb{R}^n)$ norm of $\pi_\lambda(z,w)\Phi_\alpha^\lambda$ can be calculated in terms of the special Hermite functions

$$\Phi_{\alpha,\beta}^\lambda(x,u)=(2\pi)^{-n/2}\big(\pi_\lambda(x,u)\Phi_\alpha^\lambda,\Phi_\beta^\lambda\big).$$

The special Hermite functions $\Phi_{\alpha,\beta}^\lambda(x,u)$ are expressible in terms of Laguerre functions. As a result, these functions can also defined for $(z,w)\in\mathbb{C}^{2n}$ as entire functions. For our purposes, we just recall one important formula. Let $L_k^{n-1}(r), r>0$ stand for Laguerre polynomials of type $(n-1)$. We let

$$\varphi_{k,\lambda}^{n-1}(x,u)=L_k^{n-1}\big(\frac{1}{2}|\lambda|(|x|^2+|u|^2)\big)e^{-\frac{1}{4}|\lambda|(|x|^2+|u|^2)}$$

which has a natural holomorphic extension to $\mathbb{C}^n\times\mathbb{C}^n$. We will make use of the formula

$$(2\pi)^{-n/2}\,\varphi_{k,\lambda}^{n-1}(z,w)=\sum_{|\alpha|=k}\Phi_{\alpha,\alpha}^\lambda(z,w)$$

which expresses the Laguerre function $\varphi_{k,\lambda}^{n-1}(z,w)$ in terms of special Hermite functions.

The special Hermite functions $\Phi^\lambda_{\alpha,\beta}, \alpha, \beta\in\mathbb{N}^n$ form an orthonormal basis for $L^2(\mathbb{R}^{2n})$. The special Hermite expansion of any function $g\in L^2(\mathbb{R}^{2n})$ can be put in the compact form

$$g(x,u)=(2\pi)^{-n}\,|\lambda|^n\sum_{k=0}^{\infty} g*_\lambda\varphi^{n-1}_{k,\lambda}(x,u).$$

In terms of this expansion, the inversion formula for the Fourier transform on $\mathbb{H}^n$ can be written in the form

$$f(x,u,t)=(2\pi)^{-n-1}\int_{-\infty}^{\infty} e^{-i\lambda t}\Big(\sum_{k=0}^{\infty} f^\lambda *_\lambda\varphi^{n-1}_{k,\lambda}(x,u)\Big)|\lambda|^n\,d\lambda.$$

2.3. Gutzmer's formula

As mentioned in the introduction, we let $\sigma(x,u)$ stand for the action of $K=Sp(n,\mathbb{R})\cap O(2n,\mathbb{R})$ on $\mathbb{R}^{2n}$. The symplectic form $[(x,u),(y,v)]=(u\cdot y-v\cdot x)$ on $\mathbb{R}^{2n}$ is invariant under the action of K. The action of this group has a natural extension to $\mathbb{C}^n\times\mathbb{C}^n$ and so it makes sense to talk about $\sigma(z,w)$ for $z,w\in\mathbb{C}^n$. In Theorem 1.2 we have imposed the condition

$$\int_K \|\pi_\lambda(\sigma(z,w))^*\hat{f}(\lambda)\|^2_{HS}\,d\sigma<\infty \tag{2.4}$$

on the Fourier transform $\hat{f}(\lambda)$. From this we would like to conclude that $f^\lambda(x,u)$ can be extended to $\mathbb{C}^{2n}$ as a holomorphic function. The integral appearing above can be explicitly computed using Gutzmer's formula for special Hermite expansions proved in [18]; see Theorem 6.2 (Caution: the integral over K is missing there!)

Theorem 2.1. (Gutzmer) *Let $g\in L^2(\mathbb{R}^{2n})$ has a holomorphic extension G to $\mathbb{C}^{2n}$. Then we have*

$$\begin{aligned}\int_K\int_{\mathbb{R}^{2n}} |G((x,u)+i\sigma(y,v))|^2 e^{\lambda(u\cdot y-v\cdot x)}\,dx\,du\,d\sigma\\ =c_n\sum_{k=0}^{\infty}\frac{k!(n-1)!}{(k+n-1)!}\varphi^{n-1}_{k,\lambda}(2iy,2iv)\|g*_\lambda\varphi^{n-1}_{k,\lambda}\|_2^2.\end{aligned}$$

Using the above result, we can calculate the integral (2.4).

Proposition 2.2. *For any $f\in L^2(\mathbb{H}^n)$ we have the identity*

$$\int_K \|\pi_\lambda(\sigma(z,w))\hat{f}(\lambda)\|^2_{HS}\,d\sigma$$

$$= e^{-\lambda(u\cdot y - x\cdot v)} \sum_{k=0}^{\infty} \frac{k!(n-1)!}{(k+n-1)!} \varphi_{k,\lambda}^{n-1}(2iy, 2iv) \|f^\lambda *_\lambda \varphi_{k,\lambda}^{n-1}\|_2^2$$

with the understanding that either the left hand side or the right hand side is finite.

Proof. In order to calculate (2.4) let us write it as the sum

$$\sum_{k=0}^{\infty} \int_K \|\pi_\lambda(\sigma(z,w))^* \hat{f}(\lambda) P_k(\lambda)\|_{HS}^2 \, d\sigma.$$

As $\hat{f}(\lambda) = \pi_\lambda(f^\lambda)$ and $\pi_\lambda(\varphi_{k,\lambda}^{n-1}) = (2\pi)^n |\lambda|^{-n} P_k(\lambda)$ (see [18], Proposition 2.3.3) we have

$$\hat{f}(\lambda) P_k(\lambda) = (2\pi)^{-n} |\lambda|^n \pi_\lambda(f^\lambda *_\lambda \varphi_{k,\lambda}^{n-1}).$$

We note that the function $g(x,u) = f^\lambda *_\lambda \varphi_{k,\lambda}^{n-1}(x,u)$ extends to $\mathbb{C}^{2n}$ as a homomorphic function and hence the relation

$$\pi_\lambda(x,u)^* \pi_\lambda(g) = \pi_\lambda(\tau_{(x,u)}^\lambda g), \quad \tau_{(x,u)}^\lambda g(x',u') = g(x'-x, u'-u) e^{-i\frac{\lambda}{2}(u\cdot x' - u'\cdot x)}$$

continues to hold even when (x,u) is replaced by (z,w). Therefore, by the Plancherel theorem for the Weyl transform, we have

$$(2\pi)^{-n} |\lambda|^n \int_{\mathbb{R}^{2n}} |\tau_{(z,w)}^\lambda g(x',u')|^2 \, dx' du' = \|\pi_\lambda((z,w))^* \hat{f}(\lambda) P_k(\lambda)\|_{HS}^2.$$

Recalling the definition of $\tau_{(z,w)}^\lambda g$ and making some change of variables, we see that

$$\begin{aligned} &\|\pi_\lambda((z,w))^* \hat{f}(\lambda) P_k(\lambda)\|_{HS}^2 \\ &\quad = (2\pi)^{-n} |\lambda|^n e^{-\lambda(u\cdot y - v\cdot x)} \int_{\mathbb{R}^{2n}} |g(x'-iy, u'-iv)|^2 e^{-\lambda(u'\cdot y - x'\cdot v)} \, dx' \, du'. \end{aligned}$$

As the Lebesgue measure on $\mathbb{R}^{2n}$ as well the symplectic form $[(x,u),(y,v)] = (u\cdot y - v\cdot x)$ is invariant under the action of $U(n)$ the integral in the above equation becomes

$$\int_{\mathbb{R}^{2n}} |g\big((x',u') + i\sigma(-y,-v)\big)|^2 e^{-\lambda(u'\cdot y - x'\cdot v)} \, dx' \, du'.$$

We can now appeal to Theorem 2.1 and use the relation

$$\varphi_{k,\lambda}^{n-1} *_\lambda \varphi_{j,\lambda}^{n-1} = \delta_{jk} (2\pi)^n |\lambda|^{-n} \varphi_{k,\lambda}^{n-1}$$

to arrive at the identity

$$\begin{aligned} &\|\pi_\lambda((z,w))^* \hat{f}(\lambda) P_k(\lambda)\|_{HS}^2 \\ &\quad = e^{-\lambda(u\cdot y - x\cdot v)} \frac{k!(n-1)!}{(k+n-1)!} \varphi_{k,\lambda}^{n-1}(2iy, 2iv) \|f^\lambda *_\lambda \varphi_{k,\lambda}^{n-1}\|_2^2. \end{aligned}$$

This completes the proof of the proposition. □

2.4. The sublaplacian and the associated heat kernel

We let $\mathfrak{h}_n$ stand for the Heisenberg Lie algebra consisting of left invariant vector fields on $\mathbb{H}^n$. A basis for $\mathfrak{h}_n$ is provided by the $2n+1$ vector fields

$$X_j = \frac{\partial}{\partial x_j} + \frac{1}{2} u_j \frac{\partial}{\partial t}, \quad U_j = \frac{\partial}{\partial u_j} - \frac{1}{2} x_j \frac{\partial}{\partial t}, \quad j = 1, 2, ..., n$$

and $T = \frac{\partial}{\partial t}$. These correspond to certain one parameter subgroups of $\mathbb{H}^n$. The sublaplacian on $\mathbb{H}^n$ is defined by $\mathcal{L} := -\sum_{j=1}^{\infty} (X_j^2 + U_j^2)$ which is given explicitly by

$$\mathcal{L} = -\Delta_{\mathbb{R}^{2n}} - \frac{1}{4} |(x, u)|^2 \frac{\partial^2}{\partial t^2} + N \frac{\partial}{\partial t}$$

where $\Delta_{\mathbb{R}^{2n}}$ stands for the Laplacian on $\mathbb{R}^{2n}$ and N is the rotation operator defined by

$$N = \sum_{j=1}^{n} \left(x_j \frac{\partial}{\partial u_j} - u_j \frac{\partial}{\partial x_j} \right).$$

This is a sub-elliptic operator which is homogeneous of degree 2 with respect to the non-isotropic dilation given by $\delta_r(x, u, t) = (rx, ru, r^2 t)$.

Along with the sublaplacian, we also consider the special Hermite operator L_λ defined by the relation $(\mathcal{L} f)^\lambda(x, u) = L_\lambda f^\lambda(x, u)$. It follows that L_λ is explicitly given by

$$L_\lambda = -\Delta_{\mathbb{R}^{2n}} + \frac{1}{4} \lambda^2 |(x, u)|^2 + i\lambda N.$$

It turns out that $f^\lambda *_\lambda \varphi_{k,\lambda}(x, u)$ are eigenfunctions of L_λ with eigenvalues $(2k+n)|\lambda|$ and hence we have the following expansion

$$L_\lambda f^\lambda(x, u) = (2\pi)^{-n} |\lambda|^n \sum_{k=0}^{\infty} ((2k+n)|\lambda|)\, f^\lambda *_\lambda \varphi_{k,\lambda}^{n-1}(x, u)$$

leading to the following spectral decomposition of $\mathcal{L}$:

$$\mathcal{L} f(x, u, t) = (2\pi)^{-n-1} \int_{-\infty}^{\infty} e^{-i\lambda t} \Big(\sum_{k=0}^{\infty} ((2k+n)|\lambda|)\, f^\lambda *_\lambda \varphi_{k,\lambda}^{n-1}(x, u) \Big) |\lambda|^n \, d\lambda. \tag{2.5}$$

The heat kernel $p_a(x, u, t), a > 0$ associated to the sublaplacian is defined by

$$p_a(x, u, t) = (2\pi)^{-n-1} \int_{-\infty}^{\infty} e^{-i\lambda t} \Big(\sum_{k=0}^{\infty} e^{-a(2k+n)|\lambda|} \varphi_{k,\lambda}^{n-1}(x, u) \Big) |\lambda|^n \, d\lambda.$$

Though the heat kernel p_a is not known explicitly, $p_a^\lambda(x,u)$ can be calculated. In fact,

$$p_a^\lambda(x,u)=(2\pi)^{-n}\sum_{k=0}^{\infty}e^{-a(2k+n)|\lambda|}\varphi_{k,\lambda}^{n-1}(x,u)$$

and the above series can be summed using a generating function identity for Laguerre functions to get

$$p_a^\lambda(x,u)=(4\pi)^{-n}\Big(\frac{\lambda}{\sinh a\lambda}\Big)^n e^{-\frac{1}{4}\lambda(\coth a\lambda)(|x|^2+|u|^2)}. \tag{2.6}$$

In terms of this, the semigroup generated by L_λ is given by $e^{-aL_\lambda}g(x,u)=g*_\lambda p_a^\lambda(x,u)$.

From the explicit formula for p_a^λ it is easy to see that $g*_\lambda p_a^\lambda(x,u)$ has a holomorphic extension to $\mathbb{C}^{2n}$. The problem of characterising the image of $L^2(\mathbb{R}^{2n})$ under the transform $T_a^\lambda g(z,w)=g*_\lambda p_a^\lambda(z,w)$ has been studied in Krötz et al. [12] where the authors have proved the following result.

Theorem 2.3. (Krötz et al.) *An entire function $G(z,w)$ on $\mathbb{C}^{2n}$ is square integrable with respect to the weight function $e^{\lambda(u\cdot y-v\cdot x)}p_{2a}^\lambda(2y,2v)$ if and only if $G(x,u)=g*_\lambda p_a^\lambda(x,u)$ for some $g\in L^2(\mathbb{R}^{2n})$. Moreover, for any $g\in L^2(\mathbb{R}^{2n})$ we have*

$$\int_{\mathbb{C}^{2n}}|g*_\lambda p_a^\lambda(z,w)|^2 e^{\lambda(u\cdot y-v\cdot x)}p_{2a}^\lambda(2y,2v)\,dz\,dw=c_\lambda\int_{\mathbb{R}^{2n}}|g(x,u)|^2\,dx\,du.$$

Thus we are led to consider the twisted Bergman space $\widetilde{\mathcal{B}}_\lambda(\mathbb{C}^{2n},w_\lambda)$ consisting of all entire functions $G(z,w)$ on $\mathbb{C}^{2n}$ which are square integrable with respect to the weight function $p_{2a}^\lambda(2y,2v)e^{\lambda(u\cdot y-v\cdot x)}$. In the above setting the map T_a^λ is an isometry up to a constant multiple from $L^2(\mathbb{R}^{2n})$ onto $\widetilde{\mathcal{B}}_\lambda(\mathbb{C}^{2n},w_\lambda)$.

We now consider the spaces $\widetilde{\mathcal{B}}_\lambda(\mathbb{C}^{2n},w_\lambda)$ associated to the weight function $w_\lambda(y,v)$. From the form of the weight, it is easy to see that these spaces are invariant under twisted translations. These are reproducing kernel Hilbert spaces and hence possess a reproducing kernel which we denote by $K((z,w),(z',w'))$. From general theory it follows that every $G\in\widetilde{\mathcal{B}}_\lambda(\mathbb{C}^{2n},w_\lambda)$ satisfies the point-wise estimate $|G(z,w)|^2\le CK((z,w),(z,w))$. In the case of the space associated to the transform T_a^λ the reproducing kernel is given by

$$e^{-i\frac{\lambda}{2}(w\cdot z'-z\cdot w')}p_a^\lambda*_\lambda p_a^\lambda(z-\overline{z'},w-\overline{w'})=e^{-i\frac{\lambda}{2}(w\cdot z'-z\cdot w')}p_{2a}^\lambda(z-\overline{z'},w-\overline{w'}).$$

Thus for any $g\in L^2(\mathbb{R}^{2n})$ we get the estimate

$$|g*_\lambda p_a^\lambda(z,w)|^2\le C\,p_{2a}^\lambda(2iy,2iv).$$

If we assume that $\widetilde{\mathcal{B}}_\lambda(\mathbb{C}^{2n}, w_\lambda)$ occurs as the image of a transform $T_{q_\lambda} g = g *_\lambda q_\lambda$, then we can calculate the reproducing kernel in terms of q_λ and we can prove the estimate

$$|g *_\lambda q_\lambda(z,w)|^2 \leq C\, q_\lambda *_\lambda q_\lambda(2iy, 2iv) e^{\lambda(u\cdot y - v\cdot x)}.$$

This is precisely what motivated us to formulate Theorem 1.11.

2.5. Constructing examples of twisted Bergman spaces

In the introduction we have mentioned how one can construct a weighed Bergman space on $\mathbb{C}^n$ which is invariant under the action of $\mathbb{R}^n$ starting from a weight function w. In this subsection, we describe such a procedure to construct twisted Bergman spaces on $\mathbb{C}^{2n}$. For that purpose, as in the Euclidean case, we make use of the Gutzmer's formula described in the previous subsection.

We start with a positive weight function $w_\lambda(y, v)$ on $\mathbb{R}^{2n}$ which we assume to be radial. Consider the sequence $C_\lambda(k)$ defined by

$$C_\lambda(k) = \frac{k!(n-1)!}{(k+n-1)!} \int_{\mathbb{R}^{2n}} w_\lambda(y,v) \varphi_{k,\lambda}^{n-1}(2iy, 2iv)\, dy\, dv.$$

Under the assumption that $C_\lambda(k)$ has enough decay, we can define the radial function

$$q_\lambda(x,u) = \sum_{k=0}^{\infty} C_\lambda(k)^{-1/2}\, \varphi_{k,\lambda}^{n-1}(x,u).$$

Recall that these functions $\varphi_{k,\lambda}^{n-1}(x,u)$ extend to $\mathbb{C}^{2n}$ as entire functions and satisfy the estimates

$$|\varphi_{k,\lambda}^{n-1}(z,w)|^2 \leq C_t\, e^{2t(2k+n)|\lambda|}\, p_{2t}^\lambda(2iy, 2iv)$$

for any $t>0$, (see [12].) If we know that $C_\lambda(k)^{-1} \leq c_\lambda\, e^{-t(2k+n)^s |\lambda|^s}$ for some $t>0$, $s>1$, then the function q_λ defines a holomorphic function on the whole of $\mathbb{C}^{2n}$. It then follows that for any $f \in L^2(\mathbb{R}^{2n})$ the function $f *_\lambda q_\lambda$ has a holomorphic extension to $\mathbb{C}^n \times \mathbb{C}^n$ and by Gutzmer's formula we have

$$\int_{\mathbb{C}^{2n}} |f *_\lambda q_\lambda(z,w)|^2 w_\lambda(y,v)\, e^{\lambda(u\cdot y - v\cdot x)}\, dz\, dw$$
$$= c_{n,\lambda} \sum_{k=0}^{\infty} \|f *_\lambda \varphi_{k,\lambda}^{n-1}\|_2^2 C_\lambda(k)^{-1} \frac{k!(n-1)!}{(k+n-1)!} \int_{\mathbb{R}^{2n}} w_\lambda(y,v) \varphi_{k,\lambda}^{n-1}(2iy, 2iv)\, dy\, dv.$$

By our choice of $C_\lambda(k)$ the right hand side of the above reduces to $\|f\|_2^2$ proving that the image of $L^2(\mathbb{R}^{2n})$ under the transform $T_{q_\lambda} : f \to f *_\lambda q_\lambda(z,w)$ is a twisted Bergman space with weight function $w_\lambda(y,v)$. The reproducing kernel for this space is given by

$$K_\lambda((z,w),(z',w')) = e^{-i\frac{\lambda}{2}(w\cdot z'-z\cdot w')} q_\lambda *_\lambda q_\lambda(z-\overline{z'}, w-\overline{w'}).$$

When we take the weight function $w_\lambda(y,v)=p_{2t}^\lambda(2y,2v)$ it is known that $C_\lambda(k)= e^{2t(2k+n)|\lambda|}$ and hence $q_\lambda(x,u)=p_t^\lambda(x,u)$ which is coming from the Segal-Bargmann transform $T_t f = f*p_t$ on $\mathbb{H}^n$. By taking superpositions of p_t^λ we can construct other examples of w_λ and the kernels q_λ. For example, for any $1<s<\infty$, let us define

$$w_\lambda(y,v) = \int_0^\infty t^n e^{-\frac{1}{s}t^s} p_t^\lambda(2y,2v)\,dt.$$

We recall the following result proved in [18], (see Lemma 6.3):

$$\frac{k!(n-1)!}{(k+n-1)!}\int_{\mathbb{R}^{2n}} p_t^\lambda(iy,iv)\varphi_{k,\lambda}^{n-1}(iy,iv)\,dy\,dv = c_n\, e^{t(2k+n)|\lambda|}.$$

Integrating w_λ against $\varphi_{k,\lambda}^{n-1}(2iy,2iv)$ and making use of the above formula we see that

$$\frac{k!(n-1)!}{(k+n-1)!}\int_{\mathbb{R}^{2n}} w_\lambda(y,v)\varphi_{k,\lambda}^{n-1}(2iy,2iv)\,dy\,dv = c_n \int_0^\infty t^n e^{-\frac{1}{s}t^s+(2k+n)|\lambda|t}\,dt.$$

Let $A=(2k+n)|\lambda|$ and define s' as the index conjugate to s. By writing $(\frac{1}{s}t^s - At) = \left(\frac{1}{s}t^s + \frac{1}{s'}A^{s'} - At\right) - \frac{1}{s'}A^{s'}$ and making the change of variables $t \to A^{s'/s}t$ the above integral becomes

$$A^{(n+1)s'/s} e^{\frac{1}{s'}A^{s'}} \int_0^\infty t^n e^{-A^{s'}(\frac{1}{s}t^s+\frac{1}{s'}-t)}\,dt.$$

As the function $\psi(t)=(\frac{1}{s}t^s+\frac{1}{s'}-t)$ decreases from $\frac{1}{s'}$ to 0 on the interval $[0,1]$ we can choose $0<t_0<1$ such that $\psi(t_0)=\frac{1}{2s'}$ and get the lower bound

$$\int_0^\infty t^n e^{-A^{s'}(\frac{1}{s}t^s+\frac{1}{s'}-t)}\,dt \geq e^{-\frac{1}{2s'}A^{s'}} \int_{t_0}^1 t^n\,dt = c\, e^{-\frac{1}{2s'}A^{s'}}.$$

This gives us the following lower bound for the sequence $C_\lambda(k)$:

$$C_\lambda(k) \geq c_\lambda\, ((2k+n)|\lambda|)^{(n+1)(s'-s)} e^{\frac{1}{2s'}((2k+n)|\lambda|)^{s'}}.$$

Thus we see that $C_\lambda(k)^{-1/2}$ has the required decay so that the function q_λ gives rise to a twisted Bergman space.

3. Beurling's theorem and its consequences

3.1. Hedenmalm's proof

For the convenience of the reader we recall Hedenmalms's proof from his paper [7]. The idea is very simple: under the assumption that f and $\hat{f}$ satisfy (1.1), the function

$$F(\zeta)=\int_{\mathbb{R}^n}\int_{\mathbb{R}^n}\bar{f}(y)\hat{f}(\xi)e^{i\zeta(y,\xi)}\,dy\,d\xi$$

defines a holomorphic function on the strip $S=\{\zeta:|\operatorname{Im}\zeta|<1\}$ and extends continuously up to its closure. We also note that

$$F(\zeta)=(2\pi)^{n/2}\int_{\mathbb{R}^n}\bar{f}(y)\,f(\zeta y)\,dy. \tag{3.1}$$

When $\zeta=r$ is real, owing to the inversion formula for the Fourier transform, we have

$$F(r)=(2\pi)^{n/2}\int_{\mathbb{R}^n}\bar{f}(y)\,f(ry)\,dy=r^{-n}\overline{F(r^{-1})}.$$

By defining $G(\zeta)=(1+\zeta^2)^{n/2}F(\zeta)$ we see that functional relation $F(r)=r^{-n}\overline{F(r^{-1})}$ translates into $G(r)=\overline{G(r^{-1})}$. The function G is holomorphic in a neighbourhood of $|\zeta|\le 1$ save for i and $-i$. By defining $G_1(\bar{\zeta})=\overline{G(\frac{1}{\zeta})}$ we get another holomorphic function on $|\zeta|>1$. These two domains have an overlap along the real line where they coincide. Thus we see that G extends holomorphically to the whole of $\mathbb{C}$ save for i and $-i$. However, these are removable singularities as F remains bounded in neighbourhoods of these points. Therefore, G is entire and bounded and hence reduces to a constant. But as F remains bounded on the closure of S we conclude that $c=0$ and hence $F(\zeta)=0$ for all ζ. As $F(1)=\|f\|_2^2=0$, this proves the theorem.

Remark 3.1. If we already know that f has a holomorphic extension to $\mathbb{C}^n$ then the Beurling's condition can be replaced by the modified one (1.2). All we have to do is to define $F(\zeta)$ as in (3.1) and the above proof remains valid without any problem. In particular, this proves Theorem 1.10 since the function $f*q$ satisfies the modified condition (1.2) and hence by the above argument we get $f*q(y)=0$. As $f*q$ is entire, we have $f*q(z)=0$ for all $z\in\mathbb{C}^n$ and from the isometry (1.12) we conclude that $f=0$.

3.2. Beurling's theorem on $\mathbb{H}^n$

We are now ready for proving Theorem 1.2. As in the Euclidean case we start with defining the function

$$F_\lambda(\zeta)=\int_{\mathbb{R}^{2n}}\overline{f^\lambda(y,v)}f^\lambda(\zeta y,\zeta v)\,dy\,dv$$

for complex values of ζ. For this to make sense, we have to first check if $f^\lambda(x,u)$ can be holomorphically extended to some domain in $\mathbb{C}^{2n}$ and the integral converges for all ζ in some other domain in $\mathbb{C}$. Our hypothesis (1.5) when combined with Proposition 2.2 leads to

$$\sum_{k=0}^{\infty} \frac{k!(n-1)!}{(k+n-1)!} \varphi_{k,\lambda}^{n-1}(2iy,2iv)\,\|f^\lambda *_\lambda \varphi_{k,\lambda}^{n-1}\|_2^2 < \infty. \tag{3.2}$$

We recall the following asymptotic behaviour of the Laguerre polynomials: Perron's formula for Laguerre polynomials in the complex domain (see Theorem 8.22.3 in Szego [14]) states

$$L_k^\alpha(s) = \frac{1}{2}\pi^{-1/2}e^{s/2}(-s)^{-\alpha/2-1/4}k^{\alpha/2-1/4}e^{2\sqrt{-sk}}\big(1+O(k^{-1/2})\big) \tag{3.3}$$

valid for s in the complex plane cut along the positive real axis. From this we get

$$C_1(\rho)(2k+n)^{\gamma_1}e^{c_1\rho\sqrt{2k+n}} \le L_k^{n-1}(-2\rho^2)e^{\rho^2} \le C_2(\rho)(2k+n)^{\gamma_2}e^{c_2\rho\sqrt{2k+n}}$$

as k tends to infinity. In view of this, the finiteness of (3.2) for all (y,v) implies that for every $\rho>0$,

$$\|f^\lambda *_\lambda \varphi_{k,\lambda}^{n-1}\|_2 \le Ce^{-\rho\sqrt{(2k+n)|\lambda|}}.$$

This allows us to write $f^\lambda = g_\lambda *_\lambda P_\rho^\lambda$ where $g_\lambda \in L^2(\mathbb{R}^{2n})$ and

$$P_\rho^\lambda(x,u) = (2\pi)^{-n}\sum_{k=0}^{\infty} e^{-\rho\sqrt{(2k+n)|\lambda|}}\,\varphi_{k,\lambda}^{n-1}(x,u)$$

is the Poisson kernel associated to the operator L_λ. Observe that P_ρ^λ extends holomorphically to a tube domain $\Omega_{\gamma\rho} = \{(z,w): |(y,v)| < \gamma\rho\}$ for some $\gamma>0$. This is a consequence of the estimate

$$|\varphi_{k,\lambda}^{n-1}(z,w)|^2 \le C\frac{(k+n-1)!}{k!(n-1)!}e^{\lambda(u\cdot y - v\cdot x)}\varphi_{k,\lambda}^{n-1}(2iy,2iv)$$

(see Proposition 3.1 in [18]). Therefore, from the factorisation $f^\lambda = g_\lambda *_\lambda P_\rho^\lambda$ we infer that f^λ extends holomorphically to a tube domain $\Omega_{\gamma\rho}$. As this is true for any ρ we conclude that f^λ has a holomorphic extension to the whole of $\mathbb{C}^{2n}$.

Thus the function $\zeta \to f^\lambda(\zeta y, \zeta v)$ is holomorphic, but still the integral defining $F_\lambda(\zeta)$ need not converge. From the inversion formula for the Weyl transform, we have

$$f^\lambda(y,v) = (2\pi)^{-n}|\lambda|^n \operatorname{tr}\big(\pi_\lambda(y,v)^*\hat{f}(\lambda)\big),$$

from which we obtain, with $\zeta=r+is$, the following expression:

$$f^\lambda(\zeta y,\zeta v)=(2\pi)^{-n}|\lambda|^n \operatorname{tr}\big(\pi_\lambda(r(y,v))^*\pi_\lambda(is(y,v))\hat f(\lambda)\big).$$

As $\pi_\lambda(r(y,v))^*$ are unitary operators, the above gives the estimate

$$|f^\lambda(\zeta y,\zeta v)|=(2\pi)^{-n}|\lambda|^n|\operatorname{tr}(\pi_\lambda(r(y,v))^*\,\pi_\lambda(is(y,v))\hat f(\lambda))|\le c_\lambda\|\pi_\lambda(is(y,v))\,\hat f(\lambda)\|_1.$$

By considering the radial majorant of the above, for $|s|\le1$, we get the estimate

$$|f^\lambda(\zeta y,\zeta v)|\le c_\lambda \sup_{|(y',v')|\le|(y,v)|}\|\pi_\lambda((i(y',v'))\hat f(\lambda)\|_1.$$

Therefore, in view of the hypothesis (1.6), it follows that $F_\lambda(\zeta)$ defines a holomorphic function on the strip $S=\{\zeta:|\operatorname{Im}\zeta|<1\}$ which is continuous up to the boundary. We also note that for $r>0$ the relation $F_\lambda(r)=r^{-2n}\overline{F_\lambda(r^{-1})}$. From this point we just need to repeat Hedenmalm's proof to conclude that $f^\lambda=0$. As this is true true for all $\lambda\in\mathbb{R}^*$, Theorem 1.2 is proved. It is also clear that the hypothesis (1.6) can be replaced by (1.9).

3.3. Hardy and Cowling-Price theorems

We can now easily deduce Hardy and Cowling-Price theorems on the Heisenberg group from Theorem 1.2. In doing so, we will make use of the following theorem of Douglas [5]. (Here we have stated only part of the theorem which is relevant for us).

Theorem 3.2. (Douglas) *Let A and B be two bounded linear operators on a Hilbert space $\mathcal{H}$. Then $AA^*\le c^2BB^*$ if and only if there is a bounded linear operator C such that $A=BC$.*

In view of this result, the second assumption $\hat f(\lambda)^*\hat f(\lambda)\le Ce^{-2bH(\lambda)}$ of Hardy's theorem gives us $\hat f(\lambda)=B_\lambda e^{-bH(\lambda)}$ where B_λ are bounded. As $a<b$ we can choose b' such that $a<b'<b$ and write $\hat f(\lambda)=B'_\lambda e^{-b'H(\lambda)}$ where now B'_λ is Hilbert-Schmidt. This allows us to write $f^\lambda=g_\lambda*_\lambda p^\lambda_{b'}$ with $g_\lambda\in L^2(\mathbb{R}^{2n})$. Thus, f^λ belongs to the twisted Bergman space with weight function $p^\lambda_{2b'}(2y,2v)$. This leads us to the estimate

$$|f^\lambda(z,w)|^2\le c_\lambda e^{\lambda(u\cdot y-v\cdot x)}p^\lambda_{2b'}(2iy,2iv).$$

Thus we can take $F_\lambda(y,v)^2=p^\lambda_{2b'}(2iy,2iv)$ and see if this verifies the condition (1.9). The assumption on f namely $|f(x,u,t)|\le Cp_a(x,u,t)$ gives the estimate

$$|f^\lambda(y,v)|\le C\int_{-\infty}^{\infty}p_a(y,v,t)\,dt\le C_a e^{-\frac{1}{4a}(|y|^2+|v|^2)}$$

which is not good enough for verifying (1.9) for all values of λ but only for small enough values. However, it serves the purpose when combined with an analyticity argument.

The following estimate for the heat kernel is known (see e.g Proposition 2.8.2 in [17]):

$$|p_a(y,v,t)| \le c_n a^{-n-1} e^{-\frac{d}{a}(|(y,v)|^2+|t|)}$$

for some $d>0$. As a consequence, the estimate $|f(y,v,t)| \le C_a e^{-\frac{d}{a}(|(y,v)|^2+|t|)}$ allows us to extend $\lambda \to f^\lambda$ as an $L^2(\mathbb{R}^{2n})$ valued holomorphic function on the strip $|\operatorname{Im}(\lambda)| < d/a$. Thus, we only need to show that $f^\lambda = 0$ for all $0<\lambda<\delta$ for some $\delta>0$. Consider the integral

$$\int_{\mathbb{R}^{2n}} |f^\lambda(y,v)| F_\lambda(y,v)\, dy\, dv \le C_a \int_{\mathbb{R}^{2n}} e^{-\frac{1}{4a}(|y|^2+|v|^2)} \big(p^\lambda_{2b'}(2iy,2iv)\big)^{1/2}\, dy\, dv.$$

Recalling the expression for $p^\lambda_{2b'}(2iy,2iv)$ we are led to check the finiteness of the integral

$$\int_{\mathbb{R}^{2n}} e^{-\frac{1}{4a}(|y|^2+|v|^2)} e^{\frac{1}{2}\lambda(\coth 2b'\lambda)(|y|^2+|v|^2)}\, dy\, dv.$$

As $\lambda(\coth 2b'\lambda)$ tends to $(2b')^{-1} < (2a)^{-1}$ as λ goes to 0 we can choose $\delta>0$ such that $\lambda(\coth 2b'\lambda) < (2b)^{-1}$ for $0<\lambda<\delta$. For such δ the above integral is finite. Therefore we can appeal to Beurling's theorem to complete the proof.

In the case of Cowling-Price theorem we have $f^\lambda(y,v) = g_\lambda *_\lambda p^\lambda_b(y,v)$ and hence we can take $F_\lambda(y,v)^2 = p^\lambda_{2b}(2iy,2iv)$. As before, the hypothesis $f = h\, p_a$ allows us to extend f^λ holomorphically to $|\operatorname{Im}(\lambda)| < d/a$. Moreover,

$$|f^\lambda(y,v)| \le \int_{-\infty}^{\infty} |h(y,v,t)| p_a(y,v,t)\, dt \le \|h(y,v,\cdot)\|_2 \Big(\int_{-\infty}^{\infty} (p^\lambda_a(y,v))^2\, d\lambda\Big)^{1/2}.$$

From the explicit expression for $p^\lambda_a(y,v)$, using the fact that $t\cosh t \ge \sinh t$ for $t \ge 0$, we get

$$\int_{-\infty}^{\infty} (p^\lambda_a(y,v))^2 d\lambda \le C e^{-\frac{1}{2a}(|y|^2+|v|^2)} \int_{-\infty}^{\infty} \Big(\frac{\lambda}{\sinh a\lambda}\Big)^{2n} d\lambda.$$

Thus we are led to consider the finiteness of the integral

$$\int_{\mathbb{R}^{2n}} \|h(y,v,\cdot)\|_2\, e^{-\frac{1}{4a}(|y|^2+|v|^2)} e^{\frac{1}{2}\lambda(\coth 2b\lambda)(|y|^2+|v|^2)}\, dy\, dv.$$

As before, the above integral is finite for small enough λ which completes the proof of Theorem 1.5.

3.4. Beurling's theorem for the Hermite expansions

We begin with some preparations. Recall that the Hermite projections P_kf are given by

$$P_kf(x) = \sum_{|\alpha|=k} (f, \Phi_\alpha)\, \Phi_\alpha(x) = \int_{\mathbb{R}^n} f(y)\Phi_k(x, y)\, dy.$$

It is clear that the kernel $\Phi_k(x, y) = \sum_{|\alpha|=k} \Phi_\alpha(x)\, \Phi_\alpha(y)$ extends to $\mathbb{C}^n \times \mathbb{C}^n$ holomorphically and $P_kf(x)$ to $\mathbb{C}^n$. Since $P_k^2 = P_k$ we have

$$|P_kf(z)| = |\int_{\mathbb{R}^n} P_kf(x')\Phi_k(x', z)\, dx'| \le ||P_kf||_2 \|\Phi_k(\cdot, z)\|_2.$$

In order to estimate $\|\Phi_k(\cdot, z)\|_2$ we observe that

$$\|\Phi_k(\cdot, z)\|_2^2 = \sum_{|\alpha|=k} \Phi_\alpha(z)\Phi_\alpha(\bar{z}) = \Phi_k(z, \bar{z}).$$

We now recall the following explicit formula for $\Phi_k(z, w)$ proved in [19], Lemma 2.3.

$$\Phi_k(z, w) = \pi^{-n/2} \sum_{j=0}^{k} (-1)^j L_j^{n/2-1}(\frac{1}{2}(z+w)^2) L_{k-j}^{n/2-1}(\frac{1}{2}(z-w)^2) e^{-\frac{1}{2}(z^2+w^2)}$$

so that $\Phi_k(z, \bar{z}) = \pi^{-n/2} \sum_{j=0}^{k} (-1)^j L_j^{n/2-1}(2|x|^2) e^{-|x|^2} L_{k-j}^{n/2-1}(-2|y|^2) e^{|y|^2}$. We now make use of the estimate (see 1.1.39 in [15])

$$|L_j^{n/2-1}(2|x|^2)| e^{-|x|^2} \le L_j^{n/2-1}(0) = \frac{\Gamma(j+n/2)}{\Gamma(j+1)\Gamma(n/2)}$$

and the formula (see 1.1.51 in [15])

$$\sum_{j=0}^{k} \frac{\Gamma(j+n/2)}{\Gamma(j+1)\Gamma(n/2)} L_{k-j}^{n/2-1}(|x|^2) e^{-\frac{1}{2}|x|^2} = L_k^{n-1}(|x|^2) e^{-\frac{1}{2}|x|^2}$$

in estimating $\Phi_k(z, \bar{z})$. We have thus proved $\Phi_k(z, \bar{z}) \le \pi^{-n/2} \varphi_k^{n-1}(2iy)$ and hence

$$|P_kf(z)| \le ||P_kf||_2 \|\Phi_k(\cdot, z)\|_2 \le \pi^{-n/2} ||P_kf||_2 \sqrt{\varphi_k^{n-1}(2iy)}. \tag{3.4}$$

With these preparations, we can now launch a proof of Theorem 1.7.

As usual, we define the function F on the strip S by

$$F(\zeta) = \int_{\mathbb{R}^n} f(\zeta y)\overline{f(y)}\, dy = \int_{\mathbb{R}^n} \big(\sum_{k=0}^{\infty} P_kf(\zeta y)\big)\, \overline{f(y)}\, dy.$$

In view of the estimate (3.4) and the fact that $\varphi_k^{n-1}(2iy)$ is an increasing radial function we see that under the given assumption on f

$$|F(\zeta)| \leq C \sum_{k=0}^{\infty} \int_{\mathbb{R}^n} |f(y)| \, \|P_k f\|_2 \sqrt{\varphi_k^{n-1}(2iy)} \, dy < \infty.$$

Thus $F(\zeta)$ is holomorphic on the strip S and the rest of the proof goes as before.

In order to deduce Hardy's theorem from Theorem 1.7 we need to check if Beurling's condition is verified. Given $a \tanh b > 1$, we can choose $0 < b' < b$ such that $a \tanh b' > 1$ still holds. By applying Cauchy-Schwarz and using the assumption $\|P_k f\|_2 \leq C \, e^{-\frac{1}{2} b(2k+n)}$ we get

$$\sum_{k=0}^{\infty} \|P_k f\|_2 \left(\varphi_k^{n-1}(2iy)\right)^{1/2} \leq C_b \left(\sum_{k=0}^{\infty} e^{-b'(2k+n)} \varphi_k^{n-1}(2iy) \right)^{1/2}.$$

Since $\varphi_k^{n-1}(2iy) = \varphi_{k,1}^{n-1}(2iy, 0)$, the above sum is nothing but $p_{b'}^1(2iy, 0)$ which is known explicitly. Thus we see that

$$\sum_{k=0}^{\infty} \|P_k f\|_2 \left(\varphi_k^{n-1}(2iy)\right)^{1/2} \leq C_b' \, e^{\frac{1}{2}(\coth b')|y|^2}.$$

Consequently, as $|f(y)| \leq c \, e^{-\frac{1}{2} a |y|^2}$ and $a \tanh b' > 1$, we obtain

$$\sum_{k=0}^{\infty} \int_{\mathbb{R}^n} |f(y)| \, \|P_k f\|_2 \sqrt{\varphi_k^{n-1}(2iy)} \, dy \leq C \int_{\mathbb{R}^n} e^{-\frac{1}{2}(a - \coth b')|y|^2} \, dy < \infty.$$

Thus Beurling's condition is verified and Hardy's theorem is proved.

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