

(Anti-)symmetrizing wave functions

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ABSTRACT

The construction of fully (anti-)symmetric states with many particles, when the single particle state carries multiple quantum numbers, is a problem that seems to have not been systematically addressed in the literature. A quintessential example is the construction of ground state baryon wave functions where the color singlet condition reduces the problem to just two (flavor and spin) quantum numbers. In this paper, we address the general problem by noting that it can be re-interpreted as an eigenvalue equation and provide a formalism that applies to the generic number of particles and the generic number of quantum numbers. As an immediate result, we find a complete solution to the two quantum number case, from which the baryon wave function problem with an arbitrary number of flavors follows. As a more elaborate illustration that reveals complications not visible in the two quantum number case, we present the complete class of states possible for a system of five fermionic particles with three quantum numbers each. Our formalism makes systematic use of properties of the symmetric group and Young tableaux. Even though our motivations to consider this question have their roots in Sachdev-Ye-Kitaev-like tensor models and holography, the problem and its solution should have broader applications.

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I. THE PROBLEM

The goal of this paper is to provide a systematic procedure for constructing all possible states made of a fixed number of fermionic¹³ tensors of the form $\psi^{i \cdots j}$. The various indices on these tensors can be thought of as proxies for various quantum numbers that these fermions carry, and the number of tensors can be viewed as the number of particles. Due to the fermionic nature of ψ 's, only the representations that are antisymmetric under the exchange of any two fermions will arise. In other words, the problem we wish to solve is closely related to the question of finding the fully anti-symmetric multi-particle representations of the group $G_i \times \cdots \times G_j$, where each index (i) of the tensor transforms in the fundamental of the corresponding group (G_i). We will think of the groups G_i as $U(N)$ with possibly distinct N 's in each slot, but our strategy should be adaptable to arbitrary groups with minor modifications.

We will explicitly find multi-particle states for fermions carrying two and three quantum numbers as an illustration of our approach. The two quantum number case has some extra simplifications. As a more elaborate illustration of our technology, we will also present explicit results for the cases with four and five particles, each carrying three quantum numbers. It is conceptually straightforward, but possibly computationally challenging,^{1,14} to extend it to a higher number of particles and quantum numbers per particle. But we will formulate the problem as an eigenvalue problem, so we emphasize that, in principle, it is tractable in full generality. However, our aim in Secs. II–IV will be to find nice results at low levels and a small number of quantum numbers.

Throughout this paper, we will mostly deal with fermionic systems for concreteness,¹⁵ but we will present one bosonic case. This will be the bosonic case with two quantum numbers, and it is closely related to the problem of the construction of ground state wave functions for baryons: ground state means that we take the two independent orbital angular momenta in the 3-quark system to be vanishing ($\ell = \ell' = 0$). This is a problem well-known from introductory particle physics courses, but let us quickly review it here for completeness. The relevant quantum numbers in the $\ell = \ell' = 0$ state are color, flavor, and spin, and because we expect baryons to be color singlets, the problem effectively reduces to a two quantum number problem. Since an $SU(3)$ color singlet made from three fundamentals is fully anti-symmetric, the problem reduces to the construction of states with two quantum numbers (flavor and spin) that are symmetric under the interchange of any two particles. Therefore the bosonic

two quantum number case that we will write down using our approach subsumes the solution to the baryon wave function problem.¹⁶

Our motivation for considering this problem arose from investigations of certain classes of quantum mechanical tensor models where the symmetry group above arises as a global or gauged symmetry.^{3,4} It was noticed in Refs. 5–8 that for low values of the rank of the group, these models can potentially be solved at least on a computer. Our discussions in this paper are directly relevant to solving the ungauged models following the approach of Ref. 7, but we will not further discuss this application in this paper and merely restrict our attention to the mathematical problem. See also some discussions in the gauged theory, which use loosely similar group theory techniques.^{9,10}

Let us consider the 3-index fermions of the form ψ^{ijk} . The indices $\{i, j, k\}$ can be taken to belong to the group $SU(n_1)_i \times SU(n_2)_j \times SU(n_3)_k$ and therefore take values from 1 to $n_{1,2,3}$. More precisely, ψ^{ijk} transform under the vector representation of each of $SU(n_i)$, i.e.,

$$\psi^{ijk} \rightarrow M_1^{ii'} M_2^{jj'} M_3^{kk'} \psi^{i'j'k'}, \tag{1}$$

where M_1, M_2, M_3 belong to the three $SU(n_i)$'s, respectively. A general state involving n fermions is of the form

$$\psi^{i_1 j_1 k_1} \psi^{i_2 j_2 k_2}, \dots, \psi^{i_n j_n k_n}. \tag{2}$$

This state is antisymmetric under exchange of any two fermions, i.e.,

$$\psi^{i_1 j_1 k_1}, \dots, \psi^{i_a j_a k_a}, \dots, \psi^{i_n j_n k_n} = -\psi^{i_a j_a k_a}, \dots, \psi^{i_1 j_1 k_1}, \dots, \psi^{i_n j_n k_n}. \tag{3}$$

The states can be organized in terms of irreducible representations of $SU(n_1)_i \times SU(n_2)_j \times SU(n_3)_k$. Because of the fermionic nature, some of the representations become trivially zero. Our goal is to find a systematic way to find all the non-trivial representations that the fermionic states fall into.

This question is most easily answered in terms of Young tableaux. In the language of Young tableaux, a general state at level n can be written as

$$\left(\boxed{i_1} \otimes \boxed{i_2} \otimes \dots \otimes \boxed{i_n}, \boxed{j_1} \otimes \boxed{j_2} \otimes \dots \otimes \boxed{j_n}, \boxed{k_1} \otimes \boxed{k_2} \otimes \dots \otimes \boxed{k_n} \right)$$

The number of quantum numbers becomes the number of slots, and the number of particles (which we will sometimes call the level) becomes the number of boxes in each slot. The representation content of each slot can be figured out by decomposing the tensor products into various irreducible representations via Littlewood–Richardson rules. The question we want to answer is what are the irreducible representations that survive in the full object after we impose anti-symmetry under exchange of particles.

We will answer this question by working with permutation groups $S_n^{(i)} \times S_n^{(j)} \times S_n^{(k)}$ (where S_n stands for the permutation group with n elements) instead of $G_i \times G_j \times G_k$. If we wish to work with a specific group, we can impose further constraints on the allowed representations (also known as Young tableaux) that show up. Let us illustrate this with a simple example: let us consider as case where we are working with $U(3)$ groups but looking at levels ≥ 4 . In this case, because there are not enough indices to soak up all the slots in the tableaux, for example, some of the representations will be zero. So the general problem we solve together with the specific restrictions on Young tableaux that arise for the specific group will be the complete solution of our problem for that group. In Sec. IV, we will use an argument based on the group $SU(n)$, as a useful sanity check of our results. We have collected some useful facts about the symmetric group and its representations in Appendixes.

Let R_i, R_j , and R_k denote the irreducible representations (as can be captured by Young patterns) of the corresponding permutation groups. Then their tensor product $R_i \times R_j \times R_k$ are irreducible representations under $S_n^{(i)} \times S_n^{(j)} \times S_n^{(k)}$. We need the irreducible representations of $S_n^{(i)} \times S_n^{(j)} \times S_n^{(k)}$ such that they are antisymmetric under exchange of any two objects. More operationally, the required irreducible representations need to satisfy the following equation:¹⁷

$$\begin{aligned} D(g)_{R_i} \otimes D(g)_{R_j} \otimes D(g)_{R_k} \left[\sum_{i,j,k} \alpha_{ijk}^{(R_i, R_j, R_k)} |i\rangle_{R_i} \otimes |j\rangle_{R_j} \otimes |k\rangle_{R_k} \right] \\ = \pm \sum_{i,j,k} \alpha_{ijk}^{(R_i, R_j, R_k)} |i\rangle_{R_i} \otimes |j\rangle_{R_j} \otimes |k\rangle_{R_k}. \end{aligned} \tag{4}$$

This is our main equation, and by writing this equation, we have translated our problem into an eigen-problem. Here g is one of the transpositions¹⁸ (2-cycles) of the form $(i, i + 1)$ for $i = 1, \dots, (n - 1)$. $D(g)_{R_i}, D(g)_{R_j}$, and $D(g)_{R_k}$ are the matrix forms of g in the representations R_i, R_j , and R_k , respectively. $|i\rangle_{R_i}, |j\rangle_{R_j}$, and $|k\rangle_{R_k}$ denote the standard Young tableaux of the representations R_i, R_j , and R_k , respectively, and the summation is taken over all the standard Young tableaux. We have written the equation for the three slot/index case, but it should be clear that this equation straightforwardly generalizes to more indices.

The claim is that solving the above equation for α 's will accomplish the solution to the problem that we stated in the beginning of this section. Note that once formulated in this manner in the language of symmetric groups and its representations, we have reduced the problem to a fully tractable question with an algorithmic solution. With this, in principle, now the problem can be placed on a computer. In the rest of the paper, we move on to some comments about solving Eq. (4) using two different methods. For fermions and bosons carrying only two indices, we are able to find a simple solution to the problem. For a higher number of indices, we did not find such a simple approach, but nonetheless we list the classes of states in the antisymmetric case up to level $n = 5$ for the three index case. By direct counting, we have verified that the states add up to the expected result for the total number of anti-symmetric states.

Before discussing our formalism and results further, let us briefly compare our methods with the ones previously studied in the literature (see Chap. 7 of Ref. 12, for example. Also see the first two references in Ref. 11). To describe their approach, let us first label $i_a j_a k_a \equiv I_a$. Then, we see that any fermionic state of the form $\psi^{i_1 j_1 k_1}, \dots, \psi^{i_n j_n k_n} \equiv \psi^1, \dots, \psi^n$ is in the completely antisymmetric representation $[1^n]$ of $S_{n_1 n_2 n_3}$. Then the antisymmetric states of $S_n^{(i)} \times S_n^{(j)} \times S_n^{(k)}$ can be obtained by finding the irreducible representations a, b, c , respectively, of $S_n^{(i)}, S_n^{(j)}, S_n^{(k)}$ such that the inner product of the irreducible representations a, b, c contains the completely antisymmetric representation. This method involves two steps:

- The Young patterns along with their multiplicities should be identified by using the characters of the irreducible representations under consideration. This step is straightforward though becomes tedious with increasing order of the permutation group.
- After that, we need to determine the Clebsch-Gordon coefficients for the inner/Kronecker product of irreducible representations of the permutation group. Even though there exists some recursion formulas to find these CG coefficients (see the first reference in Ref. 11, for an example), it is not clear to us whether such a general formula is known. In any event, even if it exists, finding these CG coefficients explicitly is, in general, tedious.

The method we discuss in this paper reformulates the question as an eigenvalue problem and aims to get the complete result in a single shot. Hence, as a byproduct of our formalism to find antisymmetric states, we can read off the CG coefficients of the inner product of some of the irreducible representations of S_n . But it should be mentioned that for higher levels, our approach also will choke in practice due to the large Young tableaux involved; this seems to be an insurmountable problem because of the factorials involved.

II. TWO SLOTS

Let us start by treating Eq. (4) as a set of linear equations and we solve them sequentially starting from $g_1 = (12)$ until $g_{n-1} = (n-1, n)$. The number of linear equations are $d_{R_i} d_{R_j} d_{R_k}$, where d_{R_i}, d_{R_j} , and d_{R_k} are the dimensions of R_i, R_j , and R_k irreducible representations respectively. We work with Young-Yamanouchi orthonormal basis¹⁹ in the rest of the section.

Before going to the general case, we will attack a simpler problem of finding antisymmetric states of $S_n \otimes S_n$. This corresponds to the case with two quantum numbers. As we show below, we can find a simple solution for this two-index case. But the strategy we employ here takes advantage of specific features limited to this particular case.

A. Fermions

The equation that gives us the antisymmetric states of $S_n \times S_n$ is

$$D(g)_a \otimes D(g)_b \left[\sum_{ij} \alpha_{ij}^{(a,b)} |i\rangle_a \otimes |j\rangle_b \right] = - \sum_{ij} \alpha_{ij}^{(a,b)} |i\rangle_a \otimes |j\rangle_b, \quad (5)$$

where a and b label the representations of the first and second S_n 's, respectively. g_i is one of the transpositions (2-cycles) of the form $(i, i+1)$ for $i = 1, \dots, (n-1)$. $D(g)_a$ and $D(g)_b$ are the matrix forms of g in the representations a and b of S_n , respectively.

Now, we take an inner product with some specific basis state of the form $|i'\rangle_a \otimes |j'\rangle_b$ to obtain the following:

$$\sum_{ij} \alpha_{ij}^{(a,b)} {}_a \langle i' | D(g)_a | i \rangle_a {}_b \langle j' | D(g)_b | j \rangle_b = - \alpha_{i'j'}^{(a,b)}. \quad (6)$$

The action of $D(g)$ on the states $|i'\rangle$ and $|j'\rangle$ is as follows:

$$\begin{aligned} D(g)_a |i'\rangle_a &= -p_a^{i'}(g) |i'\rangle_a + \sqrt{1 - (p_a^{i'}(g))^2} |i''\rangle_a, \\ D(g)_a |i''\rangle_a &= +\sqrt{1 - (p_a^{i'}(g))^2} |i'\rangle_a + p_a^{i'}(g) |i''\rangle_a, \end{aligned} \quad (7)$$

where $|i''\rangle_a$ is another standard Young tableau that is obtained by exchanging i and $(i + 1)$ in $|i'\rangle_a$ when we are working with $g = (i, i + 1)$. Here $p_a^{i'}(g)$ is the axial distance²⁰ between i and $(i + 1)$ when $g = (i, i + 1)$. The basic point about the above equations is that for a given 2-cycle g they mix only two of the states $|i'\rangle_a$ and $|i''\rangle_a$. Similarly, we have

$$\begin{aligned} D(g)_b |j'\rangle_a &= -p_b^{j'}(g) |j'\rangle_b + \sqrt{1 - (p_b^{j'}(g))^2} |j''\rangle_b, \\ D(g)_b |j''\rangle_a &= +\sqrt{1 - (p_b^{j'}(g))^2} |j'\rangle_b + p_b^{j'}(g) |j''\rangle_b. \end{aligned} \tag{8}$$

Substituting these expressions in (6) and noting that $|i\rangle$'s form an orthonormal basis, we get

$$\begin{aligned} p_a^{i'}(g)p_b^{j'}(g) \alpha_{i'j'}^{(a,b)} - \sqrt{1 - (p_a^{i'}(g))^2} p_b^{j'}(g) \alpha_{i'j''}^{(a,b)} - p_a^{i'}(g)\sqrt{1 - (p_b^{j'}(g))^2} \alpha_{i'j''}^{(a,b)} \\ + \sqrt{1 - (p_a^{i'}(g))^2} \sqrt{1 - (p_b^{j'}(g))^2} \alpha_{i''j''}^{(a,b)} = -\alpha_{i'j'}^{(a,b)}. \end{aligned} \tag{9}$$

Taking an inner product with $|i''\rangle_a \otimes |j'\rangle_b$, we get

$$\begin{aligned} -\sqrt{1 - (p_a^{i'}(g))^2} p_b^{j'}(g) \alpha_{i'j'}^{(a,b)} - p_a^{i'}(g)p_b^{j'}(g) \alpha_{i'j''}^{(a,b)} \\ + \sqrt{1 - (p_a^{i'}(g))^2} \sqrt{1 - (p_b^{j'}(g))^2} \alpha_{i'j''}^{(a,b)} + p_a^{i'}(g)\sqrt{1 - (p_b^{j'}(g))^2} \alpha_{i''j''}^{(a,b)} = -\alpha_{i'j'}^{(a,b)}. \end{aligned} \tag{10}$$

Taking an inner product with $|i'\rangle_a \otimes |j''\rangle_b$, we get

$$\begin{aligned} -p_a^{i'}(g)\sqrt{1 - (p_b^{j'}(g))^2} \alpha_{i'j'}^{(a,b)} + \sqrt{1 - (p_a^{i'}(g))^2} \sqrt{1 - (p_b^{j'}(g))^2} \alpha_{i'j''}^{(a,b)} \\ - p_a^{i'}(g)p_b^{j'}(g) \alpha_{i'j''}^{(a,b)} + \sqrt{1 - (p_a^{i'}(g))^2} p_b^{j'}(g) \alpha_{i''j''}^{(a,b)} = -\alpha_{i'j''}^{(a,b)}. \end{aligned} \tag{11}$$

Taking an inner product with $|i''\rangle_a \otimes |j''\rangle_b$, we get

$$\begin{aligned} \sqrt{1 - (p_a^{i'}(g))^2} \sqrt{1 - (p_b^{j'}(g))^2} \alpha_{i'j'}^{(a,b)} + p_a^{i'}(g)\sqrt{1 - (p_b^{j'}(g))^2} \alpha_{i'j''}^{(a,b)} \\ + \sqrt{1 - (p_a^{i'}(g))^2} p_b^{j'}(g) \alpha_{i'j''}^{(a,b)} + p_a^{i'}(g)p_b^{j'}(g) \alpha_{i''j''}^{(a,b)} = -\alpha_{i''j''}^{(a,b)}. \end{aligned} \tag{12}$$

Our goal is to find constraints between $p_a^{i'}(g)$ and $p_b^{j'}(g)$ so that the above four equations have a non-trivial solution for α 's. Before proceeding further, we recall that the α 's are independent of the 2-cycle g .

By solving (6), we obtain α 's in terms of p_a and p_b . But our aim is to constrain p_a and p_b themselves using Eq. (6). We can possibly obtain such constraints by demanding the existence of non-trivial solutions to Eqs. (9)–(12). If we write Eqs. (9)–(12) as $Ax = 0$ schematically, then this is the same as demanding that the determinant of A is zero. But it can be checked that the determinant of A is trivially zero if all the α 's are nonzero. So it seems that we cannot constrain p_a and p_b .

But, we now argue that we can indeed constrain $p_a^{(g)}$ and $p_b^{(g)}$. For any given g_i , we claim that only two of the four α 's that occur in Eqs. (9)–(12) are non-zero because of the constraints imposed by g_1, \dots, g_{i-1} . Before giving evidence to support the claim, we discuss its implications. Once we accept the claim, by demanding that the α 's have a non-trivial solution, we get²¹ $p_a^{(g)} = \pm p_b^{(g)}$. This condition translates to the statement that the distance between i and $(i + 1)$ in the Young tableaux in the first slot is equal to positive/negative²² of the distance between i and $(i + 1)$ in the second Young tableaux. Equivalently, we start by filling 1 and 2 and then pick a spot for 3 in the first tableaux and then the position of 3 in the second tableaux is fixed by the above condition of distances. We continue this process to obtain the entire tableaux in the second slot corresponding to a tableau in the first slot, and this solves the problem.

But this argument depends on the uniqueness of the second tableaux for a given tableau in the first slot. We will not prove this statement, but we have checked that it is true for the first six levels, and we present some of the details below. We believe this is true generally.

Suppose that we have filled the Young tableaux in both the slots from 1 to i such that it is a part of the antisymmetric state. We now want to fill the $(i + 1)$ in both the tableaux so that it forms a part of antisymmetric state. If we fix the position of $(i + 1)$ in the first tableaux, then we have (at most) four states that are obtained by permuting the i and $(i + 1)$ indices. We have represented these four states in Sec. II as

$$|i'\rangle \otimes |j'\rangle; \quad |i''\rangle \otimes |j'\rangle; \quad |i'\rangle \otimes |j''\rangle; \quad |i''\rangle \otimes |j''\rangle \tag{13}$$

Note that while permuting i and $(i + 1)$, we do change the distance between $(i - 1)$ and i . As a result, for some fixed positions of $(i - 1)$ and i in the first Young tableaux, we have two different sets of positions of $(i - 1)$ and i in the second Young tableaux. But below, we will see that only one of them in fact appears at low levels.

Let us start with $g_1 = (12)$. In this case, there are only two antisymmetric states at level 2 and they are given by

$$\left(\begin{array}{|c|c|} \hline 1 & 2 \\ \hline \end{array}, \begin{array}{|c|} \hline 1 \\ \hline 2 \\ \hline \end{array} \right); \left(\begin{array}{|c|} \hline 1 \\ \hline 2 \\ \hline \end{array}, \begin{array}{|c|c|} \hline 1 & 2 \\ \hline \end{array} \right)$$

Now we move on to the Young tableaux involving three boxes. It is easy to check that $p_a = \pm 1$; $p_b = \pm 1$ satisfy Eqs. (9)–(12). That is, following are antisymmetric states:

$$\left(\begin{array}{|c|c|c|} \hline 1 & 2 & 3 \\ \hline \end{array}, \begin{array}{|c|} \hline 1 \\ \hline 2 \\ \hline 3 \\ \hline \end{array} \right); \left(\begin{array}{|c|c|c|} \hline 1 & 2 & 3 \\ \hline \end{array}, \begin{array}{|c|} \hline 1 \\ \hline 2 \\ \hline 3 \\ \hline \end{array} \right)$$

Consider the following states corresponding to the mixed symmetry Young tableau:

$$\left(\begin{array}{|c|c|} \hline 1 & 3 \\ \hline 2 & \end{array}, \begin{array}{|c|c|} \hline 1 & 3 \\ \hline 2 & \end{array} \right); \left(\begin{array}{|c|c|} \hline 1 & 2 \\ \hline 3 & \end{array}, \begin{array}{|c|c|} \hline 1 & 2 \\ \hline 3 & \end{array} \right); \left(\begin{array}{|c|c|} \hline 1 & 3 \\ \hline 2 & \end{array}, \begin{array}{|c|c|} \hline 1 & 2 \\ \hline 3 & \end{array} \right); \left(\begin{array}{|c|c|} \hline 1 & 2 \\ \hline 3 & \end{array}, \begin{array}{|c|c|} \hline 1 & 3 \\ \hline 2 & \end{array} \right)$$

We can choose i' and j' to be mixed symmetric tableaux with 1 and 2 in the same row. Then from $g_1 = (12)$, we see that $\alpha_{i'j'} = 0 = \alpha_{i'j}$. So only two α 's survive as expected. The antisymmetric state under (12) and (23) is

$$\left(\begin{array}{|c|c|} \hline 1 & 3 \\ \hline 2 & \end{array}, \begin{array}{|c|c|} \hline 1 & 2 \\ \hline 3 & \end{array} \right) - \left(\begin{array}{|c|c|} \hline 1 & 2 \\ \hline 3 & \end{array}, \begin{array}{|c|c|} \hline 1 & 3 \\ \hline 2 & \end{array} \right)$$

We can see that the Young tableaux in the second slot are unique with respect to the Young tableaux in the first slot.

As a level 4 example, consider the following set of states:

$$\left(\begin{array}{|c|c|c|} \hline 1 & 2 & 3 \\ \hline 4 & \end{array}, \begin{array}{|c|c|} \hline 1 & 4 \\ \hline 2 & \\ \hline 3 & \end{array} \right); \left(\begin{array}{|c|c|c|} \hline 1 & 2 & 4 \\ \hline 3 & \end{array}, \begin{array}{|c|c|} \hline 1 & 3 \\ \hline 2 & \\ \hline 4 & \end{array} \right); \\ \left(\begin{array}{|c|c|c|} \hline 1 & 2 & 4 \\ \hline 3 & \end{array}, \begin{array}{|c|c|} \hline 1 & 4 \\ \hline 2 & \\ \hline 3 & \end{array} \right); \left(\begin{array}{|c|c|c|} \hline 1 & 2 & 3 \\ \hline 4 & \end{array}, \begin{array}{|c|c|} \hline 1 & 3 \\ \hline 2 & \\ \hline 4 & \end{array} \right)$$

We choose i' and j' to be the Young tableaux such that 1, 2, and 3 are in the same row in i' and in the same column in j' . From $g_2 = (23)$, we can see that only $\alpha_{i'j'}$ and $\alpha_{i'j}$ are non-zero. The antisymmetric state under (34) is

$$\left(\begin{array}{|c|c|c|} \hline 1 & 2 & 3 \\ \hline 4 & \end{array}, \begin{array}{|c|c|} \hline 1 & 4 \\ \hline 2 & \\ \hline 3 & \end{array} \right) - \left(\begin{array}{|c|c|c|} \hline 1 & 2 & 4 \\ \hline 3 & \end{array}, \begin{array}{|c|c|} \hline 1 & 3 \\ \hline 2 & \\ \hline 4 & \end{array} \right)$$

For it to be a antisymmetric state, we need to add one more term following the level-3 antisymmetric states and it is given by

$$\left(\begin{array}{|c|c|c|} \hline 1 & 2 & 3 \\ \hline 4 & & \\ \hline \end{array}, \begin{array}{|c|c|} \hline 1 & 4 \\ \hline 2 & \\ \hline 3 & \\ \hline \end{array} \right) - \left(\begin{array}{|c|c|c|} \hline 1 & 2 & 4 \\ \hline 3 & & \\ \hline \end{array}, \begin{array}{|c|c|} \hline 1 & 3 \\ \hline 2 & \\ \hline 4 & \\ \hline \end{array} \right) + \left(\begin{array}{|c|c|c|} \hline 1 & 3 & 4 \\ \hline 2 & & \\ \hline \end{array}, \begin{array}{|c|c|} \hline 1 & 2 \\ \hline 3 & \\ \hline 4 & \\ \hline \end{array} \right)$$

Here again, the Young tableaux in the second slot are unique with respect to the Young tableaux in the first slot. We have checked this uniqueness explicitly for Young tableaux with up to six boxes (i.e., level 6), and we expect it to be true in general. Also, the solution obtained here passes the counting check we have described in Sec. IV.

B. Bosons

As a simple corollary of our approach, we can construct baryonic wave functions. As explained in the introduction, this requires us to consider the bosonic case. In this subsection, we make a digression to do so. The results are quite parallel to the two-slot fermionic case. We present a solution to a generalized version of this problem, i.e., we find the fully symmetric representations of the group $G_i \times G_j$, where we take $G_{i,j}$ to be $U(n_{i,j})$ for concreteness. More operationally, we solve our main Eq. (4) for the two index case with a + sign on the RHS. In the rest of the section, we work with the Young-Yamanouchi representation.

The equation that gives us the symmetric states of $S_n \times S_n$ is given by

$$D(g)_a \otimes D(g)_b \left[\sum_{i,j} \alpha_{ij}^{(a,b)} |i\rangle_a \otimes |j\rangle_b \right] = + \sum_{i,j} \alpha_{ij}^{(a,b)} |i\rangle_a \otimes |j\rangle_b, \tag{14}$$

where a and b are certain representations of the first and second S_n 's, respectively. g is one of the transpositions (2-cycles) of the form $(i, i + 1)$ for $i = 1, \dots, (n - 1)$. $D(g)_a$ and $D(g)_b$ are the matrix forms of g in the representations a and b of S_n , respectively.

We now take an inner product on both sides of Eq. (14) with a specific basis state $|i'\rangle_a \otimes |j'\rangle_b$ to obtain the following:

$$p_a^{i'}(g)p_b^{j'}(g) \alpha_{i'j'}^{(a,b)} - \sqrt{1 - (p_a^{i'}(g))^2} p_b^{j'}(g) \alpha_{i''j'}^{(a,b)} - p_a^{i'}(g) \sqrt{1 - (p_b^{j'}(g))^2} \alpha_{i'j''}^{(a,b)} + \sqrt{1 - (p_a^{i'}(g))^2} \sqrt{1 - (p_b^{j'}(g))^2} \alpha_{i''j''}^{(a,b)} = + \alpha_{i'j'}^{(a,b)}, \tag{15}$$

where $|i''\rangle_a$ is another standard Young tableau that is obtained by exchanging i and $(i + 1)$ in $|i'\rangle_a$ when we are working with $g = (i, i + 1)$. $p_a^{i'}(g)$ is the axial distance between i and $(i + 1)$.

Taking inner products with $|i''\rangle_a \otimes |j'\rangle_b$, $|i'\rangle_a \otimes |j''\rangle_b$, and $|i''\rangle_a \otimes |j''\rangle_b$ gives the following equations:

$$-\sqrt{1 - (p_a^{i'}(g))^2} p_b^{j'}(g) \alpha_{i'j''}^{(a,b)} - p_a^{i'}(g) p_b^{j'}(g) \alpha_{i''j''}^{(a,b)} + \sqrt{1 - (p_a^{i'}(g))^2} \sqrt{1 - (p_b^{j'}(g))^2} \alpha_{i''j''}^{(a,b)} + p_a^{i'}(g) \sqrt{1 - (p_b^{j'}(g))^2} \alpha_{i''j''}^{(a,b)} = + \alpha_{i''j''}^{(a,b)}, \tag{16}$$

$$-p_a^{i'}(g) \sqrt{1 - (p_b^{j'}(g))^2} \alpha_{i'j''}^{(a,b)} + \sqrt{1 - (p_a^{i'}(g))^2} \sqrt{1 - (p_b^{j'}(g))^2} \alpha_{i''j''}^{(a,b)} - p_a^{i'}(g) p_b^{j'}(g) \alpha_{i''j''}^{(a,b)} + \sqrt{1 - (p_a^{i'}(g))^2} p_b^{j'}(g) \alpha_{i''j''}^{(a,b)} = + \alpha_{i''j''}^{(a,b)}, \tag{17}$$

$$\sqrt{1 - (p_a^{i'}(g))^2} \sqrt{1 - (p_b^{j'}(g))^2} \alpha_{i''j''}^{(a,b)} + p_a^{i'}(g) \sqrt{1 - (p_b^{j'}(g))^2} \alpha_{i''j''}^{(a,b)} + \sqrt{1 - (p_a^{i'}(g))^2} p_b^{j'}(g) \alpha_{i''j''}^{(a,b)} + p_a^{i'}(g) p_b^{j'}(g) \alpha_{i''j''}^{(a,b)} = + \alpha_{i''j''}^{(a,b)}. \tag{18}$$

As in the case of anti-symmetrizing states, we claim that the Young tableaux in the second slot are unique for a given Young tableaux in the first slot. This claim about uniqueness implies that we need²³ $p_a^{i'}(g) = \pm p_b^{j'}(g)$ so that Eqs. (15)–(18) have a non-trivial solution.

Now, we give examples to support the above claim. At level 2, we have only two symmetric representations and are given by

$$\left(\begin{array}{|c|c|} \hline 1 & 2 \\ \hline \end{array}, \begin{array}{|c|c|} \hline 1 & 2 \\ \hline \end{array} \right); \left(\begin{array}{|c|} \hline 1 \\ \hline 2 \\ \hline \end{array}, \begin{array}{|c|} \hline 1 \\ \hline 2 \\ \hline \end{array} \right)$$

Moving on to level 3, we have the following symmetric representations:

$$\left(\begin{array}{|c|c|c|} \hline 1 & 2 & 3 \\ \hline \end{array}, \begin{array}{|c|c|c|} \hline 1 & 2 & 3 \\ \hline \end{array} \right); \left(\begin{array}{|c|} \hline 1 \\ \hline 2 \\ \hline 3 \\ \hline \end{array}, \begin{array}{|c|} \hline 1 \\ \hline 2 \\ \hline 3 \\ \hline \end{array} \right); \left(\begin{array}{|c|c|} \hline 1 & 3 \\ \hline 2 & \\ \hline \end{array}, \begin{array}{|c|c|} \hline 1 & 3 \\ \hline 2 & \\ \hline \end{array} \right) + \left(\begin{array}{|c|c|} \hline 1 & 2 \\ \hline 3 & \\ \hline \end{array}, \begin{array}{|c|c|} \hline 1 & 2 \\ \hline 3 & \\ \hline \end{array} \right)$$

The discussion is parallel to the anti-symmetric case, so we will not belabor it. We have checked this uniqueness up to level-6, and we expect it to work at an arbitrary level. Also, the symmetric representations we obtain here pass the counting check that we describe in Sec. IV.

The results we find here, when interpreted as flavor and spin quantum numbers, provide the solution to the baryon wave function problem. The solution to this problem for the case of three (u, d, s) flavors can be found in Ref. 2, for example.

III. THREE SLOTS

In the case with three kinds of quantum numbers, we will stick to the fermionic case. The bosonic case is analogous, but since there is no immediate physical application that we have in mind (unlike the baryon wave functions in the two slot case), we will not spell it out explicitly.

The equation that we intend to solve to obtain the antisymmetric states in 3-index case is given by

$$D(g)_a \otimes D(g)_b \otimes D(g)_c \left[\sum_{i,j,k} \alpha_{ijk}^{(a,b,c)} |i\rangle_a \otimes |j\rangle_b \otimes |k\rangle_c \right] = - \sum_{i,j,k} \alpha_{ijk}^{(a,b,c)} |i\rangle_a \otimes |j\rangle_b \otimes |k\rangle_c. \tag{19}$$

Taking an inner product with a certain basis state $|i'\rangle_a \otimes |j'\rangle_b \otimes |k'\rangle_c$, we get

$$\sum_{i,j,k} \alpha_{ijk}^{(a,b,c)} \langle i' | D(g)_a | i \rangle_a \langle j' | D(g)_b | j \rangle_b \langle k' | D(g)_c | k \rangle_c = - \alpha_{i'j'k'}^{(a,b,c)}, \tag{20}$$

As in Sec. II, we take

$$\begin{aligned} D(g)_a |i'\rangle_a &= -p_a^{i'}(g) |i'\rangle_a + \sqrt{1 - (p_a^{i'}(g))^2} |i''\rangle_a, \\ D(g)_a |i''\rangle_a &= +\sqrt{1 - (p_a^{i'}(g))^2} |i'\rangle_a + p_a^{i'}(g) |i''\rangle_a, \end{aligned} \tag{21}$$

$$\begin{aligned} D(g)_b |j'\rangle_b &= -p_b^{j'}(g) |j'\rangle_b + \sqrt{1 - (p_b^{j'}(g))^2} |j''\rangle_b, \\ D(g)_b |j''\rangle_b &= +\sqrt{1 - (p_b^{j'}(g))^2} |j'\rangle_b + p_b^{j'}(g) |j''\rangle_b, \end{aligned} \tag{22}$$

$$\begin{aligned} D(g)_c |k'\rangle_c &= -p_c^{k'}(g) |k'\rangle_c + \sqrt{1 - (p_c^{k'}(g))^2} |k''\rangle_c, \\ D(g)_c |k''\rangle_c &= +\sqrt{1 - (p_c^{k'}(g))^2} |k'\rangle_c + p_c^{k'}(g) |k''\rangle_c, \end{aligned} \tag{23}$$

where $|i''\rangle_a$, $|j''\rangle_b$, and $|k''\rangle_c$ are the basis states of the a , b , and c irreducible representations respectively. Putting these expressions back into (20), we get

$$\begin{aligned}
 -\alpha_{i'j'k'} = & -p_a \left[\alpha_{i'j'k'} p_b p_c - \alpha_{i'j''k'} \sqrt{1-p_b^2} p_c \right. \\
 & \left. -\alpha_{i'j'k''} p_b \sqrt{1-p_c^2} + \alpha_{i'j''k''} \sqrt{1-p_b^2} \sqrt{1-p_c^2} \right] \\
 & + \sqrt{1-p_a^2} \left[\alpha_{i''j'k'} p_b p_c - \alpha_{i''j''k'} \sqrt{1-p_b^2} p_c \right. \\
 & \left. -\alpha_{i''j'k''} p_b \sqrt{1-p_c^2} + \alpha_{i''j''k''} \sqrt{1-p_b^2} \sqrt{1-p_c^2} \right], \tag{24}
 \end{aligned}$$

where we have dropped various subscripts and superscripts to avoid clutter of notation. In a similar way, we get seven more equations by taking an inner product with various states and those equations can be listed as follows:

$$\begin{aligned}
 -\alpha_{i''j'k'} = & \sqrt{1-p_a^2} \left[\alpha_{i'j'k'} p_b p_c - \alpha_{i'j''k'} \sqrt{1-p_b^2} p_c \right. \\
 & \left. -\alpha_{i'j'k''} p_b \sqrt{1-p_c^2} + \alpha_{i'j''k''} \sqrt{(1-p_b^2)(1-p_c^2)} \right] \\
 & + p_a \left[\alpha_{i''j'k'} p_b p_c - \alpha_{i''j''k'} \sqrt{1-p_b^2} p_c \right. \\
 & \left. -\alpha_{i''j'k''} p_b \sqrt{1-p_c^2} + \alpha_{i''j''k''} \sqrt{1-p_b^2} \sqrt{1-p_c^2} \right], \tag{25}
 \end{aligned}$$

$$\begin{aligned}
 -\alpha_{i'j''k'} = & -p_a \left[-\alpha_{i'j'k'} \sqrt{1-p_b^2} p_c - \alpha_{i'j''k'} p_b p_c \right. \\
 & \left. +\alpha_{i'j'k''} \sqrt{1-p_b^2} \sqrt{1-p_c^2} + \alpha_{i'j''k''} p_b \sqrt{1-p_c^2} \right] \\
 & + \sqrt{1-p_a^2} \left[-\alpha_{i''j'k'} \sqrt{1-p_b^2} p_c - \alpha_{i''j''k'} p_b p_c \right. \\
 & \left. +\alpha_{i''j'k''} \sqrt{1-p_b^2} \sqrt{1-p_c^2} + \alpha_{i''j''k''} p_b \sqrt{1-p_c^2} \right], \tag{26}
 \end{aligned}$$

$$\begin{aligned}
 -\alpha_{i'j'k''} = & \sqrt{1-p_a^2} \left[-\alpha_{i'j'k'} \sqrt{1-p_b^2} p_c - \alpha_{i'j''k'} p_b p_c \right. \\
 & \left. +\alpha_{i'j'k''} \sqrt{1-p_b^2} \sqrt{1-p_c^2} + \alpha_{i'j''k''} p_b \sqrt{1-p_c^2} \right] \\
 & + p_a \left[-\alpha_{i''j'k'} \sqrt{1-p_b^2} p_c - \alpha_{i''j''k'} p_b p_c \right. \\
 & \left. +\alpha_{i''j'k''} \sqrt{1-p_b^2} \sqrt{1-p_c^2} + \alpha_{i''j''k''} p_b \sqrt{1-p_c^2} \right], \tag{27}
 \end{aligned}$$

$$\begin{aligned}
 -\alpha_{i'j'k''} = & -p_a \left[-\alpha_{i'j'k'} p_b \sqrt{1-p_c^2} + \alpha_{i'j''k'} \sqrt{1-p_b^2} \sqrt{1-p_c^2} \right. \\
 & \left. -\alpha_{i'j'k''} p_b p_c + \alpha_{i'j''k''} \sqrt{1-p_b^2} p_c \right] \\
 & + \sqrt{1-p_a^2} \left[-\alpha_{i''j'k'} p_b \sqrt{1-p_c^2} + \alpha_{i''j''k'} \sqrt{1-p_b^2} \sqrt{1-p_c^2} \right. \\
 & \left. -\alpha_{i''j'k''} p_b p_c + \alpha_{i''j''k''} \sqrt{1-p_b^2} p_c \right], \tag{28}
 \end{aligned}$$

$$\begin{aligned}
 -\alpha_{i'j'k''} = & \sqrt{1-p_a^2} \left[-\alpha_{i'j'k'} p_b \sqrt{1-p_c^2} + \alpha_{i'j''k'} \sqrt{1-p_b^2} \sqrt{1-p_c^2} \right. \\
 & \left. -\alpha_{i'j'k''} p_b p_c + \alpha_{i'j''k''} \sqrt{1-p_b^2} p_c \right] \\
 & + p_a \left[-\alpha_{i'j'k'} p_b \sqrt{1-p_c^2} + \alpha_{i'j''k'} \sqrt{1-p_b^2} \sqrt{1-p_c^2} \right. \\
 & \left. -\alpha_{i'j'k''} p_b p_c + \alpha_{i'j''k''} \sqrt{1-p_b^2} p_c \right], \tag{29}
 \end{aligned}$$

$$\begin{aligned}
 -\alpha_{i'j''k''} = & -p_a \left[\alpha_{i'j'k'} \sqrt{1-p_b^2} \sqrt{1-p_c^2} + \alpha_{i'j''k'} p_b \sqrt{1-p_c^2} \right. \\
 & \left. + \alpha_{i'j'k''} \sqrt{1-p_b^2} p_c + \alpha_{i'j''k''} p_b p_c \right] \\
 & + \sqrt{1-p_a^2} \left[\alpha_{i'j'k'} \sqrt{1-p_b^2} \sqrt{1-p_c^2} + \alpha_{i'j''k'} p_b \sqrt{1-p_c^2} \right. \\
 & \left. + \alpha_{i'j'k''} \sqrt{1-p_b^2} p_c + \alpha_{i'j''k''} p_b p_c \right], \tag{30}
 \end{aligned}$$

$$\begin{aligned}
 -\alpha_{i'j''k''} = & \sqrt{1-p_a^2} \left[\alpha_{i'j'k'} \sqrt{1-p_b^2} \sqrt{1-p_c^2} + \alpha_{i'j''k'} p_b \sqrt{1-p_c^2} \right. \\
 & \left. + \alpha_{i'j'k''} \sqrt{1-p_b^2} p_c + \alpha_{i'j''k''} p_b p_c \right] \\
 & + p_a \left[\alpha_{i'j'k'} \sqrt{1-p_b^2} \sqrt{1-p_c^2} + \alpha_{i'j''k'} p_b \sqrt{1-p_c^2} \right. \\
 & \left. + \alpha_{i'j'k''} \sqrt{1-p_b^2} p_c + \alpha_{i'j''k''} p_b p_c \right]. \tag{31}
 \end{aligned}$$

Just as in the two slot case, one can check that if all the α 's are assumed to be non-vanishing, demanding nontrivial solutions to these equations via a determinant condition does not constrain p_a , p_b , and p_c . But unlike in the two slot case, we have not found a simple approach to setting certain α 's to zero that leads to a useful way to enumerate the solutions. At a practical level, this is because for given tableaux in the first two slots, the tableau in the third slot need not be unique. Of course, one can solve these equations by explicit calculations, and in Appendixes, we give (examples of) antisymmetric states at levels 2, 3, and 4.

But we can proceed further by approaching the problem from a different angle, and that is what we turn to next. This alternate approach gives a fairly simple way to find the form of the Young patterns that show up in the anti-symmetric states.²⁴ Note that all the specific statements we are making in this section and Sec. II are specific simplifications, we do not claim absolute generality with these methods (beyond the fact that the original equations themselves yield an eigenvalue problem which is obviously tractable with infinite computing power).

A. Auxiliary eigenvalue problems

In this section, we treat Eq. (4) as a set of $(n - 1)$ eigenvalue equations. Our goal is to find eigenvector(s) (corresponding to eigenvalue of -1) that is common to all the $(n - 1)$ matrices of the form $D^a(g_i) \otimes D^b(g_i) \otimes D^c(g_i)$. Here g_i is a group element of S_n and denotes a 2-cycle of the form $(i, i + 1)$, where i runs from 1 to $(n - 1)$. The superscripts a, b, c denote the particular irreducible representations of S_n that we are dealing with.

As in Sec. III, we work with Young-Yamanouchi orthonormal representation and each of the standard Young tableaux is given by column matrices of the form

$$\begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}; \begin{bmatrix} 0 \\ 1 \\ \vdots \\ 0 \end{bmatrix}; \dots \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 1 \end{bmatrix}.$$

Adding all these equations gives us

$$Ax \equiv (A_1 + A_2 + A_3 + \dots + A_{n-1})x = -(n - 1)x. \tag{36}$$

This means that (some of) the eigenvectors of this eigenvalue equation are the common eigenvectors x . Note that we need to solve only a single eigenvalue equation in this case.

One advantage of this method is that we have a simple way to get a bound on the number of common eigenvectors of the matrices A_i . We can do this as follows: Starting from (36), we note that a non-zero eigenvector can exist only if

$$\det(A + (n - 1)I) = 0. \tag{37}$$

If this determinant is not zero, then x has to be zero and thus there is no common eigenvector to the matrices A_i corresponding to the eigenvalue -1 . Also, the number of common eigenvectors to all A_i is less than or equal to the number of zero eigenvalues of the matrix $(A + (n - 1)I)$. In terms of rank of the matrix, the last statement implies the following:

$$\# \text{ of common eigenvectors} \leq \text{Order}(A + (n - 1)I) - \text{Rank}(A + (n - 1)I) \tag{38}$$

This is a useful relation when checking for common eigenvectors using Mathematica.

Are there any other diagnostics (i.e., the ones that do not involve calculating determinants or eigenvalues) to find whether the matrices A_i have common eigenvectors? There is another way which is the most efficient while using a computer. We start by observing that S_n is generated by only two elements: (12) and (12, . . . , n). Our goal now is to find a common eigenvector between $D^a(12) \otimes D^b(12) \otimes D^c(12)$ and $D^a(12, \dots, n) \otimes D^b(12, \dots, n) \otimes D^c(12, \dots, n)$ corresponding to eigenvalues -1 and $(-1)^{n-1}$, respectively. The technology developed above, of counting the rank and order, can be applied here as well. But instead of dealing with $(n - 1)$ matrices as in the previous case, we deal here with only two²⁶ matrices for any S_n . More precisely, the bound on the common eigenvectors is as follows:

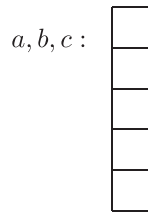
$$\begin{aligned} \# \text{ of common eigenvectors} \leq & \\ & \text{Order}(D^a(12) \otimes D^b(12) \otimes D^c(12) + (-1)^n D^a(12, \dots, n) \otimes D^b(12, \dots, n) \otimes D^c(12, \dots, n) + 2I_{d_a d_b d_c}) \\ & - \text{Rank}(D^a(12) \otimes D^b(12) \otimes D^c(12) + (-1)^n D^a(12, \dots, n) \otimes D^b(12, \dots, n) \otimes D^c(12, \dots, n) + 2I_{d_a d_b d_c}), \end{aligned} \tag{39}$$

where $d_{a,b,c}$ are dimensions of the irreducible representations a, b, c . Even though this method is strictly only an upper bound on the number of common eigenvectors, we found that at level 5 (which is the maximum level up to which we have done explicit calculations), the bounds are saturated. The allowed Young patterns²⁷ for levels 4 and 5 are presented in Appendixes. The results are quite non-trivial, especially for the level 5 case, and we do not believe they can be obtained without the formalism arising from our main Eq. (4) in Sec. I.

We conclude this section by giving a couple of examples on bounding the number of common eigenvectors at level 5. At level 5, we need to find common eigenvector(s) between $D^a(12) \otimes D^b(12) \otimes D^c(12)$ and $D^a(12\ 345) \otimes D^b(12\ 345) \otimes D^c(12\ 345)$ corresponding to eigenvalues -1 and 1 , respectively. Note that the matrices $D^r(12)$ (and more generally the matrices $D^r(i, i + 1)$) can be constructed easily for any representation $r \in S_5$ following Appendix B and the matrices $D^r(12\ 345)$ can be obtained as follows:

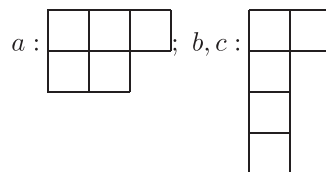
$$D^r(12\ 345) = D^r(12).D^r(23).D^r(34).D^r(45). \tag{40}$$

Let us start by choosing all of a, b, c to be the sign representation of S_5 , i.e.,



Sign representation is a one dimensional representation, and in this representation, we have $D^a(12) = D^b(12) = D^c(12) = -1$ and $D^a(12\ 345) = D^b(12\ 345) = D^c(12\ 345) = 1$. Hence, there may be a common eigenvector in this case and we find that all of a, b, c being the sign representation is an antisymmetric state with multiplicity 1.²⁸

Next, we move on to a slightly complicated example. Let us take



“a” is a 5 dimensional representation and “b, c” are 4 dimensional representations. Following Appendix B, we can write the following representation matrices:

$$\begin{aligned}
 D^a(12) &= \text{diag}(1, 1, 1, -1, -1), & D^{b,c}(12) &= \text{diag}(-1, -1, -1, 1) \\
 D^a(12\ 345) &= \begin{pmatrix} -\frac{1}{3} & -\frac{\sqrt{2}}{3} & \sqrt{\frac{2}{3}} & 0 & 0 \\ -\frac{\sqrt{2}}{3} & \frac{1}{12} & -\frac{1}{\sqrt{48}} & -\frac{\sqrt{3}}{4} & \frac{3}{4} \\ 0 & -\frac{\sqrt{3}}{4} & -\frac{1}{4} & -\frac{3}{4} & -\frac{\sqrt{3}}{4} \\ -\sqrt{\frac{2}{3}} & \frac{1}{\sqrt{48}} & -\frac{1}{4} & \frac{1}{4} & -\frac{\sqrt{3}}{4} \\ 0 & -\frac{3}{4} & -\frac{\sqrt{3}}{4} & \frac{\sqrt{3}}{4} & \frac{1}{4} \end{pmatrix}, & D^{b,c}(12\ 345) &= \begin{pmatrix} -\frac{1}{4} & -\frac{\sqrt{15}}{4} & 0 & 0 \\ \sqrt{\frac{5}{48}} & -\frac{1}{12} & -\frac{2\sqrt{2}}{3} & 0 \\ -\sqrt{\frac{5}{24}} & \frac{1}{\sqrt{72}} & -\frac{1}{6} & -\frac{\sqrt{3}}{2} \\ \sqrt{\frac{5}{8}} & -\frac{1}{\sqrt{24}} & \frac{1}{\sqrt{12}} & -\frac{1}{2} \end{pmatrix}.
 \end{aligned} \tag{41}$$

By explicit computation, we see that

$$\begin{aligned}
 &\text{Order}(D^a(12) \otimes D^b(12) \otimes D^c(12) - D^a(12\ 345) \otimes D^b(12\ 345) \otimes D^c(12\ 345) + 2I_{80}) \\
 &= \text{Rank}(D^a(12) \otimes D^b(12) \otimes D^c(12) - D^a(12\ 345) \otimes D^b(12\ 345) \otimes D^c(12\ 345) + 2I_{80}),
 \end{aligned} \tag{42}$$

and hence there are no common eigenvectors between $D^a(12) \otimes D^b(12) \otimes D^c(12)$ and $D^a(12\ 345) \otimes D^b(12\ 345) \otimes D^c(12\ 345)$ corresponding to eigenvalues -1 and 1 . Hence, this choice of a, b, c does not lead to an antisymmetric state. In a similar way, we can check for all the possible choices of a, b, c to find the possible Young patterns that can be antisymmetric states. The complete set of Young patterns that are antisymmetric at levels 4 and 5 is listed in the Appendixes.

A sanity check of our results is to compare the sum of the number states in all these representations together, with the total number of fully anti-symmetric states at that level, where we treat the quantum numbers as belonging to specific groups. We explain this in Sec. IV.

IV. COUNTING STATES AS A SANITY CHECK

In this section, we present a way to verify the antisymmetric states that we found are correct by counting the dimensions of the irreducible representations of the groups to which the quantum numbers belong. Consider the fermions of the form ψ^{ijk} carrying three quantum numbers corresponding to $SU(n_1)_i \times SU(n_2)_j \times SU(n_3)_k$. Denoting the indices $ijk \equiv I$, we can see that any state with multiple fermions is in completely antisymmetric representation of $SU(n_1 n_2 n_3)$, i.e., a state at level n is given by the following representation of $SU(n_1 n_2 n_3)$:

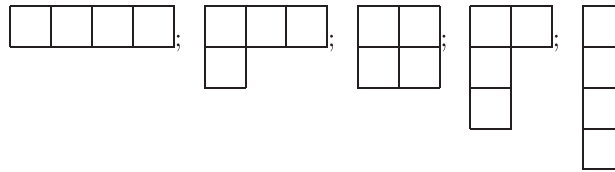
$$\begin{array}{|c|} \hline I_1 \\ \hline I_2 \\ \hline \vdots \\ \hline I_n \\ \hline \end{array}$$

The number of rows here is equal to the level that we are considering, namely, n . The dimension of this representation for $SU(n_1 n_2 n_3)$ is trivial to calculate using the hook rule (say). This dimension should be exactly equal to the sums of dimensions of various antisymmetric representations (also known as Young patterns) of $SU(n_1)_i \times SU(n_2)_j \times SU(n_3)_k$ that we find by solving our main Eq. (4); in particular, this should apply for the level 4 and level 5 cases we have listed in the Appendixes. This provides a non-trivial check, and all the representations we find for two and three-index cases do pass this check.

As an illustration, we present here how the counting works for the level 4 in the 3-slot case. For simplicity, we choose $n_1 = n_2 = n_3 = n$. That is, we explicitly show that the dimensions of the completely antisymmetric representation of $SU(n^3)$ at level 4 is equal to the $SU(n)_i \times SU(n)_j \times SU(n)_k$ representations listed in Appendix E. Using the hook rule, the dimension of completely antisymmetric representation of $SU(n^3)$ at level-4 is given by $\binom{n^3}{4}$. The sum of dimensions of the representations in the same order as in Appendix E are given as follows:

$$\begin{aligned}
 &6 \times d_{a_2} d_{a_1} d_{a_4} + 6 \times d_{a_2} d_{a_3} d_{a_4} + 3 \times d_{a_2} d_{a_2} d_{a_5} + 3 \times d_{a_4} d_{a_4} d_{a_5} + 3 \times d_{a_1} d_{a_1} d_{a_5} + 3 \times d_{a_3} d_{a_3} d_{a_5} + 3 \times d_{a_3} d_{a_3} d_{a_1} \\
 &+ 3 \times d_{a_3} d_{a_2} d_{a_2} + 3 \times d_{a_3} d_{a_4} d_{a_4} + 3 \times d_{a_2} d_{a_2} d_{a_4} + 3 \times d_{a_4} d_{a_4} d_{a_2} + d_{a_2} d_{a_2} d_{a_2} + d_{a_4} d_{a_4} d_{a_4} + d_{a_3} d_{a_3} d_{a_3} + d_{a_5} d_{a_5} d_{a_5},
 \end{aligned} \tag{43}$$

where d_{a_1}, \dots, d_{a_5} are, respectively, the dimensions of the irreducible representations of $SU(n)$ at level-4



From the hook rule, the dimensions of these tableaux can be computed to be

$$d_{a_1} = \frac{n(n+1)(n+2)(n+3)}{24}; \quad d_{a_2} = \frac{n(n+1)(n+2)(n-1)}{8}; \quad d_{a_3} = \frac{n^2(n+1)(n-1)}{12}$$

$$d_{a_4} = \frac{n(n-1)(n-2)(n+1)}{8}; \quad d_{a_5} = \frac{n(n-1)(n-2)(n-3)}{24}. \tag{44}$$

Substituting these values, we get the above sum in (43) to be $\binom{n^3}{4}$ which matches with the antisymmetric representation of $SU(n^3)$ at level 4 as expected. Note that the counting works even for different n_1, n_2, n_3 .

Furthermore, we have verified that the counting works for the 3-slot case up to level 5 for antisymmetric representations and for the 2-slot case up to level 6 for both symmetric and antisymmetric representations.

ACKNOWLEDGMENTS

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APPENDIX A: SYMMETRIC GROUPS AND YOUNG TABLEAUX: MINI REVIEW

The symmetric group S_n represents all the permutations that can be performed on a finite set of n symbols.¹¹ It is composed of cycles, of length $l \leq n$, which preserve $(n - l)$ elements in the set invariant. Two different cycles of the same length are conjugate to each other. Hence, the length of the cycles defines the conjugacy classes of the symmetric group. We can take a set of integers λ_i such that $\sum_i \lambda_i = n$, which is the partition of n , which defines the conjugacy class of that particular set of cycles.

The symmetric group, as mentioned above, is characterized by the cycles. Thus, the cycles can be understood as the generating set of the group. The group S_n can be generated by different set of cycles. For $n \geq 3$, any cycle of length greater than two can be generated by an appropriate product 2-cycles (also called transpositions). The 2-cycles are represented as $(i_j i_{j+1})$, and we have

$$(i_1 i_2, \dots, i_k) = (i_1 i_2)(i_2 i_3), \dots, (i_{k-1} i_k). \tag{A1}$$

We can generate all cycles using this method, which in turn means that we can generate the entire group S_n . In this method, the generating set is made up of ${}^n C_2 = n(n - 1)/2$ transpositions.

We could also generate all the cycles using $n - 1$ transpositions chosen in two different ways, for $n \geq 3$,

$$(12), (13), \dots, (1n) \tag{A2}$$

$$(12), (23), \dots, (n - 1n). \tag{A3}$$

The generating set can also be constructed out of just two elements in the following way, again for $n \geq 3$:

$$(12), (123, \dots, n) \tag{A4}$$

$$(12), (23, \dots, n). \tag{A5}$$

As mentioned above, the symmetric group is characterized entirely by the cycles, which in turn are unique only upto their conjugacy class. The conjugacy classes are defined only by the lengths of the cycles that it contains, and all such classes are given by the partition of n . This is where the Young tableau enters the picture. The Young tableau is a nice graphic way of representing partitions. Two crucial facts are as follows:

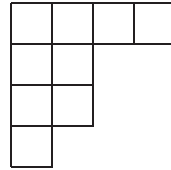
- The irreducible representations of the group S_n are labelled by the Young patterns of n .
- The basis elements of a given irrep (also known as Young pattern) are labelled by the standard Young tableaux corresponding to that pattern.

We define and discuss the words in the above paragraph now, in some detail. From the partitions of n , the first thing one can define is a *Young pattern* $[\lambda]$ in the following way:

$$[\lambda] = [\lambda_1, \lambda_2, \dots, \lambda_m], \quad \text{such that } \sum_{k=1}^m \lambda_k = n, \tag{A6}$$

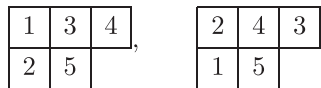
$$\lambda_1 \geq \lambda_2 \geq \dots, \lambda_m \geq 0. \tag{A7}$$

The Young pattern $[\lambda]$ can be graphically represented by a set of left justified boxes, where the i -th row contains λ_i boxes. The last line in the above relation will then enforce the condition that the number of boxes in any row is at least as many as the number of boxes in the row just below it and that the number of boxes in any column is at least as many as the number of boxes in the column just to the right of it. For example, the Young pattern $[\lambda] = [4, 2, 2, 1]$ is graphically represented as



It is not allowed to have a Young pattern in the form $[\lambda] = [4, 2, 1, 2]$, although it is a partition of 9, as it will not satisfy the rules.

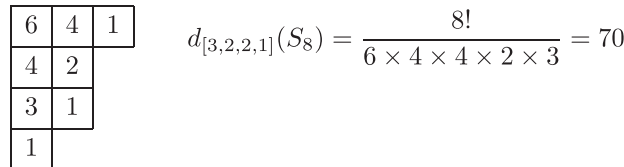
A Young pattern with the each of the boxes being assigned a number $1, 2, \dots, n$ is called a *Young tableau*. It is evident that there will be $n!$ Young tableaux for any given Young pattern. If the numbers assigned to the boxes obey the rule, the values increase as we go left to right along every row and top to bottom along every column, then it is called a *standard Young tableau*. Consider the following Young tableaux, for example:



Although both are valid Young tableaux, only the first one qualifies to be a standard Young tableau. The number of standard Young tableaux for a given Young pattern can be easily determined by the hook length formula. The hook length h_{ij} of a box at i -th row j -th column is given by the sum of the number of boxes to its right in that column and the number of boxes below it in that column added to 1. The number of standard tableaux in the Young pattern is then given by

$$d_{[\lambda]}(S_n) = n! \prod_{ij} \frac{1}{h_{ij}}. \tag{A8}$$

For example, the hook lengths and the corresponding number of standard tableaux for the Young pattern $[\lambda] = [3, 2, 2, 1]$ are



Consider two permutation groups on a finite set of $n + m$ elements, the first one S_n acting on the first n objects and the second one S_m acting on the last m objects. The two permutations act on two different subsets, which means they commute. This particular permutation action can be understood as the $S_n \otimes S_m$ subgroup of the group S_{n+m} . We can represent the subgroup as a sum of the irreducible representations of the group S_{n+m} , using the *Littlewood-Richardson Rule*. That is, a Young pattern $[\lambda]$ of S_n and $[\xi]$ of S_m can be combined into a sum over representations $[\mu]$ of S_{n+m} as

$$[\lambda] \otimes [\xi] = \bigoplus_{[\mu]} C_{[\lambda][\xi]}^{[\mu]} [\mu], \tag{A9}$$

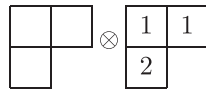
where $C_{[\lambda][\xi]}^{[\mu]}$ is the multiplicity with which $[\mu]$ occurs.

For combining representations, start by labelling the i -th row of $[\xi]$ with the digit i . Then attach these boxes of ξ row by row, ordered top to bottom, to the Young pattern $[\lambda]$ following the rules:

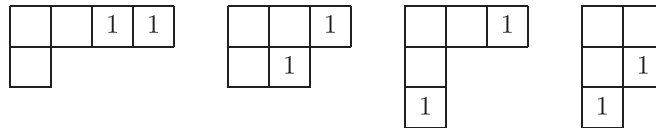
- Each time a row of $[\xi]$ is attached to $[\lambda]$, the resultant pattern is a valid Young pattern.
- None of the columns in the resultant pattern has a repeated digit.

- After all the boxes in $[\xi]$ have been attached to $[\lambda]$, the number of boxes with a smaller digit is never less than the number of boxes with a larger digit when read right to left, in each row.

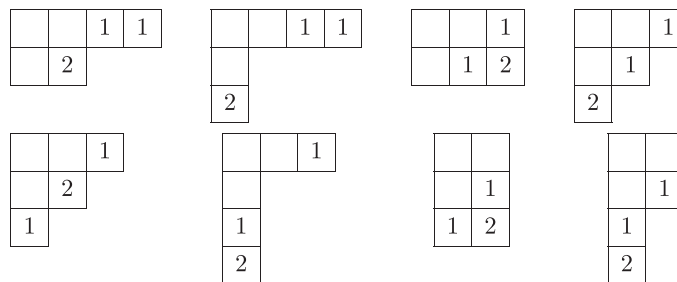
As an example, consider combining the Young patterns $[2, 1] \otimes [2, 1]$. First we label the rows in the second Young pattern



The first row is attached to the Young pattern $[\lambda]$ to get



Then attach the second row, which by the rules, gives the allowed patterns:



For clarity, the rules do not allow for a box with digit 2 attached to the first row and hence such Young patterns are discarded here. The multiplicities are given by

$$C_{[\lambda][\xi]}^{[\mu]} = \begin{cases} 2, & \text{for } [\mu] = [3, 2, 1] \\ 1, & \text{otherwise.} \end{cases} \quad (\text{A10})$$

APPENDIX B: YOUNG-YAMANOUCI BASIS

Young tableaux are a useful way to describe various irreducible representations of the symmetry group and also of unitary and orthogonal groups. We define standard Young tableaux as a tableau where the numbers in a row increase from left to right and the numbers in a column increase from top to bottom. Also, we use a convention that a state corresponding to a standard Young tableau is obtained by first symmetrizing along the rows and then antisymmetrizing along the columns. For example,

$$\begin{array}{|c|c|} \hline 1 & 2 \\ \hline 3 & \\ \hline \end{array} = |123\rangle + |213\rangle - |321\rangle - |231\rangle$$

Furthermore, it can be shown that the standard Young tableaux form a basis and hence any state can be expanded in terms of standard tableaux. This fact is important to write down Eq. (4).

In the Young-Yamanouchi basis, we represent all the standard Young tableaux of a particular representation by column matrices, where only one of the quantities is 1 and the others are zero. Now, the matrix elements of the form $\langle a' | D(g) | a \rangle$ are constructed where $D(g)$ is the matrix corresponding to the 2-cycle g in the specific representation that we are dealing with. Note that we need to only find the matrices corresponding to 2-cycles of the form $(i, i + 1)$ as all the entire permutation group can be generated by these transpositions.

The matrix elements corresponding to $D(i, i + 1)$ are quite simple and are given as follows: If the standard Young tableaux $|a\rangle$ and $|a'\rangle$ are such that they can be obtained by exchanging i and $(i + 1)$, then we have

$$D(i, i + 1) | a \rangle = -\rho_{(i, i+1)} | a \rangle + \sqrt{1 - \rho_{(i, i+1)}^2} | a' \rangle, \quad (\text{B1})$$

$$D(i, i + 1) | a' \rangle = +\sqrt{1 - \rho_{(i, i+1)}^2} | a \rangle + \rho_{(i, i+1)} | a' \rangle, \quad (\text{B2})$$

where $\rho_{(i, i+1)}$ is the inverse of the distance between i and $(i + 1)$, i.e., the number of steps taken from i to reach $(i + 1)$. When the steps are counted left or down, we take them to be positive distance. Right or upward steps contribute to negative distance. If $|a\rangle$ and $|a'\rangle$ are two standard tableaux such that they are not obtained by the exchange of i and $(i + 1)$, then $\langle a' | D(i, i + 1) | a \rangle = 0$.

If i and $(i + 1)$ are in the same row and adjacent to each other in a certain standard tableau then $\rho_{(i,i+1)} = -1$ and we put 1 in the corresponding position and the rest of the entries in that column and row are zeroes. Similarly, for i and $(i + 1)$ in the same column, we put -1 in the corresponding position and the rest of the entries are zeroes.

As an example, consider the following standard tableaux at level 4:

$$\begin{array}{|c|c|c|} \hline 1 & 2 & 3 \\ \hline 4 & & \\ \hline \end{array}; \begin{array}{|c|c|c|} \hline 1 & 2 & 4 \\ \hline 3 & & \\ \hline \end{array}; \begin{array}{|c|c|c|} \hline 1 & 3 & 4 \\ \hline 2 & & \\ \hline \end{array}$$

We assign the following column matrices to the above tableaux as follows:

$$\begin{array}{|c|c|c|} \hline 1 & 2 & 3 \\ \hline 4 & & \\ \hline \end{array} : \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}; \begin{array}{|c|c|c|} \hline 1 & 2 & 4 \\ \hline 3 & & \\ \hline \end{array} : \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}; \begin{array}{|c|c|c|} \hline 1 & 3 & 4 \\ \hline 2 & & \\ \hline \end{array} : \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$

Following the above rules, we get the matrices corresponding to (12), (23), and (34) as

$$D(12) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix}; D(23) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -\frac{1}{2} & \frac{\sqrt{3}}{2} \\ 0 & \frac{\sqrt{3}}{2} & \frac{1}{2} \end{pmatrix}; D(34) = \begin{pmatrix} -\frac{1}{3} & \frac{\sqrt{8}}{3} & 0 \\ \frac{\sqrt{8}}{3} & \frac{1}{3} & 0 \\ 0 & 0 & 1 \end{pmatrix} \tag{B3}$$

APPENDIX C: TWO SLOTS: SYMMETRIC AND ANTI-SYMMETRIC STATES

The Young patterns of the symmetric and anti-symmetric states with two slots have a simple structure. Instead of explaining the structure with a thousand words, we will present the pictures of the corresponding patterns at levels 4 and 5, which have enough structure to illustrate the idea. It is easy to check that this structure holds at all levels and that the total dimensionalities of each of these representations add up to the total number of symmetric and anti-symmetric states expected at each level.

1. Level 4

Symmetric states

$$\left(\begin{array}{|c|c|c|c|} \hline & & & \\ \hline & & & \\ \hline & & & \\ \hline & & & \\ \hline \end{array}, \begin{array}{|c|c|c|c|} \hline & & & \\ \hline & & & \\ \hline & & & \\ \hline & & & \\ \hline \end{array} \right), \left(\begin{array}{|c|c|c|} \hline & & \\ \hline & & \\ \hline & & \\ \hline & & \\ \hline \end{array}, \begin{array}{|c|c|c|} \hline & & \\ \hline & & \\ \hline & & \\ \hline & & \\ \hline \end{array} \right),$$

$$\left(\begin{array}{|c|c|} \hline & \\ \hline & \\ \hline & \\ \hline & \\ \hline \end{array}, \begin{array}{|c|c|} \hline & \\ \hline & \\ \hline & \\ \hline & \\ \hline \end{array} \right), \left(\begin{array}{|c|c|} \hline & \\ \hline & \\ \hline & \\ \hline & \\ \hline \end{array}, \begin{array}{|c|c|} \hline & \\ \hline & \\ \hline & \\ \hline & \\ \hline \end{array} \right), \left(\begin{array}{|c|c|} \hline & \\ \hline & \\ \hline & \\ \hline & \\ \hline \end{array}, \begin{array}{|c|c|} \hline & \\ \hline & \\ \hline & \\ \hline & \\ \hline \end{array} \right).$$

Anti-Symmetric states

$$\left(\begin{array}{|c|c|c|c|} \hline & & & \\ \hline & & & \\ \hline & & & \\ \hline & & & \\ \hline \end{array}, \begin{array}{|c|} \hline \\ \hline \\ \hline \\ \hline \\ \hline \end{array} \right), \left(\begin{array}{|c|c|c|} \hline & & \\ \hline & & \\ \hline & & \\ \hline & & \\ \hline \end{array}, \begin{array}{|c|c|} \hline & \\ \hline & \\ \hline & \\ \hline & \\ \hline \end{array} \right), \left(\begin{array}{|c|c|} \hline & \\ \hline & \\ \hline & \\ \hline & \\ \hline \end{array}, \begin{array}{|c|c|} \hline & \\ \hline & \\ \hline & \\ \hline & \\ \hline \end{array} \right),$$

$$\left(\begin{array}{|c|c|} \hline & \\ \hline & \\ \hline & \\ \hline & \\ \hline \end{array}, \begin{array}{|c|c|c|} \hline & & \\ \hline & & \\ \hline & & \\ \hline & & \\ \hline \end{array} \right), \left(\begin{array}{|c|} \hline \\ \hline \\ \hline \\ \hline \\ \hline \end{array}, \begin{array}{|c|c|c|c|} \hline & & & \\ \hline & & & \\ \hline & & & \\ \hline & & & \\ \hline \end{array} \right).$$

APPENDIX D: THREE SLOTS UP TO FOUR LEVELS

In this Appendix, we list the fully anti-symmetric states for the case with three quantum numbers up to level 4. In level 4, we only show a sample state for brevity.

At level 2, we have four different antisymmetric states:

$$\left(\begin{array}{|c|c|} \hline 1 & 2 \\ \hline \end{array}, \begin{array}{|c|c|} \hline 1 & 2 \\ \hline \end{array}, \begin{array}{|c|} \hline 1 \\ \hline 2 \\ \hline \end{array} \right); \left(\begin{array}{|c|c|} \hline 1 & 2 \\ \hline \end{array}, \begin{array}{|c|} \hline 1 \\ \hline 2 \\ \hline \end{array}, \begin{array}{|c|c|} \hline 1 & 2 \\ \hline \end{array} \right);$$

$$\left(\begin{array}{|c|} \hline 1 \\ \hline 2 \\ \hline \end{array}, \begin{array}{|c|c|} \hline 1 & 2 \\ \hline \end{array}, \begin{array}{|c|c|} \hline 1 & 2 \\ \hline \end{array} \right); \left(\begin{array}{|c|} \hline 1 \\ \hline 2 \\ \hline \end{array}, \begin{array}{|c|} \hline 1 \\ \hline 2 \\ \hline \end{array}, \begin{array}{|c|} \hline 1 \\ \hline 2 \\ \hline \end{array} \right)$$

At level 3, we have the following antisymmetric states:

$$\left(\begin{array}{|c|c|c|} \hline 1 & 2 & 3 \\ \hline \end{array}, \begin{array}{|c|c|c|} \hline 1 & 2 & 3 \\ \hline \end{array}, \begin{array}{|c|} \hline 1 \\ \hline 2 \\ \hline 3 \\ \hline \end{array} \right); \left(\begin{array}{|c|} \hline 1 \\ \hline 2 \\ \hline 3 \\ \hline \end{array}, \begin{array}{|c|} \hline 1 \\ \hline 2 \\ \hline 3 \\ \hline \end{array}, \begin{array}{|c|} \hline 1 \\ \hline 2 \\ \hline 3 \\ \hline \end{array} \right);$$

$$\left(\begin{array}{|c|} \hline 1 \\ \hline 2 \\ \hline 3 \\ \hline \end{array}, \begin{array}{|c|c|} \hline 1 & 3 \\ \hline \end{array}, \begin{array}{|c|c|} \hline 1 & 3 \\ \hline \end{array} \right) + \left(\begin{array}{|c|} \hline 1 \\ \hline 2 \\ \hline 3 \\ \hline \end{array}, \begin{array}{|c|c|} \hline 1 & 2 \\ \hline \end{array}, \begin{array}{|c|c|} \hline 1 & 2 \\ \hline \end{array} \right);$$

$$\left(\begin{array}{|c|c|c|} \hline 1 & 2 & 3 \\ \hline \end{array}, \begin{array}{|c|c|} \hline 1 & 2 \\ \hline 3 \\ \hline \end{array}, \begin{array}{|c|c|} \hline 1 & 3 \\ \hline 2 \\ \hline \end{array} \right) - \left(\begin{array}{|c|c|c|} \hline 1 & 2 & 3 \\ \hline \end{array}, \begin{array}{|c|c|} \hline 1 & 3 \\ \hline 2 \\ \hline \end{array}, \begin{array}{|c|c|} \hline 1 & 2 \\ \hline 3 \\ \hline \end{array} \right);$$

$$\left(\begin{array}{|c|c|} \hline 1 & 2 \\ \hline 3 \\ \hline \end{array}, \begin{array}{|c|c|} \hline 1 & 3 \\ \hline 2 \\ \hline \end{array}, \begin{array}{|c|c|} \hline 1 & 2 \\ \hline 3 \\ \hline \end{array} \right) + \left(\begin{array}{|c|c|} \hline 1 & 3 \\ \hline 2 \\ \hline \end{array}, \begin{array}{|c|c|} \hline 1 & 2 \\ \hline 3 \\ \hline \end{array}, \begin{array}{|c|c|} \hline 1 & 2 \\ \hline 3 \\ \hline \end{array} \right)$$

$$+ \left(\begin{array}{|c|c|} \hline 1 & 2 \\ \hline 3 \\ \hline \end{array}, \begin{array}{|c|c|} \hline 1 & 2 \\ \hline 3 \\ \hline \end{array}, \begin{array}{|c|c|} \hline 1 & 3 \\ \hline 2 \\ \hline \end{array} \right) - \left(\begin{array}{|c|c|} \hline 1 & 3 \\ \hline 2 \\ \hline \end{array}, \begin{array}{|c|c|} \hline 1 & 3 \\ \hline 2 \\ \hline \end{array}, \begin{array}{|c|c|} \hline 1 & 3 \\ \hline 2 \\ \hline \end{array} \right)$$

At level 4, the antisymmetric states are complicated and we will settle for showing just one of the states:

$$\begin{aligned}
 & \left[\left(\begin{array}{|c|c|c|} \hline 1 & 2 & 3 \\ \hline 4 & & \\ \hline \end{array}, \begin{array}{|c|c|c|} \hline 1 & 2 & 4 \\ \hline 3 & & \\ \hline \end{array}, \begin{array}{|c|c|} \hline 1 & 3 \\ \hline 2 & \\ \hline 4 & \\ \hline \end{array} \right) - \left(\begin{array}{|c|c|c|} \hline 1 & 2 & 3 \\ \hline 4 & & \\ \hline \end{array}, \begin{array}{|c|c|c|} \hline 1 & 3 & 4 \\ \hline 2 & & \\ \hline \end{array}, \begin{array}{|c|c|} \hline 1 & 2 \\ \hline 3 & \\ \hline 4 & \\ \hline \end{array} \right) \right] \\
 & + \left[\left(\begin{array}{|c|c|c|} \hline 1 & 2 & 4 \\ \hline 3 & & \\ \hline \end{array}, \begin{array}{|c|c|c|} \hline 1 & 2 & 3 \\ \hline 4 & & \\ \hline \end{array}, \begin{array}{|c|c|} \hline 1 & 3 \\ \hline 2 & \\ \hline 4 & \\ \hline \end{array} \right) - \left(\begin{array}{|c|c|c|} \hline 1 & 3 & 4 \\ \hline 2 & & \\ \hline \end{array}, \begin{array}{|c|c|c|} \hline 1 & 2 & 3 \\ \hline 4 & & \\ \hline \end{array}, \begin{array}{|c|c|} \hline 1 & 2 \\ \hline 3 & \\ \hline 4 & \\ \hline \end{array} \right) \right] \\
 & - \left[\left(\begin{array}{|c|c|c|} \hline 1 & 2 & 4 \\ \hline 3 & & \\ \hline \end{array}, \begin{array}{|c|c|c|} \hline 1 & 2 & 4 \\ \hline 3 & & \\ \hline \end{array}, \begin{array}{|c|c|} \hline 1 & 4 \\ \hline 2 & \\ \hline 3 & \\ \hline \end{array} \right) + \left(\begin{array}{|c|c|c|} \hline 1 & 3 & 4 \\ \hline 2 & & \\ \hline \end{array}, \begin{array}{|c|c|c|} \hline 1 & 3 & 4 \\ \hline 2 & & \\ \hline \end{array}, \begin{array}{|c|c|} \hline 1 & 4 \\ \hline 2 & \\ \hline 3 & \\ \hline \end{array} \right) \right] \\
 & -\sqrt{2} \left[\left(\begin{array}{|c|c|c|} \hline 1 & 2 & 4 \\ \hline 3 & & \\ \hline \end{array}, \begin{array}{|c|c|c|} \hline 1 & 2 & 4 \\ \hline 3 & & \\ \hline \end{array}, \begin{array}{|c|c|} \hline 1 & 3 \\ \hline 2 & \\ \hline 4 & \\ \hline \end{array} \right) + \left(\begin{array}{|c|c|c|} \hline 1 & 2 & 4 \\ \hline 3 & & \\ \hline \end{array}, \begin{array}{|c|c|c|} \hline 1 & 3 & 4 \\ \hline 2 & & \\ \hline \end{array}, \begin{array}{|c|c|} \hline 1 & 2 \\ \hline 3 & \\ \hline 4 & \\ \hline \end{array} \right) \right] \\
 & + \left(\begin{array}{|c|c|c|} \hline 1 & 3 & 4 \\ \hline 2 & & \\ \hline \end{array}, \begin{array}{|c|c|c|} \hline 1 & 2 & 4 \\ \hline 3 & & \\ \hline \end{array}, \begin{array}{|c|c|} \hline 1 & 2 \\ \hline 3 & \\ \hline 4 & \\ \hline \end{array} \right) - \left(\begin{array}{|c|c|c|} \hline 1 & 3 & 4 \\ \hline 2 & & \\ \hline \end{array}, \begin{array}{|c|c|c|} \hline 1 & 3 & 4 \\ \hline 2 & & \\ \hline \end{array}, \begin{array}{|c|c|} \hline 1 & 3 \\ \hline 2 & \\ \hline 4 & \\ \hline \end{array} \right) \right] \\
 & + 2 \left(\begin{array}{|c|c|c|} \hline 1 & 2 & 3 \\ \hline 4 & & \\ \hline \end{array}, \begin{array}{|c|c|c|} \hline 1 & 2 & 3 \\ \hline 4 & & \\ \hline \end{array}, \begin{array}{|c|c|} \hline 1 & 4 \\ \hline 2 & \\ \hline 3 & \\ \hline \end{array} \right)
 \end{aligned}$$

This state is an example where (p_a, p_b, p_c) corresponding to $g_3 = (34)$ are not constrained.

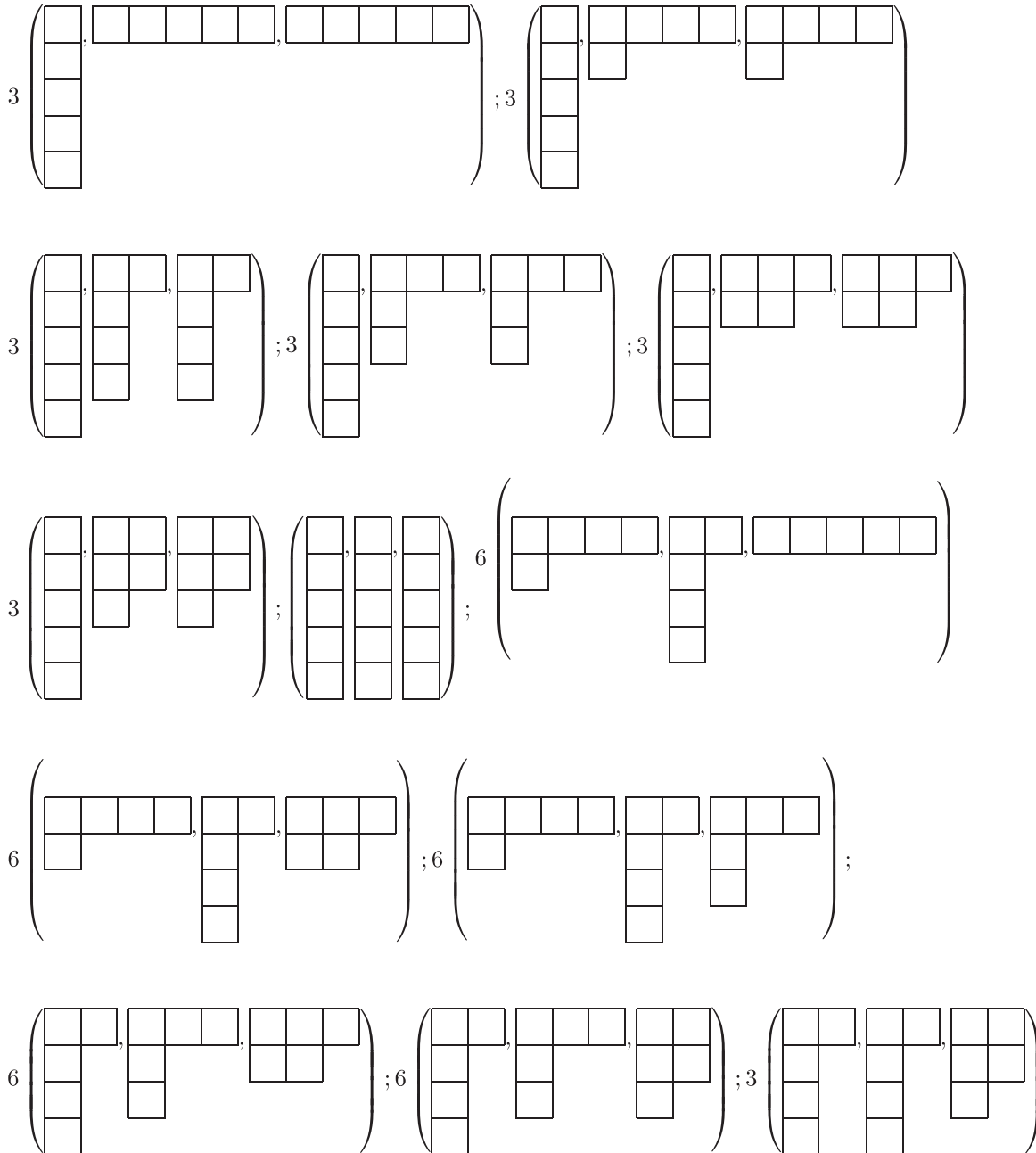
APPENDIX E: COMPLETE LIST OF ANTI-SYMMETRIC YOUNG PATTERNS AT LEVEL 4

The multiplicities in front of the representations below are a short hand way of capturing the permutations of the Young pattern among the three slots. In particular, they are not meant to suggest actual multiplicities of the same representation.

APPENDIX F: COMPLETE LIST OF ANTI-SYMMETRIC YOUNG PATTERNS AT LEVEL 5

As in level 4, (most of) the multiplicities in front of the representations below are a short hand way of capturing the permutations of the Young pattern among the three slots. In particular, (mostly) they are not meant to suggest actual multiplicities of the same representation.

But there are two exceptions to this. These are the two representations with bold face **6**'s: the number of permutations between the slots in each of those cases is 3. The extra factor of 2 actually denotes a multiplicity. This corresponds to the fact that there are two common eigenvectors that fall into those representations, in the language of Sec. III.



$$\begin{aligned}
& 3 \left(\begin{array}{|c|c|c|} \hline \square & \square & \square \\ \hline \square & \square & \square \\ \hline \end{array}, \begin{array}{|c|c|c|} \hline \square & \square & \square \\ \hline \square & \square & \square \\ \hline \end{array}, \begin{array}{|c|c|} \hline \square & \square \\ \hline \square & \square \\ \hline \end{array} \right); 3 \left(\begin{array}{|c|c|c|} \hline \square & \square & \square \\ \hline \square & \square & \square \\ \hline \square & & \square \\ \hline \end{array}, \begin{array}{|c|c|c|} \hline \square & \square & \square \\ \hline \square & \square & \square \\ \hline \end{array}, \begin{array}{|c|c|c|} \hline \square & \square & \square \\ \hline \square & \square & \square \\ \hline \end{array} \right) \\
& 3 \left(\begin{array}{|c|c|c|} \hline \square & \square & \square \\ \hline \square & \square & \square \\ \hline \square & & \square \\ \hline \end{array}, \begin{array}{|c|c|c|} \hline \square & \square & \square \\ \hline \square & \square & \square \\ \hline \end{array}, \begin{array}{|c|c|c|} \hline \square & \square & \square \\ \hline \square & \square & \square \\ \hline \end{array} \right); 6 \left(\begin{array}{|c|c|c|c|c|} \hline \square & \square & \square & \square & \square \\ \hline \square & \square & \square & \square & \square \\ \hline \end{array}, \begin{array}{|c|c|c|} \hline \square & \square & \square \\ \hline \square & \square & \square \\ \hline \end{array}, \begin{array}{|c|c|c|} \hline \square & \square & \square \\ \hline \square & \square & \square \\ \hline \end{array} \right) \\
& 6 \left(\begin{array}{|c|c|c|} \hline \square & \square & \square \\ \hline \square & \square & \square \\ \hline \end{array}, \begin{array}{|c|c|c|} \hline \square & \square & \square \\ \hline \square & \square & \square \\ \hline \end{array}, \begin{array}{|c|c|c|c|} \hline \square & \square & \square & \square \\ \hline \square & \square & \square & \square \\ \hline \end{array} \right); 6 \left(\begin{array}{|c|c|c|} \hline \square & \square & \square \\ \hline \square & \square & \square \\ \hline \end{array}, \begin{array}{|c|c|c|} \hline \square & \square & \square \\ \hline \square & \square & \square \\ \hline \end{array}, \begin{array}{|c|c|c|} \hline \square & \square & \square \\ \hline \square & \square & \square \\ \hline \square & \square & \square \\ \hline \end{array} \right) \\
& 6 \left(\begin{array}{|c|c|c|c|} \hline \square & \square & \square & \square \\ \hline \square & \square & \square & \square \\ \hline \square & \square & \square & \square \\ \hline \end{array}, \begin{array}{|c|c|c|} \hline \square & \square & \square \\ \hline \square & \square & \square \\ \hline \end{array}, \begin{array}{|c|c|c|c|c|} \hline \square & \square & \square & \square & \square \\ \hline \square & \square & \square & \square & \square \\ \hline \end{array} \right); 6 \left(\begin{array}{|c|c|c|c|} \hline \square & \square & \square & \square \\ \hline \square & \square & \square & \square \\ \hline \square & \square & \square & \square \\ \hline \end{array}, \begin{array}{|c|c|c|c|} \hline \square & \square & \square & \square \\ \hline \square & \square & \square & \square \\ \hline \square & \square & \square & \square \\ \hline \end{array}, \begin{array}{|c|c|c|c|} \hline \square & \square & \square & \square \\ \hline \square & \square & \square & \square \\ \hline \square & \square & \square & \square \\ \hline \end{array} \right) \\
& 6 \left(\begin{array}{|c|c|c|c|} \hline \square & \square & \square & \square \\ \hline \square & \square & \square & \square \\ \hline \square & \square & \square & \square \\ \hline \end{array}, \begin{array}{|c|c|c|c|} \hline \square & \square & \square & \square \\ \hline \square & \square & \square & \square \\ \hline \square & \square & \square & \square \\ \hline \end{array}, \begin{array}{|c|c|c|c|} \hline \square & \square & \square & \square \\ \hline \square & \square & \square & \square \\ \hline \square & \square & \square & \square \\ \hline \end{array} \right); 3 \left(\begin{array}{|c|c|c|c|} \hline \square & \square & \square & \square \\ \hline \square & \square & \square & \square \\ \hline \square & \square & \square & \square \\ \hline \end{array}, \begin{array}{|c|c|c|} \hline \square & \square & \square \\ \hline \square & \square & \square \\ \hline \end{array}, \begin{array}{|c|c|c|} \hline \square & \square & \square \\ \hline \square & \square & \square \\ \hline \end{array} \right) \\
& 3 \left(\begin{array}{|c|c|c|c|} \hline \square & \square & \square & \square \\ \hline \square & \square & \square & \square \\ \hline \square & \square & \square & \square \\ \hline \end{array}, \begin{array}{|c|c|c|c|} \hline \square & \square & \square & \square \\ \hline \square & \square & \square & \square \\ \hline \square & \square & \square & \square \\ \hline \end{array}, \begin{array}{|c|c|c|c|} \hline \square & \square & \square & \square \\ \hline \square & \square & \square & \square \\ \hline \square & \square & \square & \square \\ \hline \end{array} \right)
\end{aligned}$$

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- ¹³We will mostly deal with fermions for concreteness, but an entirely analogous discussion holds for bosons as well.

¹⁴Calculations involving Young tableaux are claimed¹ to be non-trivial even on a computer.

¹⁵See also our discussion in the next paragraph for our motivations for considering this problem.

¹⁶For three flavors, the baryon wave function problem is discussed in many introductory books on particle physics, e.g., Ref. 2. But we have not found a discussion in the literature which includes more flavors.

¹⁷Let us emphasize once again that for most of the paper we will stick to the negative sign on the right-hand side, which corresponds to fermions.

¹⁸We note that the any other 2-cycle (and therefore all elements of the group) can be obtained from (12), (23), ..., (n - 1, n).

¹⁹See Appendixes for some relevant definitions and explanations.

²⁰It is given by the inverse of number of steps to reach (i + 1) starting from i in a standard Young tableau. So $p_a^i(g)$ is of the form (positive integer)⁻¹.

²¹The \pm sign depends on our choices of Young tableaux.

²²The distance is equal to the number of steps to reach from i to (i + 1). When the steps are counted left or down, we take them to be positive distance. Right or upward steps contribute to negative distance. Furthermore, note that the positions of i and i + 1 are exchanged in $|i'\rangle$ to obtain $|i''\rangle$.

²³As before, the sign depends on the choice of Young tableaux.

²⁴If we wish, we can further go ahead and solve the problem more completely and find the answer that includes the precise combinations of Young tableaux that show up in the Young patterns. But to present the answer in a reasonably compact form, we limit ourselves to just listing the patterns in the Appendixes.

²⁵The subscript p_i is supposed to index the degeneracy in the eigenvectors of $D^a(g_i) \otimes D^b(g_i) \otimes D^c(g_i)$. The superscript (i) is for emphasis and is not strictly necessary since the p_i contains the information about i.

²⁶There is a simple way to check whether two matrices share a common eigenvector. If two matrices B and C share a common eigenvector, then we need to have $\det[B, C] = 0$.

²⁷Remember that the Young patterns capture the representation and the standard Young tableaux list the basis elements.

²⁸By multiplicity, we mean the number of times a set of Young patterns appear in the set of antisymmetric states at a particular level.