# Hermite Multipliers on Modulation Spaces

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**Abstract.** We study multipliers associated to the Hermite operator $H = -\Delta + |x|^2$ on modulation spaces $M^{p,q}(\mathbb{R}^d)$. We prove that the operator $m(H)$ is bounded on $M^{p,q}(\mathbb{R}^d)$ under standard conditions on $m$, for suitable choice of $p$ and $q$. As an application, we point out that the solutions to the free wave and Schrödinger equations associated to $H$ with initial data in a modulation space will remain in the same modulation space for all times. We also point out that Riesz transforms associated to $H$ are bounded on some modulation spaces.

**Keywords:** Hermite multipliers · Modulation spaces · Wave and Schrödinger equations

## 1 Introduction

The main aim of this article is to study the boundedness properties of Hermite multipliers on modulation spaces. We quickly recall the setup in order to state our results and we refer to Sect. 2 for details. The spectral decomposition of the Hermite operator $H = -\Delta + |x|^2$ on $\mathbb{R}^d$ is given by $H = \sum_{k=0}^{\infty}(2k+d)P_k$ where $P_k$ stands for the orthogonal projectionof $L^2(\mathbb{R}^d)$ onto the eigenspace corresponding to the eigenvalue $(2k+d)$. Given a bounded function $m$ defined on the set of all natural numbers, we define the operator $m(H)$ simply by setting $m(H) = \sum_{k=0}^{\infty} m(2k+d)P_k$. We say that $m$ is an $L^p$ multiplier for the Hermite expansions if $m(H)$ extends to $L^p$ as a bounded operator. Sufficient conditions on $m$ are known so that $m$ is an $L^p$ multiplier, see e.g. [6,7,10,11]. In this article we are interested in multipliers $m$ which define bounded operators on the modulation spaces $M^{p,q}(\mathbb{R}^d)$.

Recall that a tempered distribution $f$ on $\mathbb{R}^d$ belongs to the modulation space $M^{q,p}(\mathbb{R}^d)$ if the Fourier-Wigner transform of $f$ and the Gaussian $\Phi_0(\xi) = \pi^{-d/2}e^{-\frac{1}{2}|\xi|^2}$ defined by

$$\langle \pi(x+iy)f, \Phi_0\rangle = \int_{\mathbb{R}^d} e^{i(x\cdot\xi+\frac{1}{2}x\cdot y)} f(\xi+y)\Phi_0(\xi)d\xi$$


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belongs to the mixed norm space $L^p(\mathbb{R}^d, L^q(\mathbb{R}^d))$ consisting of functions $F(x, y)$ for which the norms

$$\|F\|_{L^{p,q}} = \left( \int_{\mathbb{R}^d} \|F(x, \cdot)\|_q^p dx \right)^{1/p}$$

are finite (See Sect. 2.5 below). These spaces have several interesting properties not shared by the $L^p$ spaces. For example, $M^{p,p}(\mathbb{R}^d)$ are invariant under the Fourier transform and $M^{q,1}(\mathbb{R}^d)$ are algebras under pointwise multiplication.

For the multiplier operators $m(D)$, the problem of establishing sufficient conditions on $m$ that make the operator $m(D)$ bounded on $L^p$ has a long history. As it appears often in various applications, like solving linear dispersive PDE, e.g., wave/Schrödinger equations, for more detail, we refer to [1, 11, 12, 20] and the reference therein. It is well known that the operator (See Definition 4 below) with Fourier multiplier $e^{i|\xi|^\alpha}$ $(\alpha > 2)$ is bounded on $L^p(\mathbb{R}^d)$ if and only if $p = 2$.

The study of Fourier multiplier operators, which are of the form $m(-\Delta)$ in the context of modulation space $M^{2,1}(\mathbb{R}^d)$ was initiated in the works of Wang-Zhao-Guo [19]. In fact, in the consequent year Bényi-Gröchenig-Okoudjou-Rogers [1] have shown that the Fourier multiplier operator with multiplier $e^{i|\xi|^\alpha}$ $(\alpha \in [0, 2])$ is bounded on $M^{p,q}(\mathbb{R}^d)$ for all $1 \leq p, q \leq \infty$. The cases $\alpha = 1$ and $\alpha = 2$ are particularly interesting and have been studied intensively in PDE, because they occur in the time evolution of the wave equation $(\alpha = 1)$ and the free Schrödinger operator $(\alpha = 2)$. Thus, the Schödinger and wave propagators are not $L^p (p \neq 2)$-bounded but $M^{p,q}$- bounded for all $1 \leq p, q \leq \infty$. In fact, this leads to fixed-time estimates for Schödinger and wave propagators and some of their applications to well-posedness results on modulation spaces $M^{p,q}(\mathbb{R}^d)$. Modulation spaces have turned out to be very fruitful in numerous applications in various problems in analysis and PDE. And yet there has been a lot of ongoing interest in these spaces from the harmonic analysis and PDE points of view. We refer to the recent survey [12] and the references therein.

Coming back to the Hermite operator, we note that Thangavelu [18] (See also [15, Theorem 4.2.1]) has proved an analogue of the Hörmander-Mikhlin type multiplier theorem for Hermite expansions on $L^p(\mathbb{R}^d)$. Specifically, he showed that under certain conditions on $m$, the operator $m(H)$ is bounded on $L^p(\mathbb{R}^d)(1 < p < \infty)$. It is well known that the harmonic oscillator $H = -\Delta + |x|^2$ appears in various applications. We refer to the recent article [5] and the reference therein for details.

Taking all these considerations into account, we are motivated to study Hermite multipliers $m(H)$ on modulation spaces $M^{p,q}(\mathbb{R}^d)$. The conditions we impose on the multiplier $m$ is the standard one in terms of local Sobolev spaces. We assume that $m$ is defined on the whole of $\mathbb{R}$. Let $0 \neq \psi \in C_0^\infty(\mathbb{R}^+)$ be a fixed cut-off function with support contained in the interval $[\frac{1}{2}, 1]$, and define the scale invariant localized Sobolev norm of order $\beta$ of $m \in L^\infty(\mathbb{R}^+)$ by

$$\|m\|_{L^2_{\beta,sloc}} = \sup_{t>0} \|\psi m(t\cdot)\|_{L^2_\beta}.$$

We then have the following theorem.

**Theorem 1.** *Let $1 < p \leq q \leq 2$ or $2 \leq q \leq p < \infty$. Suppose that $\|m\|_{L^2_{\beta,sloc}} < \infty$ for some $\beta > (2d+1)/2$. Then the operator $m(H)$ is bounded on $M^{q,p}(\mathbb{R}^d)$.*

We remark that this theorem is not sharp. For $m$ to be an $L^p$ multiplier it is sufficient to assume the condition on $m$ with $\beta > d/2$. We believe the same is true in the case of multipliers on modulation spaces though our method of proof requires a stronger assumption on $m$. However, it is worth noting that $M^{p,p}(\mathbb{R}^d)$ for $p > 2$ is a much wider class than $L^p(\mathbb{R}^d)$ (See Lemma 1(3) below). Thus, Theorem 1 shows that the Hörmander-Mikhlin multiplier type theorem is true for a much wider class than $L^p(\mathbb{R}^d)$.

We deduce Theorem 1 from a corresponding result on the polarised Heisenberg group $\mathbb{H}^d_{pol} = \mathbb{R}^d \times \mathbb{R}^d \times \mathbb{R}$ which is of Euclidean dimension $(2d+1)$. Let $\tilde{\mathcal{L}}$ stand for the sublaplacian on $\mathbb{H}^d_{pol}$ and define $m(\tilde{\mathcal{L}})$ using spectral theorem (See Subsect. 2.2 below for precise definitions). We then have the following transference result.

**Theorem 2.** *Let $1 < p \leq q \leq 2$ or $2 \leq q \leq p < \infty$. Then $m(H)$ is bounded on $M^{q,p}(\mathbb{R}^d)$ whenever $m(\tilde{\mathcal{L}})$ is bounded on $L^p(\mathbb{H}^d_{pol})$.*

The above theorem allows us to deduce some interesting corollaries for Hermite multipliers. For example, let $R_j = (-\frac{\partial}{\partial \xi_j} + \xi_j)H^{-1/2}, j = 1, 2, ..., d$ be the Riesz transforms associated to $H$. It is well known that these Riesz transforms are bounded on $L^p(\mathbb{R}^d), 1 < p < \infty$ but their behavior on modulation spaces have not been studied. By considering Riesz transforms on $G^d$ we obtain

**Corollary 1.** *Let $1 < p \leq q \leq 2$ or $2 \leq q \leq p < \infty$. Then the Riesz transforms $R_j$ are bounded on $M^{q,p}(\mathbb{R}^d)$.*

Another interesting corollary is the following result about solutions of the wave equation associated to $H$. Consider the following Cauchy problem:

$$\partial_t^2 u(x,t) = -Hu(x,t), \quad u(x,0) = 0, \quad \partial_t u(x,0) = f(x)$$

whose solution is given by $u(x,t) = H^{-1/2} \sin(tH^{1/2}) f(x)$.

**Corollary 2.** *Let $u$ be the solution of the above Cauchy problem for $H$. Then for $1 < p \leq q \leq 2$ or $2 \leq q \leq p < \infty$ we have the estimate $\|u(\cdot,t)\|_{M^{q,p}} \leq C_t \|f\|_{M^{q,p}}$ provided $|\frac{1}{p} - \frac{1}{2}| < \frac{1}{2d}$.*

When $p = q$ Theorem 2 as well as Corollary 2 can be improved. This will be achieved using transference as before, but now the transference is from multiplier theorems for multiple Fourier series. Given a function $m$ on $\mathbb{R}$ we define a Fourier multiplier $T_m$ on $L^p(\mathbb{T}^d)$ where $\mathbb{T}^d$ is the $d-$dimensional torus by

$$T_m f(x) = \sum_{\mu \in \mathbb{Z}^d} m(|\mu|) \hat{f}(\mu) e^{i\mu \cdot x}.$$

Here $\hat{f}(\mu)$ are the Fourier coefficients of $f$ and $|\mu| = \sum_{j=1}^d |\mu_j|$. Using the connection (See Proposition 1 below) between Fourier multipliers $T_m$ on $L^p(\mathbb{T}^d)$ and $m(H)$ on $M^{p,p}$ we prove

**Theorem 3.** *Let $1 < p < \infty$. Suppose that $\|m\|_{L^2_{\beta,sloc}} < \infty$ for some $\beta > d/2$. Then the operator $m(H)$ is bounded on $M^{p,p}(\mathbb{R}^d)$.*

We also have the following improvement of Corollary 2. More generally, we consider multipliers of the form

$$m(2k+d) = \frac{e^{i(2k+d)^{\gamma}}}{(2k+d)^{\beta}}, \quad (\beta > 0, \gamma > 0). \tag{1}$$

**Theorem 4.** *Let $1 \leq p < \infty$, and $|\frac{1}{p} - \frac{1}{2}| < \frac{\beta}{d\gamma}$, and let $m$ be given by (1). Then $m(H)$ is bounded on $M^{p,p}(\mathbb{R}^d)$. In particular, Corollary 2 is valid on the bigger range $|\frac{1}{p} - \frac{1}{2}| < \frac{1}{d}$ when $p = q$.*

We now turn our attention to a multiplier which occurs in the time evolution of the free Schrödinger equation associated to $H$. Specifically, we consider the Cauchy problem for the Schrödinger equation associated to $H$ :

$$i\partial_t u(x,t) - Hu(x,t) = 0, \; u(x,0) = f(x)$$

whose solution is given by $u(x,t) = e^{itH} f(x)$.

**Theorem 5.** *The Schrödinger propagator $m(H) = e^{itH}$ is bounded on $M^{p,p}(\mathbb{R}^d)$ for all $1 \leq p < \infty$.*

Recently Cordero-Nicola [2, Sect. 5.1] have studied the operator $m(H) = e^{itH}$ on Wiener amalgam spaces (closely related to modulation spaces). Later Kato-Kobayashi-Ito [8] have given a refinement of Cordero-Nicola's results in the context of Wiener amalgam spaces. We have studied (Theorem 5) the boundeness of $m(H) = e^{itH}$ in the context of modulation spaces. Here we would like to point out that our method of proof is completely different from the method used in the context of Wiener amalgam spaces. We also believe that our method of proof is much simpler than the proofs available in the literature. Our proof relies on properties of Hermite and special Hermite functions and illustrates the importance of these functions in the study of such problems.

Finally, we note that modulation spaces have been used as regularity classes for initial data associated to Cauchy problems for nonlinear dispersive equations (eg., NLS, NLW, etc..) but so far mainly for the nonlinear dispersive equations associated to the Laplacian without potential ($D = \Delta$) [12,19,20]. There is also an ongoing interest to use harmonic analysis tools (specifically, multiplier results) to solve modern nonlinear PDE problems (See e.g., [5]). Thus, we strongly believe that our results will be useful in the future for studying nonlinear dispersive equations associated to $H$ in the realm of modulation spaces.

The paper is organized as follows. In Sect. 2, we introduce notations and preliminaries which will be used in the sequel. Specifically, in Subsect. 2.2, we introduce the Heisenberg and polarised Heisenberg groups, the sublaplacians corresponding to these groups, and spectral multipliers associated to these sub-laplacians. In Subsect. 2.4, we prove the transference result which connects spectral multiplier on polarised Heisenberg groups and reduced polarised Heisenberg groups. In Subsect. 2.5, we introduce modulation spaces and recall some of their basic properties. In Sect. 3, we prove Theorems 1 and 2. In Sect. 4, we prove Theorems 3 and 4. In Sect. 5, we prove Theorem 5.

## 2 Notation and Preliminaries

### 2.1 Notations

The notation $A \lesssim B$ means $A \leq cB$ for some constant $c > 0$. The symbol $A_1 \hookrightarrow A_2$ denotes the continuous embedding of the topological linear space $A_1$ into $A_2$. If $\alpha = (\alpha_1, ..., \alpha_d) \in \mathbb{N}^d$ is a multi-index, we set $|\alpha| = \sum_{j=1}^{d} \alpha_j, \alpha! = \prod_{j=1}^{d} \alpha_j!$. If $z = (z_1, ..., z_d) \in \mathbb{C}^d$, we put $z^\alpha = \prod_{j=1}^{d} z_j^{\alpha_j}$. We denote the $d-$dimensional torus by $\mathbb{T}^d \equiv [0, 2\pi)^d$, and the $L^p(\mathbb{T}^d)-$norm by

$$\|f\|_{L^p(\mathbb{T}^d)} = \left( \int_{[0,2\pi)^d} |f(t)|^p dt \right)^{1/p}.$$

The class of trigonometric polynomials on $\mathbb{T}^d$ is denoted by $\mathcal{P}(\mathbb{T}^d)$. The mixed $L^p(\mathbb{R}^d, L^q(\mathbb{R}^d))$ norm is denoted by

$$\|f\|_{L^{p,q}} = \left( \int_{\mathbb{R}^d} \left( \int_{\mathbb{R}^d} |f(x,y)|^q dy \right)^{p/q} dx \right)^{1/p} \quad (1 \leq p, q < \infty),$$

the $L^\infty(\mathbb{R}^d)$ norm is $\|f\|_{L^\infty} = \text{ess.sup}_{x \in \mathbb{R}^d} |f(x)|$. The Schwartz class is denoted by $\mathcal{S}(\mathbb{R}^d)$ (with its usual topology), and the space of tempered distributions is denoted by $\mathcal{S}'(\mathbb{R}^d)$. For $x = (x_1, \cdots, x_d), y = (y_1, \cdots, y_d) \in \mathbb{R}^d$, we put $x \cdot y = \sum_{i=1}^{d} x_i y_i$. Let $\mathcal{F} : \mathcal{S}(\mathbb{R}^d) \to \mathcal{S}(\mathbb{R}^d)$ be the Fourier transform defined by

$$\mathcal{F}f(\xi) = \widehat{f}(\xi) = (2\pi)^{-d/2} \int_{\mathbb{R}^d} f(x) e^{-ix\cdot\xi} dx, \ \xi \in \mathbb{R}^d.$$

Then $\mathcal{F}$ is a bijection and the inverse Fourier transform is given by

$$\mathcal{F}^{-1}f(x) = f^\vee(x) = (2\pi)^{-d/2} \int_{\mathbb{R}^d} f(\xi)\, e^{ix\cdot\xi} d\xi, \quad x \in \mathbb{R}^d.$$

It is well known that the Fourier transform can be uniquely extended to $\mathcal{F} : \mathcal{S}'(\mathbb{R}^d) \to \mathcal{S}'(\mathbb{R}^d)$.

## 2.2 Heisenberg Group and Fourier Multipliers

We consider the Heisenberg group $\mathbb{H}^d = \mathbb{C}^d \times \mathbb{R}$ with the group law

$$(z,t)(w,s) = (z+w, t+s+\frac{1}{2}\mathrm{Im}(z\cdot\bar{w})).$$

Sometimes we use real coordinates $(x,y,t)$ instead of $(z,t)$ if $z = x+iy, x,y \in \mathbb{R}^d$. In order to define the (group) Fourier transform on the Heisenberg group we briefly recall the following family of irreducible unitary representations. For each non-zero real $\lambda$, we have a representation $\pi_\lambda$ realised on $L^2(\mathbb{R}^d)$ as follows:

$$\pi_\lambda(z,t)\varphi(\xi) = e^{i\lambda t}e^{i(x\cdot\xi+\frac{1}{2}x\cdot y)}\varphi(\xi+y)$$

where $\varphi \in L^2(\mathbb{R}^d)$. For $f \in L^1(\mathbb{H}^d)$, one defines the operator

$$\hat{f}(\lambda) = \int_{\mathbb{H}^d} f(z,t)\pi_\lambda(z,t)dzdt.$$

The operator valued function $f \to \hat{f}(\lambda)$ is called the group Fourier transform of $f$ on $\mathbb{H}^d$. We refer to [16] for more about the group Fourier transform.

The Fourier transform initially defined on $L^1(\mathbb{H}^d)\cap L^2(\mathbb{H}^d)$ can be extended to the whole of $L^2(\mathbb{H}^d)$ and we have a version of Plancherel theorem. Specifically, when $f \in L^1 \cap L^2(\mathbb{H}^d)$, it can be shown that $\hat{f}(\lambda)$ is a Hilbert-Schmidt operator and the Plancherel theorem holds:

$$\int_{\mathbb{H}^d} |f(z,t)|^2 dzdt = \frac{2^{d-1}}{\pi^{d+1}}\int_{-\infty}^{\infty} \|\hat{f}(\lambda)\|_{HS}^2|\lambda|^d d\lambda$$

where $\|\cdot\|_{HS}$ is the Hilbert-Schmidt norm given by $\|T\|_{HS}^2 = tr(T^*T)$, for $T$ a bounded operator, $T^*$ being the adjoint operator of $T$. Under the assumption that $\hat{f}(\lambda)$ is of trace class we have the inversion formula

$$f(z,t) = (2\pi)^{-d-1}\int_{-\infty}^{\infty} tr(\pi_\lambda(z,t)^*\hat{f}(\lambda))|\lambda|^d d\lambda.$$

Let $f^\lambda$ stand for the inverse Fourier transform of $f$ in the *central variable* $t$

$$f^\lambda(z) = \int_{-\infty}^{\infty} f(z,t)e^{i\lambda t}dt. \tag{2}$$

By taking the Euclidean Fourier transform of $f^\lambda(z)$ in the variable $\lambda$, we obtain

$$f(z,t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-i\lambda t}f^\lambda(z)\,d\lambda. \tag{3}$$

Recalling the definition of $\pi_\lambda$ we see that we can write the group Fourier transform as

$$\hat{f}(\lambda) = \pi_\lambda(f^\lambda)$$

where $\pi_\lambda(f^\lambda) = \int_{\mathbb{C}^d} f^\lambda(z)\pi_\lambda(z,0)dz$. The operator which takes a function $g$ on $\mathbb{C}^d$ into the operator

$$\int_{\mathbb{C}^d} g(z)\pi_\lambda(z,0)dz$$

is called the Weyl transform of $g$ and is denoted by $W_\lambda(g)$. Thus $\widehat{f}(\lambda) = W_\lambda(f^\lambda)$. With these notations we can rewrite the inversion formula as

$$f(z,t) = (2\pi)^{-d-1}\int_{-\infty}^{\infty} e^{-i\lambda t} tr(\pi_\lambda(z,0)^*\pi_\lambda(f^\lambda))|\lambda|^d d\lambda.$$

On the Heisenberg group, we have the vector fields

$$T = \frac{\partial}{\partial t}, \quad X_j = \frac{\partial}{\partial x_j} + \frac{1}{2}y_j\frac{\partial}{\partial t}, \quad Y_j = \frac{\partial}{\partial y_j} - \frac{1}{2}x_j\frac{\partial}{\partial t}, \quad j = 1, ..., d,$$

and they form a basis for the Lie algebra of left invariant vector fields on the Heisenberg group. The second-order operator

$$\mathcal{L} = -\sum_{j=1}^{d}(X_j^2 + Y_j^2)$$

is called the sublaplacian which is self-adjoint and nonnegative and hence admits a spectral decomposition

$$\mathcal{L} = \int_0^\infty \lambda \; dE_\lambda.$$

Given a bounded function $m$ defined on $(0,\infty)$ one can define the operator $m(\mathcal{L})$ formally by setting

$$m(\mathcal{L})f = \int_0^\infty m(\lambda)\; dE_\lambda f.$$

It can be shown that $\widehat{m(\mathcal{L})f}(\lambda) = \hat{f}(\lambda)m(H(\lambda))$ where $H(\lambda) = -\Delta + |\lambda|^2|x|^2$ are the scaled Hermite operators. Hence, the operators $m(\mathcal{L})f$ are examples of Fourier multipliers on the Heisenberg group. More generally, (right) Fourier multipliers on the Heisenberg group are operators defined by $\widehat{T_M f}(\lambda) = \hat{f}(\lambda)M(\lambda)$ where $M(\lambda)$, called the multiplier is a family of bounded linear operators on $L^2(\mathbb{R}^d)$.

The boundedness properties of these operators have been studied by several authors, see e.g. [6,7,11]. We make use of the following result. Let $0 \neq \psi \in C_0^\infty(\mathbb{R}^+)$ be a fixed cut-off function with support contained in the interval $[\frac{1}{2}, 1]$, and define the scale invariant localized Sobolev norm of order $\beta$ of $m$ by

$$\|m\|_{L^2_{\beta,sloc}} = \sup_{t>0} \|\psi m(t\cdot)\|_{L^2_\beta}.$$

**Theorem 6 (Müller-Stein, Hebisch [7,11]).** *If* $\|m\|_{L^2_{\beta,sloc}} < \infty$ *for some* $\beta > (2d+1)/2$, *then* $m(\mathcal{L})$ *is bounded on* $L^p(\mathbb{H}^d)$ *for* $1 < p < \infty$, *and of weak type* $(1,1)$.

We make use of this theorem in proving our main result. Actually we need an analogue of the above result in the context of polarised Heisenberg group $\mathbb{H}^d_{pol}$ which is just $\mathbb{R}^d \times \mathbb{R}^d \times \mathbb{R}$ with the group law

$$(x, y, t)(x', y', t') = (x + x', y + y', t + t' + x' \cdot y).$$

A basis for the algebra of left invariant vectors fields on this group are given by

$$T = \frac{\partial}{\partial t}, \quad \tilde{X}_j = \frac{\partial}{\partial x_j} + y_j \frac{\partial}{\partial t}, \quad \tilde{Y}_j = \frac{\partial}{\partial y_j}, \quad j = 1, ..., d.$$

The sublaplacian is then defined as the second-order operator

$$\tilde{\mathcal{L}} = -\sum_{j=1}^{d} (\tilde{X}_j^2 + \tilde{Y}_j^2).$$

The group $\mathbb{H}^d_{pol}$ is isomorphic to $\mathbb{H}^d$ and the isomorphism is given by the map $\Phi : \mathbb{H}^d \to \mathbb{H}^d_{pol}, \quad \Phi(x, y, t) = (x, y, t + \frac{1}{2} x \cdot y)$. Note that $\Phi$ is measure preserving and it is easy to check that

$$\mathcal{L}(f \circ \Phi^{-1}) = \tilde{\mathcal{L}} f \circ \Phi$$

for reasonable functions $f$ on $\mathbb{H}^d_{pol}$. In view of this, an analogue of Theorem 6 is true for $m(\tilde{\mathcal{L}})$. This also follows from the fact that the above theorem is valid in a more general context of H-type groups, see [6,7,10].

Using the isomorphism $\Phi$ we can define the following family of representations for $\mathbb{H}^d_{pol}$. For each non zero real $\lambda$, the representations $\rho_\lambda = \pi_\lambda \circ \Phi^{-1}$ are irreducible and unitary. We can use them to define Fourier transform on the group $\mathbb{H}^d_{pol}$.

## 2.3 Hermite and Special Hermite Functions

The spectral decomposition of $H = -\Delta + |x|^2$ is given by the Hermite expansion. Let $\Phi_\alpha(x), \ \alpha \in \mathbb{N}^d$ be the normalized Hermite functions which are products of one dimensional Hermite functions. More precisely, $\Phi_\alpha(x) = \Pi_{j=1}^d h_{\alpha_j}(x_j)$ where

$$h_k(x) = (\sqrt{\pi} 2^k k!)^{-1/2} (-1)^k e^{\frac{1}{2}x^2} \frac{d^k}{dx^k} e^{-x^2}.$$

The Hermite functions $\Phi_\alpha$ are eigenfunctions of $H$ with eigenvalues $(2|\alpha| + d)$ where $|\alpha| = \alpha_1 + ... + \alpha_d$. Moreover, they form an orthonormal basis for $L^2(\mathbb{R}^d)$. The spectral decomposition of $H$ is then written as

$$H = \sum_{k=0}^{\infty} (2k + d) P_k, \qquad P_k f(x) = \sum_{|\alpha| = k} \langle f, \Phi_\alpha \rangle \Phi_\alpha$$

where $\langle \cdot, \cdot \rangle$ is the inner product in $L^2(\mathbb{R}^d)$. Given a function $m$ defined and bounded on the set of all natural numbers we can use the spectral theorem to define $m(H)$. The action of $m(H)$ on a function $f$ is given by

$$m(H)f = \sum_{\alpha \in \mathbb{N}^d} m(2|\alpha| + d)\langle f, \Phi_\alpha \rangle \Phi_\alpha = \sum_{k=0}^{\infty} m(2k + d) P_k f.$$

This operator $m(H)$ is bounded on $L^2(\mathbb{R}^d)$. This follows immediately from the Plancherel theorem for the Hermite expansions as $m$ is bounded.

On the other hand, the mere boundedness of $m$ is not sufficient to imply the $L^p$ boundedness of $m(H)$ for $p \neq 2$. So, we need to impose some conditions $m$ to ensure that $m(H)$ is bounded on $L^p(\mathbb{R}^d)$. The boundedness results of $m(H)$ on $L^p(\mathbb{R}^d)$ have been studied by several authors, see e.g. [15,18]. Our aim in this paper is to study the boundedness of $m(H)$ on modulation spaces.

In the sequel, we make use of some properties of special Hermite functions $\Phi_{\alpha,\beta}$ which are defined as follows. Let $\pi(z) = \pi_1(z, t)$ and define

$$\Phi_{\alpha,\beta}(z) = (2\pi)^{-d/2} \langle \pi(z) \Phi_\alpha, \Phi_\beta \rangle. \tag{4}$$

Then it is well known that these so called special Hermite functions form an orthonormal basis for $L^2(\mathbb{C}^d)$. Moreover, they are eigenfunctions of the special Hermite operator $L$ which is defined by the equation

$$\mathcal{L}(f(z)e^{it}) = e^{it} Lf(z).$$

Indeed, we can show ([15, Theorem 1.3.3]) that

$$L\Phi_{\alpha,\beta} = (2|\alpha| + d)\Phi_{\alpha,\beta}.$$

For a function $f \in L^2(\mathbb{C}^d)$ we have the eigenfunction expansion

$$f(z) = \sum_{\alpha \in \mathbb{N}^d} \sum_{\beta \in \mathbb{N}^d} \langle f, \Phi_{\alpha,\beta} \rangle \Phi_{\alpha,\beta}(z)$$

which is called the special Hermite expansion.

Let $f$ and $g$ be two measurable functions on $\mathbb{C}^d$. We recall that the $\lambda-$twisted convolution $(0 \neq \lambda \in \mathbb{R})$ of $f$ and $g$ is the function $f *_\lambda g$ defined by

$$f *_\lambda g(z) = \int_{\mathbb{C}^d} f(z - w) g(w) e^{i \frac{\lambda}{2} \mathrm{Im}(z \cdot \bar{w})} dw$$

for all $z$ such that the integral exits. When $\lambda = 1$, we simply call them twisted convolution and denote them by $f \times g$. It is well known that the twisted convolution of special Hermite functions satisfies [15, Proposition 1.3.2] the following relation

$$\Phi_{\alpha,\beta} \times \Phi_{\mu,v} = (2\pi)^{d/2} \delta_{\beta,\mu} \Phi_{\alpha,v} \tag{5}$$

where $\delta_{\beta,\mu} = 1$ if $\beta = \mu$, otherwise it is 0. Using the identity (5), we can show that the special Hermite expansions can be written ([15, Sect. 2.1]) in the compact form as follows:

$$\begin{aligned} f(z) &= (2\pi)^{-d/2} \sum_{\alpha \in \mathbb{N}^d} f \times \Phi_{\alpha,\alpha}(z) \\ &= (2\pi)^{-d/2} \sum_{k=0}^{\infty} f \times \left( \sum_{|\alpha|=k} \Phi_{\alpha,\alpha}(z) \right) \\ &= (2\pi)^{-d} \sum_{k=0}^{\infty} f \times \phi_k(z) \end{aligned}$$

where $\varphi_k$ are the Laguerre functions of type $(d-1)$:

$$\varphi_k(z) = (2\pi)^{d/2} \sum_{|\alpha|=k} \Phi_{\alpha,\alpha}(z) = L_k^{d-1}(\frac{1}{2}|z|^2) e^{-\frac{1}{4}|z|^2}.$$

We note that $f \times \Phi_{\alpha,\alpha}$ is an eigenfunction of the operator $L$ with the eigenvalue $(2|\alpha| + d)$. Hence $(2\pi)^{-d} f \times \phi_k$ is the projection of $f$ onto the eigenspace corresponding to the eigenvalue $(2k+d)$. The spectral decomposition of the operator $L$ is given by the special Hermite functions and we can write the same in a compact form as

$$Lf(z) = (2\pi)^{-d} \sum_{k=0}^{\infty} (2k+d) f \times \varphi_k(z).$$

As in the case of the Hermite operators, one can define and study special Hermite multipliers $m(L)$, see [15]. The functions $\Phi_{\alpha,\beta}$ can be expressed in terms of Laguerre functions. In particular, we have ([15, Theorem 1.3.5])

$$\Phi_{\alpha,0}(z) = (2\pi)^{-d/2} (\alpha!)^{-1/2} \left( \frac{i}{\sqrt{2}} \right)^{|\alpha|} \bar{z}^{\alpha} e^{-\frac{1}{4}|z|^2}. \tag{6}$$

We also have to deal with the family of operators $H(\lambda) = -\Delta + \lambda^2 |x|^2$ whose eigenfunctions are given by scaled Hermite functions. For $\lambda \in \mathbb{R}^*$ and each $\alpha \in \mathbb{N}^d$, we define the family of scaled Hermite functions

$$\Phi_\alpha^\lambda(x) = |\lambda|^{\frac{d}{4}} \Phi_\alpha(\sqrt{|\lambda|} x), \quad x \in \mathbb{R}^d.$$

They form an orthonormal basis for $L^2(\mathbb{R}^d)$. The spectral decomposition of the scaled Hermite operator $H(\lambda) = -\Delta + |\lambda|^2 |x|^2$ is then written as

$$H(\lambda) = \sum_{k=0}^{\infty} (2k+d) |\lambda| P_k(\lambda), \tag{7}$$

for $\lambda \in \mathbb{R}^*$, where $P_k(\lambda)$ are the (finite-dimensional) orthogonal projections defined on $L^2(\mathbb{R}^d)$ by

$$P_k(\lambda)f = \sum_{|\alpha|=k} \langle f, \Phi_\alpha^\lambda \rangle \Phi_\alpha^\lambda,$$

where $f \in L^2(\mathbb{R}^d)$. We can now define scaled special Hermite functions

$$\Phi_{\alpha,\beta}^\lambda(z) = (2\pi)^{-d/2}|\lambda|^{d/2}\langle \pi_\lambda(z)\Phi_\alpha^\lambda, \Phi_\beta^\lambda \rangle. \tag{8}$$

They form a complete orthonormal system in $L^2(\mathbb{C}^d)$. For every $f \in L^2(\mathbb{C}^d)$, we have the special Hermite expansion

$$f = \sum_{\alpha \in \mathbb{N}^d} \sum_{\beta \in \mathbb{N}^d} \langle f, \Phi_{\alpha,\beta}^\lambda \rangle \Phi_{\alpha,\beta}^\lambda. \tag{9}$$

We now define the (scaled) special Hermite operator (twisted Laplacian) by the relation

$$\mathcal{L}(e^{i\lambda t} f(z)) = e^{i\lambda t} L_\lambda f(z). \tag{10}$$

We note that the operators $L_\lambda$ and $H(\lambda)$ are related via the Weyl transform:

$$\pi_\lambda(L_\lambda f) = \pi_\lambda(f)H(\lambda).$$

In fact, the scaled Hermite functions are eigenfunctions of the operator $L_\lambda$ :

$$L_\lambda \Phi_{\alpha,\beta}^\lambda(z) = (2|\alpha| + d)|\lambda| \Phi_{\alpha,\beta}^\lambda(z). \tag{11}$$

Note that the eigenvalues of $L_\lambda$ are of the form $(2k+d)|\lambda|, k = 0, 1, 2, ...,$ and the $k^{th}$ eigenspace corresponding to the eigenvalue $(2k+d)|\lambda|$ is infinite-dimensional being the span of $\{\Phi_{\alpha,\beta}^\lambda : |\alpha| = k, \beta \in \mathbb{N}^d\}$. It is well known that the special Hermite expansion can be written in the compact form as follows:

$$f(z) = (2\pi)^{-d}|\lambda|^d \sum_{k=0}^{\infty} f *_\lambda \varphi_k^\lambda(z) \tag{12}$$

where $\varphi_k^\lambda$ is the scaled Laguerre functions of type $(d-1)$

$$\varphi_k^\lambda(z) = L_k^{n-1}\Big(\frac{1}{2}|\lambda||z|^2\Big)e^{-\frac{1}{4}|\lambda||z|^2}. \tag{13}$$

The spectral decomposition of $L_\lambda$ is then written as

$$L_\lambda f(z) = (2\pi)^{-d}|\lambda|^d \sum_{k=0}^{\infty}(2k + d)|\lambda| f *_\lambda \varphi_k^\lambda(z).$$

In particular, recalling $f^\lambda$ (see(2)), we have

$$L_\lambda f^\lambda(z) = (2\pi)^{-d}|\lambda|^d \sum_{k=0}^{\infty}(2k + d)|\lambda| f^\lambda *_\lambda \varphi_k^\lambda(z). \tag{14}$$

Taking the Fourier transform in $\lambda$ variable in equation (14), and using (10) and (3), we obtain the spectral decomposition of the sublaplacian $\mathcal{L}$ on $\mathbb{H}^d$ as follows:

$$\mathcal{L}f(z,t) = (2\pi)^{-d-1}\int_{-\infty}^{\infty}\Big(\sum_{k=0}^{\infty}(2k+d)|\lambda| f^{\lambda} *_{\lambda} \varphi_k^{\lambda}(z)\Big)e^{-i\lambda t}|\lambda|^d d\lambda. \tag{15}$$

For more on spectral theory of sublaplacian on Heisenberg group, we refer to [17, Chap. 2].

### 2.4 A Transference Theorem for $m(\mathcal{L})$

Let $\Gamma_0$ be the subgroup of $\mathbb{H}^d$ consisting of elements of the form $(0, 0, 2\pi k)$ where $k \in \mathbb{Z}$. Then $\Gamma_0$ is a central subgroup and the quotient $\mathbb{H}^d/\Gamma_0$ whose underlying manifold is $\mathbb{R}^d \times \mathbb{R}^d \times \mathbb{T}$ is called the Heisenberg group with compact center.(See [17, Chap. 4]) Let $\mathcal{L}_0$ be the sublaplacian on $\mathbb{H}^d/\Gamma_0$. Note that $\mathcal{L}_0 f = \mathcal{L}f$, considering $f$ as a function on $\mathbb{H}^d$.

Note that on functions $g$ which are independent of $t$, $\mathcal{L}_0 g(z) = -\Delta g(z)$ where $\Delta$ is the Euclidean Laplacian on $\mathbb{C}^d$. The spectral decomposition of the sublaplacian $\mathcal{L}_0$ on this group is given by

$$\mathcal{L}_0 f(z,t) = (-\Delta)f^0(z) + (2\pi)^{-d-1}\sum_{k\in\mathbb{Z}\setminus\{0\}}\Big(\sum_{j=0}^{\infty}e^{-ikt}((2j+d)|k|)f^k *_k \varphi_j^k(z)\Big)|k|^d$$

where $\varphi_j^k(z) = \varphi_j(\sqrt{|k|}z)$ and $f^k$ is defined by the equation $f * e_j^k(z,t) = e^{-ikt}f^k *_k \varphi_j^k(z)$, $e_j^k(z,t) = e^{ikt}\varphi_j^k(z)$. More generally,

$$m(\mathcal{L}_0)f(z,t) = m(-\Delta)f^0(z) + (2\pi)^{-d-1}\sum_{k\in\mathbb{Z}\setminus\{0\}}\Big(\sum_{j=0}^{\infty}e^{-ikt}m((2j+d)|k|)f^k *_k \varphi_j^k(z)\Big)|k|^d.$$

There is a transference theorem which connects $m(\mathcal{L})$ on $\mathbb{H}^d$ with $m(\mathcal{L}_0)$ on $\mathbb{H}^d/\Gamma_0$ which is the analogue of the classical de Leeuw's theorem (See Theorem 10) on the real line.

**Theorem 7.** *Let $1 < p < \infty$. Suppose $m(\mathcal{L})$ is a bounded operator on $L^p(\mathbb{H}^d)$. Then the transferred operator $m(\mathcal{L}_0)$ is bounded on $L^p(\mathbb{H}^d/\Gamma_0)$.*

This theorem has been proved and used in [13]. The idea is to realise $m(\mathcal{L})$ ( $m(\mathcal{L}_0)$ ) as an operator valued Fourier multiplier for $\mathbb{R}$ (resp. $\mathbb{T}$.). Indeed, writing the Fourier inversion as

$$f(z,t) = (2\pi)^{-1}\int_{-\infty}^{\infty}e^{-i\lambda t}f^{\lambda}(z)d\lambda$$

and recalling that $L_\lambda$ is defined by the equation $\mathcal{L}(f(z)e^{i\lambda t}) = e^{i\lambda t}L_\lambda f(z)$ we have

$$m(\mathcal{L})f(z,t) = (2\pi)^{-1}\int_{-\infty}^{\infty}e^{-i\lambda t}m(L_\lambda)f^{\lambda}(z)d\lambda$$

where $m(L_\lambda)$ is the multiplier transform associated to the elliptic operator $L_\lambda$ which has an explicit spectral decomposition. By identifying $L^p(\mathbb{H}^d)$ with $L^p(\mathbb{R}, L^p(\mathbb{C}^d))$ we see that, in view of the above equation, $m(\mathcal{L})$ can be considered as an operator valued Fourier multiplier for the Euclidean Fourier transform on $\mathbb{R}$. Similarly, $m(\mathcal{L}_0)$ can be thought about as an operator valued multiplier for the Fourier series on $\mathbb{T}$. Now, it is an easy matter to imitate the proof of de Leeuw's theorem to prove the above result.

We need the following analogue of Theorem 7 for the reduced polarised Heisenberg group. Let $\tilde{\mathcal{L}}_0$ be the sublaplacian on $G^d = \mathbb{H}^d_{pol}/\Gamma_0$. We note that $\tilde{\mathcal{L}}_0 f = \tilde{\mathcal{L}} f$ considering $f$ as a function on $\mathbb{H}^d_{pol}$.

**Theorem 8.** *Let $1 < p < \infty$. Suppose $m(\tilde{\mathcal{L}})$ is a bounded operator on $L^p(\mathbb{H}^d_{pol})$. Then the transferred operator $m(\tilde{\mathcal{L}}_0)$ is bounded on $L^p(\mathbb{H}^d_{pol}/\Gamma_0)$.*

### 2.5 Modulation Spaces

In 1983, Feichtinger [3] introduced a class of Banach spaces, the so called modulation spaces, which allow a measurement of space variable and Fourier transform variable of a function or distribution on $\mathbb{R}^d$ simultaneously using the short-time Fourier transform(STFT). The STFT of a function $f$ with respect to a window function $g \in \mathcal{S}(\mathbb{R}^d)$ is defined by

$$V_g f(x,y) = (2\pi)^{-d/2} \int_{\mathbb{R}^d} f(t)\overline{g(t-x)}\, e^{-iy\cdot t}\, dt,\ \ (x,y) \in \mathbb{R}^{2d}$$

whenever the integral exists. For $x, y \in \mathbb{R}^d$ the translation operator $T_x$ and the modulation operator $M_w$ are defined by $T_x f(t) = f(t-x)$ and $M_y f(t) = e^{iy\cdot t} f(t)$. In terms of these operators the STFT may be expressed as

$$V_g f(x,y) = \langle f, M_y T_x g\rangle$$

where $\langle f, g\rangle$ denotes the inner product for $L^2$ functions, or the action of the tempered distribution $f$ on the Schwartz class function $g$. Thus $V : (f,g) \to V_g(f)$ extends to a bilinear form on $\mathcal{S}'(\mathbb{R}^d) \times \mathcal{S}(\mathbb{R}^d)$ and $V_g(f)$ defines a uniformly continuous function on $\mathbb{R}^d \times \mathbb{R}^d$ whenever $f \in \mathcal{S}'(\mathbb{R}^d)$ and $g \in \mathcal{S}(\mathbb{R}^d)$.

**Definition 1 (modulation spaces).** Let $1 \le p, q \le \infty$, and $0 \ne g \in \mathcal{S}(\mathbb{R}^d)$. The modulation space $M^{p,q}(\mathbb{R}^d)$ is defined to be the space of all tempered distributions $f$ for which the following norm is finite:

$$\|f\|_{M^{p,q}} = \left( \int_{\mathbb{R}^d} \left( \int_{\mathbb{R}^d} |V_g f(x,y)|^p dx \right)^{q/p} dy \right)^{1/q},$$

for $1 \le p, q < \infty$. If $p$ or $q$ is infinite, $\|f\|_{M^{p,q}}$ is defined by replacing the corresponding integral by the essential supremum.

*Remark 1.* The definition of the modulation space given above, is independent of the choice of the particular window function. See [4, Proposition 11.3.2(c), p.233].

Next, we shall see how the Fourier-Wigner transform and the STFT are related. Let $\pi$ be the Schrödinger representation of the Heisenberg group with the parameter $\lambda = 1$ which is realized on $L^2(\mathbb{R}^d)$ and explicitly given by

$$\pi(x, y, t)\phi(\xi) = e^{it} e^{i(x\cdot\xi + \frac{1}{2}x\cdot y)}\phi(\xi + y)$$

where $x, y \in \mathbb{R}^d, t \in \mathbb{R}, \phi \in L^2(\mathbb{R}^d)$. The Fourier-Wigner transform of two functions $f, g \in L^2(\mathbb{R}^d)$ is defined by

$$W_g f(x, y) = (2\pi)^{-d/2}\langle \pi(x, y, 0)f, g\rangle.$$

For $z = x + iy$, we put $\pi(x, y, 0) = \pi(z, 0) = \pi(z)$. We may rewrite the Fourier-Wigner transform as

$$W_g f(x, y) = \langle f, \pi^*(z)g\rangle,$$

where $\langle f, g\rangle$ denotes the inner product for $L^2$ functions, or the action of the tempered distribution $f$ on the Schwartz class function $g$. We recall the representation $\rho_1$ (See Sect. 2.2) of $\mathbb{H}^d_{pol}$, $\rho_1 = \rho = \pi \circ \Phi^{-1}$, $\rho(x, y, e^{it})$ acting on $L^2(\mathbb{R}^d)$ is given by

$$\rho(x, y, e^{it})\phi(\xi) = e^{it} e^{ix\cdot\xi}\phi(\xi + y), \quad \phi \in L^2(\mathbb{R}^d).$$

We now write the Fourier-Wigner transform in terms of the STFT: Specifically, we put $\rho(x, y)\phi(\xi) = e^{ix\cdot\xi}\phi(\xi + y)$, and have

$$\langle \pi(x, y)f, g\rangle = e^{\frac{i}{2}x\cdot y}\langle \rho(x, y)f, g\rangle = e^{-\frac{i}{2}x\cdot y} V_g f(y, -x). \tag{16}$$

This useful identity (16) reveals that the definition of modulation spaces we have introduced in the introduction and in the present section is essentially the same.

The following basic properties of modulation spaces are well-known and for the proof we refer the reader to [3,4].

**Lemma 1.** *1. The space $M^{p,q}(\mathbb{R}^d)(1 \le p \le \infty)$ is a Banach space.*
*2. $M^{p,p}(\mathbb{R}^d) \hookrightarrow L^p(\mathbb{R}^d)$ for $1 \le p \le 2$, and $L^p(\mathbb{R}^d) \hookrightarrow M^{p,p}(\mathbb{R}^d)$ for $2 \le p \le \infty$.*
*3. If $q_1 \le q_2$ and $p_1 \le p_2$, then $M^{p_1,q_1}(\mathbb{R}^d) \hookrightarrow M^{p_2,q_2}(\mathbb{R}^d)$.*
*4. $\mathcal{S}(\mathbb{R}^d)$ is dense in $M^{p,q}(\mathbb{R}^d)(1 \le p, q < \infty)$.*

We also refer to Gröchenig's book [4] for the basic definitions and further properties of modulation spaces. Finally, we note that there is also an equivalent definition of modulation spaces using frequency-uniform decomposition techniques (which is quite similar in the spirit of Besov spaces), independently studied by Wang et al. in [19], which has turned out to be very fruitful in PDE, see [20].

## 3 Hermite Multipliers via Transference Theorems

As mentioned in the introduction we prove our main results via transference techniques. We first investigate the connection between $m(H)$ acting on $M^{q,p}(\mathbb{R}^d)$ and $m(\mathcal{L}_0)$ acting on $L^p(G^d)$ where $G^d = \mathbb{H}^d_{pol}/\Gamma_0$. Recall that $f \in M^{q,p}(\mathbb{R}^d)$ if and only if $\langle \rho(x,y)f, \Phi_0\rangle \in L^p(\mathbb{R}^d, L^q(\mathbb{R}^d))$. Moreover, for any left invariant vector filed $X$ on $G^d$ we have

$$X\langle \rho(x,y,t)f, \Phi_0\rangle = \langle \rho(x,y,t)\rho^*(X)f, \Phi_0\rangle$$

where $\rho^*(X)\varphi = \frac{d}{dt}|_{t=0}\rho(\exp(tX))\varphi$. A simple calculation shows that $\rho^*(X_j) = i\xi_j$ and $\rho^*(Y_j) = \frac{\partial}{\partial \xi_j}$ for $j = 1, 2, ..., d$. Consequently, we get $\mathcal{L}\langle \rho(x,y,t)f, \Phi_0\rangle = \langle \rho(x,y,t)Hf, \Phi_0\rangle$ which leads to the identity

$$m(\tilde{\mathcal{L}}_0)\langle \rho(\cdot)f, \Phi_0\rangle = \langle \rho(x,y,t)m(H)f, \Phi_0\rangle$$

via spectral theorem (See [16, Sect. 2.3] for details). Thus we see that

$$\|m(\tilde{\mathcal{L}}_0)\langle \rho(\cdot)f, \Phi_0\rangle\|_{L^p(\mathbb{R}^d, L^q(\mathbb{R}^d\times\mathbb{T}))} = \|m(H)f\|_{M^{q,p}(\mathbb{R}^d)}. \tag{17}$$

Consequently, the boundedness of $m(H)$ on $M^{q,p}(\mathbb{R}^d)$ is implied by the boundedness of $m(\tilde{\mathcal{L}}_0)$ on $L^p(\mathbb{R}^d, L^q(\mathbb{R}^d \times \mathbb{T}))$. We can now prove Theorem 2.

*Proof of Theorem 2.* We do this in two steps. Assuming that $m(\tilde{\mathcal{L}})$ is bounded on $L^p(\mathbb{H}^d_{pol})$ it follows from Theorem 8 that $m(\tilde{\mathcal{L}}_0)$ is bounded on $L^p(\mathbb{H}^d_{pol}/\Gamma)$. As the underlying manifold of $\mathbb{H}^d_{pol}/\Gamma$ is $\mathbb{R}^d\times\mathbb{R}^d\times\mathbb{T}$ we have the boundedness of $m(\tilde{\mathcal{L}}_0)$ on the space $L^p(\mathbb{R}^d, L^p(\mathbb{R}^d\times\mathbb{T}))$. If we can show that under the assumptions on $p$ and $q$ stipulated in Theorem 2, the operator $m(\tilde{\mathcal{L}}_0)$ is bounded on the mixed norm space $L^p(\mathbb{R}^d, L^q(\mathbb{R}^d\times\mathbb{T}))$ then we are done simply by applying $m(\tilde{\mathcal{L}}_0)$ to $\langle \pi(z,t)f, \Phi_0\rangle$ (see (17)). To this end, we make use of the following result of Herz and Riviére [9].

**Theorem 9** ***(Herz-Rivière*** *[9]****).*** *Let $G = \Gamma H$ be the semi-direct product of an amenable group $H$ by a locally compact group $\Gamma$. Assume that a bounded linear operator $T : L^p(G) \to L^p(G)$ commutes with left-translations. Take $q$ such that $1 \le p \le q \le 2$ or $2 \le q \le p < \infty$. Then for any complex-valued continuous function $f$ of compact support on $G$, we have*

$$\|Tf\|_{L^p(\Gamma, L^q(H))} \lesssim \|f\|_{L^p(\Gamma, L^q(H))}.$$

*Consequently, $T$ has a bounded extension to the mixed norm spaces $L^p(\Gamma, L^q(H))$.*

In view of the above theorem all we have to do is to realise $G^d$ as a semidirect product of $\mathbb{R}^d$ with $\mathbb{R}^d\times\mathbb{T}$. As $m(\mathcal{L}_0)$ is invariant under left translations, we can then apply the above theorem to arrive at the required conclusion.

Let us briefly recall the definition of a semidirect product for the convenience of the readers. Let $H$ be a topological group whose operation is written as $+$

and $\varGamma$ another topological group, written as multiplicatively, such that there is a continuous map $\varGamma \times H \to H$: $(\sigma, x) \mapsto \sigma x$ with $\sigma(x+y) = \sigma x + \sigma y$ and $\tau(\sigma x) = (\tau\sigma)x$. In this situation, we say that $\varGamma$ acts on $H$. The semi-direct product $\varGamma H$ is then the topological space $\varGamma \times H$ with the group operation

$$(\sigma, x) \cdot (\tau, y) = (\sigma\tau, \tau x + y).$$

Let $\varGamma = (\mathbb{R}^d, +)$ be the additive group and $H = (\mathbb{R}^d \times \mathbb{T}, \cdot)$ be the group with following group law:

$$(y, e^{it}) \cdot (y', e^{it'}) = (y + y', e^{i(t+t')}).$$

We define a map $\varGamma \times H \to H$ as $(x, (y, e^{it})) \mapsto (y, e^{i(t+xy)})$. We note that via this map, $\varGamma$ acts on $H$. If there is no confusion, we write $(x, (y, e^{it})) = (x, y, e^{it})$ for $(x, (y, e^{it})) \in \varGamma \times H$. Forming the semi-direct product $G = \varGamma H$ we see that the group law is given by

$$(x, y, e^{it})(x', y', e^{it'}) = (x + x', y + y', e^{i(t+t'+x'.y)}).$$

This is precisely the group $G^d = \mathbb{H}^d_{pol}/\varGamma$. Hence, by Theorem 9 we get the boundedness of $m(\mathcal{L}_0)$ on the mixed norm space $L^p(\mathbb{R}^d, L^q(\mathbb{R}^d \times \mathbb{T}))$. This completes the proof of Theorem 2.

*Proof of Theorem* 1. Using Theorems 6 and 8, we conclude that $m(\tilde{\mathcal{L}}_0)$ is bounded on $L^p(\mathbb{H}^d_{pol})$. We now apply Theorem 2, to complete the proof.

## 4 Hermite Multipliers via Fourier Multipliers on Torus

In the last section, we have proved the boundedness of Hermite multiplier on modulation spaces via transference results. Specifically, we have proved that the boundedness of multiplier operators on Heisenberg groups ensures the boundeness of corresponding Hermite multiplier operators on modulation spaces.

In this section, we first we prove the transference result which connects Hermite multipliers on modulation spaces and Fourier multipliers on $L^p(\mathbb{T}^d)$. Specifically, our result (Proposition 1) states that boundedness of Fourier multipliers on torus guarantees the boundedness of Hermite multipliers on modulation spaces. In order to find a fruitful application of our result, we need to know the boundedness of Fourier multiplier on $L^p(\mathbb{T}^d)$. To this end, we prove(Proposition 2) boundedness of Fourier multiplier on $L^p(\mathbb{R}^d)$−and then use the celebrated theorem of de Leeuw to come back to the Fourier multiplier on $L^p(\mathbb{T}^d)$. Combining these results, finally in this section we prove Theorem 4. For the sake of convenience of the reader, we recall definitions:

**Definition 2 (Hermite multipliers on $M^{p,q}(\mathbb{R}^d)$).** Let $m$ be a bounded function on $\mathbb{N}^d$. We say that $m$ is a Hermite multiplier on the space $M^{p,q}(\mathbb{R}^d)$ if the linear operator defined by

$$T_m f = \sum_{\alpha \in \mathbb{N}^d} m(\alpha) \langle f, \varPhi_\alpha \rangle \varPhi_\alpha, \ \ (f \in \mathcal{S}(\mathbb{R}^d))$$

extends to a bounded linear operator from $M^{p,q}(\mathbb{R}^d)$ into itself, that is, $\|T_m f\|_{M^{p,q}} \lesssim \|f\|_{M^{p,q}}$.

**Definition 3 (Fourier multiplier on $L^p(\mathbb{T}^d)$).** Let $m$ be a bounded measurable function defined on $\mathbb{Z}^d$. We say that $m$ is a Fourier multiplier on $L^p(\mathbb{T}^d)$ if the linear operator $T_m$ defined by

$$\widehat{(T_m f)}(\alpha) = m(\alpha)\hat{f}(\alpha), \ (f \in \mathcal{P}(\mathbb{T}^d), \alpha \in \mathbb{Z}^d)$$

where $\hat{f}(\alpha) = \int_{\mathbb{T}^d} f(\theta)e^{-i\theta\cdot\alpha}d\theta$ are the Fourier coefficients of $f$, extends to a bounded linear operator from $L^p(\mathbb{T}^d)$ into itself, that is, $\|T_m f\|_{L^p(\mathbb{T}^d)} \lesssim \|f\|_{L^p(\mathbb{T}^d)}$.

Now we prove a transference result for Fourier multiplier on torus and Hermite multiplier on modulation spaces−which is of interest in itself. Specifically, we have the following proposition.

**Proposition 1.** *Let $1 \leq p < \infty$. If $m : \mathbb{Z}^d \to \mathbb{C}$ is a Fourier multiplier on $L^p(\mathbb{T}^d)$, then $m|_{\mathbb{N}^d}$, the restriction of $m$ to $\mathbb{N}^d$, is a Hermite multiplier on $M^{p,p}(\mathbb{R}^d)$.*

*Proof.* Since $\mathcal{S}(\mathbb{R}^d)$ is dense (Lemma 1(4)) in $M^{p,p}(\mathbb{R}^d)$, it would be sufficient to prove

$$\|T_m f\|_{M^{p,p}} \lesssim \|f\|_{M^{p,p}} \tag{18}$$

for $f \in \mathcal{S}(\mathbb{R}^d)$. In fact, by density argument, it follows that $T_m$ has a bounded extension to $M^{p,p}(\mathbb{R}^d)$, that is, inequality (18) holds true for $f \in M^{p,p}(\mathbb{R}^d)$.

Let $f \in \mathcal{S}(\mathbb{R}^d)$. Then using special Hermite functions (4) and their property (6), we compute the modulation space norm (See Sect. 2.5 ) of Hermite multiplier operator (See Definition 2), and we obtain

$$\begin{aligned}\|T_m f\|^p_{M^{p,p}} &= (2\pi)^{-dp/2}\int_{\mathbb{R}^{2d}} |\langle \pi(x,y)T_m f, \Phi_0\rangle|^p dy\ dx \\ &= (2\pi)^{-dp/2}\int_{\mathbb{R}^{2d}} \left|\sum_{\alpha\in\mathbb{N}^d} m(\alpha)\langle f, \Phi_\alpha\rangle\langle \pi(x,y)\Phi_\alpha, \Phi_0\rangle\right|^p dy\ dx \\ &= \int_{\mathbb{C}^d} \left|\sum_{\alpha\in\mathbb{N}^d} m(\alpha)\langle f, \Phi_\alpha\rangle \Phi_{\alpha,0}(z)\right|^p dz \\ &= (2\pi)^{-dp/2}\int_{\mathbb{C}^d} \left|\sum_{\alpha\in\mathbb{N}^d} m(\alpha)\langle f, \Phi_\alpha\rangle \frac{i^{|\alpha|}\bar{z}^\alpha}{\sqrt{\alpha!}\,2^{|\alpha|/2}} e^{-\frac{1}{4}|z|^2}\right|^p dz. \end{aligned} \tag{19}$$

By using polar coordinates $z_j = r_j e^{i\theta_j}$, $r_j := |z_j| \in [0,\infty), z_j \in \mathbb{C}$ and $\theta_j \in [0, 2\pi)$, we get

$$z^\alpha = r^\alpha e^{i\alpha\cdot\theta} \ \text{ and } \ dz = r_1 r_2 \cdots r_d d\theta dr \tag{20}$$

where $r = (r_1, \cdots, r_d)$, $\theta = (\theta_1, \cdots, \theta_d)$, $dr = dr_1 \cdots dr_d$, $d\theta = d\theta_1 \cdots d\theta_d$, $|r| = \sqrt{\sum_{j=1}^d r_j^2}$.
By writing the integral over $\mathbb{C}^d = \mathbb{R}^{2d}$ in polar coordinates in each time-frequency pair and using (20), we have

$$\int_{\mathbb{C}^d} \left| \sum_{\alpha \in \mathbb{N}^d} m(\alpha) \langle f, \Phi_\alpha \rangle \frac{i^{|\alpha|} \bar{z}^\alpha}{\sqrt{\alpha!}\, 2^{|\alpha|/2}} e^{-\frac{1}{4}|z|^2} \right|^p dz \tag{21}$$

$$= \prod_{j=1}^d \int_{\mathbb{R}+} \int_{[0,2\pi]} \left| \sum_{\alpha \in \mathbb{N}^d} m(\alpha) \left( \langle f, \Phi_\alpha \rangle \frac{i^{|\alpha|} r^\alpha}{\sqrt{\alpha!}\, 2^{|\alpha|/2}} e^{-\frac{1}{4}|r|^2} \right) e^{-i\alpha \cdot \theta} \right|^p r_j \, d\theta_j \, dr_j .$$

We put $a_\alpha = \langle f, \Phi_\alpha \rangle \frac{i^{|\alpha|} r^\alpha}{\sqrt{\alpha!}\, 2^{|\alpha|/2}} e^{-\frac{1}{4}|r|^2}$. Since $|\langle f, \Phi_\alpha \rangle| \leq \|f\|_{L^2} \|\Phi_\alpha\|_{L^2} \leq \|f\|_2$, the series $\sum_{\alpha \in \mathbb{N}^d} |a_\alpha|$ converges. Thus there exists a continuous function $g \in L^p(\mathbb{T}^d)$ with Fourier coefficients $\hat{g}(\alpha) = a_\alpha$ for $\alpha \in \mathbb{N}^d$ and $\hat{g}(\alpha) = 0$ for $\alpha \in \mathbb{Z}^d \setminus \mathbb{N}^d$. In fact, the Fourier series of this $g$ is absolutely convergent, and therefore we may write

$$g(e^{i\theta}) = \sum_{\alpha \in \mathbb{Z}^d} a_\alpha e^{-i\alpha \cdot \theta}. \tag{22}$$

Since $m$ is a Fourier multiplier on $L^p(\mathbb{T}^d)$, (22) gives

$$\int_{[0,2\pi]^d} \left| \sum_{\alpha \in \mathbb{N}^d} m(\alpha) a_\alpha e^{i\alpha \cdot \theta} \right|^p d\theta \lesssim \int_{[0,2\pi]^d} \left| \sum_{\alpha \in \mathbb{N}^d} a_\alpha e^{-i\alpha \cdot \theta} \right|^p d\theta. \tag{23}$$

Now taking (19), (21), and (23) into account, we obtain

$$\begin{aligned} \|T_m f\|_{M^{p,p}} &\lesssim (2\pi)^{-d/2} \left( \prod_{j=1}^d \int_{\mathbb{R}+} \int_{[0,2\pi]} \left| \sum_{\alpha \in \mathbb{N}^d} \left( \langle f, \Phi_\alpha \rangle \frac{i^{|\alpha|} r^\alpha}{\sqrt{\alpha!}\, 2^{|\alpha|/2}} e^{-\frac{1}{4}|r|^2} \right) e^{-i\alpha \cdot \theta} \right|^p r_j \, d\theta_j \, dr_j \right)^{\frac{1}{p}} \\ &= (2\pi)^{-d/2} \left( \int_{\mathbb{C}^d} \left| \sum_{\alpha \in \mathbb{N}^d} \langle f, \Phi_\alpha \rangle \frac{i^{|\alpha|} \bar{z}^\alpha}{\sqrt{\alpha!}\, 2^{|\alpha|/2}} e^{-\frac{1}{4}|z|^2} \right|^p dz \right)^{\frac{1}{p}} \\ &= \left( \int_{\mathbb{C}^d} \left| \sum_{\alpha \in \mathbb{N}^d} \langle f, \Phi_\alpha \rangle \Phi_{\alpha,0}(z) \right|^p dz \right)^{\frac{1}{p}} \\ &= (2\pi)^{-d/2} \left( \int_{\mathbb{R}^{2d}} \left| \sum_{\alpha \in \mathbb{N}^d} \langle f, \Phi_\alpha \rangle \langle \pi(x,y) \Phi_\alpha, \Phi_0 \rangle \right|^p dy \, dx \right)^{\frac{1}{p}} \\ &= \|f\|_{M^{p,p}}. \end{aligned} \tag{24}$$

Thus, we conclude that $\|T_m f\|_{M^{p,p}} \lesssim \|f\|_{M^{p,p}}$ for $f \in \mathcal{S}(\mathbb{R}^d)$. This completes the proof of Proposition 1.

Next we recall the celebrated theorem of de Leeuw, which gives the relation between Fourier multipliers on Euclidean spaces and tori. To do this, we start with

**Definition 4 (Fourier multiplier on $L^p(\mathbb{R}^d)$).** Let $m$ be a bounded measurable function defined on $\mathbb{R}^d$. We say that $m$ is a Fourier multiplier on $L^p(\mathbb{R}^d)$ if the linear operator $T_m$ defined by

$$\widehat{(T_m f)}(\xi) = m(\xi) \hat{f}(\xi), \;\; (f \in \mathcal{S}(\mathbb{R}^d), \;\; \xi \in \mathbb{R}^d)$$

where $\hat{f}$ is the Fourier transform, extends to a bounded linear operator from $L^p(\mathbb{R}^d)$ into itself, that is, $\|T_m f\|_{L^p} \lesssim \|f\|_{L^p}$.

**Theorem 10** ***(de Leeuw).*** *If $m : \mathbb{R}^d \to \mathbb{C}$ is continuous and Fourier multiplier on $L^p(\mathbb{R}^d)(1 \leq p < \infty)$, then $m|_{\mathbb{Z}^d}$, the restriction of $m$ to $\mathbb{Z}^d$, is a Fourier multiplier on $L^p(\mathbb{T}^d)$.*

In order prove Theorem 4, we also need the boundedness of a following Fourier multiplier

$$m(\xi) = \frac{e^{i(2|\xi|_1+d)^\gamma}}{(2|\xi|_1+d)^\beta} \quad (\beta > 0, \gamma > 0, \xi \in \mathbb{R}^d) \tag{25}$$

where $|\xi|_1 := \sum_{j=1}^d |\xi_j|$, on $L^p(\mathbb{R}^d)$. This we shall prove in the next proposition. We note that in [14, Theorem 1], it is proved that the function $m(\xi)$ which is 0 near the origin and $|\xi|^{-\beta} e^{i|\xi|^\alpha}$ $(|\xi| = \sqrt{\sum_{j=1}^d \xi_j^2})$ outside a compact set is a Fourier multiplier on $L^p(\mathbb{R}^d)$ for a suitable choice of $\alpha, \beta, p, d$. Our proof of Proposition 2 is motivated by this result.

**Proposition 2.** *Let $m$ be given by (25). Then $m$ is a Fourier multiplier on $L^p(\mathbb{R}^d)$ for $|\frac{1}{p} - \frac{1}{2}| < \frac{\beta}{d\gamma}$.*

To prove this proposition we need the following technical lemma. We will prove this lemma at the end.

**Lemma 2.** *Let $\sigma > 0, \gamma > 0$ and*

$$k_\sigma(x) = \frac{1}{(2\pi)^{d/2}} \int_{\mathbb{R}^d} e^{i(2|\xi|_1+d)^\gamma} e^{-\sigma(2|\xi|_1+d)^\gamma} e^{ix\cdot\xi}\, d\xi \;\; (x \in \mathbb{R}^d). \tag{26}$$

*Then*

$$\|k_\sigma\|_{L^1} \lesssim \sigma^{-d/2} e^{-\frac{1}{2}\sigma d^\gamma}.$$

Proof of Proposition 2 Performing a simple change of variables in the gamma function, we write

$$(2|\xi|_1 + d)^{-\beta} = \frac{1}{\Gamma(\beta/\gamma)} \int_0^\infty \sigma^{\frac{\beta}{\gamma}-1} \exp(-\sigma(2|\xi|_1+d)^\gamma)\, d\sigma. \tag{27}$$

In view of Definition 4, (25) and (27), we write

$$\begin{aligned} T_m f(x) &= (2\pi)^{-d/2} \int_{\mathbb{R}^d} m(\xi) \hat{f}(\xi) e^{ix\cdot\xi} d\xi \\ &= \frac{1}{\Gamma(\beta/\gamma)} \int_0^\infty \sigma^{(\beta/\gamma)-1} (k_\sigma * f)(x)\, d\sigma \end{aligned} \tag{28}$$

where $k_\sigma$ is as defined in Lemma 2. Using (28), we have

$$\|T_m f\|_{L^p} \leqslant \frac{1}{\Gamma(\beta/\gamma)} \int_0^\infty \sigma^{(\beta/\gamma)-1} \|k_\sigma * f\|_{L^p}\, d\sigma. \tag{29}$$

The boundedness of the multiplier operator $T_m f$ on $L^p$ will follow if we could show that the operator $f \to k_\sigma * f$ is bounded on $L^p$. We shall achieve this by using Riesz-Thorin interpolation theorem and the standard duality argument. To this end, we use Lemma 2 and Young's inequality, to obtain

$$\|k_\sigma * f\|_{L^1} \lesssim \sigma^{-d/2} e^{-\frac{1}{2}\sigma d^\gamma} \|f\|_{L^1}. \tag{30}$$

Since $|\hat{k}_\sigma(\xi)| \leqslant e^{-\frac{1}{2}\sigma d^\gamma}$, Plancherel theorem gives

$$\|k_\sigma * f\|_{L^2} \leqslant e^{-\frac{1}{2}\sigma d^\gamma} \|f\|_{L^2}. \tag{31}$$

Taking (30), and (31) into account, Riesz-Thorin interpolation theorem gives

$$\|k_\sigma * f\|_{L^p} \leqslant C_1 \sigma^{-\lambda} e^{-\frac{1}{2}\sigma d^\gamma} \|f\|_{L^p} \tag{32}$$

where $\lambda = d\left(\frac{1}{p} - \frac{1}{2}\right)$, for $1 \leq p \leq 2$. Finally using (32) and (29), we see that $T_m$ is bounded from $L^p(\mathbb{R}^d) \to L^p(\mathbb{R}^d)$ if

$$\int_0^\infty \sigma^{\beta/\gamma - 1 - \lambda} e^{-\frac{1}{2}\sigma d^\gamma}\, d\sigma < \infty,$$

which happens if and only if $\frac{1}{p} - \frac{1}{2} < \frac{\beta}{d\gamma}$. This proves the theorem for the case when $1 < p < 2$ and the case $p \geqslant 2$ follows from the duality.

Now we shall prove our Lemma 2.

*Proof of Lemma 2.* Since $k_\sigma$ (see (26)) is the inverse Fourier transform of the function $\exp\left((i-\sigma)(2|\xi|_1 + d)^\gamma\right)$, we have

$$\widehat{k_\sigma}(\xi) = e^{(i-\sigma)(2|\xi|_1 + d)^\gamma}. \tag{33}$$

Let $\alpha = (\alpha_1, \cdots, \alpha_d) \in \mathbb{N}^d$ be a multi-index of length $l$, that is, $\sum_{j=1}^d \alpha_j = l$, and put $D_\xi^\alpha = \left(\frac{\partial}{\partial \xi_1}\right)^{\alpha_1} \cdots \left(\frac{\partial}{\partial \xi_d}\right)^{\alpha_d}, \ \ \xi = (\xi_1, ..., \xi_d) \in \mathbb{R}^d$. Then in view of (33), we have

$$|D_\xi^\alpha \widehat{k_\sigma}(\xi)| \lesssim (2|\xi|_1 + d)^{-l(1-\gamma)} e^{-\frac{1}{2}\sigma(2|\xi|_1 + d)^\gamma} e^{-\frac{1}{2}\sigma d^\gamma}. \tag{34}$$

To obtain inequality (34), we have used these ideas: After taking partial derivatives of higher order for $\widehat{k_\sigma}$, we have estimated all the powers of $(2|\xi|_1 + d)$ by the highest appearing power−which is $(2|\xi|_1 + d)^{-l(1-\gamma)}$, this we could do because $(2|\xi|_1 + d) \geqslant 1$. We have also dominated $e^{-\frac{1}{2}\sigma(2|\xi|_1 + d)^\gamma}$ by $e^{-\frac{1}{2}\sigma d^\gamma}$, which is obvious.

Next, by Plancherel's theorem and (34), we get

$$\||x|^l k_\sigma\|_{L^2} \lesssim \left(\int_{\mathbb{R}^d} (2|\xi|_1 + d)^{-2l(1-\gamma)} e^{-\sigma(2|\xi|_1 + d)^\gamma}\, d\xi\right)^{\frac{1}{2}} e^{-\frac{1}{2}\sigma d^\gamma}.$$

Performing a change of variable, we get

$$\||x|^l k_\sigma\|_{L^2} \lesssim \sigma^{-\frac{d}{2\gamma} + \frac{l(1-\gamma)}{\gamma}} e^{-\frac{1}{2}\sigma d^\gamma} \left(\int_{\mathbb{R}^d} g(2|\xi|_1 + \sigma^{1/\gamma} d)\, d\xi\right)^{1/2},$$

where $g(t) = t^{-2l(1-\gamma)}e^{-t^\gamma}$ $(t > 0)$. Noting that $\int g(2|\xi|_1 + \sigma^{1/\gamma}d)\,d\xi < \infty$ for all $\gamma$ and $l$, we conclude

$$\||x|^l k_\sigma\|_{L^2} \lesssim \sigma^{-\frac{d}{2\gamma}+\frac{l(1-\gamma)}{\gamma}} e^{-\frac{1}{2}\sigma d^\gamma}.$$

Next we use the fact that for $h \in L^2(\mathbb{R}^d) \cap L^1(\mathbb{R}^d)$, $l > d/2$ and $R > 0$, we have

$$\|h\|_{L^1} \lesssim R^{d/2}(\|h\|_{L^2} + R^{-l}\||x|^l h\|_{L^2})$$

for all $R > 0$. Taking $h = k_\sigma$ and $R = \sigma^{\frac{1-\gamma}{\gamma}}$, we obtain

$$\|k_\sigma\|_{L^1} \leqslant C\sigma^{-d/2}e^{-\frac{1}{2}\sigma d^\gamma}.$$

This completes the proof.

We now use Proposition 2 and Theorem 10 to obtain the following corollary.

**Corollary 3.** *Let* $\left|\frac{1}{p} - \frac{1}{2}\right| < \frac{\beta}{d\,\gamma}$, $\beta > 0, \gamma > 0$, $|\alpha|_1 = \sum_{j=1}^d |\alpha_j|$. *Then the sequence* $\{(2|\alpha|_1 + d)^{-\beta}\exp(i(2|\alpha|_1 + d)^\gamma)\}_{\alpha\in\mathbb{Z}^d}$ *defines a multiplier on* $L^p(\mathbb{T}^d)$.

Proof of Theorem 4 Using Corollary 3 and Proposition 2, we may deduce that $m(H)$ is bounded on $M^{p,p}(\mathbb{R}^d)$. This completes the proof.

## 5 Hermite Multiplier for Schrödinger Propagator

In this section, we prove the boundedness of Schrödinger propagator $m(H) = e^{itH}$ using the properties of Hermite and special Hermite functions. Our approach of proof illustrates how these functions nicely fit into modulation spaces−and prove useful estimate.

*Proof of Theorem 5.* Let $f \in \mathcal{S}(\mathbb{R}^d)$. Then we have the Hermite expansion of $f$ as follows:

$$f = \sum_{\alpha\in\mathbb{N}^d} \langle f, \Phi_\alpha\rangle \Phi_\alpha. \tag{35}$$

Now using (35) and (4), we obtain

$$\begin{aligned}\langle \pi(z)f, \Phi_0\rangle &= \sum_{\alpha\in\mathbb{N}^d} \langle f, \Phi_\alpha\rangle \langle \pi(z)\Phi_\alpha, \Phi_0\rangle \\ &= \sum_{\alpha\in\mathbb{N}^d} \langle f, \Phi_\alpha\rangle \Phi_{\alpha,0}(z). \end{aligned}\tag{36}$$

Since $\{\Phi_\alpha\}$ forms an orthonormal basis for $L^2(\mathbb{R}^d)$, (36) gives

$$\begin{aligned}\langle \pi(z)m(H)f, \Phi_0\rangle &= \sum_{\alpha\in\mathbb{N}^d} \langle m(H)f, \Phi_\alpha\rangle \Phi_{\alpha,0}(z) \\ &= \sum_{\alpha\in\mathbb{N}^d} m(2|\alpha| + d)\langle f, \Phi_\alpha\rangle \Phi_{\alpha,0}(z).\end{aligned}$$

Therefore, for $m(H) = e^{itH}$, we have

$$\begin{aligned}\langle \pi(z)e^{itH} f, \Phi_0\rangle &= e^{itd} \sum_{\alpha\in\mathbb{N}^d} e^{2it|\alpha|}\langle f, \Phi_\alpha\rangle \Phi_{\alpha,0}(z) \\ &= e^{itd}(2\pi)^{-d/2} \sum_{\alpha\in\mathbb{N}^d} e^{2it|\alpha|}\langle f, \Phi_\alpha\rangle (\alpha!)^{-1/2} \left(\frac{i}{\sqrt{2}}\right)^{|\alpha|} \bar{z}^\alpha e^{-\frac{1}{4}|z|^2}. \end{aligned} \tag{37}$$

In view of (16) and (37), we have

$$\|e^{itH} f\|_{M^{p,p}}^p = \frac{1}{(2\pi)^{d/2}} \int_{\mathbb{C}^d} \left| \sum_{\alpha\in\mathbb{N}^d} e^{2it|\alpha|}\langle f, \Phi_\alpha\rangle (\alpha!)^{-1/2} \left(\frac{i}{\sqrt{2}}\right)^{|\alpha|} \bar{z}^\alpha e^{-\frac{1}{4}|z|^2} \right|^p dz. \tag{38}$$

Using polar coordinates as above (see (20)), we have

$$\begin{aligned}&\int_{\mathbb{C}^d} \left| \sum_{\alpha\in\mathbb{N}^d} e^{2it|\alpha|}\langle f, \Phi_\alpha\rangle (\alpha!)^{-1/2} \left(\frac{i}{\sqrt{2}}\right)^{|\alpha|} \bar{z}^\alpha e^{-\frac{1}{4}|z|^2} \right|^p dz \\ &= \prod_{j=1}^d \int_{\mathbb{R}+} \int_{[0,2\pi]} \left| \sum_{\alpha\in\mathbb{N}^d} \langle f, \Phi_\alpha\rangle (\alpha!)^{-1/2} \left(\frac{i}{\sqrt{2}}\right)^{|\alpha|} r^\alpha e^{i\sum_{j=1}^d (2t-\theta_j)\alpha_j} e^{-\frac{1}{4}|r|^2} \right|^p r_j dr_j d\theta_j. \end{aligned} \tag{39}$$

By a simple change of variable $(\theta_j - 2t) \to \theta_j$, we obtain

$$\begin{aligned}&\prod_{j=1}^d \int_{\mathbb{R}+} \int_{[0,2\pi]} \left| \sum_{\alpha\in\mathbb{N}^d} \langle f, \Phi_\alpha\rangle (\alpha!)^{-1/2} \left(\frac{i}{\sqrt{2}}\right)^{|\alpha|} r^\alpha e^{i\sum_{j=1}^d (2t-\theta_j)\alpha_j} e^{-\frac{1}{4}|r|^2} \right|^p r_j dr_j d\theta_j \\ &= \prod_{j=1}^d \int_{\mathbb{R}+} \int_{[0,2\pi]} \left| \sum_{\alpha\in\mathbb{N}^d} \langle f, \Phi_\alpha\rangle (\alpha!)^{-1/2} \left(\frac{i}{\sqrt{2}}\right)^{|\alpha|} r^\alpha e^{-i\theta\cdot\alpha} e^{-\frac{1}{4}|r|^2} \right|^p r_j dr_j d\theta_j. \end{aligned} \tag{40}$$

Combining (38), (39), (40), and Lemma 1(4), we have $\|e^{itH} f\|_{M^{p,p}} = \|f\|_{M^{p,p}}$ for $f \in M^{p,p}(\mathbb{R}^d)$. This completes the proof of Theorem 5.

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